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COMMUTATION AND NORMAL ORDERING FOR OPERATORS ON SYMMETRIC FUNCTIONS

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Dedicated to Ira Gessel on the occasion of his retirement

Abstract. We study the commutation relations and normal ordering between families of operators on symmetric functions. These operators can be naturally defined by the operations of multiplication, Kronecker product, and their adjoints. As applications, we give a new proof of the skew Littlewood–Richardson rule and prove an identity about the Kronecker product with a skew Schur function.

1. Introduction

Due to their connection to representation theory, Schubert calculus, and their beautiful combinatorial description, Schur functions are ubiquitous. For this reason, it is not surprising that identities involving Schur functions greatly improve the understanding of other subjects.

There are two important products defined on symmetric functions: the ordinary and the Kronecker product. The well-known Littlewood–Richardson coefficients are the structure coefficients for the ordinary product, while the elusive Kronecker coefficients are the structure coefficients for the Kronecker product. They naturally define linear operators on symmetric functions.

Let f be a symmetric function. Then, define U_f to be the operator "multiplication by f", and K_f be "Kronecker multiplication by f." Explicitly, the operators act on a symmetric function g by

$$U_f(g) = fg, \qquad\qquad K_f(g) = f * g.$$

The Hall inner product allows us to define the adjoint of U_f, which is denoted by D_f, and sometimes called the skewing operator. With respect to this inner product, K_f is self-adjoint. We will also consider another intriguing operator related to the Kronecker product, $\overline{K}_\lambda$, defined on the Schur basis

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as follows: let λ be a partition and g any homogeneous symmetric function of degree n. Then

$$\overline{K}_\lambda(g) = s_{(n-|\lambda|,\lambda)} * g\,,$$

where $(n-|\lambda|,\lambda) := (n-|\lambda|,\lambda_1,\lambda_2,\ldots)$. This sequence of integers is not always decreasing. To deal with this issue, we define the Schur function $s_{(n-|\lambda|,\lambda)}$ by means of the Jacobi–Trudi formula:

$$s_{(\alpha_1,\alpha_2,\ldots,\alpha_N)} = \det(h_{\alpha_i+j-i})_{i,j=1\ldots N}.$$

This determinant coincides with the Schur function s_α when α is weakly decreasing (i.e., is a partition) but makes sense even when α is not. Using linearity, we can define $\overline{K}_f$ for any symmetric function f.

In this paper, we study identities involving four families of operators on the ring of symmetric functions, *Sym*: U_λ, D_λ, K_λ, and $\overline{K}_\lambda$. We seek to address the following questions: what are the commutation relations between these operators? Given a word involving these operators, how can we put it in a normal form? Is this expression unique? Is it possible to express some of these operators in terms of the other ones?

For a motivating example, let us look at the operators $U_{(1)}$ and $D_{(1)}$. They are well-known to satisfy the commutation relation $D_{(1)}U_{(1)} = U_{(1)}D_{(1)} + 1$. That is Leibniz's rule for multiplication, when $U_{(1)}$ is multiplication by x, and $D_{(1)} = \frac{\partial}{\partial x}$. This identity is the defining relation for the algebra of Weyl, and the building identity for Stanley's theory of differential posets [16].

Our first result is the following theorem that gives beautiful commutation relations for the operators.

Theorem 1.1. *For any partitions α and β we have the following identities (where λ, τ and ν each run over the set of all partitions).*

$$D_\beta U_\alpha = \sum_\lambda U_{\alpha/\lambda} D_{\beta/\lambda} \tag{1.1}$$

$$U_\alpha D_\beta = \sum_\lambda (-1)^{|\lambda|} D_{\beta/\lambda'} U_{\alpha/\lambda} \tag{1.2}$$

$$K_\beta U_\alpha = \sum_\lambda U_{s_{\beta/\lambda} * s_\alpha} K_\lambda \tag{1.3}$$

$$D_\alpha K_\beta = \sum_\lambda K_\lambda D_{s_{\beta/\lambda} * s_\alpha} \tag{1.4}$$

$$\overline{K}_\beta U_\alpha = \sum_{\tau,\nu} U_{(s_{\beta/\nu} * s_\tau) s_{\alpha/\tau}} \overline{K}_\nu \tag{1.5}$$

$$D_\alpha \overline{K}_\beta = \sum_{\tau,\nu} \overline{K}_\nu D_{(s_{\beta/\nu} * s_\tau) s_{\alpha/\tau}} \tag{1.6}$$

The operators K_μ and $\overline{K}_\nu$ commute. To the best of our knowledge, the commutation relations for the pairs of operators (U, K), $(U, \overline{K})$, (K, D), and $(\overline{K}, D)$ are new results. Section 2.7 discusses the historical context of the results stated in Theorem 1.1.

With this paper, we aim to show the important role that generating functions can play, providing unified tools to deal with symmetric functions. Towards this end, we associate a generating series with each of our operators. This allows us to obtain a uniform and elegant method to derive our results.

More precisely, let P be any of the operators U, D, K and $\overline{K}$. We associate with P a formal series of operators $\sum_\lambda s_\lambda[A]P_\lambda$. We call this series the Schur generating series of P. The Schur generating series of P defines a linear map that sends any symmetric function $g \in Sym(X)$ to the expression $\sum_\lambda s_\lambda[A]P_\lambda(g)$. Then, we first study the existing relations between the Schur generating functions obtained in this way. For example, to derive our first expression we show that $D_{\sigma[BX]}U_{\sigma[AX]} = \sigma[AB]U_{\sigma[AX]}D_{\sigma[BX]}$.

Unexpectedly, given the common nature of these results, and of their proofs, the different pairs of operators behave quite differently with regards to the question of uniqueness. In Proposition 2.6, we show that finite expansions with respect to the ordered pairs of operators (U, D), (D, U), (U, K), (K, D), $(U, \overline{K})$ and $(\overline{K}, D)$ are unique. In contrast, observe that expansions with respect to (K, U) or to (D, K) are not unique. For instance we have the relation $K_{p_2}U_{p_1} = 0$, where p_i denotes the power sum symmetric function $\sum_j x_j^i$, which is equivalent to the relation $K_2U_1 = K_{1,1}U_1$. Taking adjoints, we have the relation $D_1K_2 = D_1K_{1,1}$. An open problem is to determine what happens for the remaining pairs of operators: are the corresponding expansions unique?

The charming identity $\overline{K}_{(1)} = U_{(1)}D_{(1)} - 1$ describes a relation between the operators $\overline{K}, U$, and D. In the second part of this work, we will see how it translates to one of the few known cases where it is possible to give a combinatorial description for a Kronecker product. Note that this identity shows that $\overline{K}_{(1)}$ can be rewritten in terms of the much simpler operators $U_{(1)}$ and $D_{(1)}$. In Proposition 2.10 we vastly generalize this observation, and show that for any symmetric function f the operator $\overline{K}_f$ lies in the subalgebra of $\mathrm{End}(Sym)$ generated by the operators U_g and D_g.

Dealing with naturally defined objects, like our families of operators, one is bound to recover some classical results. A testimony of the elegance of this approach is that both Foulkes' (Eq. (2.22)) and Littlewood's identities (Eq. (2.23)) can be easily derived from Theorem 2.11. However, a new identity of the same nature (Eq. (2.24)) relating the skewing and the Kronecker operators is also obtained, as described in the following table that shows how the identities fit in relation to the different products and coproducts of symmetric functions. This is discussed in Section 2.7.

Product	Coproduct	equivalent identities
Ordinary, $\cdot$	Adjoint of the ordinary product	(1.1) and (2.22)
Kronecker, $*$	Adjoint of the ordinary product	(1.3) and (2.23)
Ordinary, $\cdot$	Adjoint of the Kronecker product	(1.4) and (2.24)

Table 1.1. The three bialgebra structures on *Sym* and the corresponding identities $\partial_x \circ \mu = \mu \circ \partial_{\Delta(x)}$.

The second part of our work is of combinatorial nature. Written for the reader with a combinatorial mind, it explores some applications of our results. The first application is a new proof of the *skew Littlewood–Richardson rule*, a combinatorial rule that gives the product of two skew Schur functions as a linear combination of skew Schur functions, based on counting Young tableaux (Theorem 3.1). This rule was conjectured in [1], and proved in [7]. Our proof relies on the normal ordering relation that decomposes the products $U_\alpha D_\beta$ as linear combinations of products of the form $D_{\beta/\lambda'} U_{\alpha/\lambda}$. It generalizes the algebraic proof given by Thomas Lam of the skew Pieri rule (a particular case of the skew Littlewood–Richardson rule) in the appendix of [1]. Indeed, Lam's proof relies on the same normal ordering relation, in the special case of β having only one part.

The second application exploits our normal ordering relation for the products $\overline{K}_1 D_\lambda$. We extend the combinatorial rule for the expansion in the Schur basis of the Kronecker product of $s_{(n-1,1)}$ with a Schur function to the Kronecker product of $s_{(n-1,1)}$ with any skew Schur function (Theorem 3.2). This might be the only known instance of a combinatorial rule for the Kronecker product of a Schur function with a skew Schur function. We also give a combinatorial proof of this result.

2. Part I: Algebraic results

In this section, we prove all six identities of Theorem 1.1. We begin with an overview of the method of proof. Let P be any of the families of operators U, D, K and $\overline{K}$. These operators act on $Sym = Sym(X)$, the symmetric functions in some alphabet X. Introduce an auxiliary alphabet A. The Schur generating series of P is defined as

$$\sum_\lambda s_\lambda[A] P_\lambda.$$

This can be interpreted as the linear map that sends any symmetric function $g \in \mathit{Sym}(X)$ to the expression

$$\sum_{\lambda} s_\lambda[A] P_\lambda(g).$$

For each of the four operators under consideration, the effect of the Schur generating function operator is described nicely by means of operations on alphabets (Lemma 2.2). The identities of Theorem 1.1 are derived at the level of generating series. The result is then recovered by extracting coefficients by means of the appropriate scalar product.

2.1. **Preliminaries: operations on alphabets.** Let $X = \{x_1, x_2, \ldots\}$ be the underlying alphabet for the symmetric functions in *Sym*. Any infinite alphabet A gives rise to a copy $\mathit{Sym}(A)$ of *Sym*. In this copy, the corresponding scalar product will be denoted by $\langle\ |\ \rangle_A$, and the element corresponding to $f \in \mathit{Sym}$, by $f[A]$. Accordingly, the scalar product $\langle\ |\ \rangle$ of *Sym* and elements $f \in \mathit{Sym}$ will be denoted sometimes by $\langle\ |\ \rangle_X$ and $f[X]$.

If A and B are two alphabets, the tensor product $\mathit{Sym}(A) \otimes \mathit{Sym}(B)$ is endowed with the induced scalar product $\langle\ |\ \rangle_{A,B}$.

Both the Kronecker product $*$ and the adjoint D_f of the operator of multiplication by a symmetric function f will be only considered with respect to $\mathit{Sym} = \mathit{Sym}(X)$.

Given an algebra homomorphism A from *Sym* to some commutative algebra $\mathcal{R}$, it will be convenient to write it as $f \mapsto f[A]$ (rather than $f \mapsto A(f)$) for any *algebra homomorphism* and consider it as a "specialization at the virtual alphabet A."

Since the power sum symmetric functions p_k ($k \geqslant 1$) generate *Sym* and are algebraically independent, the map

$$A \mapsto (p_1[A], p_2[A], \ldots) \tag{2.1}$$

is a bijection from the set of all algebra homomorphisms from *Sym* to $\mathcal{R}$ to the set of infinite sequences of elements from $\mathcal{R}$. This set of sequences is endowed with its operations of component-wise sum and product, and multiplication by a scalar. The bijection (2.1) is used to lift these operations to the set of homomorphisms from *Sym* to $\mathcal{R}$. This defines expressions like $f[A+B]$ and $f[AB]$, where f is a symmetric function and A and B are two "virtual alphabets," and more general expressions $f[P(A, B, \ldots)]$, where $P(A, B, \ldots)$ is a polynomial in several virtual alphabets A, B ... with coefficients in the base field. Note that, by definition, for any power sum p_k ($k \geqslant 1$), virtual alphabets A and B, and scalar z,

$$p_k[A+B] = p_k[A] + p_k[B], \quad p_k[AB] = p_k[A] \cdot p_k[B], \quad p_k[zA] = z\, p_k[A].$$

In our calculations below, the homomorphism $f \mapsto f[1]$ will appear: it is the specialization at $x_1 = 1$, $x_2 = 0$, $x_3 = 0 \ldots$, sending each p_k to 1. The homomorphism $f \mapsto f[X]$ is just the identity of *Sym*. The homomorphism $f \mapsto f[X^\perp] = f^\perp = D_f$ associates to f the adjoint of the operator U_f.

Let σ be the generating series for the complete homogeneous symmetric functions h_n, meaning $\sigma = \sum_{n=0}^{\infty} h_n$, where $h_0 = 1$. Recall from [9, I.§2] that we also have

$$\sigma = \exp\left(\sum_{k=1}^{\infty} \frac{p_k}{k}\right) = \sum_{\lambda} \frac{p_\lambda}{z_\lambda},$$

where the last sum is over all partitions λ. We will make use of the following well–known identities.

Lemma 2.1. *Let A and B be two alphabets, and f and g be two symmetric functions. Then we have the following identities:*

$$\sigma[A+B] = \sigma[A]\sigma[B] \quad , \tag{2.2}$$

$$\sigma[AB] = \sum_{\lambda} s_\lambda[A]s_\lambda[B], \qquad \textit{(Cauchy identity)} \tag{2.3}$$

$$\sigma[-AB] = \sum_{\lambda} (-1)^{|\lambda|} s_{\lambda'}[A]s_\lambda[B], \tag{2.4}$$

$$D_{\sigma[AX]}(f[X]) = f[X+A], \tag{2.5}$$

$$\sigma[AX] * f[X] = f[AX], \tag{2.6}$$

$$\langle\, f[AX] \,|\, g[X] \,\rangle_X = (f * g)[A], \tag{2.7}$$

$$\langle\, \sigma[AB] \,|\, g[B] \,\rangle_B = g[A]. \qquad \textit{(Reproducing Kernel)} \tag{2.8}$$

2.2. **Generating series.** Let P be any of the families of operators U, D, K and $\overline{K}$, and let A be an auxiliary alphabet. We introduce the generating series for P as

$$\sum_{\lambda} s_\lambda[A]P_\lambda.$$

Using the linearity of $f \mapsto P_f$ and the Cauchy identity (2.3), we may simplify this expression as follows:

$$\sum_{\lambda} s_\lambda[A]P_\lambda = \sum_{\lambda} s_\lambda[A]P_{s_\lambda[X]} = P_{\sum_\lambda s_\lambda[A]s_\lambda[X]} = P_{\sigma[AX]}.$$

Note that any operator P_λ can be recovered from the generating series by a coefficient extraction using a scalar product:

$$P_\lambda(f) = \left\langle\, P_{\sigma[AX]}(f[X]) \,\middle|\, s_\lambda[A] \,\right\rangle_A .$$

The generating series $P_{\sigma[AX]}$ also acts linearly on symmetric functions. The following lemma describes the effect of all four generating series $U_{\sigma[AX]}$, $D_{\sigma[AX]}$, $K_{\sigma[AX]}$ and $\overline{K}_{\sigma[AX]}$.

Lemma 2.2. *Let $f[X]$ be any symmetric function. Then*

$$U_{\sigma[AX]}(f[X]) = \sigma[AX] \cdot f[X], \tag{2.9}$$

$$D_{\sigma[AX]}(f[X]) = f[X + A], \tag{2.10}$$

$$K_{\sigma[AX]}(f[X]) = f[AX], \tag{2.11}$$

$$\overline{K}_{\sigma[AX]}(f[X]) = \sigma[-A] \cdot f[X(A + 1)]. \tag{2.12}$$

Proof. Equation (2.9) follows by definition. Equation (2.10) is (2.5). Equation (2.11) is (2.6). We prove (2.12). For any symmetric functions f and g, $\overline{K}_f(g) = \Gamma_1 f * g$, where Γ_1 is the vertex operator

$$\Gamma_1 = \left(\sum_{i=0}^{\infty} U_{(i)}\right)\left(\sum_{j=0}^{\infty}(-1)^j D_{(1^j)}\right). \tag{2.13}$$

We will make use of the following identity (see [13, §3]; for a combinatorial approach to this identity see [11]):

$$\Gamma_1 f = \sigma[X]f[X - 1]. \tag{2.14}$$

Therefore, we have

$$\begin{aligned}
\overline{K}_{\sigma[AX]}(f) &= (\Gamma_1\, \sigma[AX]) * f[X] \\
&= \sigma[X + A(X - 1)] * f[X] && \text{by (2.2),} \\
&= \sigma[-A]\ \ (\sigma[X(A + 1)] * f[X]) && \text{by (2.2),} \\
&= \sigma[-A] \cdot f[X(A + 1)] && \text{by (2.6).}
\end{aligned}$$

□

2.3. **Operators U and D.** We now prove (1.1) and (1.2) of Theorem 1.1.

The following lemma establishes the commutation relation between U and D.

Lemma 2.3. *Let A and B be two alphabets. We have*

$$D_{\sigma[BX]}U_{\sigma[AX]} = \sigma[AB]U_{\sigma[AX]}D_{\sigma[BX]}, \tag{2.15}$$

$$U_{\sigma[AX]}D_{\sigma[BX]} = \sigma[-AB]D_{\sigma[BX]}U_{\sigma[AX]}. \tag{2.16}$$

Proof. Notice that (2.16) follows from (2.15) and the fact that $\sigma[-AB]$ is the inverse of $\sigma[AB]$ (see (2.2)). We prove (2.15) using Lemmas 2.1 and 2.2. For any symmetric function $f[X]$, we have

$$\begin{aligned}
D_{\sigma[BX]}U_{\sigma[AX]}(f[X]) &= \sigma[A(X + B)]f[X + B] && \text{by (2.10),} \\
&= \sigma[AB]\sigma[AX]f[X + B] && \text{by (2.2),} \\
&= \sigma[AB]\sigma[AX]D_{\sigma[BX]}(f[X]) && \text{by (2.10),} \\
&= \sigma[AB]U_{\sigma[AX]}D_{\sigma[BX]}(f[X]).
\end{aligned}$$

□

Proof of (1.1). We use that, since U and $D : Sym \to \mathrm{End}(Sym)$ are algebra homomorphisms, for any symmetric function f we may write that $U_f = f[U]$ and $D_f = f[D]$. In particular, for the generating series we have $U_{\sigma[AX]} = \sigma[AU]$ and $D_{\sigma[BX]} = \sigma[BD]$.

In (2.15), the operator $D_\beta U_\alpha$ is the coefficient of $s_\alpha[A]s_\beta[B]$ in the expansion in the Schur basis of $\sigma[AB]U_{\sigma[AX]}D_{\sigma[BX]}$, which is extracted by performing the scalar product with $s_\alpha[A]s_\beta[B]$. Thus,

$$\begin{aligned}
D_\beta U_\alpha &= \left\langle \sigma[AB]U_{\sigma[AX]}D_{\sigma[BX]} \,\middle|\, s_\alpha[A]s_\beta[B] \right\rangle_{A,B} \\
&= \sum_\lambda \left\langle s_\lambda[A]s_\lambda[B]U_{\sigma[AX]}D_{\sigma[BX]} \,\middle|\, s_\alpha[A]s_\beta[B] \right\rangle_{A,B} && \text{by (2.3),} \\
&= \sum_\lambda \left\langle \sigma[AU] \,\middle|\, s_{\alpha/\lambda}[A] \right\rangle_A \left\langle \sigma[BD] \,\middle|\, s_{\beta/\lambda}[B] \right\rangle_B \\
&= \sum_\lambda s_{\alpha/\lambda}[U]s_{\beta/\lambda}[D] = \sum_\lambda U_{\alpha/\lambda}D_{\beta/\lambda}. && \square
\end{aligned}$$

Identity (1.2) is derived from (2.16) analogously.

2.4. **Operators U and K.**

We now turn to the proof of (1.3). Identity (1.4) follows from (1.3) by taking adjoints.

The following lemma gives the commutation relations for the generating series of K and U.

Lemma 2.4. *Let A and B be two alphabets. We have*

$$K_{\sigma[BX]}U_{\sigma[AX]} = U_{\sigma[ABX]}K_{\sigma[BX]}. \tag{2.17}$$

Proof. For any symmetric function f, we have

$$\begin{aligned}
K_{\sigma[BX]}U_{\sigma[AX]}(f[X]) &= K_{\sigma[BX]}(\sigma[AX]f[X]) \\
&= \sigma[ABX]f[BX] && \text{by (2.11),} \\
&= U_{\sigma[ABX]}K_{\sigma[BX]}(f[X]) && \text{by (2.11).} \quad \square
\end{aligned}$$

Proof of (1.3). We get $K_\beta U_\alpha$ from (2.17) by extracting the coefficient of $s_\beta[B]s_\alpha[A]$ in its expansion in terms of Schur functions:

$$\begin{aligned}
K_\beta U_\alpha &= \left\langle U_{\sigma[ABX]}K_{\sigma[BX]} \,\middle|\, s_\beta[B]s_\alpha[A] \right\rangle_{A,B} \\
&= \sum_\lambda \left\langle U_{\sigma[ABX]}s_\lambda[B] \,\middle|\, s_\beta[B]s_\alpha[A] \right\rangle_{A,B} K_\lambda \\
&= \sum_\lambda \left\langle s_\alpha[BU] \,\middle|\, s_{\beta/\lambda}[B] \right\rangle_B K_\lambda && \text{by (2.8),} \\
&= \sum_\lambda (s_\alpha * s_{\beta/\lambda})[U]K_\lambda && \text{by (2.7),}
\end{aligned}$$

$$= \sum_{\lambda} U_{s_\alpha * s_{\beta/\lambda}} K_\lambda. \qquad \square$$

2.5. **Operators U and $\overline{K}$.** We now proceed to proving (1.5). Identity (1.6) is deduced by taking adjoints.

Again, we compute commutation relations for the generating series of the families of operators involved.

Lemma 2.5. *Let A and B be two alphabets. We have*

$$\overline{K}_{\sigma[BX]} U_{\sigma[AX]} = U_{\sigma[A(B+1)X]} \overline{K}_{\sigma[BX]}. \tag{2.18}$$

Proof. For any symmetric function f, we have

$$\begin{aligned} \overline{K}_{\sigma[BX]} U_{\sigma[AX]}(f) &= \overline{K}_{\sigma[BX]}(\sigma[AX] f[X]) \\ &= \sigma[-B]\sigma[AX(B+1)] f[X(B+1)] && \text{by (2.12)}, \\ &= \sigma[AX(B+1)] \overline{K}_{\sigma[BX]}(f) && \text{by (2.12)}, \\ &= U_{\sigma[A(B+1)X]} \overline{K}_{\sigma[BX]}(f). \end{aligned} \qquad \square$$

Proof of (1.5). From (2.18) we extract the term $\overline{K}_\beta U_\alpha$ by taking the scalar product with $s_\alpha[A]\, s_\beta[B]$. This yields

$$\overline{K}_\beta U_\alpha = \left\langle U_{\sigma[A(B+1)X]} \overline{K}_{\sigma[BX]} \,\middle|\, s_\alpha[A] s_\beta[B] \right\rangle_{A,B}.$$

Expanding the generating function $\overline{K}_{\sigma[BX]}$ in the scalar product and simplifying, we get

$$\begin{aligned} \overline{K}_\beta U_\alpha &= \sum_\nu \left\langle U_{\sigma[A(B+1)X]} s_\nu[B] \,\middle|\, s_\alpha[A] s_\beta[B] \right\rangle_{A,B} \overline{K}_\nu, \\ &= \sum_\nu \left\langle \left\langle U_{\sigma[A(B+1)X]} \,\middle|\, s_{\alpha[A]} \right\rangle_A s_\nu[B] \,\middle|\, s_\beta[B] \right\rangle_B \overline{K}_\nu \\ &= \sum_\nu \left\langle s_{\alpha[(B+1)U]} s_\nu[B] \,\middle|\, s_\beta[B] \right\rangle_B \overline{K}_\nu && \text{by (2.8)}, \\ &= \sum_{\nu,\tau} \left\langle s_\tau[BU] \,\middle|\, s_{\beta/\nu}[B] \right\rangle_B s_{\alpha/\tau}[U] \overline{K}_\nu \\ &= \sum_{\nu,\tau} (s_\tau * s_{\beta/\nu})[U] s_{\alpha/\tau}[U] \overline{K}_\nu && \text{by (2.7)}, \\ &= \sum_{\nu,\tau} U_{(s_\tau * s_{\beta/\nu}) s_{\alpha/\tau}} \overline{K}_\nu\,. \end{aligned} \qquad \square$$

In Section 3.5 we present as an application a combinatorial rule for the Kronecker product of any skew Schur function by $s_{(n-1,1)}$. Other interesting particular cases of (1.5) correspond to the cases when $\lambda = (k)$ (Kronecker product with a two-row shape) and $\lambda = (1^k)$ (Kronecker product with a hook), where we get

$$KB(k)U_\alpha = \sum_{j=0}^{k}\left(\sum_{\rho\vdash k-j} U_{\alpha/\rho}U_\rho\right)\overline{K}_{(j)},$$
$$\overline{K}_{(1^k)}U_\alpha = \sum_{j=0}^{k}\left(\sum_{\rho\vdash k-j} U_{\alpha/\rho}U_{\rho'}\right)\overline{K}_{(1^j)}.$$

Setting $n = |\alpha|$ and $m = |\gamma|$, the same identities can be restated as

$$s_{(n+m-k,k)} * (s_\alpha s_\gamma) = \sum_{j=0}^{k}\left(\sum_{\rho\vdash k-j} s_{\alpha/\rho}s_\rho\right)(s_\gamma * s_{(m-j,j)}),$$
$$s_{(n+m-k,1^k)} * (s_\alpha s_\gamma) = \sum_{j=0}^{k}\left(\sum_{\rho\vdash k-j} s_{\alpha/\rho}s_{\rho'}\right)(s_\gamma * s_{(m-j,1^j)}).$$

The terms of the form $\sum_{\rho\vdash q} s_{\alpha/\rho}s_\rho$ and $\sum_{\rho\vdash q} s_{\alpha/\rho}s_{\rho'}$ in these identities can be rewritten using Littlewood's identity (2.21) as follows:

$$\sum_{\rho\vdash q} s_{\alpha/\rho}s_\rho = s_\alpha * h_{(n-q,q)}, \qquad \text{and likewise} \qquad \sum_{\rho\vdash q} s_{\alpha/\rho}s_{\rho'} = s_\alpha * (h_{n-q}e_q).$$

2.6. **Uniqueness of expansions.** The identities appearing in Theorem 1.1 and Corollary 2.11 express some operators as linear combinations of operators $U_\mu D_\nu$, $D_\nu U_\mu$, $U_\mu K_\nu$, etc. Are such expressions unique?

To answer this question, we associate with any pair (P,Q) of linear maps from *Sym* to End(*Sym*) a generating series depending on four independent alphabets X, A, B, T. This generating series is

$$\sum_{\alpha,\beta,\lambda} P_\alpha(Q_\beta(s_\lambda[X]))s_\alpha[A]s_\beta[B]s_\lambda[T] = P_{\sigma[AX]}Q_{\sigma[BX]}(\sigma[XT]).$$

We also associate with the pair (P,Q) the linear map $\Phi_{P,Q}$ from $Sym(A)\otimes_{\mathbb{Q}} Sym(B)$ to the set of formal series of the form $\sum_{\alpha,\beta} a_{\alpha,\beta}s_\alpha[X]s_\beta[T]$, defined on simple tensors by

$$\Phi_{P,Q}(f[A]g[B]) = \left\langle P_{\sigma[AX]}Q_{\sigma[BX]}(\sigma[XT]) \,\middle|\, f[A]g[B]\right\rangle_{A,B}.$$

Let us say that *finite expansions with respect to* (P,Q) *are unique* if, for any $M \in \mathrm{End}(Sym)$, there is at most one expansion $M = \sum_{\alpha,\beta} a_{\alpha,\beta}P_\alpha Q_\beta$. That is, finite expansions with respect to (P,Q) are unique when the operators $P_\alpha Q_\beta$, for α and β partitions, are linearly independent.

Proposition 2.6. *Finite expansions with respect to the pairs of operators* (U,D), (D,U), (U,K), (K,D), $(U,\overline{K})$ *and* $(\overline{K},D)$ *are unique.*

Example 2.7. In contrast, observe that expansions with respect to (K, U) or to (D, K) are not unique. For instance, we have the relation $K_{p_2}U_{p_1} = 0$, that is straightforwardly equivalent to the relation $K_2U_1 = K_{1,1}U_1$. Taking adjoints, we have the relation $D_1K_2 = D_1K_{1,1}$.

Proposition 2.6 will be proved using the following lemma.

Lemma 2.8. *Let P and Q be linear maps from Sym to* $\mathrm{End}(Sym)$*.*

(1) *Finite expansions with respect to (P, Q) are unique if and only if $\Phi_{P,Q}$ is injective.*
(2) *Let $M \in \mathrm{End}(Sym)$.*
   - *For f, $g \in Sym$, the operator M is in the linear span of the operators P_fQ_g, if and only if $M(\sigma[XT])$ lies in the image of $\Phi_{P,Q}$.*
   - *If $M(\sigma[XT]) = \Phi_{P,Q}(F)$ then M is the image of F under the linear map defined on the Schur basis by $s_\alpha[A]s_\beta[B] \mapsto P_\alpha Q_\beta$.*

Proof. We start with a computation: let $M \in \mathrm{End}(Sym)$. We have that $M = \sum_{\alpha,\beta} a_{\alpha,\beta}P_\alpha Q_\beta$ if and only if for any partition λ,

$$\begin{aligned} &M(s_\lambda[X]) = \sum_{\alpha,\beta} a_{\alpha,\beta}P_\alpha Q_\beta(s_\lambda[X]) \\ \Leftrightarrow &\sum_\lambda M(s_\lambda[X])s_\lambda[T] = \sum_{\alpha,\beta,\lambda} a_{\alpha,\beta}P_\alpha Q_\beta(s_\lambda[X])s_\lambda[T] \\ \Leftrightarrow &M(\sigma[XT]) = \sum_{\alpha,\beta} a_{\alpha,\beta}P_\alpha Q_\beta(\sigma[XT]) \\ \Leftrightarrow &M(\sigma[XT]) = \Phi_{P,Q}\left(\sum_{\alpha,\beta} a_{\alpha,\beta}s_\alpha[A]s_\beta[B]\right). \end{aligned}$$

This proves (2), since the linear span of the operators P_fQ_g, for f, $g \in Sym$, is also the linear span of the operators $P_\alpha Q_\beta$, for α and β partitions.

To obtain (1), take $M = 0$ in the above equivalence. ☐

Proof of Proposition 2.6. For each of the pairs (U, D), (D, U), (U, K) and $(U, \overline{K})$, we use Lemma 2.2 in order to compute the corresponding generating series. Next we deduce a description of the corresponding map Φ to show that it is injective. The uniqueness of finite expansions follows then from (1) in Lemma 2.8.

For (U, D), the details of the calculation are as follows. The generating series is

$$U_{\sigma[AX]}D_{\sigma[BX]}(\sigma[XT]) = U_{\sigma[AX]}(\sigma[(X + B)T] = \sigma[AX + XT + BT].$$

We deduce from this a formula for $\Phi_{U,D}$. Let f and g be two symmetric functions. We have

$$\begin{aligned}\Phi_{U,D}(f[A]g[B]) &= \langle\, \sigma[AX+XT+BT] \,|\, f[A]g[B]\,\rangle_{A,B}\\ &= \sigma[XT]\langle\, \sigma[AX]\sigma[BT] \,|\, f[A]g[B]\,\rangle_{A,B}\\ &= \sigma[XT]f[X]g[T] && \text{by (2.8).}\end{aligned}$$

This shows that $\Phi_{U,D}$ is injective, since the series $\sigma[XT]$ is invertible. For the other three pairs, we skip the details of the calculations.

The generating series for (D,U) is $D_{\sigma[BX]}U_{\sigma[AX]}(\sigma[XT]) = \sigma[(A+T)(B+X)]$. The map $\Phi_{D,U}$ is given by $f[A]g[B] \mapsto \sigma[XT]\sigma[X^{\perp}T^{\perp}](f[X]gT])$.

The map $\Phi_{D,U}$ is injective, since the series $\sigma[XT]$ is invertible, and the operator $\sigma[X^{\perp}T^{\perp}]$ is invertible as well (its inverse is $\sigma[-X^{\perp}T^{\perp}]$).

The generating series for (U,K) is given by $U_{\sigma[AX]}K_{\sigma[BX]}(\sigma[XT]) = \sigma[B(X+A)T]$. The map $\Phi_{U,K}$ is $f[A]g[B] \mapsto f[X]g[XT]$. To check that $\Phi_{U,K}$ is injective, post–compose it with the specialization of T at T/X: the map obtained is $f[A]g[B] \mapsto f[X]g[T]$, which is injective. Thus $\Phi_{U,K}$ is injective.

The generating series for $(U,\overline{K})$ is $U_{\sigma[AX]}\overline{K}_{\sigma[BX]}(\sigma[XT]) = \sigma[AX + B(XT-1) + XT]$. The map $\Phi_{U,\overline{K}}$ is $f[A]g[B] \mapsto \sigma[XT]f[X]g[XT-1]$. This map $\Phi_{U,\overline{K}}$ is injective. Indeed, post–composing first with the product with the inverse of $\sigma[XT]$, and next by the specialization of T at $(T+1)/X$, we obtain the map $f[A]g[B] \mapsto f[X]g[T]$, which is injective.

This proves the uniqueness of finite expansions with respect to (U,D), (D,U), (U,K) and $(U,\overline{K})$. The uniqueness of finite expansions with respect to (K,D) and $(\overline{K},D)$ is obtained by taking adjoints. $\square$

Remark 2.9. The uniqueness of finite expansions with respect to (U,D) and (D,U) can alternatively be proved by switching to the basis of power sums. The algebra generated by the operators U_g and D_g is also generated by 1, the U_{p_k} and the D_{p_k} $(k>0)$. The following maps define an isomorphism between this algebra and, firstly, the bosonic creation and annihilation operator algebra (this appears for instance in [6]) and, secondly, the Weyl algebra in infinitely many generators.

$$U_{p_k} \mapsto a_k^{\dagger} \mapsto \widehat{x_k}\,, \qquad\qquad D_{p_k} \mapsto k\ a_k \mapsto k\,\frac{\partial}{\partial x_k}\,,$$

where the a_k are the creation operators, and the $a_k^{\dagger}$ are the annihilation operators. It is well-known that, in the bosonic creation and annihilation operator algebra, the monomials in normal order $(a_1^{\dagger})^{m_1}(a_2^{\dagger})^{m_2}\cdots a_1^{n_1}a_2^{n_2}\cdots$ as well as the monomials in antinormal order $a_1^{n_1}a_2^{n_2}\cdots(a_1^{\dagger})^{m_1}(a_2^{\dagger})^{m_2}\cdots$ are linearly independent. This shows that the operators $U_{p_\lambda}D_{p_\mu}$ are linearly independent, and so are the operators $D_{p_\mu}U_{p_\lambda}$. From this, one deduces that finite expansions with respect to (U, D) and with respect to (D, U) are unique.

We finish this section with an expansion of the operators $\overline{K}_f$ in terms of operators U_g and D_g. From (2) in Lemma 2.8, we get the following result.

Proposition 2.10. *Let f be a symmetric function. The operator $\overline{K}_f$ lies in the subalgebra of* End(*Sym*) *generated by the operators U_g and D_g (for $g \in Sym$). More precisely,*

$$\overline{K}_f = \sum_{\lambda} U_{f[X-1]*s_\lambda} D_\lambda.$$

Proof. The subalgebra of End(*Sym*) generated by the operators U_g and D_g, for $g \in Sym$, is the linear span of the operators $U_\alpha D_\beta$, for α and β partitions. According to (2) in Lemma 2.8, and the calculations in the proof of Proposition 2.6, it is the set of operators M such that $M(\sigma[XT])$ is $\sigma[XT]$ times an element of $Sym(X) \otimes_{\mathbb{Q}} Sym(T)$.

According to (2.14), $\overline{K}_f(\sigma[XT]) = \sigma[X]f[X-1] * \sigma[XT]$, which is equal, according to (2.6), to $\sigma[XT]f[XT-1]$. This proves Proposition 2.10.

To get an explicit decomposition of $\overline{K}_f$, we decompose $f[XT-1]$ as an element of $Sym(X) \otimes_{\mathbb{Q}} Sym(T)$. We start with $f[XT-1] = f[X-1] * \sigma[XT]$. (The Kronecker product $*$ is relative to the symmetric functions in X). Thus,

$$f[XT-1] = f[X-1] * \sum_{\lambda} s_\lambda[X] s_\lambda[T] = \sum_{\lambda} f[X-1] * s_\lambda[X] s_\lambda[T].$$

Therefore, $\overline{K}_f(\sigma[XT]) = \Phi_{U,D}\left(\sum_\lambda f[A-1] * s_\lambda[A] s_\lambda[B]\right)$. We conclude the proof by applying the second part of (2) in Lemma 2.8. ☐

For instance, for $f = h_k$ we have: $\overline{K}_{(k)} = \sum_{\lambda \vdash k} U_\lambda D_\lambda - \sum_{\lambda \vdash k-1} U_\lambda D_\lambda$, since $h_k[X-1] = h_k - h_{k-1}$. The simplest case is $k = 1$. Here we obtain

$$\overline{K}_{(1)} = U_{(1)} D_{(1)} - 1\,. \tag{2.19}$$

By the Pieri rule, for any partition α,

$$U_{(1)} D_{(1)} s_\alpha = \sum_{\beta} s_\beta,$$

where each term in the sum corresponds to a choice of a corner in the diagram of α that is removed, and then a choice of a box that is added, to give the diagram of a partition β. There are two cases: the box can be added where the corner was removed, or not. Accordingly the sum splits: $U_{(1)} D_{(1)} s_\alpha = \#\text{corners}(\alpha) s_\alpha + \sum_{\beta \in \alpha^{\mp}} s_\beta$, where $\alpha^{\mp}$ denotes the set of partitions not equal to α that can be obtained by removing a corner of α and then adding a box to the result. Therefore,

$$\overline{K}_{(1)} s_\alpha = U_{(1)} D_{(1)} s_\alpha - s_\alpha = (\#\mathrm{corners}(\alpha) - 1) s_\alpha + \sum_{\beta \in \alpha^{\mp}} s_\beta \,,$$

which will reappear in (3.6).

2.7. **Historical context.** Theorem 1.1 can be rewritten in an equivalent way using the Littlewood–Richardson coefficients $c^{\alpha}_{\lambda,\mu}$ and the Kronecker coefficients $g_{\alpha,\lambda,\mu}$ (the structural constants for the product and the Kronecker product of Schur functions), see [9].

Theorem 2.11. *For any partitions α and β, we have the following identities:*

$$D_\beta U_\alpha = \sum_{\mu,\nu} \left(\sum_{\lambda} c^{\alpha}_{\lambda,\mu}\, c^{\beta}_{\lambda,\nu} \right) U_\mu D_\nu,$$

$$U_\alpha D_\beta = \sum_{\mu,\nu} \left(\sum_{\lambda} (-1)^{|\lambda|} c^{\alpha}_{\lambda,\mu}\, c^{\beta}_{\lambda',\nu} \right) D_\nu U_\mu,$$

$$K_\beta U_\alpha = \sum_{\mu,\nu} \left(\sum_{\lambda} g_{\alpha,\lambda,\mu}\, c^{\beta}_{\lambda,\nu} \right) U_\mu K_\nu,$$

$$D_\alpha K_\beta = \sum_{\mu,\nu} \left(\sum_{\lambda} g_{\alpha,\lambda,\mu}\, c^{\beta}_{\lambda,\nu} \right) K_\nu D_\mu,$$

$$\overline{K}_\beta U_\alpha = \sum_{\mu,\nu} \left(\sum_{\lambda,\sigma,\tau,\theta} g_{\lambda,\tau,\theta}\, c^{\beta}_{\lambda,\nu}\, c^{\alpha}_{\tau,\sigma}\, c^{\mu}_{\theta,\sigma} \right) U_\mu \overline{K}_\nu,$$

$$D_\alpha \overline{K}_\beta = \sum_{\mu,\nu} \left(\sum_{\lambda,\sigma,\tau,\theta} g_{\lambda,\tau,\theta}\, c^{\beta}_{\lambda,\nu}\, c^{\alpha}_{\tau,\sigma}\, c^{\mu}_{\theta,\sigma} \right) \overline{K}_\nu D_\mu.$$

Identities (1.1) and (1.3) are avatars of well-known identities of Foulkes and Littlewood. Indeed, if we apply the operators in (1.1) and (1.3) to the Schur function s_γ, we get

$$D_\beta(s_\alpha s_\gamma) = \sum_{\lambda} s_{\alpha/\lambda} D_{\beta/\lambda}(s_\gamma), \tag{2.20}$$

$$s_\beta * (s_\alpha s_\gamma) = \sum_{\lambda} (s_{\beta/\lambda} * s_\alpha)(s_\lambda * s_\gamma). \tag{2.21}$$

By linearity, we can replace s_α and s_γ with arbitrary symmetric functions f and g. Moreover, if we expand $s_{\beta/\lambda} = \sum_\mu c^\beta_{\lambda,\mu} s_\mu$, we obtain

$$D_\beta(fg) = \sum_{\lambda,\mu} c^\beta_{\lambda,\mu} D_\lambda(f) D_\mu(g), \tag{2.22}$$

$$s_\beta * (fg) = \sum_{\lambda,\mu} c^\beta_{\lambda,\mu} (s_\mu * f)(s_\lambda * g). \tag{2.23}$$

Formula (2.22) was obtained by Foulkes ([3, §3.b], also mentioned in [9, I.§5, Ex. 25.(d)], while (2.23) is due to Littlewood ([8, Theorem III], see also [9, I.§7, Ex. 23.(c)]).

An expression similar to (2.22) and (2.23) can be derived in the same way from (1.4). This is,

$$D_\beta(f * g) = \sum_\lambda g_{\beta,\lambda,\mu} D_\lambda(f) * D_\mu(g). \tag{2.24}$$

The similarity between (2.22), (2.23) and (2.24) has a nice explanation. It is provided by Thibon for (2.22) and (2.24), see [17, p. 554], [18, Proposition 6.4], and [19], but applies as well to (2.23). The explanation is as follows: let B be a bialgebra with product $\odot$ and coproduct Δ. The coproduct Δ induces a product μ on the dual space B^*. For any $x \in B$ or in $B \otimes B$, let ∂_x be the adjoint of the $\odot$–product by x. Then, for any $x \in B$,

$$\partial_x \circ \mu = \mu \circ \partial_{\Delta(x)}. \tag{2.25}$$

See the aforementioned references by Thibon for a proof.

Consider *Sym* with a product that is either the ordinary product or the Kronecker product, and a coproduct that is either the adjoint of the ordinary product or the adjoint of the ordinary coproduct. This gives four possibilities, but only three of them make *Sym* a bialgebra. The three identities (2.22), (2.23) and (2.24) are obtained by applying (2.25) to these three bialgebra structures, with $x = s_\beta$. See Table 1.1. Note that, for any symmetric function f, the operator K_f is its own adjoint.

In [19], Thibon also relates Identity (2.23) to Mackey's formula in group theory (see [5, 18.15]). Let us mention that Cummins [2, Identity (13)] also derives an identity very close to (1.4).

Formula (1.2) can be restated as

$$s_\alpha s_{\gamma/\beta} = \sum_\lambda (-1)^{|\lambda|} D_{\beta/\lambda'}(s_{\alpha/\lambda} s_\gamma). \tag{2.26}$$

In Section 3.1, we will show that Formula (1.2) happens to be closely related to the *skew Littlewood–Richardson rule*. In [7, Lemma 1.1], Lam, Lauve, and Sottile obtain a more general version of Formula (1.2) valid for arbitrary pairs of dual Hopf algebras.

As mentioned in the introduction, Ira Gessel [4] established special cases of (1.1) and (1.2) when the Schur functions are indexed by one-row or one-column shapes. He showed that

$$\begin{aligned} D_n U_m &= \sum_i U_{m-i} D_{n-i}\,, \\ U_m D_n &= D_n U_m - D_{n-1} U_{m-1}\,, \\ D_{(1^n)} U_m &= U_m D_{(1^n)} + U_{m-1} D_{(1^{n-1})}. \end{aligned}$$

Since K_β and $\overline{K}_\beta$ are self-adjoint, and U_α and D_α are adjoints of each other, (1.4) and (1.6) are obtained from (1.3) and (1.5), respectively, by taking adjoints.

An interesting and elegant way of stating some of the results of Theorem 1.1 is in terms of commutators.

Corollary 2.12. *For any two partitions α and β, we have*

$$\begin{aligned} [D_\beta, U_\alpha] &= \sum_{\lambda \neq \varnothing} U_{\alpha/\lambda} D_{\beta/\lambda} = \sum_{\lambda \neq \varnothing} (-1)^{|\lambda|-1} D_{\beta/\lambda'} U_{\alpha/\lambda}\,, \\ [\overline{K}_\beta, U_\alpha] &= \sum_{(\tau,\nu)\neq(\varnothing,\beta)} U_{(s_{\beta/\nu} * s_\tau) s_{\alpha/\tau}} \overline{K}_\nu\,, \\ [D_\alpha, \overline{K}_\beta] &= \sum_{(\tau,\nu)\neq(\varnothing,\beta)} \overline{K}_\nu D_{(s_{\beta/\nu} * s_\tau) s_{\alpha/\tau}}\,, \end{aligned}$$

where $\varnothing$ denotes the empty partition.

3. Part II: Combinatorial applications

3.1. **Application to the skew Littlewood–Richardson rule.** In this section, we present our first application of Theorem 1.1: a new proof of the skew Littlewood–Richardson rule as conjectured by Assaf and the second author [1] and proved by Lam, Lauve and Sottile [7]. As in [7], our starting point is (1.2). In [7], first an "algebraic skew Littlewood–Richardson rule" is derived, involving sums of products of Littlewood–Richardson coefficients. Then, the combinatorial skew Littlewood–Richardson rule is obtained by interpreting these Littlewood–Richardson coefficients as the number of semistandard Young tableaux with given rectification. Our proof fits more closely with the statement of the skew Littlewood–Richardson rule: we avoid going through the algebraic skew Littlewood–Richardson rule and use the interpretation of the Littlewood–Richardson coefficients as the number of semistandard Young tableaux with certain constraints on their content and reverse reading words.

Our proof appears in Subsections 3.3 and 3.4 and is largely combinatorial.

For a positive integer k and a partition γ, the classical Pieri rule [10] gives a simple and beautiful expression for the product $s_{(k)}s_\gamma$ as a sum of Schur functions. A *k-horizontal* (respectively *k-vertical*) *strip* is a skew shape with k boxes that has at most one box in each column (respectively row). The Pieri rule states that

$$s_{(k)}s_\gamma = \sum_{\widehat{\gamma}} s_{\widehat{\gamma}}\,,$$

where the sum is over all partitions $\widehat{\gamma}$ such that $\widehat{\gamma}/\gamma$ is a k-horizontal strip. In [1], Assaf and the second author generalized the Pieri rule to the setting of skew shapes as follows:

$$s_{(k)}s_{\gamma/\beta} = \sum_{i=0}^{k}(-1)^i \sum_{\widehat{\gamma},\breve{\beta}} s_{\widehat{\gamma}/\breve{\beta}}\,, \tag{3.1}$$

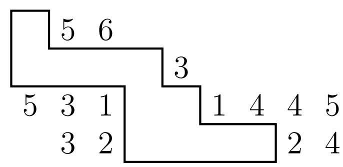
where the sum is over all partitions $\widehat{\gamma}$ and $\breve{\beta}$ such that $\widehat{\gamma}/\gamma$ is a $(k-i)$-horizontal strip, and $\beta/\breve{\beta}$ is an i-vertical strip. We will use the skew Pieri rule with $k = 1$ in Section 3.5.

For the next level of generality, it is natural to ask for a similarly combinatorial expression for $s_\alpha s_{\beta/\gamma}$ for any partition α. Equation (2.26) gives one expression, but it does not mimic (3.1) in the sense that it does not give the answer as a signed sum of skew Schur functions. Instead, the skew Littlewood–Richardson rule [7] gives an expression for the even more general product $s_{\alpha/\delta}s_{\beta/\gamma}$ as a signed sum of skew Schur functions. In this section, we will derive the skew Littlewood–Richardson rule from (1.2) in the following way. In Subsection 3.3, we will use a combinatorial approach to obtain from (1.2) the skew Littlewood–Richardson rule in the case when δ is empty, and then we will use a linearity argument to derive the result for general δ in Subsection 3.4.

3.2. **The combinatorial skew Littlewood–Richardson rule.** In order to state the skew Littlewood–Richardson rule, we first need some terminology. As usual, a sequence of positive integers ω is said to be a *lattice permutation* if any prefix of ω contains at least as many appearances of i as $i+1$, for all $i \geqslant 1$. For a partition δ, we say that ω is a δ-lattice permutation if the word obtained by prefixing ω with δ_1 copies of 1 followed by δ_2 copies of 2, etc., is a lattice permutation.

We draw our Young tableaux in French notation, implying that the entries of an SSYT weakly increase along the rows and strictly increase up the columns. An *anti-semistandard Young tableau* (ASSYT) T_1 of shape α/β is a filling of the boxes of α/β so that the entries strictly decrease along the rows and weakly decrease up the columns. Equivalently, T_1 is an ASSYT if the tableau $(T_1')^r$ obtained by transposing T_1 and then rotating it $180°$ is an

SSYT. The *reverse reading word of an SSYT* T_2 is defined as usual as the word obtained by reading right-to-left along the rows of T_2, taking the rows from bottom to top. In contrast, the *reverse reading word of an ASSYT* T_1 is the word obtained by reading up the columns of T_1, taking the columns from right-to-left. Equivalently, we can take the usual reverse reading word of the SSYT $(T_1')^r$. Given a pair of tableaux (T_1, T_2), where T_1 is an ASSYT and T_2 is an SSYT, we define the *reverse reading word of the pair* as the concatenation of the reading word of T_1 with that of T_2. We will encounter such pairs as in (3.2) below, where the entries in the bottom left form an ASSYT, and the entries above or to the right of the outlined skew shape form an SSYT.



(3.2)

The reverse reading word of (T_1, T_2) shown in (3.2) is 21335425441365, which is certainly not a lattice permutation but is a 5321-lattice permutation.

We are now ready to state the skew Littlewood–Richardson rule.

Theorem 3.1 ([1, Conjecture 6.1]; [7, Theorem 3.2 and Remark 3.3(ii)])**.** *For skew shapes α/δ and γ/β,*

$$s_{\alpha/\delta} s_{\gamma/\beta} = \sum_{\substack{T_1 \in \mathrm{ASSYT}(\beta/\check{\beta}) \\ T_2 \in \mathrm{SSYT}(\widehat{\gamma}/\gamma)}} (-1)^{|\beta/\check{\beta}|} s_{\widehat{\gamma}/\check{\beta}}\,, \tag{3.3}$$

where the sum is over all ASSYT T_1 of shape $\beta/\check{\beta}$ for some $\check{\beta} \subseteq \beta$, and SSYT T_2 of shape $\widehat{\gamma}/\gamma$ for some $\widehat{\gamma} \supseteq \gamma$, with the following properties:

(a) *the combined content of T_1 and T_2 is the component-wise difference $\alpha - \delta$, and*
(b) *the reverse reading word of (T_1, T_2) is a δ-lattice permutation.*

For example, the ASSYT and SSYT pair of (3.2) contribute $-s_{9953/1}$ to the product $s_{755431/5321} s_{7541/33}$. Note that, when β and δ are empty, we recover the classical Littlewood–Richardson rule. Indeed, this case corresponds to counting SSYT whose reverse reading words are lattice permutations; we call such SSYT *LR-fillings*.

3.3. **Recovering a special case of the combinatorial skew Littlewood–Richardson rule.** Our first step to reproving Theorem 3.1 is to start with (1.2) and show that it implies Theorem 3.1 in the case when $\delta = \varnothing$, the empty partition. Instead of (1.2), we work with the equivalent identity in (2.26):

$$s_\alpha s_{\gamma/\beta} = \sum_\lambda (-1)^{|\lambda|} D_{\beta/\lambda'}(s_{\alpha/\lambda} s_\gamma).$$

First, let us examine the product $s_{\alpha/\lambda} s_\gamma$ from the right–hand side, and expand it in terms of Schur functions. Note that only those s_ν with $\nu \supseteq \gamma$ will appear in the Schur expansion with nonzero coefficient. Thus we can write

$$s_{\alpha/\lambda} s_\gamma = \sum_{\widehat{\gamma} \supseteq \gamma} a_{\widehat{\gamma}} s_{\widehat{\gamma}}\,,$$

for some coefficients $a_{\widehat{\gamma}}$. We have

$$a_{\widehat{\gamma}} = \left\langle\, s_{\widehat{\gamma}} \,\middle|\, s_{\alpha/\lambda} s_\gamma \,\right\rangle = \left\langle\, s_{\widehat{\gamma}/\gamma} \,\middle|\, s_{\alpha/\lambda} \,\right\rangle = \left\langle\, s_{\widehat{\gamma}/\gamma} s_\lambda \,\middle|\, s_\alpha \,\right\rangle.$$

The product $s_{\widehat{\gamma}/\gamma} s_\lambda$ is equal to the skew Schur function of the shape $(\widehat{\gamma}/\gamma) \oplus \lambda$. (The notation $\oplus$ means that $\widehat{\gamma}/\gamma$ is positioned so that its bottom-right corner box is immediately northwest of the top-left corner box of λ.) Therefore, the coefficient $a_{\widehat{\gamma}}$ is equal to the number of LR-fillings of that skew shape that have content α. Any LR-filling of that shape must just fill the ith row of λ with the number i, for all i. Thus $a_{\widehat{\gamma}}$ equals the number of SSYT of shape $\widehat{\gamma}/\gamma$ whose reverse reading word is a λ-lattice permutation and whose content is the component-wise difference $\alpha - \lambda$. Hence (2.26) is equivalent to

$$s_\alpha s_{\gamma/\beta} = \sum_\lambda (-1)^{|\lambda|} D_{\beta/\lambda'} \sum_{T_2} s_{\widehat{\gamma}}\,, \tag{3.4}$$

where the second sum is over all SSYT T_2 of shape $\widehat{\gamma}/\gamma$ for some $\widehat{\gamma} \supseteq \gamma$, and content $\alpha - \lambda$, whose reverse reading word is a λ-lattice permutation.

Next, we examine the term $s_{\beta/\lambda'}$. The coefficient of s_ν in this term is exactly the Littlewood–Richardson coefficient $c^\beta_{\lambda'\nu}$, which is only nonzero if $\nu \subseteq \beta$. Thus we wish to determine the coefficient $c^\beta_{\lambda'\breve{\beta}}$ of $s_{\breve{\beta}}$ in $s_{\beta/\lambda'}$ when $\breve{\beta} \subseteq \beta$, which equals the number of LR-fillings of $\beta/\breve{\beta}$ of content λ'. We claim that such fillings T are in a shape-preserving bijection with ASSYT whose reverse readings word is a lattice permutation and whose content is λ (as opposed to content λ' previously). Indeed the bijection ψ is defined by mapping the ith appearance (in the reverse reading word of the SSYT T) of the number j to the number i, for all i and j. For example,

$$\begin{array}{|c|c|c|c|c|c}\cline{1-2} 3 & 4 & \multicolumn{4}{c}{} \\ \cline{1-3} 2 & 3 & 3 & \multicolumn{3}{c}{} \\ \cline{1-4} \multicolumn{1}{c|}{} & 1 & 2 & 2 & \multicolumn{2}{c}{} \\ \cline{2-6} \multicolumn{2}{c|}{} & 1 & 1 & 1 & \multicolumn{1}{c|}{1} \\ \cline{3-6}\end{array} \quad\longrightarrow\quad \begin{array}{|c|c|c|c|c|c}\cline{1-2} 3 & 1 & \multicolumn{4}{c}{} \\ \cline{1-3} 3 & 2 & 1 & \multicolumn{3}{c}{} \\ \cline{1-4} \multicolumn{1}{c|}{} & 5 & 2 & 1 & \multicolumn{2}{c}{} \\ \cline{2-6} \multicolumn{2}{c|}{} & 4 & 3 & 2 & \multicolumn{1}{c|}{1} \\ \cline{3-6}\end{array}\,.$$

Then one can check that ψ has the following necessary properties:

- The inverse of ψ is given by the ASSYT analogue of ψ: map the jth appearance (in the reverse reading word, now in the ASSYT sense) of the number i to the number j, for all i and j.
- The image $\psi(T)$ of an LR-filling T is indeed an ASSYT whose reverse reading word is a lattice permutation.
- Such a $\psi(T)$ maps to an LR-filling under the inverse map.
- Both ψ and its inverse transpose the content partition.

Thus (3.4) is equivalent to

$$s_\alpha s_{\gamma/\beta} = \sum_\lambda (-1)^{|\lambda|} \sum_{T_1} D_{\breve{\beta}} \sum_{T_2} s_{\widehat{\gamma}} = \sum_\lambda (-1)^{|\lambda|} \sum_{T_1,T_2} s_{\widehat{\gamma}/\breve{\beta}}\,,$$

where the relevant sums are over all T_1 and T_2 such that

- T_1 is an ASSYT of content λ, whose reverse reading word is a lattice permutation, and with shape $\beta/\breve{\beta}$ for some $\breve{\beta} \subseteq \beta$, and
- T_2 is an SSYT of content $\alpha - \lambda$, whose reverse reading word is a λ-lattice permutation, and with shape $\widehat{\gamma}/\gamma$ for some $\widehat{\gamma} \supseteq \gamma$.

Note that T_1 tells us that $|\lambda| = |\beta/\breve{\beta}|$, and we have arrived at Theorem 3.1 in the case when $\delta = \varnothing$.

3.4. **Recovering the full combinatorial skew Littlewood–Richardson rule.** Our second step is to use a linearity argument to derive Theorem 3.1 for general δ. For this, observe that the coefficient of $(-1)^{|\beta/\breve{\beta}|} s_{\widehat{\gamma}/\breve{\beta}}$ on the right-hand side of (3.3) is the number of pairs of tableaux (T_1, T_2) with T_1 an ASSYT and T_2 a SSYT, fulfilling conditions (a) and (b) in Theorem 3.1. But T_1 being an ASSYT is equivalent to $(T_1')^r$ being an SSYT, and the reverse reading word of the ASSYT T_1 is defined so that $(T_1')^r$ has the same reverse reading word as an SSYT. Therefore, the coefficient of $(-1)^{|\beta/\breve{\beta}|} s_{\widehat{\gamma}/\breve{\beta}}$ on the right-hand side of (3.3) equals the number of SSYT of shape $(\widehat{\gamma}/\gamma) \oplus (\beta'/\breve{\beta}')^r$ and content α whose reverse reading word is a δ–lattice permutation. This is the number of SSYT of shape $(\widehat{\gamma}/\gamma) \oplus (\beta'/\breve{\beta}')^r \oplus \delta$ and content α whose reverse reading word is a lattice permutation. By the Littlewood–Richardson rule, this quantity equals the coefficient of s_α in the Schur expansion of $s_{(\widehat{\gamma}/\gamma)\oplus(\beta'/\breve{\beta}')^r\oplus\delta}$. This skew Schur function being equal to the product $s_{\widehat{\gamma}/\gamma} s_{(\beta'/\breve{\beta}')^r} s_\delta$ means that the coefficient of s_α is

$$\left\langle s_{\widehat{\gamma}/\gamma} s_{(\beta'/\breve{\beta}')^r} s_\delta \,\middle|\, s_\alpha \right\rangle ,$$

which is equal to

$$\left\langle s_{\widehat{\gamma}/\gamma} s_{(\beta'/\breve{\beta}')^r} \,\middle|\, s_{\alpha/\delta} \right\rangle .$$

Therefore, (3.3) is equivalent to

$$s_{\alpha/\delta}s_{\gamma/\beta} = \sum_{\widehat{\gamma},\check{\beta}'}(-1)^{|\beta/\check{\beta}|}\left\langle s_{\widehat{\gamma}/\gamma}s_{(\beta'/\check{\beta}')^r} \,\middle|\, s_{\alpha/\delta}\right\rangle s_{\widehat{\gamma}/\check{\beta}}\,, \tag{3.5}$$

where the sums are over all partitions $\widehat{\gamma}$ and $\check{\beta}'$ such that $\widehat{\gamma} \supseteq \gamma$ and $\check{\beta} \subseteq \beta$.

The key observation is that (3.5) is linear in $s_{\alpha/\delta}$. Since the Schur functions form a basis for the space of symmetric functions, Equation (3.5) is equivalent to

$$f \cdot s_{\gamma/\beta} = \sum_{\widehat{\gamma},\check{\beta}'}(-1)^{|\beta/\check{\beta}|}\left\langle s_{\widehat{\gamma}/\gamma}s_{(\beta'/\check{\beta}')^r} \,\middle|\, f\right\rangle s_{\widehat{\gamma}/\check{\beta}}\ ,$$

for any symmetric function f, and any partitions β and γ. Hence, we see that in order to prove (3.3), it suffices to check the above equation for $f = s_\alpha$, for all partitions α. This is what we have done in Subsection 3.3.

3.5. **A combinatorial interpretation for the Kronecker product of a skew Schur function by $s_{(n-1,1)}$.** As another application of the identities of Section 2.7, our goal for this section is to derive a combinatorial formula for Kronecker products involving skew Schur functions. Let α be a partition of n, and let us speak of partitions and their Young diagrams interchangeably.

A well-known case of $\overline{K}_\lambda$ is when $\lambda = (1)$. We pause to describe an identity that we will generalize in Section 3.5 using the normal ordering relations. A *corner* of α is a box of α whose removal results in another partition, and we denote by #corners(α) the number of corners of α. Denote by α^- the set of partitions that result from removing a corner of α. Similarly, α^+ will denote the set of those partitions β such that $\alpha \in \beta^-$. We use $\alpha^\mp$ to denote the set of partitions not equal to α that can be obtained by removing a corner of α and then adding a box to the result. Equivalently, $\alpha^\mp$ is the set of partitions that can be obtained from α by first adding a box and then removing a different box. For example, (31) has two corners, and $(31)^\mp = \{(4),(22),(211)\}$. We finished Subsection 2.6 by relating the combinatorial identity

$$\overline{K}_{(1)}s_\alpha = (\#\text{corners}(\alpha) - 1)s_\alpha + \sum_{\beta\in\alpha^\mp} s_\beta \tag{3.6}$$

to the decomposition of the operators $\overline{K}_f$ in terms of the operators $U_{(1)}$ and $D_{(1)}$.

We aim to generalize (3.6) to skew Schur functions. This leads to our next use of the identities of Section 2. Corollary 2.12 implies the relation $[D_\theta, \overline{K}_{(1)}] = D_{s_{\theta/(1)}s_1}$, which gives

$$\overline{K}_{(1)}D_\theta(s_\alpha) = D_\theta\overline{K}_{(1)}(s_\alpha) - D_{s_{\theta/(1)}s_1}(s_\alpha).$$

Applying (3.6) and the fact that

$$s_{\theta/(1)} s_1 = \#\text{corners}(\theta) s_\theta + \sum_{\phi \in \theta^{\mp}} s_\phi,$$

we get

$$s_{\alpha/\theta} * s_{(n-|\theta|-1,1)} = (\#\text{corners}(\alpha) - \#\text{corners}(\theta) - 1) s_{\alpha/\theta} + \sum_{\beta \in \alpha^{\mp}} s_{\beta/\theta} - \sum_{\phi \in \theta^{\mp}} s_{\alpha/\phi}\ ,$$

Thus we have an algebraic proof of the following result.

Theorem 3.2. *Suppose $\alpha \vdash n$ and $\theta \vdash k$ with $\theta \subseteq \alpha$. Then*

$$s_{\alpha/\theta} * s_{(n-k-1,1)} = (\#\text{corners}(\alpha) - \#\text{corners}(\theta) - 1) s_{\alpha/\theta} + \sum_{\beta \in \alpha^{\mp}} s_{\beta/\theta} - \sum_{\phi \in \theta^{\mp}} s_{\alpha/\phi}\ .$$

This identity begs for a combinatorial proof. We offer a proof which is "two-thirds" combinatorial. The part which is non-combinatorial makes use of (2.19), which in turn is proved using Littlewood's identity (2.23).

Second proof of Theorem 3.2. The proof proceeds in three stages. In the short first stage, which is the non-combinatorial one, we apply (2.19) to express $s_{\alpha/\theta} * s_{(n-k-1,1)}$ in a form (3.7) not involving any Kronecker products. Then, using the skew Pieri rule (3.1), we reduce the problem to showing an identity (3.8) that is effectively purely about SSYT. This identity is proved in the third stage using jeu de taquin (first defined in [14]; can also be found in [12, 15], for example).

First, we need to determine the result of applying $D_{(1)}$ to the skew Schur function $s_{\alpha/\theta}$. We have

$$\left\langle s_\beta \,\middle|\, D_{(1)} s_{\alpha/\theta} \right\rangle = \left\langle s_\beta s_{(1)} \,\middle|\, s_{\alpha/\theta} \right\rangle = \sum_{\delta \in \theta^+} \left\langle s_\beta s_\delta \,\middle|\, s_\alpha \right\rangle = \left\langle s_\beta \,\middle|\, \sum_{\delta \in \theta^+} s_{\alpha/\delta} \right\rangle,$$

and so $D_{(1)} s_{\alpha/\theta} = \sum_{\delta \in \theta^+} s_{\alpha/\delta}$. Applying (2.19) to $s_{\alpha/\theta}$, we immediately deduce

$$s_{\alpha/\theta} * s_{(n-k-1,1)} = s_{(1)} \sum_{\delta \in \theta^+} s_{\alpha/\delta} - s_{\alpha/\theta}\ . \tag{3.7}$$

We next wish to apply the skew Pieri rule to $s_{(1)} \sum_{\delta \in \theta^+} s_{\alpha/\delta}$, but it will prove worthwhile to perform a preliminary step. By definition, $s_{\alpha/\delta} = 0$ unless $\delta \subseteq \alpha$, so it suffices to sum over the δ's with $\delta \in \theta^+$ and $\delta \subseteq \alpha$. We write $\delta \in \theta^{+\alpha}$ if δ satisfies these two conditions. So we now apply the skew Pieri rule (3.1) to

$$s_{\alpha/\theta} * s_{(n-k-1,1)} = s_{(1)} \sum_{\delta \in \theta^{+\alpha}} s_{\alpha/\delta} - s_{\alpha/\theta}$$

to obtain

$$s_{\alpha/\theta} * s_{(n-k-1,1)} = \sum_{\substack{\gamma\in\alpha^+\\ \delta\in\theta^{+\alpha}}} s_{\gamma/\delta} - \sum_{\substack{\delta\in\theta^{+\alpha}\\ \phi\in\delta^-}} s_{\alpha/\phi} - s_{\alpha/\theta}\ .$$

Let us examine the second sum. For any δ in $\theta^{+\alpha}$, we can choose $\phi = \theta$. We see that the other ϕ's that arise will be exactly those elements of $\theta^{\mp}$ that are contained in α. Therefore,

$$s_{\alpha/\theta} * s_{(n-k-1,1)} = \sum_{\substack{\gamma\in\alpha^+\\ \delta\in\theta^{+\alpha}}} s_{\gamma/\delta} - |\theta^{+\alpha}|s_{\alpha/\theta} - \sum_{\phi\in\theta^{\mp}} s_{\alpha/\phi} - s_{\alpha/\theta}\ .$$

Thus, to prove Theorem 3.2, it remains to show that

$$\sum_{\substack{\gamma\in\alpha^+\\ \delta\in\theta^{+\alpha}}} s_{\gamma/\delta} - |\theta^{+\alpha}|s_{\alpha/\theta} = (\#\mathrm{corners}(\alpha) - \#\mathrm{corners}(\theta))s_{\alpha/\theta} + \sum_{\beta\in\alpha^{\mp}} s_{\beta/\theta}\ . \tag{3.8}$$

Our main tool for proving the above identity is jeu de taquin but, as with our application of the skew Pieri rule, it is worthwhile to rewrite (3.8) in a slightly different form. Observe that, for any partition α, we have $\#\mathrm{corners}(\alpha) = |\alpha^+| - 1$. For $\theta \subseteq \alpha$, denote the set of elements of θ^+ that are not contained in α by $\theta^{+\alpha^c}$; we can check that $\theta^{+\alpha^c}$ can be non-empty only if α/θ has some empty rows or columns.

We can now rewrite (3.8) as

$$\sum_{\substack{\gamma\in\alpha^+\\ \delta\in\theta^{+\alpha}}} s_{\gamma/\delta} = (|\alpha^+| - |\theta^{+\alpha^c}|)s_{\alpha/\theta} + \sum_{\beta\in\alpha^{\mp}} s_{\beta/\theta}\ . \tag{3.9}$$

For intuition, we call the positions of those boxes of the form λ/α for some $\lambda \in \alpha^+$ the *outside corners* of α. Then the term $|\alpha^+| - |\theta^{+\alpha^c}|$ is the number of outside corners of α, excluding those that are also outside corners of θ. See Example 3.3 below for a fully worked out example of the remainder of the proof.

To prove (3.9) using jeu de taquin (jdt), consider an SSYT T that contributes to the left-hand side, meaning that T has shape γ/δ, where $\gamma \in \alpha^+$ and $\delta \in \theta^{+\alpha}$. Notice that the unique box b of δ/θ is not an element of T, and that the unique box c of γ/α is in T. Perform a jdt slide of T into b, and let T' denote the resulting SSYT. There are three possibilities that can arise:

(a) $T' = T$, meaning that there is no way to fill b under a jdt slide of T.
(b) T' contains b, and the single vacated box under the jdt slide is not c.
(c) T' contains b, and the single vacated box under the jdt slide is c.

By definition of jdt, Case (a) can happen if and only if b is a corner of γ. Since $b \in \delta \subseteq \alpha \subset \gamma$, the box b must be a corner of α. Therefore, since b is the unique box of δ/θ and is not an element of T' while c is the unique box of γ/α and is in T', the shape γ/δ of T' can be written in the equivalent form

β/θ, where β is obtained from α by removing b and adding c. In particular, $\beta \in \alpha^{\mp}$, and so such T's contribute part of the sum on the right of (3.9).

We claim that Case (b) contributes the rest of the sum on the right of (3.9). We see that the shape of T' is obtained from the shape of T by making exactly two changes: T' contains b, and a box c' different from c has been vacated. As a result, T' has shape β/θ for some $\beta \in \alpha^{\mp}$.

To prove our claim from the start of the previous paragraph, let T' be an SSYT of shape β/θ, where $\beta \in \alpha^{\mp}$. We wish to show that T' arises as the image under jdt of exactly one T from Cases (a) and (b). In short, the reason is that jdt slides are reversible, but let us be more precise. Suppose β is obtained from α by removing a box d and adding a different box e. Perform a jdt slide of T' into d. (Some references would call this a *reverse* jdt slide, since d is outside β.) There are two possibilities that can arise:

(i) There is no way to fill d under a jdt slide of T'. This can happen if and only if d is an outside corner of θ. Thus β/θ can be written in the equivalent form γ/δ, where γ equals α with e added, and δ equals θ with d added. Since d is a corner of α, it is also a corner of γ. These are exactly the conditions for T' to arise as an image under jdt of a T from Case (a) (where in fact $T = T'$ and our d here corresponds to b in Case (a)).

(ii) Performing a jdt slide of T' into d fills d and vacates an outside corner of θ. This is exactly the reverse of the jdt slide from Case (b) (where d is playing the role of c').

Thus T' arises as the image of a single T from Cases (a) and (b).

It remains to consider Case (c). Note that all T' in Case (c) are of shape α/θ. We would like to show that each T' is the image under jdt of k distinct T, where $k = (|\alpha^+| - |\theta^{+\alpha^c}|)$. This would show that the T's from Case (c) together contribute the term $(|\alpha^+| - |\theta^{+\alpha^c}|)s_{\alpha/\theta}$ from the right-hand side of (3.9), and (3.9) would be proved.

So pick a T' from Case (c). Pick an outside corner c of α, and perform a (reverse) jdt slide of T' into c. If c is not an outside corner of θ, then this jdt slide will fill c, and the result will be an SSYT T of shape γ/δ with $\gamma \in \alpha^+$ and $\delta \in \theta^{+\alpha}$. These are exactly the conditions for T' to arise in Case (c).

On the other hand, if c is an outside corner of α and also of θ, then T' will remain fixed under the (reverse) jdt slide. Thus any T that maps to such a T' under a jdt slide must also have shape α/θ. Such a T from the left-hand side of (3.9) does not exist, since T would contain the single box γ/α whereas T' does not. We conclude that each T' in Case (c) is the image of exactly k distinct Ts, as required. $\square$

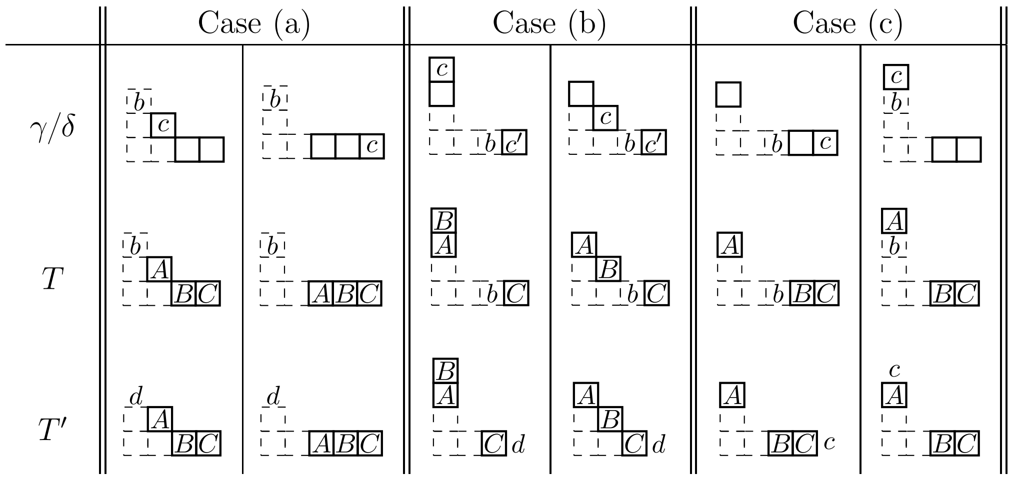
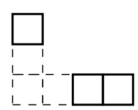
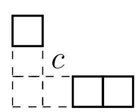


Table 3.1. The full set of skew shapes and SSYTs for Example 3.3. In the first row, the dashed boxes are those in δ, and the solid boxes are those in γ/δ. Lowercase letters correspond to notation in the proof of (3.9), while the uppercase A, B and C denote entries of the SSYTs that we assume satisfy the necessary inequalities but are otherwise any positive integers.

Example 3.3. Suppose $\alpha = (4,1,1)$ and $\theta = (2,1)$. We have

$\alpha/\theta =$ 

The complete set of shapes γ/δ, and SSYT T and T' from the proof of (3.9) are shown in Table 3.1. Deliberately omitted from the table is the scenario from the last paragraph of the proof, where c is an outside corner of both α and θ, which does not contribute to either side of (3.9). In this example, there is $1 = |\theta^{+\alpha^c}|$ such situation, shown below.

$\alpha/\theta =$ 

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