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ON THE ASYMPTOTICS OF KRONECKER COEFFICIENTS, 2

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Abstract. Kronecker coefficients encode the tensor products of complex irreducible representations of symmetric groups. Their stability properties have been considered recently by several authors (Vallejo, Pak and Panova, Stembridge). In [*J. Alg. Combin.* **42** (2015), 999–1025], we described a geometric method, based on Schur–Weyl duality, that allows one to produce huge series of instances of this phenomenon. In this note, we show how to go beyond these so-called additive triples. We show that the set of stable triples defines a union of faces of the cone generated by the supports of the nonzero Kronecker coefficients. Moreover, these faces may have different dimensions, and many of them have codimension one.

1. Introduction

The complex representation theory of symmetric groups is well understood: the irreducible representations, usually called Specht modules, are indexed by partitions, and their dimensions are given by the famous hook-length formula. However, the multiplicative structure of the representation ring has always remained elusive. The multiplicities in tensor products of Specht modules are called Kronecker coefficients. They are poorly understood and notoriously hard to compute. We refer to the introduction of [3] for a discussion of some of the most basic questions about Kronecker coefficients whose answers remain out of reach.

That Kronecker coefficients enjoy certain stability properties has been observed by Murnaghan in 1938 [4, 5]. Such properties are extremely surprising in that they involve representations of different groups, but they become less mysterious once translated in terms of representations of general linear groups, via Schur–Weyl duality. More stability phenomena have been discovered during the last twenty years, and the wealth of examples we are now aware of makes the need to understand and organize them better more urgent. This is one of the goals of this note.

We use the following terminology, taken from [7] and [3]. We denote by $[\lambda]$ the Specht module associated with the partition λ. This is an irreducible representation of the symmetric group S_n, if λ is a partition of n. Kronecker coefficients are defined by the identity

$$[\lambda]\otimes[\mu]=\bigoplus_{\nu} g(\lambda,\mu,\nu)\,[\nu].$$

They are symmetric in λ,μ,ν and, of course, non-negative.

Key words and phrases. Symmetric group, Kronecker coefficient, stability, Schur–Weyl duality, Borel–Weil theorem, face, facet, simplicial.

Definition. A triple of partitions (λ,μ,ν) is *stable* if the Kronecker coefficients satisfy

$$g(k\lambda,k\mu,k\nu)=1 \quad \text{for all } k\geq 1.$$

Equivalently, (λ,μ,ν) is stable if $g(\lambda,\mu,\nu)\neq 0$ and for any triple (α,β,γ), the sequence of Kronecker coefficients $g(\alpha+k\lambda,\beta+k\mu,\gamma+k\nu)$ is bounded independently of k, or equivalently, eventually constant. We call the asymptotic value of this coefficient a stable Kronecker coefficient.

These two properties were called weak stability and stability in [7], where it was proved that stability implies weak stability. The reverse implication has recently been established in [6].

In order to get nice finiteness properties we restrict to partitions whose length, rather than size, is bounded; the length $\ell(\lambda)$ of a partition λ is the number of non-zero parts. We would then like to understand stability phenomena in relation with the Kronecker semigroup and the Kronecker polyhedron. The former is

$$\mathrm{Kron}_{a,b,c}:=\{(\lambda,\mu,\nu),\ \ell(\lambda)\leq a,\ \ell(\mu)\leq b,\ \ell(\nu)\leq c,\ g(\lambda,\mu,\nu)\neq 0\}.$$

This is a finitely generated semigroup. A more precise version of the semigroup property is the elementary, but useful *monotonicity property*: if $g(\lambda,\mu,\nu)\neq 0$, then, for any triple (α,β,γ),

$$g(\alpha+\lambda,\beta+\mu,\gamma+\nu)\geq g(\alpha,\beta,\gamma).$$

The semigroup $\mathrm{Kron}_{a,b,c}$ lives inside a codimension two sublattice of $\mathbf{Z}^{a+b+c}$, because of the obvious condition $|\lambda|=|\mu|=|\nu|$ for a Kronecker coefficient $g(\lambda,\mu,\nu)$ to be non-zero. We call this lattice the *weight lattice*. The cone generated by $\mathrm{Kron}_{a,b,c}$ is a rational polyhedral cone $\mathrm{PKron}_{a,b,c}$, that we call the Kronecker polyhedron. It is defined by some finite list of linear inequalities, corresponding to the equations of its facets (the maximal faces, of codimension one). The number of facets is huge already for small values of the parameters, and certainly grows exponentially with a,b,c (see [1] and [9]).

In [2, 3], we showed that all lattice points in certain minimal faces of the Kronecker polyhedron are stable triples. These minimal faces were defined in terms of certain standard tableaux with the *additivity property*, and we call them *additive faces*. Let us suppose for simplicity that $c=ab$ (this is not a restriction since it is well known that, for a Kronecker coefficient $g(\lambda,\mu,\nu)$ to be non-zero, the condition $\ell(\lambda)\leq\ell(\mu)\ell(\nu)$ on the lengths is required). Consider a standard tableau T of rectangular shape $a\times b$. Such a tableau is *additive* if there exist increasing sequences $x_1<\cdots<x_a$ and $y_1<\cdots<y_b$ of real (or rational) numbers such that

$$T(i,j)<T(k,l) \quad \text{if and only if} \quad x_i+y_j<x_k+y_l.$$

The main stability result in [3] was the following.

Proposition 1. *Let T be an additive standard tableau of rectangular shape $a\times b$. For a partition $\lambda=(\lambda_1,\dots,\lambda_{ab})$, define two partitions $a_T(\lambda),b_T(\lambda)$ by*

$$a_T(\lambda)_i=\sum_{j=1}^{b}\lambda_{T(i,j)} \quad \text{and} \quad b_T(\lambda)_j=\sum_{i=1}^{a}\lambda_{T(i,j)}.$$

Then $(\lambda,a_T(\lambda),b_T(\lambda))$ is a stable triple.

Moreover, the set of these additive triples, for a fixed T, is exactly the set of lattice points inside a minimal face f_T of $\mathrm{PKron}_{a,b,ab}$ defined by this standard tableau.

We want to stress here that the fact that stable triples can be related to faces of the Kronecker polyhedron is by no means a surprise. A general statement is the following.

Proposition 2. *The set* $\mathrm{SKron}_{a,b,c}$ *of weakly stable triples in* $\mathrm{Kron}_{a,b,c}$ *is the intersection of* $\mathrm{Kron}_{a,b,c}$ *with a union of faces of* $\mathrm{PKron}_{a,b,c}$.

More generally, one can associate with any face of the Kronecker polyhedron a positive integer, which gives the order of growth of the stretched Kronecker coefficients on the interior of the face. This will be discussed in the last section of this paper.

Concentrating on (weakly) stable triples, it is natural to try to describe the faces of $\mathrm{PKron}_{a,b,c}$ that are maximal in $\mathrm{SKron}_{a,b,c}$. We will call the faces of $\mathrm{PKron}_{a,b,c}$ whose intersection with $\mathrm{Kron}_{a,b,c}$ is contained in $\mathrm{SKron}_{a,b,c}$ the *stable faces*, and those that are maximal in $\mathrm{SKron}_{a,b,c}$ the *maximal stable faces* (which we do not expect a priori to be maximal faces in $\mathrm{PKron}_{a,b,c}$, or facets). Among many other questions, we can ask: what is the maximal dimension of such a face? What can be their dimensions? Could they all be of the same dimension? Can the additive stable faces be maximal in $\mathrm{SKron}_{a,b,c}$? More generally, what are the stable faces containing a given additive stable face?

The main goal of this note is to answer some of these questions, in particular the last one, and draw some unexpected consequences. In [3] we explained how to describe the local structure of the Kronecker polyhedron around an additive face. Among the faces that contain such an additive face, we will distinguish those that have a property that we will call *strong simpliciality*. We will prove the following result.

Theorem 1. *Among the faces of* $\mathrm{PKron}_{a,b,c}$ *that contain an additive face, the stable ones are exactly those that are strongly simplicial.*

The fact that strongly simplicial faces are stable will be proved in Theorem 2, and Theorem 3 will provide the reverse implication.

A priori, we would have expected the stable faces to be very special, in particular to have high codimension. Surprisingly, our theorem has the following consequence.

Corollary 1. *The polyhedral cone* $\mathrm{PKron}_{a,b,c}$ *always contains stable facets.*

This means that there exist families of stable triples of the largest possible dimension. It would be extremely interesting to have a full classification. We can give many explicit examples of strongly simplicial facets and show that there always exist many of them (Proposition 4). We can also describe their structure, which is that of a cone over a hypercube (Proposition 3). The vertices of this hypercube are in bijection with the additive faces contained in the facet.

Another striking phenomenon is the following. Consider an additive face, and the maximal stable faces that contain it. It may very well happen that these maximal faces have different dimensions! In fact, it seems quite plausible that the maximal stable faces can have all the possible dimensions between the smallest and maximal possible dimensions. In particular, the set of stable triples seems to have a very intricate structure in general.

2. Strongly simplicial faces

2.1. **The geometric method.** Let us briefly recall the main features of the geometric method used in [2, 3] in order to approach Kronecker coefficients. Let A, B be complex vector spaces of finite dimensions a, b. By Schur–Weyl duality, Kronecker coefficients are the multiplicities of the Schur powers $S_\lambda(A\otimes B)$, when decomposed into irreducible representations for $GL(A)\times GL(B)$. By the Borel–Weil theorem,

$$S_\lambda(A\otimes B)=H^0(\mathrm{Fl}(A\otimes B),L_\lambda)$$

for a suitable linearized line bundle L_λ on the variety $\mathrm{Fl}(A\otimes B)$ of complete flags in $A\otimes B$. A standard tableau T defines an embedding

$$\iota_T:\mathrm{Fl}(A)\times\mathrm{Fl}(B)\hookrightarrow\mathrm{Fl}(A\otimes B).$$

The induced map on equivariant Picard groups

$$\iota_T^*:Pic(\mathrm{Fl}(A\otimes B))\simeq\mathbf{Z}^{ab}\to Pic(\mathrm{Fl}(A)\times\mathrm{Fl}(B))\simeq\mathbf{Z}^a\times\mathbf{Z}^b$$

is precisely our map $\lambda\mapsto(a_T(\lambda),b_T(\lambda))$ when expressed in the natural basis. In particular, restriction gives a non-zero map

$$H^0(\mathrm{Fl}(A\otimes B),L_\lambda)\longrightarrow H^0(\mathrm{Fl}(A)\times\mathrm{Fl}(B),L_{a_T(\lambda)}\otimes L_{b_T(\lambda)})=S_{a_T(\lambda)}A\otimes S_{b_T(\lambda)}B,\quad(2.1)$$

implying that the Kronecker coefficient $g(\lambda,a_T(\lambda),b_T(\lambda))$ is positive. Then we can define a filtration of $H^0(\mathrm{Fl}(A\otimes B),L_\lambda)$ by the order of vanishing on $\mathrm{Fl}(A)\times\mathrm{Fl}(B)$. This allows us to define an injective map

$$H^0(\mathrm{Fl}(A\otimes B),L_\lambda)\hookrightarrow H^0(\mathrm{Fl}(A)\times\mathrm{Fl}(B),L_{a_T(\lambda)}\otimes L_{b_T(\lambda)}\otimes S^*N^*),\qquad(2.2)$$

where N denotes the normal bundle of the embedding ι_T, and S^*N^* is the symmetric algebra of the dual bundle, the conormal bundle. This map must be thought of as taking a section of L_λ to its Taylor expansion in the normal directions to $\mathrm{Fl}(A)\times\mathrm{Fl}(B)$. Moreover, if λ is strictly decreasing, the line bundle L_λ is very ample. By the usual properties of ample bundles, the previous map becomes surjective onto every finite part of S^*N^* if L_λ is sufficiently ample (that is, if the differences $\lambda_i-\lambda_{i+1}$ are large enough). This shows that the multiplicities in $S_\lambda(A\otimes B)=H^0(\mathrm{Fl}(A\otimes B),L_\lambda)$ — in alternative terms, the Kronecker coefficients — are somehow controlled by the normal bundle.

This works particularly well when the embedding ι_T is *convex*, in the sense that the weights of the normal bundle are contained in a strictly convex cone. Combinatorially, this exactly means that the tableau T is additive. Then the Kronecker coefficient $g(\alpha+k\lambda,\beta+ka_T(\lambda),\gamma+kb_T(\lambda))$ is bounded by the multiplicity of $L_{\beta-a_T(\alpha)}\otimes L_{\gamma-b_T(\alpha)}$ inside S^*N^*, and the latter is finite by convexity. This implies that we can focus on a finite part of this algebra, independently of k. However, if λ is strict and k is large enough, the surjectivity of (2.2) in finite degrees implies that we have equality between the latter multiplicity and the Kronecker coefficient. In particular, this coefficient does not depend on k, when big enough: this is the stability phenomenon. Of course, we get much more information since we are in principle able to compute the stable Kronecker coefficients, directly from the normal bundle.

Combinatorially, the weights of the conormal bundle are determined as follows. Denote by $e_1,\dots,e_a$ and $f_1,\dots,f_b$ bases of the character lattices of maximal tori in $GL(A)$ and $GL(B)$. If $T(i,j)=k$ (i.e., the box (i,j) is numbered k in T), let $g_k=e_i+f_j$. Then the weights of the normal bundle are the differences $g_\ell-g_k$ for $\ell>k$. Among these weights, the horizontal and vertical ones are those of the form $e_p - e_q$ and $f_p - f_q$. They will appear repeatedly, in fact, in the conormal bundle their multiplicities are b for the horizontal ones and a for the vertical ones. In particular, these multiplicities will be bigger than one as soon as we suppose that $a, b > 2$. All the other weights have multiplicity one. Of course, the multiplicity of $L_{\beta - a_T(\alpha)} \otimes L_{\gamma - b_T(\alpha)}$ inside S^*N^*, which gives the stable Kronecker coefficient, can be obtained as the number of ways to express the weight $(\beta - a_T(\alpha), \gamma - b_T(\alpha))$ as a non-negative linear combination of the weights of the conormal bundle, considered with their multiplicities.

Obviously, this is possible only when $(\beta - a_T(\alpha), \gamma - b_T(\alpha))$ belongs to the cone generated by these weights, which we call the *conormal cone*. This cone gives a local picture of the Kronecker polyhedron locally around f_T. In particular, any face of the latter containing f_T can be identified with a face of the conormal cone, and conversely.

Let us summarize these results, which will be used repeatedly in the remaining of the paper.

Lemma 1. (1) *The lattice points in the additive face f_T are the triples $(\lambda, a_T(\lambda), b_T(\lambda))$.*
(2) *The conormal cone is the cone generated by the vectors $g_\ell - g_k$, $\ell > k$.*
(3) *The Kronecker coefficient $g(\alpha + k\lambda, \beta + ka_T(\lambda), \gamma + kb_T(\lambda))$ is bounded by the integer $V_T(\beta - a_T(\alpha), \gamma - b_T(\beta))$, where $V_T(\sigma, \theta)$ is the number of ways of decomposing (σ, θ) as a non-negative linear combination of the vectors $g_\ell - g_k$ $\ell > k$, taking into account multiplicities.*

2.2. **Perturbations of additive triples.** Now consider a triple of the form $(\lambda, a_T(\lambda) + \sigma, b_T(\lambda) + \theta)$, where σ and θ are not necessarily partitions, but sequences (or weights) such that $a_T(\lambda) + \sigma$ and $b_T(\lambda) + \theta$ are partitions.

By the injectivity of (2.2), the Kronecker coefficient $g(k\lambda, k(a_T(\lambda) + \sigma), k(b_T(\lambda) + \theta))$ is bounded by the multiplicity of $(k\sigma, k\theta)$ as a weight of S^*N^*. If we suppose that the line generated by (σ, θ) belongs to the conormal cone, this multiplicity will eventually become positive, and we expect it to grow to infinity with k. However, this is not necessarily the case: the multiplicity will remain bounded if (σ, θ) belongs to a face of the cone which is *strongly simplicial*. By this we mean that the weights of the conormal bundle contained in the face, considered with their multiplicities, define a basis of the linear space generated by the face. Then the multiplicity will be 0 or 1, the second possibility occurring exactly when $(k\sigma, k\theta)$ belongs to the lattice generated by the latter weights.

Let us insist on the definition of strongly simplicial faces.

Definition. A face F of the Kronecker polytope is *strongly simplicial* if it contains an additive face f_T such that, locally around f_T, the face F corresponds to a face of the conormal cone which is strongly simplicial in the sense that:

(1) it is a face of dimension d generated by d vectors $g_{k_1+1} - g_{k_1}, \ldots, g_{k_d+1} - g_{k_d}$;
(2) none of these vectors is horizontal or vertical;
(3) no other vector of the form $g_p - g_q$ belongs to the face;
(4) in particular, the pairs $\{k_1, k_1 + 1\}, \ldots, \{k_d, k_d + 1\}$ do not intersect.

The structure of a strongly simplicial face is not difficult to describe. Recall that an additive tableau T is defined by parameters $x_1 < \cdots < x_a$ and $y_1 < \cdots < y_b$ such that

$T(i,j) < T(k,l)$ if and only if $x_i + y_j < x_k + y_l$. Of course, these parameters are not unique. In fact, the tableau T really corresponds to a connected component C_T of the complement of the collection of hyperplanes defined by the equations $x_i + y_j = x_k + y_l$ inside the parameter space.

Locally around the additive face f_T, the Kronecker polyhedron is, by hypothesis, the simplicial cone over the vectors $g_{k_1+1} - g_{k_1}, \ldots, g_{k_d+1} - g_{k_d}$. Let us choose one of them, say $g_{k_s+1} - g_{k_s}$. Since it is neither horizontal nor vertical, we can exchange the entries k_s and $k_s + 1$ in T and obtain another standard tableau T_s. We claim that T_s is again additive. Indeed, observe that C_T is the dual cone of the normal cone at T. Suppose that the entries k_s and k_s+1 of T appear in boxes (i,j) and (k,l), so that the half-space $x_i + x_j < x_k + x_l$ is the dual cone of the ray generated by $g_{k_s+1} - g_{k_s}$. The fact that $g_{k_s+1} - g_{k_s}$ is an extremal vector of the cone implies that the hyperplane $x_i + y_j = x_k + y_l$ is really a facet of C_T. Crossing this facet we get into a component corresponding to T_s, which is therefore also additive.

Iterating the process, we deduce that the 2^d standard tableaux obtained by considering all the possibilities to exchange the entries $(k_1, k_1+1), \ldots, (k_d, k_d+1)$, are all additive. Moreover, the Kronecker polyhedron, around each of the corresponding additive faces, is described by the same cone, up to a change of signs for the generators. Since

$$(a_{T_s}(\lambda), b_{T_s}(\lambda)) = (a_T(\lambda), b_T(\lambda)) + (\lambda_{k_s} - \lambda_{k_s+1})(g_{k_s+1} - g_{k_s}),$$

this implies that the intersection of $\mathrm{Kron}_{a,b,c}$ with our strongly simplicial face is equal to the set of triples

$$(\lambda, \mu, \nu) = (\lambda, a_T(\lambda), b_T(\lambda)) + \sum_{i=1}^{d} u_s(g_{k_s+1} - g_{k_s}), \tag{2.3}$$

with $0 \le u_s \le \lambda_{k_s} - \lambda_{k_s+1}$, these coefficients $u_1, \ldots, u_d$ being all integers. Note that it would a priori be possible that we get a triple of partitions (λ, μ, ν) given by the same expression but with rational coefficients $u_1, \ldots, u_d$, not all integral. In this case, the Kronecker coefficient $g(\lambda, \mu, \nu)$ is zero.

In other words, the identity (2.3) defines a lattice L_F, which could be a proper sublattice of the intersection of the weight lattice with the linear span of F. In this lattice, F is simply defined by the inequalities $0 \le u_s \le \lambda_{k_s} - \lambda_{k_s+1}$ for $1 \le s \le d$. Recall that d is the number of generators of the face in the normal directions of an additive face it contains. In particular, the codimension of F is the codimension of an additive face (that is, $a + b - 2$) minus d. We get the following description of strongly simplicial faces.

Proposition 3. *A strongly simplicial face F of codimension δ in the Kronecker polyhedron is a cone over a hypercube of dimension $a + b - 2 - \delta$.*

The main result of this paper is the following.

Theorem 2. *A strongly simplicial face F of the Kronecker polyhedron is stable. More precisely, a point in F is a stable triple if it belongs to L_F, and the corresponding Kronecker coefficient is zero otherwise.*

Proof. Consider a triple of the form $(k\lambda + \alpha, k(a_T(\lambda) + \sigma) + \beta, k(b_T(\lambda) + \theta)) + \gamma)$, where as before (σ, θ) belongs to the simplicial face corresponding to F in the conormal cone of the additive face f_T. As we have seen, the corresponding Kronecker coefficient is bounded by the multiplicity of $(k\sigma+\beta-a_T(\alpha), k\theta+\gamma-b_T(\alpha))$ as a weight of S^*N^*. Suppose that we have expressed this weight as a non-negative integer linear combination $t_1\eta_1+\cdots+t_N\eta_N$ of the weights $\eta_1,\ldots,\eta_N$ of the conormal bundle, considered with their multiplicities. Suppose these weights are indexed in such a way that the first d generate our simplicial face. By projection along the direction of this face, we get a relation of the form

$$p_F(\beta-a_T(\alpha),\gamma-b_T(\alpha)) = t_{d+1}p_F(\eta_{d+1})+\cdots+t_Np_F(\eta_N), \tag{2.4}$$

where p_F denotes the projection. However, the projected weights $p_F(\eta_{d+1}),\ldots,p_F(\eta_N)$ generate a strictly convex cone, so the latter equation has only finitely many non-negative integer solutions $(t_{d+1},\ldots,t_N)$. These solutions do not depend on k, and, for each of these, the original equation has at most one solution in $(t_1,\ldots,t_d)$ since it can be considered as an equation in the simplicial face F. This proves that the Kronecker coefficient $g(k\lambda+\alpha, k(a_T(\lambda)+\sigma)+\beta, k(b_T(\lambda)+\theta))+\gamma)$ is bounded independently of k.

This is precisely the definition of stability, up to the fact that the Kronecker coefficient $g(\lambda, a_T(\lambda)+\sigma, b_T(\lambda)+\theta)$ must be equal to one. Recall that by [7, Prop. 3.2], the only alternative is that it is equal to zero. So what remains to prove is that, if (λ,μ,ν) is a point of F that also belongs to the lattice L_F, the Kronecker coefficient $g(\lambda,\mu,\nu)$ cannot be zero.

To check this, we will use that (λ,μ,ν) is given by (2.3) for some integer coefficients $u_1,\ldots,u_d$ such that $0\le u_s\le \lambda_{k_s}-\lambda_{k_s+1}$ for all s. Denote the partition of t with t parts equal to one by ω_t. Recall that we denoted the standard tableau obtained by exchanging the entries k_s and k_s+1 in T by T_s. It is straightforward to check that

$$(\omega_{k_s}, a_{T_s}(\omega_{k_s}), b_{T_s}(\omega_{k_s})) - (\omega_{k_s}, a_T(\omega_{k_s}), b_T(\omega_{k_s})) = g_{k_s+1}-g_{k_s}.$$

This allows us to rewrite (2.3) as

$$(\lambda,\mu,\nu) = (\theta, a_T(\theta), b_T(\theta)) + \sum_{s=1}^{d} u_s(\omega_{k_s}, a_{T_s}(\omega_{k_s}), b_{T_s}(\omega_{k_s})),$$

where $\theta=\lambda-\sum_{s=1}^d u_s\omega_{k_s}$. Since $u_s\le\lambda_{k_s}-\lambda_{k_s+1}$ for all s, this θ is again a partition. Since T and the T_s are all additive, we know that $g(\theta,a_T(\theta),b_T(\theta))=1$ and $g(\omega_{k_s}, a_{T_s}(\omega_{k_s}), b_{T_s}(\omega_{k_s}))=1$ for all s. In particular, all these Kronecker coefficients are non-zero, and from the semigroup property we deduce that $g(\lambda,\mu,\nu)$ is positive. □

Corollary 2. *A strongly simplicial face is the non-negative integral span of the additive triples it contains. Moreover, any additive face is properly contained in some strongly simplicial face.*

Proof. The first statement means that any stable triple in F can be obtained as a linear combination with positive integer coefficients, of some stable triples in the additive faces contained in F. This is what we established in the proof of Theorem 2.

For the second statement, simply observe that at least one face of the Kronecker polyhedron that contains f_T and has dimension one more must be simplicial. Indeed, these faces correspond to the minimal generators of the conormal cone, and they are simplicial exactly for those generators that are neither horizontal nor vertical. However, the generators cannot be all horizontal or vertical, since otherwise, inside T, the integer $k+1$ would always be South-East of k, which is absurd. □

Remarks. One can wonder if there can be non-trivial arithmetic conditions on the strongly simplicial faces, for the Kronecker coefficients to be non-zero? This would mean that L_F is really a proper sublattice of the intersection of F with the weight lattice. This seems a priori possible but we have no example of such a phenomenon.

One can also wonder if stable Kronecker coefficients, when one considers strongly simplicial faces, count points in some polytopes, as they do on additive faces [3]. In the proof above, we indeed bounded the stretched Kronecker coefficients by numbers of points in some polytopes, but it is not clear that this bound coincides with the stable Kronecker coefficient. In the additive case, this follows from an ampleness argument, which does not apply in this more general situation.

2.3. **Strongly simplicial facets.** In [3] we gave a combinatorial description of the facets of the Kronecker polytope containing a given additive face f_T. These facets are in bijection with the *maximal relaxations* compatible with T, where a maximal relaxation R is given by an additive (non-standard) tableau defined by sequences $x_1 \le \cdots \le x_a$ and $y_1 \le \cdots \le y_b$ such that the sums $R(i,j) = x_i + y_j$ are not necessarily distinct. What we require is that the set of vectors $e_i + f_j - e_k - f_l$, for $R(i,j) = R(k,l)$, has maximal rank $r = a+b-3$. Such a family of vectors being given, the sequences $x_1 \le \cdots \le x_a$ and $y_1 \le \cdots \le y_b$ are uniquely defined up to translation and multiplication by the same positive number. It is convenient to define the tableau R uniquely by letting $x_1 = y_1 = 0$ and asking the two sequences to be made of integers, with no common divisor. The compatibility condition with a standard tableau T is that $R(i,j) < R(k,l)$ implies $T(i,j) < T(k,l)$. In other words, R defines a partial order on the boxes in the rectangle $a \times b$, which is refined by the total order defined by T. The equation of the facet F_R is then given by

$$\sum_{i=1}^{a} x_i \mu_i + \sum_{j=1}^{b} y_j \nu_j = \sum_{i=1}^{a} \sum_{j=1}^{b} (x_i + y_j) \lambda_{T(i,j)},$$

where T is some standard tableau compatible with R.

Can such a maximal relaxation R define a strongly simplicial facet? This would mean that R is defined by strictly increasing sequences, and that there exist exactly $r = a+b-3$ values of R appearing twice, the corresponding difference vectors being independent. In terms of the hyperplanes of equations $x_i + y_j - x_k - y_l = 0$ and the arrangement they define in the open cone defined by $0 = x_1 < \cdots < x_a$ and $0 = y_1 < \cdots < y_b$, such an R corresponds to a point where exactly r hyperplanes meet transversally. Recall that this transversality property implies that any of the 2^r standard tableaux T compatible with R is additive.

Another unexpected fact is that, in general, there exist surprisingly many strongly simplicial facets!

Proposition 4. *$PKron(a,b,ab)$ contains at least $\binom{a+b-4}{b-2}$ strongly simplicial facets.*

Proof. One can construct tableaux defining strongly simplicial facets by a simple inductive procedure: suppose that a tableau S defines a simplicial facet for the shape $a \times (b-1)$. Then we get one for the shape $a \times b$ by adding a column defined by $y_b = x_a + y_{b-1}$. Of course, this also works for rows. So starting from the tableau defining the unique simplicial face in shape 2×2, we can construct $\binom{a+b-4}{b-2}$ strongly simplicial facets in shape $a \times b$ by choosing to apply the previous process on rows or columns successively, in all possible orders. $\square$

2.4. **Examples.** Let us examine in more detail the low dimension cases.

Example 1. For $a = b = 2$ there is exactly one additive face (up to symmetry). This additive face is the intersection of two facets, one of which is strongly simplicial. On the additive face, we get

$$g((\lambda_1, \lambda_2, \lambda_3, \lambda_4), (\lambda_1 + \lambda_2, \lambda_3 + \lambda_4), (\lambda_1 + \lambda_3, \lambda_2 + \lambda_4)) = 1,$$

and for the strongly simplicial facet we get the more general statement that

$$g((\lambda_1, \lambda_2, \lambda_3, \lambda_4), (\mu_1, \mu_2), (\nu_1, \nu_2)) = 1$$

when $\mu_1 - \nu_2 = \lambda_1 - \lambda_4$ and $\lambda_1 + \lambda_3 \leq \mu_1 \leq \lambda_1 + \lambda_2$. Moreover, all these triples are stable.

Example 2. For $a = b = 3$ there exist 42 standard tableaux fitting in a square of size three, among which 36 are additive. The number of maximal relaxations is 17. They are encoded in the following tableaux:

$$F_1^+ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{matrix} \qquad F_2^+ \begin{matrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{matrix} \qquad F_3^+ \begin{matrix} 0 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 2 & 3 \end{matrix} \qquad F_4^+ \begin{matrix} 0 & 1 & 1 \\ 1 & 2 & 2 \\ 2 & 3 & 3 \end{matrix} \qquad F_5^+ \begin{matrix} 0 & 1 & 2 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{matrix}$$

$$F_6 \begin{matrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 2 \end{matrix} \qquad F_7^+ \begin{matrix} 0 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{matrix} \qquad F_8 \begin{matrix} 0 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 2 \end{matrix} \qquad F_9 \begin{matrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{matrix} \qquad F_{10}^+ \begin{matrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 4 & 5 \end{matrix}$$

and for each tableau F_i^+ there is another one denoted F_i^- obtained by a reflection in the diagonal.

Recall that additive faces have dimension four. A detailed analysis yields the following result.

Proposition 5. *For $a = b = 3$, the maximal strongly simplicial faces are, up to reflection in the diagonal:*

1. *in codimension one, F_5^+ and F_{10}^+;*
2. *in codimension two, $F_3^+ \cap F_4^+$, $F_3^+ \cap F_9$, $F_4^+ \cap F_9$;*
3. *in codimension three, $F_6 \cap F_7^+ \cap F_9$ and $F_7^+ \cap F_8 \cap F_9$.*

Let us describe the sets of triples (λ, μ, ν) on these strongly simplicial faces. We will use the notations $\lambda_{ij} = \lambda_i + \lambda_j$ and $\lambda_{ijk} = \lambda_i + \lambda_j + \lambda_k$.

F_5^+ is defined by the equation

$$2\mu_2 + 3\mu_3 + \nu_2 + 2\nu_3 = \lambda_2 + 2\lambda_3 + 2\lambda_4 + 3\lambda_5 + 3\lambda_6 + 4\lambda_7 + 4\lambda_8 + 5\lambda_9$$

and the inequalities

$$\lambda_{124} \leq \mu_1 \leq \lambda_{123},$$
$$\lambda_{123} + \lambda_{146} \leq \mu_1 + \nu_1 \leq \lambda_{123} + \lambda_{145},$$
$$\lambda_{12} - \lambda_{79} \leq \mu_1 - \nu_3 \leq \lambda_{12} - \lambda_{89}.$$

F_{10}^+ is defined by the equation

$$\mu_2+3\mu_3+\nu_2+2\nu_3=\lambda_2+\lambda_3+2\lambda_4+2\lambda_5+3\lambda_6+3\lambda_7+4\lambda_8+5\lambda_9$$

and the inequalities

$$\lambda_{13}-\lambda_{89}\le\nu_1-\mu_3\le\lambda_{12}-\lambda_{89},$$
$$\lambda_{569}+\lambda_{789}\le\mu_3+\nu_3\le\lambda_{469}+\lambda_{789},$$
$$\lambda_{789}\le\mu_3\le\lambda_{689}.$$

$F_3^+\cap F_4^+$ is defined by the two equations

$$\nu_2=\lambda_{258}\quad\text{and}\quad\mu_2+2\mu_3+\nu_3=\lambda_{369}+\lambda_{456}+2\lambda_{789}$$

and the inequalities

$$\lambda_{124}\le\mu_1\le\lambda_{123}\quad\text{and}\quad\lambda_{789}\le\mu_3\le\lambda_{689}.$$

$F_3^+\cap F_5$ is defined by the two equations

$$\nu_1=\lambda_{136}\quad\text{and}\quad\mu_1-\mu_3+\nu_2=\lambda_{124}+\lambda_{258}-\lambda_{689}$$

and the inequalities

$$\lambda_{125}\le\mu_1\le\lambda_{124}\quad\text{and}\quad\lambda_{689}\le\mu_3\le\lambda_{679}.$$

$F_4^+\cap F_5$ is defined by the two equations

$$\nu_3=\lambda_{479}\quad\text{and}\quad\mu_1-\mu_3-\nu_2=\lambda_{124}-\lambda_{258}-\lambda_{689}$$

and the inequalities

$$\lambda_{134}\le\mu_1\le\lambda_{124}\quad\text{and}\quad\lambda_{689}\le\mu_3\le\lambda_{589}.$$

$F_6\cap F_7^+\cap F_9$ is defined by the three equalities

$$\mu_1+\nu_1=\lambda_{125}+\lambda_{136},\ \mu_2=\lambda_{348},\ \nu_2=\lambda_{247}$$

and the inequalities

$$\lambda_{126}\le\mu_1\le\lambda_{125}.$$

$F_7^+\cap F_8\cap F_9$ is defined by the three equalities

$$\mu_1+\nu_1=\lambda_{124}+\lambda_{135},\ \mu_2=\lambda_{356},\ \nu_2=\lambda_{267}$$

and the inequalities

$$\lambda_{125}\le\mu_1\le\lambda_{124}.$$

There are no arithmetic constraints on these strongly simplicial faces, so the Kronecker coefficients are always equal to one, and all these triples are stable.

Note also that the additive face defined by the standard tableau

$$\begin{matrix}1&2&3\\4&5&7\\6&8&9\end{matrix}$$

is contained in both F_5^+ and $F_3^+\cap F_4^+$, showing that an additive face can be contained in two maximal strongly simplicial faces of different dimensions! This indicates that the structure of the set of additive triples must be quite intricate in general.

Example 3. For $a = b = 4$ there are 6660 additive tableaux and 457 maximal relaxations, according to [1, 9]. Among these, we know 43 strongly simplicial ones, among which

$$\begin{array}{cccc} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 7 & 8 & 9 & A \end{array} \qquad \begin{array}{cccc} 0 & 1 & 2 & 5 \\ 1 & 2 & 3 & 6 \\ 3 & 4 & 5 & 8 \\ 7 & 8 & 9 & C \end{array} \qquad \begin{array}{cccc} 0 & 1 & 2 & 5 \\ 1 & 2 & 3 & 6 \\ 3 & 4 & 5 & 8 \\ 8 & 9 & A & D \end{array} \qquad \begin{array}{cccc} 0 & 1 & 2 & 5 \\ 2 & 3 & 4 & 7 \\ 3 & 4 & 5 & 8 \\ 8 & 9 & A & D \end{array} \qquad \begin{array}{cccc} 0 & 1 & 2 & 6 \\ 2 & 3 & 4 & 8 \\ 3 & 4 & 5 & 9 \\ 5 & 6 & 7 & B \end{array}$$

$$\begin{array}{cccc} 0 & 1 & 2 & 7 \\ 1 & 2 & 3 & 8 \\ 3 & 4 & 5 & A \\ 5 & 6 & 7 & C \end{array} \qquad \begin{array}{cccc} 0 & 1 & 2 & 7 \\ 2 & 3 & 4 & 9 \\ 3 & 4 & 5 & A \\ 5 & 6 & 7 & C \end{array} \qquad \begin{array}{cccc} 0 & 1 & 4 & 6 \\ 2 & 3 & 6 & 8 \\ 3 & 4 & 7 & 9 \\ 7 & 8 & B & D \end{array} \qquad \begin{array}{cccc} 0 & 1 & 5 & 7 \\ 5 & 6 & A & C \\ 6 & 7 & B & D \\ B & C & G & I \end{array} \qquad \begin{array}{cccc} 0 & 2 & 3 & 6 \\ 3 & 5 & 6 & 9 \\ 5 & 7 & 8 & B \\ 7 & 9 & A & D \end{array}$$

(The symbols A, B, etc. stand for $10, 11$, etc.) It would be interesting to have the complete list.

2.5. **Symmetries.** There exist two natural involutions on the set of additive tableaux. Recall that an additive tableau can be defined by increasing sequences $x_1 < \cdots < x_a$ and $y_1 < \cdots < y_b$ such that the sums $x_i + y_j$ are distinct. We can replace each of these sequences by the other one, reordered increasingly. Since this preserves the set of hyperplanes of equations $x_i + y_j = x_k + y_l$, this defines two commuting involutions on the set of additive tableaux, and then also on the set of maximal relaxations, and on the subset of simplicial relaxations.

3. Non-simplicial faces

3.1. **The degree of a face.** Recall that a stretched Kronecker coefficient $g(k\lambda, k\mu, k\nu)$ is quasipolynomial: there exists a collection of polynomials $P_0, \dots, P_{p-1}$, such that

$$g(k\lambda, k\mu, k\nu) = P_i(k) \qquad \text{for} \quad k \equiv i \pmod{p}.$$

By the monotonicity property, $P_{i+j}(k+\ell) \geq P_i(k)$ as soon as $P_j(\ell) \neq 0$. This implies that, among the polynomials $P_0, \dots, P_{p-1}$, those that are not identically zero have the same degree d, and the same leading term as well. We call $d = d(\lambda, \mu, \nu)$ the degree of the triple (λ, μ, ν). For example, weakly stable triples have degree zero, and a triple of degree zero is one that has a weakly stable multiple.

Another straightforward consequence of the monotonicity property, and of the convexity of the faces, is the following statement.

Proposition 6. *Let F be a face of the Kronecker polyhedron. The degree is constant on the interior of F, and can only decrease, or remain the same, on its boundary faces.*

Definition. Let F be a face of the Kronecker polyhedron. We define its *degree* as the degree of its interior points. For example, any additive face, more generally any strongly simplicial face, has degree zero.

3.2. **The defect of simpliciality.** For a non-simplicial face containing an additive face, we will show that the degree can be read off directly from the normal bundle.

Definition. A face F of the Kronecker polyhedron is called δ-*simplicial* if there exists an additive face f_T in F such that the face f of the conormal cone corresponding to F is δ-simplicial. By this we mean that f has dimension d, but contains $d+\delta$ weights of the conormal bundle, counted with their multiplicities.

Strongly simplicial is therefore the same as 0-simplicial. Note also that, starting from a face F of the Kronecker polyhedron, the integer δ will not depend on the minimal face f_T contained in F. This is a consequence of the following statement.

Theorem 3. *A δ-simplicial face F of the Kronecker polyhedron has degree δ.*

Proof. We consider F with the additive face f_T, and we identify F with the corresponding face of the conormal cone. We consider stretched Kronecker coefficients $g(k\lambda, k(a_T(\lambda)+\sigma), k(b_T(\lambda)+\theta))$, where the weight (σ,θ) belongs to the linear span of the face. Denote this Kronecker coefficient by g_k. It may a priori happen that the lattice generated by the weights of the conormal bundle belonging to the face does not contain (σ,θ). In general, there exists a minimal integer e, depending on (σ,θ), such that $e(\sigma,\theta)$ belongs to this lattice. If k is not divisible by e, then $k(\sigma,\theta)$ does not belong to the lattice and $g_k=0$. If we restrict to those k that are divisible by e, then the number of ways to express $k(\sigma,\theta)$ as a non-negative integer linear combination of weights of the conormal bundle certainly grows like k^δ. By the injectivity of (2.2), this implies that the growth of g_k is at most of the order k^δ.

To get the required conclusion, we must control the surjectivity of (2.2). The key point is the following general statement.

Lemma 2. *Let L be an ample line bundle and M a globally generated line bundle on a smooth complex projective variety X. Let $\iota : Y \hookrightarrow X$ be the embedding of a smooth subvariety, and denote the normal bundle by N. Then there exist integers m and n, not depending on M, such that the natural map*

$$H^0(X, I_Y^d \otimes L^a \otimes M) \longrightarrow H^0(Y, S^d N^* \otimes \iota^*(L^a \otimes M))$$

is surjective when $a \geq md+n$.

If we apply this statement to ι_T, we deduce that there exist integers m_T, n_T such that (2.2) is surjective up to degree d as soon as $\lambda_i - \lambda_{i+1} \geq m_T d + n_T$ for each i. Replacing λ by $k\lambda$, we get the surjectivity up to degree $(k-n_T)/m_T$. This yields a lower bound for g_k of order $(k-n_T/m_T)^\delta$, and the claim follows. □

Proof of the lemma. To get the surjectivity it is enough to prove that

$$H^1(X, I_Y^{d+1} \otimes L^a \otimes M) = 0.$$

Let $\pi : Z \to X$ be the blow-up of Y, and E the exceptional divisor. Since $\pi_* O_Z(-iE) = I_Y^i$ and there are no higher direct images, we are reduced to proving that

$$H^1(Z, O_Z(-(d+1)E) \otimes \pi^*(L^a \otimes M)) = 0.$$

The canonical line bundle of Z is $K_Z = \pi^* K_X \otimes O_Z((c-1)E)$, if c denotes the codimension of Y in X. So we can rewrite the previous condition as

$$H^1(Z, K_Z \otimes O_Z(-(d+c)E) \otimes \pi^*(L^a \otimes M \otimes K_X^{-1})) = 0.$$

We can find an a_0 such that $L^{a_0} \otimes K_X^{-1}$ is ample. Moreover, there exists b_0 such that $I_Y \otimes L^{b_0}$ is generated by global sections, hence also $O_Z(-E) \otimes \pi^* L^{b_0}$. Then

$$O_Z(-(d+c)E) \otimes \pi^*(L^a \otimes M \otimes K_X^{-1})$$

is nef and big as soon as $a \geq a_0 + b_0(d+c)$, and the required vanishing follows from the Kawamata–Viehweg vanishing theorem. □

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