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HALL–LITTLEWOOD EXPANSIONS OF SCHUR DELTA OPERATORS AT $t = 0$

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ABSTRACT. For any Schur function s_ν, the associated *delta operator* Δ'_{s_ν} is a linear operator on the ring of symmetric functions which has the modified Macdonald polynomials as an eigenbasis. When $\nu = (1^{n-1})$ is a column of length $n-1$, the symmetric function $\Delta'_{e_{n-1}} e_n$ appears in the Shuffle Theorem of Carlsson and Mellit. More generally, when $\nu = (1^{k-1})$ is any column the polynomial $\Delta'_{e_{k-1}} e_n$ is the symmetric function side of the Delta Conjecture of Haglund, Remmel, and Wilson. We give an expansion of $\omega\Delta'_{s_\nu} e_n$ at $t = 0$ in the dual Hall–Littlewood basis for any partition ν. The Delta Conjecture at $t = 0$ was recently proven by Garsia, Haglund, Remmel, and Yoo; our methods give a new proof of this result. We give an algebraic interpretation of $\omega\Delta'_{s_\nu} e_n$ at $t = 0$ in terms of a Hom-space.

1. Introduction and Main Results

Let $\Lambda = \bigoplus_{n\geq 0} \Lambda_n$ be the ring of symmetric functions over the ground field $\mathbb{Q}(q,t)$ in an infinite variable set $\mathbf{x} = (x_1, x_2, \dots)$. Given a partition μ, let $\widetilde{H}_\mu = \widetilde{H}_\mu(\mathbf{x}; q, t)$ be the associated *modified Macdonald symmetric function*. The collection $\{\widetilde{H}_\mu \ : \ \mu \text{ a partition}\}$ forms a basis for the ring Λ.

If $f \in \Lambda$ is any symmetric function, the *(unprimed) delta operator* $\Delta_f : \Lambda \to \Lambda$ is the Macdonald eigenoperator given by

$$\Delta_f : \widetilde{H}_\mu \mapsto f(\dots, q^{i-1}t^{j-1}, \dots) \cdot \widetilde{H}_\mu, \tag{1.1}$$

where (i,j) ranges over all coordinates in the (English) Ferrers diagram of the partition μ (and all remaining variables in f are set to zero). As an example, if $\mu = (3,2)$, we fill the Ferrers diagram of μ with monomials as

1	q	q^2
t	qt	

so that $\Delta_f : \widetilde{H}_{(3,2)} \mapsto f(1, q, q^2, t, qt) \cdot \widetilde{H}_{(3,2)}$.

In this paper, we will focus on a primed version $\Delta'_f : \Lambda \to \Lambda$ of the delta operator defined by

$$\Delta'_f : \widetilde{H}_\mu \mapsto f(\dots, q^{i-1}t^{j-1}, \dots) \cdot \widetilde{H}_\mu, \tag{1.2}$$

where (i,j) range over all coordinates $\neq (0,0)$ in the Ferrers diagram of μ. If $\mu = (3,2)$ as above, we fill the Ferrers diagram of μ with monomials as

$\cdot$	q	q^2
t	qt	

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so that $\Delta'_f : \widetilde{H}_{(3,2)} \mapsto f(q, q^2, t, qt) \cdot \widetilde{H}_{(3,2)}$.

Let $k \leq n$ be positive integers. The *Delta Conjecture* of Haglund, Remmel, and Wilson [6] predicts the monomial expansion of $\Delta'_{e_{k-1}} e_n$ in terms of lattice paths. It reads

$$\Delta'_{e_{k-1}} e_n = \mathrm{Rise}_{n,k-1}(\mathbf{x}; q, t) = \mathrm{Val}_{n,k-1}(\mathbf{x}; q, t), \tag{1.3}$$

where $\mathrm{Rise}_{n,k-1}(\mathbf{x}; q, t)$ and $\mathrm{Val}_{n,k-1}(\mathbf{x}; q, t)$ are certain combinatorially defined quasisymmetric functions; see [6] for their definitions.

Various special cases of the Delta Conjecture have been proven already. When $k = n$, the Delta Conjecture reduces to the Shuffle Theorem of Carlsson and Mellit [2]. In the specialization $q = 1$, Romero [13] has proven

$$\Delta'_{e_{k-1}} e_n\big|_{q=1} = \Delta'_{e_{k-1}} e_n\big|_{t=1,q=t} = \mathrm{Rise}_{n,k}(\mathbf{x}; 1, t). \tag{1.4}$$

Zabrocki [15] has given evidence for the Delta Conjecture at $t = 1/q$ by showing that both sides coincide upon pairing with e_n under the Hall inner product. At $q = 0$, the following theorem summarizes work of Wilson and Rhoades.

Theorem 1.1 (Wilson [14], R. [11])**.** *Let $k \leq n$ be positive integers. We have*

$$\mathrm{Rise}_{n,k-1}(\mathbf{x}; q, 0) = \mathrm{Rise}_{n,k-1}(\mathbf{x}; 0, q) = \mathrm{Val}_{n,k-1}(\mathbf{x}; q, 0) = \mathrm{Val}_{n,k-1}(\mathbf{x}; 0, q). \tag{1.5}$$

Theorem 1.1 is proven by interpreting the four formal power series therein in terms of four statistics (called inv, maj, dinv, and minimaj) on ordered multiset partitions, and then proving the relevant equidistribution results. Let $C_{n,k} = C_{n,k}(\mathbf{x}; q)$ be the common symmetric function of Theorem 1.1:

$$C_{n,k} := \mathrm{Rise}_{n,k-1}(\mathbf{x}; q, 0) = \mathrm{Rise}_{n,k-1}(\mathbf{x}; 0, q) = \mathrm{Val}_{n,k-1}(\mathbf{x}; q, 0) = \mathrm{Val}_{n,k-1}(\mathbf{x}; 0, q). \tag{1.6}$$

The authors of this paper showed [7] that the image $\omega C_{n,k}$ of $C_{n,k}$ under the ω involution has the following expansion in the dual Hall–Littlewood basis:

$$\omega C_{n,k} = \sum_{\substack{\mu \vdash n \\ \ell(\mu)=k}} q^{\overline{b}(\mu)} \begin{bmatrix} k \\ m(\mu) \end{bmatrix}_q \cdot Q'_\mu. \tag{1.7}$$

Here $\left[\begin{smallmatrix} k \\ m(\mu) \end{smallmatrix} \right]_q$ is the q-multinomial coefficient corresponding to the part multiplicities of μ, the numbers $b(\mu)$ and $\overline{b}(\mu)$ are given by

$$\begin{cases} b(\mu) = \sum_i \mu_i (i-1), \\ \overline{b}(\mu) = \sum_i (\mu_i - 1)(i-1), \end{cases} \tag{1.8}$$

and $Q'_\mu = Q'_\mu(\mathbf{x}; q)$ is the dual Hall–Littlewood symmetric function related to the Schur basis by

$$Q'_\mu = \sum_\lambda K_{\lambda,\mu}(q) s_\lambda, \tag{1.9}$$

where $K_{\lambda,\mu}(q) \in \mathbb{Z}_{\geq 0}[q]$ is the Kostka–Foulkes polynomial.

Garsia, Haglund, Remmel, and Yoo [3] recently proved the Delta Conjecture at $t = 0$ by using plethystic methods and Equation (1.7) to show

$$\Delta'_{e_{k-1}} e_n\big|_{t=0} = \Delta'_{e_{k-1}} e_n\big|_{q=0,t=q} = C_{n,k}. \tag{1.10}$$

We give a new proof of Equation (1.10) using skewing operators $e_j^\perp$ on the ring Λ of symmetric functions together with ${}_3\phi_2$-hypergeometric transformations.

Finding positive Q'-basis expansions of symmetric functions is interesting for several reasons. Equation (1.9) shows that any symmetric function with a positive Q' expansion is automatically Schur positive, and thus is the Frobenius image of some module over the symmetric group $\mathfrak{S}_n$. Even better, the function Q'_μ is itself (up to a twist) the Frobenius image of the action of $\mathfrak{S}_n$ on the cohomology of the *Springer fiber* $\mathcal{B}_\mu$ or on the quotient of the polynomial ring $\mathbb{Q}[x_1, \ldots, x_n]$ by the *Tanisaki ideal* I_μ. We generalize Equation (1.10) to find the Q'-basis expansion of $\omega\Delta'_{s_\nu} e_n|_{t=0}$ for any partition ν.

Theorem 1.2. *Let ν be a partition and let $n \geq 0$. We have*

$$\omega\Delta'_{s_\nu} e_n\big|_{t=0} = \sum_{k=\ell(\nu)+1}^{|\nu|+1} P_{\nu,k-1}(q) \sum_{\substack{\mu \vdash n \\ \ell(\mu)=k}} q^{\bar{b}(\mu)} \cdot \begin{bmatrix} k \\ m(\mu) \end{bmatrix}_q \cdot Q'_\mu, \tag{1.11}$$

where

$$P_{\nu,k-1}(q) = q^{|\nu|-\binom{k}{2}} \sum_{\substack{|\rho|=|\nu| \\ \ell(\rho)=k-1}} q^{b(\rho)} \begin{bmatrix} k-1 \\ m(\rho) \end{bmatrix}_q K_{\nu,\rho}(q) \tag{1.12}$$

and $K_{\nu,\rho}(q)$ is the Kostka–Foulkes polynomial.

As operators on Λ we have the identity

$$\Delta_{s_\nu} = \sum_{\rho \subseteq \nu} \Delta'_{s_\rho},$$

where ρ ranges over all partitions obtainable from ν by removing a horizontal strip. Theorem 1.2 therefore also gives a positive expansion for $\omega\Delta_{s_\nu} e_n|_{t=0}$ in the Q'-basis, where we are using an unprimed delta operator.

Haiman proved that the symmetric function $\Delta'_{e_{n-1}} e_n = \Delta_{e_n} e_n$ (otherwise known as ∇e_n) is the bigraded Frobenius image of the diagonal coinvariant ring [8]. We mention that Zabrocki [16] has recently introduced a generalization of the diagonal coinvariant ring whose bigraded Frobenius image is conjecturally equal to the symmetric function $\Delta'_{e_{k-1}} e_n$ appearing in the Delta Conjecture.

If ν is a partition with $\ell(\nu) = n$, Haiman [8] gave an algebraic interpretation of $\Delta_{s_\nu} e_n$ as a Schur functor applied to a vector bundle over the Hilbert scheme of n points in the plane $\mathbb{C}^2$. In particular, Haiman's result implies that $\Delta_{s_\nu} e_n$ is Schur positive when $\ell(\nu) = n$. Haiman conjectured that $\Delta_{s_\nu} e_n$ is Schur positive for any partition ν. Haglund and Wilson have computational evidence that $\Delta'_{s_\nu} e_n$ is also Schur positive for any partition ν. Theorem 1.2 gives evidence for the Schur positivity of $\Delta'_{s_\nu} e_n$ (and thus also $\Delta_{s_\nu} e_n$) for arbitrary partitions ν.

In [7] the authors found an algebraic interpretation of the Delta Conjecture at $t = 0$. Let the symmetric group $\mathfrak{S}_n$ act on the polynomial ring $\mathbb{Q}[x_1, \ldots, x_n]$ in n variables. Following [7, Defn. 1.1], given positive integers $k \leq n$ we define the ideal $I_{n,k} \subseteq \mathbb{Q}[x_1, \ldots, x_n]$ by

$$I_{n,k} := \langle e_n, e_{n-1}, \ldots, e_{n-k+1}, x_1^k, x_2^k, \ldots, x_n^k \rangle \tag{1.13}$$

and let

$$R_{n,k} := \mathbb{Q}[x_1, \ldots, x_n]/I_{n,k} \tag{1.14}$$

be the corresponding quotient. When $k = n$ the ring $R_{n,k}$ reduces to the classical *coinvariant algebra* $R_n = \mathbb{Q}[x_1, \ldots, x_n]/\langle e_1, \ldots, e_n \rangle$ obtained by modding out by symmetric polynomials in $\mathbb{Q}[x_1, \ldots, x_n]$ with vanishing constant term. Just as algebraic properties of R_n are governed by combinatorial properties of permutations in $\mathfrak{S}_n$, it is shown in [7] that algebraic properties of $R_{n,k}$ are governed by *ordered set partitions* of $[n] := \{1, 2, \ldots, n\}$ with k blocks.

The ring $R_{n,k}$ has the structure of a graded $\mathfrak{S}_n$-module; in [7] it is proven that its graded Frobenius image is

$$\text{grFrob}(R_{n,k}; q) = (\text{rev}_q \circ \omega) C_{n,k}, \tag{1.15}$$

where rev_q is the operator which reverses the coefficient sequences of polynomials in q, e.g.

$$\text{rev}_q(3s_{(2,1)}q^2 + 2s_{(1,1,1)}q + s_{(3)}) = s_{(3)}q^2 + 2s_{(1,1,1)}q + 3s_{(2,1)}.$$

Thanks to the Garsia–Haglund–Remmel–Yoo Equation (1.10) we can also express Equation (1.15) as

$$\text{grFrob}(R_{n,k}; q) = (\text{rev}_q \circ \omega) \Delta'_{e_{k-1}} e_n \big|_{t=0}. \tag{1.16}$$

Informally, we think of $R_{n,k}$ as the 'coinvariant algebra' attached to the operator $\Delta'_{e_{k-1}} e_n$.

Given Equation (1.16), one could ask for a graded $\mathfrak{S}_n$-module $R_{\nu,n}$ which satisfies

$$\text{grFrob}(R_{\nu,n}; q) = (\text{rev}_q \circ \omega) \Delta'_{s_\nu} e_n \big|_{t=0} \tag{1.17}$$

for any partition ν. This would give a coinvariant algebra attached to the operator Δ'_{s_ν}. In [12] Rhoades and Wilson exhibited a quotient of $\mathbb{Q}[x_1, \ldots, x_n]$ with graded Frobenius image $(\text{rev}_q \circ \omega)\Delta_{s_\nu} e_n|_{t=0}$ when ν is a hook of the form $(r, 1^{n-1})$.

For general partitions $\nu \vdash m$, it is impossible to exhibit a module $R_{n,\nu}$ satisfying Equation (1.17) as a submodule of $\mathbb{Q}[x_1, \ldots, x_n]$; the graded components of the polynomial ring are not large enough for this purpose. Given Theorem 1.2, two artificial solutions to this problem are as follows.

- Very artificially, we could use the positive expansion of Q'_μ in the Schur basis $\{s_\lambda\}$ and define $R_{n,\nu}$ as a direct sum of $\mathfrak{S}_n$-irreducibles S^λ with appropriate grading shifts.
- Less artificially, we could use the fact that $\text{rev}_q(Q'_\mu)$ is the graded Frobenius image of the *Tanisaki quotient* $R_\mu = \mathbb{Q}[x_1, \ldots, x_n]/I_\mu$, where I_μ is the Tanisaki ideal. Theorem 1.2 then leads to a definition of $R_{n,\nu}$ as a direct sum of R_μ's with appropriate grading shifts.

The second bullet point is less artificial because the pieces R_μ which constitute the module $R_{n,\nu}$ are larger than the pieces S^λ appearing in the first bullet point.

In this paper we give a still less artificial construction for $R_{n,\nu}$ as a Hom-space. For any $n, m \geq 0$ we define a graded $\mathfrak{S}_m \times \mathfrak{S}_n$-module $V_{n,m}$ by

$$V_{n,m} := \bigoplus_{k \geq 0} (R_{m,k-1} \otimes R_{n,k})\{-mn + km + kn - n - k(k-1)\}. \tag{1.18}$$

Here $M\{-d\}$ denotes a graded module M with degree shifted up by d and we impose grading on tensor products by declaring

$$(M \otimes N)_d = \bigoplus_{i+j=d} M_i \otimes N_j.$$

If M is any $\mathfrak{S}_m$-module, the Hom-space $\text{Hom}_{\mathfrak{S}_m}(M, V_{n,m})$ has the structure of a graded $\mathfrak{S}_n$-module.

Theorem 1.3. *Let $n \geq 0$ and let $\nu \vdash m$ be a partition. Define the graded $\mathfrak{S}_n$-module $R_{n,\nu}$ by*

$$R_{n,\nu} := \mathrm{Hom}_{\mathfrak{S}_m}(S^\nu, V_{n,m})\{b(\nu)\}, \tag{1.19}$$

where $V_{n,m}$ is defined as in Equation (1.18)*. We have*

$$\mathrm{grFrob}(R_{n,\nu}; q) = (\mathrm{rev}_q \circ \omega)\Delta'_{s_\nu} e_n\big|_{t=0}. \tag{1.20}$$

The module $V_{n,m}$ only depends on the integers n and $m = |\nu|$, so that we may regard $V_{n,m}$ as a universal 'generator' for coinvariant algebras corresponding to partitions $\nu \vdash m$. Since the $R_{n,k}$ modules are 'larger' than the Tanisaki quotients R_μ, this gives a still less artificial solution to finding a 'coinvariant algebra' attached to the operator Δ'_{s_ν}. This also suggests that understanding the algebraic and geometric properties of objects related to Δ'_{s_ν} at $t = 0$ may be deduced from the case where ν is a single column.

2. Background

2.1. **Symmetric functions.** We adopt standard symmetric function terminology which may be found in e.g. [5, 9]. Given a partition λ, let

$$e_\lambda = e_\lambda(\mathbf{x}), \quad h_\lambda = h_\lambda(\mathbf{x}), \quad s_\lambda = s_\lambda(\mathbf{x}), \quad Q'_\lambda = Q'_\lambda(\mathbf{x}; q), \quad \widetilde{H}_\lambda = \widetilde{H}_\lambda(\mathbf{x}; q, t)$$

be the associated *elementary, homogeneous, Schur, dual Hall–Littlewood*, and *modified Macdonald symmetric function*. The functions Q'_μ expand positively in the Schur basis. If $\mu \vdash n$, the transition coefficients $K_{\lambda,\mu}(q) \in \mathbb{Z}_{\geq 0}[q]$ given by

$$Q'_\mu = \sum_{\lambda \vdash n} K_{\lambda,\mu}(q) s_\lambda \tag{2.1}$$

are the *Kostka–Foulkes polynomials*. The following relationship between the modified Macdonald symmetric functions and the dual Hall–Littlewood functions is well known:

$$\widetilde{H}_\lambda\big|_{t=0} = \mathrm{rev}_q(Q'_\lambda). \tag{2.2}$$

If $S \subseteq [n-1]$, we let $F_{n,S}$ be the associated *fundamental quasisymmetric function* of degree n given by

$$F_{n,S} := \sum_{\substack{i_1 \leq \cdots \leq i_n \\ (j \in S) \Rightarrow (i_j < i_{j+1})}} x_{i_1} \cdots x_{i_n}. \tag{2.3}$$

We adopt the usual q-analogs of numbers, factorials, binomial coefficients, and multinomial coefficients:

$$\begin{cases} [n]_q := 1 + q + q^2 + \cdots + q^{n-1}, & n \geq 0, \\ [n]!_q := [n]_q [n-1]_q \cdots [1]_q, & n \geq 0, \\ \left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q := \frac{[n]!_q}{[k]!_q [n-k]!_q}, & n \geq k \geq 0, \\ \left[\begin{smallmatrix} n \\ a_1, \ldots, a_k \end{smallmatrix}\right]_q := \frac{[n]!_q}{[a_1]!_q \cdots [a_k]!_q}, & a_1 + \cdots + a_k = n. \end{cases} \tag{2.4}$$

If $\lambda = (\lambda_1, \ldots, \lambda_k)$ is a partition, we let $\ell(\lambda) = k$ be the number of parts of λ, let $|\lambda| = \lambda_1 + \cdots + \lambda_k$ be the sum of the parts of λ, and set

$$b(\lambda) := \sum_{i=1}^{k} \lambda_i \cdot (i-1) \tag{2.5}$$

$$\overline{b}(\lambda) := b(\lambda) - \binom{\ell(\lambda)}{2} = \sum_{i=1}^{k} (\lambda_i - 1)(i-1). \tag{2.6}$$

We let $m_i(\lambda)$ denote the multiplicity of i as a part of λ and adopt the q-multinomial coefficient shorthand

$$\begin{bmatrix} \ell(\lambda) \\ m(\lambda) \end{bmatrix}_q := \begin{bmatrix} \ell(\lambda) \\ m_1(\lambda), m_2(\lambda), \ldots \end{bmatrix}_q. \tag{2.7}$$

Let ω be the involution on Λ which interchanges e_n and h_n. We will use the following 'twisted' version of the polynomials $C_{n,k}$ for $k \leq n$:

$$D_{n,k} := (\mathrm{rev}_q \circ \omega) C_{n,k}. \tag{2.8}$$

Let $\langle \cdot, \cdot \rangle$ be the *Hall inner product* on Λ defined by the declaring the Schur functions to be orthonormal: $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda,\mu}$. For any symmetric function f, the operator $f^\perp : \Lambda \to \Lambda$ is the dual operator to multiplication by f under the Hall inner product. Said differently, the operator $f^\perp$ is characterized by

$$\langle f^\perp g, h \rangle = \langle g, fh \rangle, \tag{2.9}$$

for all $g, h \in \Lambda$. For a proof of the following standard fact, see for example [7, Lem. 3.6].

Lemma 2.1. *Let $f, g \in \Lambda$ be symmetric functions with equal constant terms. We have $f = g$ if and only if $e_j^\perp f = e_j^\perp g$ for all $j \geq 1$.*

Lemma 2.1 will be used to form the recursions which underly our new proof of the Delta Conjecture at $t = 0$. The image of the $D_{n,k}$ functions under $e_j^\perp$ can be recursively described as follows.

Lemma 2.2 (H., R., S. [7, Lem. 3.7])**.** *Let $k \leq n$ be positive integers and let $j \geq 1$. We have*

$$e_j^\perp D_{n,k} = q^{\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q \cdot \sum_{m=\max(1,k-j)}^{\min(k,n-j)} q^{(k-m)\cdot(n-j-m)} \begin{bmatrix} j \\ k-m \end{bmatrix}_q D_{n-j,m}.$$

The irreducible representations of the symmetric group $\mathfrak{S}_n$ over $\mathbb{Q}$ are indexed by partitions $\lambda \vdash n$. If λ is a partition, we let S^λ denote the corresponding irreducible representation. If V is any finite-dimensional $\mathfrak{S}_n$-module, there exist unique integers $c_\lambda \geq 0$ such that $V \cong_{\mathfrak{S}_n} \bigoplus_{\lambda \vdash n} c_\lambda S^\lambda$. The *Frobenius image* $\mathrm{Frob}(V) \in \Lambda_n$ is the symmetric function

$$\mathrm{Frob}(V) := \sum_{\lambda \vdash n} c_\lambda s_\lambda. \tag{2.10}$$

More generally, if $V = \bigoplus_{d \geq 0} V_d$ is a graded $\mathfrak{S}_n$-module with each V_d finite-dimensional, the *graded Frobenius image* $\mathrm{grFrob}(V)$ is

$$\mathrm{grFrob}(V) = \sum_{d \geq 0} \mathrm{Frob}(V_d) \cdot q^d. \tag{2.11}$$

2.2. **Ordered set partitions.** If $\pi = \pi_1\pi_2\cdots\pi_n \in \mathfrak{S}_n$ is a permutation (written in one-line notation), the *descent set* of π is

$$\mathrm{Des}(\pi) := \{1 \leq i \leq n-1 \,:\, \pi_i > \pi_{i+1}\} \tag{2.12}$$

and the *inverse descent set* is $\mathrm{iDes}(\pi) := \mathrm{Des}(\pi^{-1})$.

Let $k \leq n$ be positive integers. An *ordered set partition of size n with k blocks* is a sequence $\sigma = (B_1 \mid \cdots \mid B_k)$ of k nonempty subsets of $[n]$ such that we have a disjoint union decomposition $[n] = B_1 \uplus \cdots \uplus B_k$. Let $\mathcal{OP}_{n,k}$ be the family of ordered set partitions of size n with k blocks. As an example, we have $\sigma = (27 \mid 135 \mid 46) \in \mathcal{OP}_{7,3}$. There is a natural identification $\mathcal{OP}_{n,n} = \mathfrak{S}_n$ of ordered set partitions of size n with n blocks and permutations in the symmetric group on n letters.

Let $\sigma = (B_1 \mid \cdots \mid B_k) \in \mathcal{OP}_{n,k}$ be an ordered set partition. An *inversion* in σ is a pair $1 \leq i < j \leq n$ such that

- i's block is strictly to the right of j's block in σ and
- i is minimal in its block.

We let $\mathrm{inv}(\sigma)$ be the number of inversions of σ. For example, if $\sigma = (27 \mid 135 \mid 46)$, the inversions of σ are $12, 17, 47$, and 45 so that $\mathrm{inv}(\sigma) = 4$.

If $\sigma = (B_1 \mid \cdots \mid B_k) \in \mathcal{OP}_{n,k}$, the *reading word* $\mathrm{rword}(\sigma)$ is the permutation $\pi_1\pi_2\cdots\pi_n \in \mathfrak{S}_n$ obtained by reading σ along 'diagonals' from left to right (where the m^{th} 'diagonal' is the set of elements which are m^{th} largest in their block). As an example, we have

$$\mathrm{rword}(27 \mid 135 \mid 45) = 5736214.$$

We have (see [6] or [7, Eq. 2.20]) the following quasisymmetric expansion of $C_{n,k}$ in terms of ordered set partitions:

$$C_{n,k} = \sum_{\sigma \in \mathcal{OP}_{n,k}} q^{\mathrm{inv}(\sigma)} F_{n,\mathrm{iDes}(\mathrm{rword}(\sigma))}. \tag{2.13}$$

2.3. **Hypergeometric functions.** Given a continuous parameter x and an integer $k \geq 0$, the *q-shifted factorial* is

$$(x)_k = (x;q)_k = (1-x)(1-xq)\cdots(1-xq^{k-1}). \tag{2.14}$$

We adopt the abbreviation

$$\prod_{i=1}^{j} (a_i)_k := (a_1, a_2, \ldots, a_k)_j. \tag{2.15}$$

If $r, s \geq 0$ are nonnegative integers and $\alpha_1, \ldots, \alpha_r$ and $\beta_1, \ldots, \beta_s$ are parameters, the corresponding *q-hypergeometric series* is

$${}_r\phi_s\left(\begin{matrix}\alpha_1, \ldots, \alpha_r \\ \beta_1, \ldots, \beta_s\end{matrix}; q, z\right) := \sum_{n=0}^{\infty} \frac{(\alpha_1;q)_n \cdots (\alpha_r;q)_n}{(\beta_1;q)_n \cdots (\beta_s;q)_n} \frac{z^n}{(q;q)_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r}. \tag{2.16}$$

In this paper we will only be concerned with the ${}_3\phi_2$-functions. See [4] for more information on hypergeometric series.

3. Polynomial identities

In this section we will prove symmetric function and hypergeometric identities which will be used in our proof of the Delta Conjecture at $t = 0$, and ultimately in our proof of Theorem 1.2. The first of these is a recursive description of the image of $C_{n,k}$ under the operator $e_j^{\perp}$.

Lemma 3.1. *Let $j \geq 1$ and $k \leq n$. We have*

$$e_j^{\perp} C_{n,k} = \sum_{r=0}^{j} q^{\binom{r}{2}} \begin{bmatrix} k \\ r \end{bmatrix}_q \begin{bmatrix} k+j-r-1 \\ j-r \end{bmatrix}_q C_{n-j,k-r}.$$

The proof of Lemma 3.1 should be compared with that of Lemma 2.2 (= [7, Lem. 3.7]).

Proof. We start with the quasisymmetric expansion of $C_{n,k}$ in terms of ordered set partitions:

$$C_{n,k} = \sum_{\sigma \in \mathcal{OP}_{n,k}} q^{\mathrm{inv}(\sigma)} F_{n,\mathrm{iDes}(\mathrm{rword}(\sigma))}. \tag{3.1}$$

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_p)$ be any (strict) composition of n. General facts about superization (see [5]) imply

$$\langle C_{n,k}, e_{\alpha_1} e_{\alpha_2} \cdots e_{\alpha_p} \rangle = \sum_{\substack{\sigma \in \mathcal{OP}_{n,k} \\ \mathrm{rword}(\sigma) \text{ is an } \alpha\text{-shuffle}}} q^{\mathrm{inv}(\sigma)}. \tag{3.2}$$

The α-shuffle condition means that the sequence $\mathrm{rword}(\sigma)$ is a shuffle of the p decreasing sequences

$$(\alpha_1, \ldots, 2, 1), (\alpha_1 + \alpha_2, \ldots, \alpha_1 + 2, \alpha_1 + 1), \ldots, (n, n-1, \ldots, \alpha_1 + \cdots + \alpha_{p-1} + 1).$$

We are interested in the case where $\alpha_1 = j$, so that

$$\langle C_{n,k}, e_{\alpha_1} e_{\alpha_2} \cdots e_{\alpha_p} \rangle = \langle C_{n,k}, e_j e_{\alpha_2} \cdots e_{\alpha_p} \rangle = \langle e_j^{\perp} C_{n,k}, e_{\alpha_2} \cdots e_{\alpha_p} \rangle. \tag{3.3}$$

As in the proof of [7, Lem. 3.7], we give a combinatorial interpretation of this expression.

Fix an index $0 \leq r \leq k-1$ and consider $\sigma \in \mathcal{OP}_{n-j,k-r}$. Let T be a way of adding the j letters $\{n-j+1, \ldots, n\}$ to σ (the *big* letters) in such a way that the resulting ordered set partition σ' has k blocks and the big letters appear in $\mathrm{rword}(\sigma')$ in the order $n, n-1, \ldots, n-j+1$. An example of such a way T for $n = 9, k = 5, j = 4, r = 2$ is shown below, with the big letters in bold:

$$(4 \mid 15 \mid 23) \rightsquigarrow (4 \mid 15\mathbf{9} \mid \mathbf{78} \mid 23 \mid \mathbf{6}).$$

Notice that exactly r of the big letters are minimal in their blocks of σ', and these minimal letters must be (from left to right)

$$n-j+r, \ldots, n-j+2, n-j+1.$$

Let us consider the effect on the inv statistic of all possible ways T of producing σ' from σ. Call the letters $1, 2, \ldots, n-j$ of σ' which are not big *small*. Following the notation of [7], let us call a letter i of σ'

- *mins* if i is small and minimal in its block,
- *minb* if i is big and minimal in its block,
- *nmins* if i is small and not minimal in its block, and
- *nminb* if i is big and not minimal in its block.

We observe the following, applying the standard interpretation of q-binomial coefficients as generating functions of inversions in binary strings (see for example [5, Ch. 1])

- The r letters $n-j+r, \ldots, n-j+1$ are precisely the *minb* letters for any way T, and they contribute amongst themselves $\binom{r}{2}$ inversions in σ'.
- The $\binom{k}{r}$ ways of distributing the *minb* letters among the blocks of σ' generate inversions with the *mins* letters, contributing to a factor of $\left[\begin{smallmatrix} k \\ r \end{smallmatrix}\right]_q$ in the generating function for $\mathrm{inv}(\sigma')$ when we sum over all ways T.
- The values of the *nminb* letters are completely determined by which block they are added to. There are $j-r$ letters which are *nminb* and they may be added to any of the k blocks of σ (upon addition of the *minb* letters), with multiplicity. This gives $\binom{k+j-r-1}{j-r}$ choices for distributing the *nminb* letters in T. The inversions contributed between the *nminb* letters and the *mins* letters generate a factor of $\left[\begin{smallmatrix} k+j-r-1 \\ j-r \end{smallmatrix}\right]_q$ to the generating function for $\mathrm{inv}(\sigma')$ when we sum over all ways T.

By the last paragraph, we have

$$\sum_{T:\sigma \rightsquigarrow \sigma'} q^{\mathrm{inv}(\sigma')} = q^{\mathrm{inv}(\sigma)+\binom{r}{2}} \begin{bmatrix} k \\ r \end{bmatrix}_q \begin{bmatrix} k+j-r-1 \\ j-r \end{bmatrix}_q, \tag{3.4}$$

where the sum is over all ways T of producing σ' from σ. If we sum this expression over all σ with $\mathrm{rword}(\sigma)$ an $(\alpha_2, \ldots, \alpha_p)$-shuffle, and then over all r, we get the inner product

$$\left\langle \sum_{r=0}^{j} q^{\binom{r}{2}} \begin{bmatrix} k \\ r \end{bmatrix}_q \begin{bmatrix} k+j-r-1 \\ j-r \end{bmatrix}_q C_{n-j,k-r}, e_{\alpha_2} \cdots e_{\alpha_p} \right\rangle, \tag{3.5}$$

which is also equal to

$$\left\langle e_j^{\perp} C_{n,k}, e_{\alpha_2} \cdots e_{\alpha_p} \right\rangle, \tag{3.6}$$

completing the proof. □

We will need the theory of hypergeometric series for our proof of Equation (1.10). In particular, we have the following transformation of the ${}_3\phi_2$ basic hypergeometric series (see [1] for background on basic hypergeometric series).

Lemma 3.2. *Let $j \in \mathbb{N}$ and $\alpha, x, y, z \in \mathbb{R}$. We have*

$$\begin{aligned} {}_3\phi_2 \left(\begin{matrix} q^{-j}, & q^{\alpha}, & q^{\alpha+z} \\ & q^{\alpha-y-j+1}, & q^{\alpha-x-j+1} \end{matrix}; q, q \right) \\ = \frac{(q^{-y-j+1})_j (q^{-x-j+1})_j}{(q^{\alpha-y-j+1})_j (q^{\alpha-x-j+1})_j} \frac{(q^{-y-z-j+1})_j}{(q^y)_j} q^{(\alpha+x+y+z+j-1)j} \\ \times {}_3\phi_2 \left(\begin{matrix} q^{-j}, & q^{x+y+z+j-1}, & q^{x-\alpha} \\ & q^x, & q^{x+z} \end{matrix}; q, q^{1+\alpha-y} \right). \end{aligned} \tag{3.7}$$

Proof. We utilize the following identities from [1, p. 525]:

$${}_3\phi_2 \left(\begin{matrix} q^{-n}, & w, & b \\ & d, & e \end{matrix}; q, q \right) = \frac{(e/w)_n w^n}{(e)_n} {}_3\phi_2 \left(\begin{matrix} q^{-n}, & w, & d/b \\ & d, & q^{1-n} w/e \end{matrix}; q, bq/e \right), \tag{3.8}$$

$${}_3\phi_2 \left(\begin{matrix} q^{-n}, & w, & b \\ & e, & f \end{matrix}; q, q \right) = \frac{(e/w)_n (f/w)_n w^n}{(e)_n (f)_n} {}_3\phi_2 \left(\begin{matrix} q^{-n}, & w, & wbq^{1-n}/ef \\ & q^{1-n}w/e, & q^{1-n}w/f \end{matrix}; q, q \right), \tag{3.9}$$

where $n \in \mathbb{N}$, and w, b, c, d, e, f are continuous parameters.

Begin by setting $j = n, w = q^\alpha, b = q^{\alpha+z}, e = q^{\alpha-y-j+1}$, and $f = q^{\alpha-x-j+1}$ in (3.9) to get

$$
\begin{aligned}
(3.10)\quad {}_3\phi_2\left(\begin{matrix} q^{-j}, & q^{\alpha}, & q^{\alpha+z} \\ & q^{\alpha-y-j+1}, & q^{\alpha-x-j+1} \end{matrix}; q, q\right) \\
= \frac{(q^{-y-j+1})_j (q^{-x-j+1})_j q^{j\alpha}}{(q^{\alpha-y-j+1})_j (q^{\alpha-x-j+1})_j} {}_3\phi_2\left(\begin{matrix} q^{-j}, & q^{\alpha}, & q^{x+y+z+j-1} \\ & q^{y}, & q^{x} \end{matrix}; q, q\right).
\end{aligned}
$$

Now apply (3.8) with $n = j, w = q^{x+y+z+j-1}, b = q^\alpha, d = q^x, e = q^y$ to the ${}_3\phi_2$ appearing in the right-hand side of (3.10) to get

$$
\begin{aligned}
(3.11)\quad {}_3\phi_2\left(\begin{matrix} q^{-j}, & q^{\alpha}, & q^{\alpha+z} \\ & q^{\alpha-y-j+1}, & q^{\alpha-x-j+1} \end{matrix}; q, q\right) \\
= \frac{(q^{-y-j+1})_j (q^{-x-j+1})_j}{(q^{\alpha-y-j+1})_j (q^{\alpha-x-j+1})_j} \frac{(q^{-x-z-j+1})_j}{(q^y)_j} q^{(\alpha+x+y+z+j-1)j} \\
\times {}_3\phi_2\left(\begin{matrix} q^{-j}, & q^{x+y+z+j-1}, & q^{x-\alpha} \\ & q^{x}, & q^{x+z} \end{matrix}; q, q^{1+\alpha-y}\right). \quad \square
\end{aligned}
$$

We express the hypergeometric transformation of Lemma 3.2 in a more convenient form involving q-binomials.

Lemma 3.3. *Let $j \leq k \leq n$ be positive integers. Let p be an integer in the range $k - j \leq p \leq n - j$. There holds the identity*

$$
\begin{aligned}
(3.12)\quad & q^{\binom{k}{2}+\binom{j}{2}} \sum_{r=p}^{p+j} (-1)^{n-r} \begin{bmatrix} r-1 \\ k-1 \end{bmatrix}_q q^{\binom{r+1}{2}-nr} \begin{bmatrix} r \\ j \end{bmatrix}_q q^{(r-p)(n-j-p)} \begin{bmatrix} j \\ r-p \end{bmatrix}_q \\
& = \sum_{r=k-p}^{j} q^{\binom{r}{2}} \begin{bmatrix} k \\ r \end{bmatrix}_q \begin{bmatrix} k+j-r-1 \\ j-r \end{bmatrix}_q q^{\binom{k-a}{2}} (-1)^{n-j-p} \begin{bmatrix} p-1 \\ k-r-1 \end{bmatrix}_q q^{\binom{p+1}{2}-(n-j)p}.
\end{aligned}
$$

Proof. The first step is to express everything in terms of hypergeometric series. We make use of the following facts, which we refer to as the 'simple identities'. Here $u, j, a \in \mathbb{Z}_{\geq 0}$ and p, x are continuous parameters.

$$
(3.13)\quad \begin{bmatrix} u+p \\ j \end{bmatrix}_q = \begin{bmatrix} p \\ j \end{bmatrix}_q \frac{(q^{p+1})_u}{(q^{p-j+1})_u}
$$

$$
(3.14)\quad \begin{bmatrix} p \\ u+a \end{bmatrix}_q = \begin{bmatrix} p \\ a \end{bmatrix}_q \frac{(q^{p-a-u+1})_u}{(q^{a+1})_u} = \begin{bmatrix} p \\ a \end{bmatrix}_q \frac{(q^{a-p})_u}{(q^{a+1})_u} (-q^{p-a})^u q^{-\binom{u}{2}}
$$

$$
(3.15)\quad \begin{bmatrix} p \\ j-u \end{bmatrix}_q = \begin{bmatrix} p \\ j \end{bmatrix}_q \frac{(q^{-j})_u}{(q^{p-j+1})_u} (-q^{j})^u q^{-\binom{u}{2}}
$$

$$
(3.16)\quad \binom{u+a}{2} = \binom{u}{2} + \binom{a}{2} + ua.
$$

Using the simple identities and setting $u = r - p$, the left-hand side of Equation (3.12) can be expressed as

$$
(3.17)\quad \begin{bmatrix} p \\ j \end{bmatrix}_q \begin{bmatrix} p-1 \\ k-1 \end{bmatrix}_q (-1)^{n-p} q^{-np+\binom{p+1}{2}+\binom{k}{2}+\binom{j}{2}} \sum_{u=0}^{j} \frac{(q^{-j}, q^{p+1}, q^p)_u}{(q, q^{p-j+1}, q^{p-k+1})_u} q^u.
$$

Similarly, using the simple identities and setting $u = j - a$, we see that the right-hand side of Equation (3.12) can be expressed as

$$q^{-p(n-j)+\binom{p+1}{2}+\binom{k-j}{2}+\binom{j}{2}}(-1)^{n+j-p}\begin{bmatrix}k\\ j\end{bmatrix}_q\begin{bmatrix}p-1\\ k-j-1\end{bmatrix}_q\sum_{u=0}^{p-k+j}\frac{(q^{-j},q^k,q^{-p+k-j})_u}{(q,q^{k-j},q^{k-j+1})_u}q^{u(p+1)}. \tag{3.18}$$

The next step is to express (3.17) and (3.18) in terms of the hypergeometric series ${}_3\phi_2$. The expression (3.17) is given by

$$\begin{bmatrix}p\\ j\end{bmatrix}_q\begin{bmatrix}p-1\\ k-1\end{bmatrix}_q(-1)^{n-p}q^{-np+\binom{p+1}{2}+\binom{k}{2}+\binom{j}{2}}{}_3\phi_2\left(\begin{matrix}q^{-j}, & q^{p+1}, & q^p\\ & q^{p-j+1}, & q^{p-k+1}\end{matrix};q,q\right) \tag{3.19}$$

whereas (3.18) is equal to

$$q^{-p(n-j)+\binom{p+1}{2}+\binom{k-j}{2}+\binom{j}{2}}(-1)^{n+j-p}\begin{bmatrix}k\\ j\end{bmatrix}_q\begin{bmatrix}p-1\\ k-j-1\end{bmatrix}_q{}_3\phi_2\left(\begin{matrix}q^{-j}, & q^k, & q^{-p+k-j}\\ & q^{k-j}, & q^{k-j+1}\end{matrix};q,q^{p+1}\right). \tag{3.20}$$

The fact that (3.19) = (3.20) is a consequence of Lemma 3.2. □

4. Proofs of the Main Results

Our starting point is the following expansion (see [5, Eqn. 2.72]) of e_n in the modified Macdonald basis:

$$e_n = \sum_{\lambda\vdash n}\frac{MB_\lambda\Pi_\lambda\widetilde{H}_\lambda}{w_\lambda}, \tag{4.1}$$

where

- $M = (1-q)(1-t)$,
- $B_\lambda = \sum_{(i,j)\in\lambda} q^{i-1}t^{j-1}$, where the sum is over all cells (i,j) in the Ferrers diagram of λ,
- $\Pi_\lambda = \prod_{(1,1)\neq(i,j)\in\lambda}(1-q^{i-1}t^{j-1})$, where the product is over all cells (i,j) in the Ferrers diagram of λ other than the northwest corner $(1,1)$, and
- $w_\lambda = \prod_{c\in\lambda}(q^{a(c)}-t^{l(c)+1})(t^{l(c)}-q^{a(c)+1})$, where the product is over all cells c in the diagram of λ and $a(c), l(c)$ are the arm and leg lengths (see [5, p. 29]) of the cell c in λ.

If we apply the operator $\Delta'_{e_{k-1}}$ to both sides of Equation (4.1), we get

$$\Delta'_{e_{k-1}}e_n = \sum_{\lambda\vdash n} e_{k-1}[B_\lambda - 1]\frac{MB_\lambda\Pi_\lambda\widetilde{H}_\lambda}{w_\lambda}. \tag{4.2}$$

Here we used the plethystic shorthand $e_{k-1}[B_\lambda - 1] = e_{k-1}(\dots, q^{i-1}t^{j-1}, \dots)$ where (i,j) range over all cells $\neq (1,1)$ in the Ferrers diagram of λ.

Recall that $\widetilde{H}_\lambda|_{t=0} = \mathrm{rev}_q Q'_\lambda$ for any partition λ. If we evaluate both sides of Equation (4.2) at $t = 0$, we get

$$\Delta'_{e_{k-1}}e_n\big|_{t=0} = \sum_{\lambda\vdash n}(-1)^{n-\ell(\lambda)}q^{\binom{k}{2}-2b(\lambda)-n+\sum_i\binom{m_i(\lambda)+1}{2}}\begin{bmatrix}\ell(\lambda)-1\\ k-1\end{bmatrix}_q\begin{bmatrix}\ell(\lambda)\\ m(\lambda)\end{bmatrix}_q\cdot\mathrm{rev}_q Q'_\lambda. \tag{4.3}$$

Here we used the evaluation

$$e_{k-1}[B_\lambda - 1] = e_{k-1}(1, q, \ldots, q^{\ell(\lambda)-1}) = q^{\binom{k}{2}} \begin{bmatrix} \ell(\lambda) - 1 \\ k-1 \end{bmatrix}_q . \tag{4.4}$$

Equation (4.3) can be expressed in terms of the D-functions $D_{n,r}$.

Lemma 4.1. *We have the identity*

$$\Delta'_{e_{k-1}} e_n \big|_{t=0} = q^{\binom{k}{2}} \sum_{r=k}^{n} (-1)^{n-r} q^{\binom{r+1}{2}-nr} \begin{bmatrix} r-1 \\ k-1 \end{bmatrix}_q D_{n,r}. \tag{4.5}$$

Proof. Starting with Equation (4.3) and grouping partitions $\lambda \vdash n$ according to their number of parts we have

$$\begin{aligned} \Delta'_{e_{k-1}} e_n \big|_{t=0} &= \sum_{\lambda \vdash n} (-1)^{n-\ell(\mu)} q^{\binom{k}{2}-2b(\mu)-n+\sum_i \binom{m_i(\lambda)+1}{2}} \begin{bmatrix} \ell(\lambda)-1 \\ k-1 \end{bmatrix}_q \begin{bmatrix} \ell(\lambda) \\ m(\mu) \end{bmatrix}_q \cdot \mathrm{rev}_q Q'_\lambda \\ &= q^{\binom{k}{2}-n} \sum_{r=k}^{n} (-1)^{n-r} \begin{bmatrix} r-1 \\ k-1 \end{bmatrix}_q \sum_{\substack{\lambda \vdash n \\ \ell(\mu)=r}} q^{-2b(\mu)+\sum_i \binom{m_i(\lambda)+1}{2}} \begin{bmatrix} r \\ m(\lambda) \end{bmatrix}_q \cdot \mathrm{rev}_q Q'_\lambda. \end{aligned}$$

We focus on the internal summand. We have

$$\begin{aligned} &\sum_{\substack{\mu \vdash n \\ \ell(\lambda)=r}} q^{-2b(\lambda)+\sum_i \binom{m_i(\lambda)+1}{2}} \begin{bmatrix} r \\ m(\lambda) \end{bmatrix}_q \cdot \mathrm{rev}_q Q'_\lambda \\ &\quad = \sum_{\substack{\lambda \vdash n \\ \ell(\lambda)=r}} q^{-2b(\mu)+\sum_i \binom{m_i(\lambda)+1}{2}+\sum_{i<j} m_i(\lambda)m_j(\lambda)-\sum_{i<j} m_i(\lambda)m_j(\lambda)} \begin{bmatrix} r \\ m(\lambda) \end{bmatrix}_q \cdot \mathrm{rev}_q Q'_\lambda \\ &\quad = q^{\binom{r+1}{2}} \sum_{\substack{\lambda \vdash n \\ \ell(\lambda)=r}} q^{-2b(\lambda)-\sum_{i<j} m_i(\lambda)m_j(\lambda)} \begin{bmatrix} r \\ m(\lambda) \end{bmatrix}_q \cdot \mathrm{rev}_q Q'_\lambda \\ &\quad = q^{\binom{r+1}{2}} \sum_{\substack{\lambda \vdash n \\ \ell(\lambda)=r}} q^{-b(\lambda)} \cdot \left[q^{\sum_{i<j} -m_i(\lambda)m_j(\lambda)} \begin{bmatrix} r \\ m(\lambda) \end{bmatrix}_q \right] \cdot \left[q^{-b(\lambda)} \mathrm{rev}_q Q'_\lambda(\mathbf{x}; q) \right] \\ &\quad = q^{\binom{r+1}{2}-\binom{r}{2}} \sum_{\substack{\lambda \vdash n \\ \ell(\lambda)=r}} q^{-\overline{b}(\lambda)} \cdot \begin{bmatrix} r \\ m(\lambda) \end{bmatrix}_{q^{-1}} \cdot Q'_\lambda(\mathbf{x}; q^{-1}) \\ &\quad = q^{\binom{r+1}{2}-\binom{r}{2}} q^{-\binom{r}{2}-(n-r)(r-1)} \cdot [q^{\binom{r}{2}+(n-r)(r-1)} \cdot \omega C'_{n,r}(\mathbf{x}; q^{-1})] \\ &\quad = q^{\binom{r+1}{2}-r(r-1)-(n-r)(r-1)} \cdot D_{n,r}(\mathbf{x}; q) \\ &\quad = q^{\binom{r+1}{2}-nr+n} \cdot D_{n,r}(\mathbf{x}; q). \end{aligned}$$

The second equality used $\sum_i \binom{m_i(\lambda)+1}{2} + \sum_{i<j} m_i(\lambda)m_j(\lambda) = \binom{\ell(\lambda)+1}{2}$. The fourth equality comes from the fact that the degree of the palindromic polynomial $\begin{bmatrix} \ell(\lambda) \\ m(\lambda) \end{bmatrix}_q$ is $\sum_{i<j} m_i(\lambda)m_j(\lambda)$ and that the q-degree of $Q'_\lambda(\mathbf{x}; q)$ is $b(\lambda)$.

Going back to the assertion of the lemma, we have

$$\begin{aligned}\Delta'_{e_{k-1}} e_n\big|_{t=0} &= q^{\binom{k}{2}-n} \sum_{r=k}^{n} (-1)^{n-r} \begin{bmatrix} r-1 \\ k-1 \end{bmatrix}_q \sum_{\substack{\lambda \vdash n \\ \ell(\lambda)=r}} q^{-2b(\mu)+\sum_i \binom{m_i(\lambda)+1}{2}} \begin{bmatrix} r \\ m(\lambda) \end{bmatrix}_q \cdot \mathrm{rev}_q Q'_\lambda \\ &= q^{\binom{k}{2}-n} \sum_{r=k}^{n} (-1)^{n-r} \begin{bmatrix} r-1 \\ k-1 \end{bmatrix}_q \cdot q^{\binom{r+1}{2}-nr+n} \cdot D_{n,r}(\mathbf{x}; q).\end{aligned}$$

Cancelation of a factor of q^n completes the proof. □

We are in a position to give our proof of Equation (1.10), and thus give a new proof of the Delta Conjecture at $t = 0$.

Theorem 4.2 (Garsia, H., Remmel, Yoo [3]). *Let $k \leq n$ be positive integers. We have*

$$\Delta'_{e_{k-1}} e_n\big|_{t=0} = \Delta'_{e_{k-1}} e_n\big|_{q=0,t=q} = C_{n,k}.$$

Proof. Let $j \geq 1$. Given Lemma 4.1 and Lemma 2.2, the symmetric function $e_j^\perp \Delta'_{e_{k-1}} e_n|_{t=0}$ has the following D-function expansion, where we adopt the convention that $D_{n,k} = 0$ if $k > n$ or if $k < 0$:

$$\begin{aligned}e_j^\perp \Delta'_{e_{k-1}} e_n\big|_{t=0} = q^{\binom{k}{2}+\binom{j}{2}} \sum_{r=k}^{n} (-1)^{n-r} \begin{bmatrix} r-1 \\ k-1 \end{bmatrix}_q q^{\binom{r+1}{2}-nr} \begin{bmatrix} r \\ j \end{bmatrix}_q \\ \times \sum_{m=r-j}^{r} q^{(r-m)(n-j-m)} \begin{bmatrix} j \\ r-m \end{bmatrix}_q D_{n-j,m}.\end{aligned} \tag{4.6}$$

If we want $e_j^\perp \Delta'_{e_{k-1}} e_n|_{t=0}$ to satisfy the recursion of Lemma 3.1, we must have

$$e_j^\perp \Delta'_{e_{k-1}} e_n\big|_{t=0} = \sum_{r=0}^{j} q^{\binom{r}{2}} \begin{bmatrix} k \\ r \end{bmatrix}_q \begin{bmatrix} k+j-r-1 \\ j-r \end{bmatrix}_q \Delta'_{e_{k-r-1}} e_{n-j}\big|_{t=0}. \tag{4.7}$$

By Lemma 4.1, we know

$$\begin{aligned}&\sum_{r=0}^{j} q^{\binom{r}{2}} \begin{bmatrix} k \\ a \end{bmatrix}_q \begin{bmatrix} k+j-r-1 \\ j-r \end{bmatrix}_q \Delta'_{e_{k-r-1}} e_{n-j}\big|_{t=0} \\ &= \sum_{r=0}^{j} q^{\binom{r}{2}} \begin{bmatrix} k \\ r \end{bmatrix}_q \begin{bmatrix} k+j-r-1 \\ j-r \end{bmatrix}_q q^{\binom{k-r}{2}} \sum_{b=k-r}^{n-j} (-1)^{n-j-b} \begin{bmatrix} b-1 \\ k-r-1 \end{bmatrix}_q q^{\binom{b+1}{2}-(n-j)b} D_{n-j,b}.\end{aligned} \tag{4.8}$$

We want to show that the right-hand side of Equation (4.6) is equal to the right-hand side of Equation (4.8). To this end, let p be an integer in the range $k - j \leq p \leq n - j$. The coefficient of $D_{n-j,p}$ in Equation (4.6) is

$$q^{\binom{k}{2}+\binom{j}{2}} \sum_{r=p}^{p+j} (-1)^{n-r} \begin{bmatrix} r-1 \\ k-1 \end{bmatrix}_q q^{\binom{r+1}{2}-nr} \begin{bmatrix} r \\ j \end{bmatrix}_q q^{(r-p)(n-j-p)} \begin{bmatrix} j \\ r-p \end{bmatrix}_q, \tag{4.9}$$

whereas the coefficient of $D_{n-j,p}$ in Equation (4.8) is

$$\sum_{r=k-p}^{j} q^{\binom{r}{2}} \begin{bmatrix} k \\ r \end{bmatrix}_q \begin{bmatrix} k+j-r-1 \\ j-r \end{bmatrix}_q q^{\binom{k-r}{2}} (-1)^{n-j-p} \begin{bmatrix} p-1 \\ k-r-1 \end{bmatrix}_q q^{\binom{p+1}{2}-(n-j)p}. \tag{4.10}$$

Theorem 4.2 will be proven if we can only establish the equality of the expressions (4.9) and (4.10). This is Lemma 3.3. □

We use Theorem 4.2 to derive the more general Theorem 1.2. In this proof we will use the notation of plethysm; see [5].

Proof of Theorem 1.2. Let $k \leq n$ be positive integers. The polynomials Q'_μ and $\mathrm{rev}_q(Q'_\mu)$ have Schur expansions

$$Q'_\mu = \sum_{\lambda \vdash n} K_{\lambda,\mu}(q) s_\lambda,$$
$$\mathrm{rev}_q(Q'_\mu) = \sum_{\lambda \vdash n} q^{b(\mu)} K_{\lambda,\mu}(1/q) s_\lambda.$$

By Equation (1.7), Equation (4.3), and the truth of the Delta Conjecture at $q = 0$ (i.e., Theorem 4.2) we have the following identity:

$$q^{k(k-1)} \sum_{\mu \vdash n} (-1)^{n-\ell(\mu)} q^{-n-b(\mu)+\sum_{i=1}^n \binom{m_i+1}{2}} \begin{bmatrix} \ell(\mu)-1 \\ k-1 \end{bmatrix}_q \begin{bmatrix} \ell(\mu) \\ m(\mu) \end{bmatrix}_q K_{\lambda,\mu}(1/q) = \sum_{\substack{\mu \vdash n \\ \ell(\mu)=k}} q^{b(\mu)} \begin{bmatrix} \ell(\mu) \\ m(\mu) \end{bmatrix}_q K_{\lambda',\mu}(q). \tag{4.11}$$

Equation (4.11) is also recorded in [3, Prop. 3.2].

Using reasoning identical to that of our derivation of Equation (4.3), we see that $\Delta'_{s_\nu} e_n|_{t=0}$ has the following expansion in the q-reversed Q'-basis:

$$\Delta'_{s_\nu} e_n \big|_{t=0} = \sum_{\mu \vdash n} (-1)^{n-\ell(\mu)} s_\nu(q, q^2, \ldots, q^{\ell(\mu)-1}) q^{-n-2b(\mu)+\sum_i \binom{m_i(\mu)+1}{2}} \begin{bmatrix} \ell(\mu) \\ m(\mu) \end{bmatrix}_q \cdot \mathrm{rev}_q(Q'_\mu). \tag{4.12}$$

Multiplying both sides of (4.11) by s_λ, summing over λ and applying ω, we get the following equivalent form of (4.11):

$$q^{k(k-1)} \sum_{\mu \vdash n} (-1)^{n-\ell(\mu)} q^{-n-b(\mu)+\sum_{i=1}^n \binom{m_i+1}{2}} \begin{bmatrix} \ell(\mu)-1 \\ k-1 \end{bmatrix}_q \begin{bmatrix} \ell(\mu) \\ m(\mu) \end{bmatrix}_q \sum_\lambda K_{\lambda',\mu}(1/q) s_\lambda = \sum_{\substack{\mu \vdash n \\ \ell(\mu)=k}} q^{b(\mu)} \begin{bmatrix} \ell(\mu) \\ m(\mu) \end{bmatrix}_q Q'_\mu \tag{4.13}$$

for all $\lambda \vdash n$ and $1 \leq k \leq n$.

Note that the sum on the right-hand side of (4.13) also occurs on the right-hand side in Theorem 1.2. By Equation (4.12), the following equation is equivalent to Theorem 1.2:

$$\begin{aligned}(4.14)\quad &\sum_{\mu\vdash n}(-1)^{n-\ell(\mu)}q^{-b(\mu)-n+\sum_i\binom{m_i(\mu)+1}{2}}\begin{bmatrix}\ell(\mu)\\ m(\mu)\end{bmatrix}_q\sum_\lambda s_\lambda K_{\lambda',\mu}(1/q)s_\nu(q,q^2,\dots,q^{\ell(\mu)-1})\\ &=\sum_{\mu\vdash n}(-1)^{n-\ell(\mu)}q^{-\binom{\ell(\mu)}{2}-\binom{\ell(\mu)}{2}+\ell(\mu)(\ell(\mu)-1)-n-b(\mu)+\sum_i\binom{m_i(\mu)+1}{2}}\begin{bmatrix}\ell(\mu)\\ m(\mu)\end{bmatrix}_q\sum_\lambda s_\lambda K_{\lambda',\mu}(1/q)\\ &\qquad\times\sum_{\substack{k,\rho\\ \ell(\rho)=k-1,\,|\rho|=|\nu|}}q^{|\nu|+b(\rho)}\begin{bmatrix}\ell(\mu)-1\\ k-1\end{bmatrix}_q\begin{bmatrix}k-1\\ m(\rho)\end{bmatrix}_q K_{\nu,\rho}(q).\end{aligned}$$

If we can show that the coefficients of $s_\lambda K_{\lambda',\mu}(1/q)$ in the inner sums on both sides of (4.14) are equal for any $\mu\vdash n$ then (4.14), and hence Theorem 1.2, will follow. Replacing $\ell(\mu)$ by $j+1$, we see that this statement can be expressed as

$$(4.15)\qquad s_\nu(1,q,q^2,\dots,q^{j-1})=\sum_{\substack{k,\rho\\ \ell(\rho)=k-1,\,|\rho|=|\nu|}}q^{b(\rho)}\begin{bmatrix}j\\ k-1\end{bmatrix}_q\begin{bmatrix}k-1\\ m(\rho)\end{bmatrix}_q K_{\nu,\rho}(q),$$

for any nonnegative integer j.

To prove (4.15), multiply both sides of (4.15) by $s_\nu = s_\nu[\mathbf{x}]$ and sum over ν. Using the Cauchy identity, (4.15) is thus equivalent to

$$(4.16)\qquad h_n\left[(1-q^j)\frac{\mathbf{x}}{1-q}\right]=\sum_{\substack{k,\rho\\ \ell(\rho)=k-1,\,|\rho|=|\nu|}}q^{b(\rho)}\begin{bmatrix}j\\ k-1\end{bmatrix}_q\begin{bmatrix}k-1\\ m(\rho)\end{bmatrix}_q\sum_\nu s_\nu[\mathbf{x}]K_{\nu,\rho}(q),$$

for any nonnegative integer j. Using [3, Eqn. 14] this can be expressed as

$$(4.17)\qquad h_n\left[(1-q^j)\frac{\mathbf{x}}{1-q}\right]=\sum_{\rho,\,|\rho|=|\nu|}q^{n(\rho)}\begin{bmatrix}j\\ \ell(\rho)\end{bmatrix}_q[\ell(\rho)]!(1-q)^{\ell(\rho)}P_\rho\left[\frac{\mathbf{x}}{1-q};q\right],$$

for any nonnegative integer j. Here P_ρ is the Hall–Littlewood P-function. Making the transformations $\mathbf{x}\mapsto\mathbf{x}/(1-q)$ and $\mathbf{y}\mapsto 1-q^j$ in [3, Eqn. 17], another expression for the left-hand side of (4.17) is

$$(4.18)\qquad \sum_{\rho,\,|\rho|=|\nu|}P_\rho\left[\frac{\mathbf{x}}{1-q};q\right]Q'_\rho[1-q^j;q].$$

By [3, Lem. 3.3] we have

$$(4.19)\qquad Q'_\rho[1-q^j;q]=q^{b(\rho)}(1-q^j)(1-q^{j-1})\cdots(1-q^{j-\ell(\rho)+1}).$$

Using (4.19) in (4.18) and simplifying, we see that the right-hand side of (4.17) is the same as (4.18), which completes the proof. □

We want to prove the algebraic interpretation of $\Delta'_{s_\nu}e_n$ at $t=0$ given in Theorem 1.3. This interpretation is based on the q-reversal of the following symmetric function identity. We consider symmetric functions in two infinite variable sets: $\mathbf{x}$ and $\mathbf{y}$. We let $\omega_{\mathbf{x}}$ and $\omega_{\mathbf{y}}$ be the ω involution acting on the $\mathbf{x}$ and $\mathbf{y}$ variables (respectively).

Proposition 4.3. *Let $n, m \geq 0$. We have*

$$\sum_{\nu \vdash m} s_\nu(\mathbf{y}) \cdot \omega_{\mathbf{x}} \Delta'_{s_\nu} e_n(\mathbf{x})\big|_{t=0} = \sum_{k \geq 0} q^{m-k+1} \omega_{\mathbf{y}} C_{m,k-1}(\mathbf{y}; q) \cdot \omega_{\mathbf{x}} C_{n,k}(\mathbf{x}; q). \tag{4.20}$$

Proof. By Theorem 1.2,

$$\begin{aligned}
&\sum_{\nu \vdash m} s_\nu(\mathbf{y}) \cdot \omega_{\mathbf{x}} \Delta'_{s_\nu} e_n(\mathbf{x})\big|_{t=0} \\
&= \sum_{\nu \vdash m} s_\nu(\mathbf{y}) \cdot \omega_{\mathbf{x}} \sum_{k=\ell(\nu)+1}^{|\nu|+1} P_{\nu,k-1}(q) \sum_{\substack{\mu \vdash n \\ \ell(\mu)=k}} q^{\overline{b}(\mu)} \begin{bmatrix} k \\ m(\mu) \end{bmatrix}_q Q'_\mu(\mathbf{x}; q) \\
&= \sum_{\nu \vdash m} s_\nu(\mathbf{y}) \cdot \omega_{\mathbf{x}} \sum_{k=\ell(\nu)+1}^{|\nu|+1} P_{\nu,k-1}(q) \cdot \omega_{\mathbf{x}} C_{n,k}(\mathbf{x}; q) \\
&= \sum_{\nu \vdash m} \sum_{k=\ell(\nu)+1}^{|\nu|+1} q^{|\nu|-\binom{k}{2}} \sum_{\substack{\rho \vdash m \\ \ell(\rho)=k-1}} q^{b(\rho)} \begin{bmatrix} k-1 \\ m(\rho) \end{bmatrix}_q K_{\nu,\rho}(q) s_\nu(\mathbf{y}) \cdot \omega_{\mathbf{x}} C_{n,k}(\mathbf{x}; q) \\
&= \sum_{\nu \vdash m} \sum_{k=\ell(\nu)+1}^{|\nu|+1} q^{m-k+1} \sum_{\substack{\rho \vdash m \\ \ell(\rho)=k-1}} q^{\overline{b}(\rho)} \begin{bmatrix} k-1 \\ m(\rho) \end{bmatrix}_q K_{\nu,\rho}(q) s_\nu(\mathbf{y}) \cdot \omega_{\mathbf{x}} C_{n,k}(\mathbf{x}; q) \\
&= \sum_{\substack{\nu \vdash m \\ k \geq 1}} q^{m-k+1} \sum_{\substack{\rho \vdash m \\ \ell(\rho)=k-1}} q^{\overline{b}(\rho)} \begin{bmatrix} k-1 \\ m(\rho) \end{bmatrix}_q K_{\nu,\rho}(q) s_\nu(\mathbf{y}) \cdot \omega_{\mathbf{x}} C_{n,k}(\mathbf{x}; q) \\
&= \sum_{k \geq 1} q^{m-k+1} \sum_{\substack{\rho \vdash m \\ \ell(\rho)=k-1}} q^{\overline{b}(\rho)} \begin{bmatrix} k-1 \\ m(\rho) \end{bmatrix}_q \left[\sum_{\nu \vdash m} K_{\nu,\rho}(q) s_\nu(\mathbf{y}) \right] \cdot \omega_{\mathbf{x}} C_{n,k}(\mathbf{x}; q) \\
&= \sum_{k \geq 1} q^{m-k+1} \sum_{\substack{\rho \vdash m \\ \ell(\rho)=k-1}} q^{\overline{b}(\rho)} \begin{bmatrix} k-1 \\ m(\rho) \end{bmatrix}_q Q'_\rho(\mathbf{y}; q) \cdot \omega_{\mathbf{x}} C_{n,k}(\mathbf{x}; q) \\
&= \sum_{k \geq 1} q^{m-k+1} \omega_{\mathbf{y}} C_{m,k-1}(\mathbf{y}; q) \cdot \omega_{\mathbf{x}} C_{n,k}(\mathbf{x}; q),
\end{aligned}$$

which is what we wanted to prove. □

We want to q-reverse the identity of Proposition 4.3.

Proposition 4.4. *Let $n, m \geq 0$. We have*

$$\sum_{\nu \vdash m} q^{b(\nu)} s_\nu(\mathbf{y}) \cdot (\mathrm{rev}_q \circ \omega_{\mathbf{x}}) \Delta'_{s_\nu} e_n(\mathbf{x})\big|_{t=0} = \sum_{k \geq 0} q^{mn-km-kn+n+k(k-1)} D_{m,k-1}(\mathbf{y}; q) \cdot D_{n,k}(\mathbf{x}; q). \tag{4.21}$$

Proof. We want to q-reverse both sides of Equation (4.20). We begin with the left-hand side.

Claim: *For any partition $\nu \vdash m$, the q-degree of $\Delta'_{s_\nu} e_n|_{t=0}$ is $(n-1)m - b(\nu)$.*

To see why the Claim is true, let $\nu \vdash m$ and consider Equation (4.12), recapitulated here:

$$\Delta'_{s_\nu} e_n\big|_{t=0} = \sum_{\mu \vdash n} (-1)^{n-\ell(\mu)} s_\nu(q, q^2, \ldots, q^{\ell(\mu)-1}) q^{-n-2b(\mu)+\sum_i \binom{m_i(\mu)+1}{2}} \begin{bmatrix} \ell(\mu) \\ m(\mu) \end{bmatrix}_q \cdot \mathrm{rev}_q(Q'_\mu).$$

We know that Q'_μ (and also $\mathrm{rev}_q(Q'_\mu)$) has q-degree $b(\mu)$. If $\mu \vdash n$ is such that $\ell(\mu) > \ell(\nu)$ so that the μ-summand on the right-hand side of Equation (4.12) does not vanish, the q-degree of this μ-summand is therefore

$$\sum_{i=j}^{\ell(\nu)} \nu_j(\ell(\mu) - j) - n - 2b(\mu) + \sum_i \binom{m_i(\mu)+1}{2} + \sum_{i<j} m_i(\mu) m_j(\mu) + b(\mu), \tag{4.22}$$

or equivalently

$$m(\ell(\mu) - 1) - b(\nu) - n - b(\mu) + \sum_i \binom{m_i(\mu)+1}{2} + \sum_{i<j} m_i(\mu) m_j(\mu). \tag{4.23}$$

It is not hard to see that Expression (4.23) is maximized uniquely when $\mu = (1^n)$, in which case it equals

$$m(n-1) - b(\nu) - n - \binom{n}{2} + \binom{n+1}{2} = m(n-1) - b(\nu), \tag{4.24}$$

which completes the proof of the Claim.

Our Claim implies that the overall q-degree of the left-hand side of Equation 4.20 (and hence also the right-hand side) is $m(n-1)$; this corresponds to the summand $\nu = (m)$ so that $b(\nu) = 0$. The q-reversal of the left-hand side of Equation (4.20) is therefore

$$q^{m(n-1)} \sum_{\nu \vdash m} q^{b(\nu)-(n-1)m} s_\nu(\mathbf{y}) \cdot (\mathrm{rev}_q \circ \omega_{\mathbf{x}}) \Delta'_{s_\nu} e_n(\mathbf{x})\big|_{t=0}, \tag{4.25}$$

which coincides with the left-hand side of Equation (4.21).

Now we q-reverse the right-hand side of Equation (4.20). Since the q-degree of $C_{m,k-1}(\mathbf{y}; q)$ is $(k-2)m - \binom{k-1}{2}$ and the q-degree of $C_{n,k}(\mathbf{x}; q)$ is $(k-1)n - \binom{k}{2}$, the q-reversal of the right-hand side of Equation (4.20) is

$$q^{m(n-1)} \sum_{k \geq 0} q^{-m+k-1} \cdot \left[q^{-(k-2)m+\binom{k-1}{2}} D_{m,k-1}(\mathbf{y}; q) \right] \cdot \left[q^{-(k-1)n+\binom{k}{2}} D_{n,k}(\mathbf{x}; q) \right], \tag{4.26}$$

which is equivalent to the right-hand side of Equation (4.21). □

We are in a position to prove Theorem 1.3.

Proof of Theorem 1.3. If $\lambda \vdash m$ and $\mu \vdash n$ are any partitions, the Frobenius image of the irreducible $\mathfrak{S}_m \times \mathfrak{S}_n$-module $S^\lambda \otimes S^\mu$ is $\mathrm{Frob}(S^\lambda \otimes S^\mu) = s_\lambda(\mathbf{y}) \cdot s_\mu(\mathbf{x})$, regarded as an element of the ring $\Lambda(\mathbf{x}) \otimes \Lambda(\mathbf{y})$ of formal power series which are separately symmetric in the $\mathbf{x}$ and $\mathbf{y}$ variables.

More generally, if V is any finite-dimensional $\mathfrak{S}_m \times \mathfrak{S}_n$-module, there exist unique integers $c_{\lambda,\mu} \geq 0$ such that

$$V \cong_{\mathfrak{S}_m \times \mathfrak{S}_n} \bigoplus_{\lambda,\mu} c_{\lambda,\mu} S^\lambda \otimes S^\mu. \tag{4.27}$$

We then set

$$\mathrm{Frob}(V) := \sum_{\lambda,\mu} c_{\lambda,\mu} s_\lambda(\mathbf{y}) s_\mu(\mathbf{x}). \tag{4.28}$$

Finally, if $V = \bigoplus_{d\geq 0} V_d$ is a graded $\mathfrak{S}_m \times \mathfrak{S}_n$-module with each graded piece V_d finite-dimensional, we set

$$\mathrm{grFrob}(V;q) := \sum_{d\geq 0} \mathrm{Frob}(V) \cdot q^d. \tag{4.29}$$

If U is a graded $\mathfrak{S}_m$-module and W is a graded $\mathfrak{S}_n$-module, we have

$$\mathrm{grFrob}(U \otimes W;q) = \mathrm{grFrob}(U;q) \cdot \mathrm{grFrob}(W;q). \tag{4.30}$$

Recall that the $\mathfrak{S}_m \times \mathfrak{S}_n$-module $V_{n,m}$ is defined by

$$V_{n,m} = \bigoplus_{k\geq 0} (R_{m,k-1} \otimes R_{n,k})\{-mn + km + kn - n - k(k-1)\}. \tag{4.31}$$

Applying Equation (1.15), we see that the right-hand side of Equation (4.21) may be expressed as

$$\sum_{k\geq 0} q^{mn-km-kn+n+k(k-1)} D_{m,k-1}(\mathbf{y};q) \cdot D_{n,k}(\mathbf{x};q) = \mathrm{grFrob}(V_{n,m};q). \tag{4.32}$$

On the other hand, for any graded $\mathfrak{S}_m \times \mathfrak{S}_n$-module V and any partition $\nu \vdash m$, we have

$$\text{coefficient of } s_\nu(\mathbf{y}) \text{ in } \mathrm{grFrob}(V;q) = \mathrm{grFrob}(\mathrm{Hom}_{\mathfrak{S}_m}(S^\nu, V);q). \tag{4.33}$$

Therefore, we have

$$(\mathrm{rev}_q \circ \omega_{\mathbf{x}}) \Delta'_{s_\nu} e_n(\mathbf{x})\big|_{t=0} = q^{-b(\nu)} \cdot \mathrm{grFrob}(\mathrm{Hom}_{\mathfrak{S}_m}(S^\nu, V_{n,m});q), \tag{4.34}$$

which is what we wanted to prove. □

5. Closing remarks

In this paper we found an expansion of $\omega\Delta'_{s_\nu} e_n|_{t=0}$ in the dual Hall–Littlewood basis for any partition $\nu \vdash m$. This led to the algebraic interpretation of $(\mathrm{rev}_q \circ \omega)\Delta'_{s_\nu} e_n|_{t=0}$ presented in Theorem 1.3 involving tensor products of $R_{n,k}$ modules. It may be interesting to find a simpler module whose graded Frobenius image is $(\mathrm{rev}_q \circ \omega)\Delta'_{s_\nu} e_n|_{t=0}$.

Let $k \leq n$ be positive integers. The ring $R_{n,k}$ has the following geometric interpretation. Denote by $\mathbb{P}^{k-1}$ the $(k-1)$-dimensional complex projective space of lines through the origin in $\mathbb{C}^k$ and let $(\mathbb{P}^{k-1})^n$ denote the n-fold Cartesian product of $\mathbb{P}^{k-1}$ with itself. In joint work with Pawlowski [10], the second author defined the open subvariety $X_{n,k} \subseteq (\mathbb{P}^{k-1})^n$ given by

$$X_{n,k} := \{(\ell_1, \ldots, \ell_n) \in (\mathbb{P}^{k-1})^n \,:\, \ell_1 + \cdots + \ell_n = \mathbb{C}^k\}. \tag{5.1}$$

A typical point in $X_{n,k}$ is an n-tuple $(\ell_1, \ldots, \ell_n)$ of one-dimensional subspaces of $\mathbb{C}^k$ which together span $\mathbb{C}^k$.

The symmetric group $\mathfrak{S}_n$ acts on $X_{n,k}$ by the rule $\pi.(\ell_1, \ldots, \ell_n) := (\ell_{\pi_1}, \ldots, \ell_{\pi_n})$ for any permutation $\pi = \pi_1 \ldots \pi_n \in \mathfrak{S}_n$. If $H^\bullet(X_{n,k})$ denotes the singular cohomology of $X_{n,k}$ with integer coefficients, this gives rise to an action of $\mathfrak{S}_n$ on $H^\bullet(X_{n,k})$. Pawlowski and the second author prove [10] that

$$H^\bullet(X_{n,k}) = \mathbb{Z}[x_1, \ldots, x_n]/\langle e_n, e_{n-1}, \ldots, e_{n-k+1}, x_1^k, x_2^k, \ldots, x_n^k \rangle. \tag{5.2}$$

The identification (5.2) may be regarded as both an isomorphism of graded rings and an isomorphism of graded $\mathbb{Z}[\mathfrak{S}_n]$-modules. The variable x_i represents the Chern class $c_1(\ell_i^*)$ of the dual to the i^{th} tautological line bundle $\ell_i \twoheadrightarrow X_{n,k}$. In particular, we have $R_{n,k} = \mathbb{Q} \otimes_{\mathbb{Z}} H^\bullet(X_{n,k})$.

The isomorphism (5.2), together with the fact that $X_{n,n}$ is homotopy equivalent to the manifold $\mathcal{F}\ell(n)$ of complete flags in $\mathbb{C}^n$, justify the statement that $X_{n,k}$ is the flag variety attached to the Delta Conjecture (i.e., the Macdonald eigenoperator $\Delta'_{e_{k-1}}$). Given an arbitrary partition ν, it would be interesting to find an analogous variety $X_{n,\nu}$ with an action of $\mathfrak{S}_n$ which would play the corresponding role for the operator Δ'_{s_ν}. That is, the cohomology ring $H^\bullet(X_{n,\nu})$ should carry an action of $\mathfrak{S}_n$ such that (upon tensoring with $\mathbb{Q}$), the graded Frobenius image of this action is $(\mathrm{rev}_q \circ \omega)\Delta'_{s_\nu} e_n|_{t=0}$. The space $X_{n,k}$ solves this problem when $\nu = (1^{k-1})$. Theorem 1.3 might be helpful in constructing such a space $X_{n,\nu}$ in general.

We close by giving a geometric interpretation of Equation (4.20), recapitulated here:

$$\sum_{\nu \vdash m} s_\nu(\mathbf{y}) \cdot \omega_{\mathbf{x}} \Delta'_{s_\nu} e_n(\mathbf{x})\big|_{t=0} = \sum_{k \geq 0} q^{m-k+1} \omega_{\mathbf{y}} C_{m,k-1}(\mathbf{y}; q) \cdot \omega_{\mathbf{x}} C_{n,k}(\mathbf{x}; q).$$

If $M = M_0 \oplus M_1 \oplus \cdots \oplus M_d$ is any graded vector space with $M_d \neq 0$, let $\widetilde{M} = \widetilde{M_0} \oplus \widetilde{M_1} \oplus \cdots \oplus \widetilde{M_d}$ be the reversed graded vector space with components

$$\widetilde{M_i} := M_{d-i}, \quad 0 \leq i \leq d. \tag{5.3}$$

In terms of reversals of R-modules, Equation (4.20) reads

$$\omega \Delta'_{s_\nu} e_n\big|_{t=0} = \mathrm{grFrob}(\mathrm{Hom}_{\mathfrak{S}_n}(S^\nu, W_{n,m})), \tag{5.4}$$

where

$$W_{n,m} := \bigoplus_{k \geq 0} (\widetilde{R}_{m,k-1} \otimes \widetilde{R}_{n,k})\{-m + k - 1\}. \tag{5.5}$$

Taking the reversal $\widetilde{R}_{n,k}$ of the quotient $R_{n,k} := \mathbb{Q}[x_1, \ldots, x_n]/I_{n,k}$ is not a natural ring-theoretic operation, but it has a geometric interpretation in terms of the variety $X_{n,k}$.

Let $X^+_{n,k} = X_{n,k} \cup \{\infty\}$ denote the one-point compactification of $X_{n,k}$, where ∞ is the adjoined point. The *Borel–Moore homology* $\bar{H}_\bullet(X_{n,k})$ of $X_{n,k}$ is the (singular) homology of the pair $(X^+_{n,k}, \{\infty\})$:

$$\bar{H}_\bullet(X_{n,k}) := H_\bullet(X^+_{n,k}, \{\infty\}). \tag{5.6}$$

The action of $\mathfrak{S}_n$ on $R_{n,k}$ is both continuous and proper, and so induces a (graded) action of $\mathfrak{S}_n$ on the Borel–Moore homology $\bar{H}_\bullet(X_{n,k})$. By Poincaré duality and Equation (5.2), we have the isomorphism of $\mathfrak{S}_n$-modules

$$\widetilde{R}_{n,k} \cong_{\mathfrak{S}_n} \mathbb{Q} \otimes_{\mathbb{Z}} \bar{H}_\bullet(X_{n,k}). \tag{5.7}$$

This yields a geometric interpretation of Equation (4.20) in terms of Borel–Moore homology.

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