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WORD BELL POLYNOMIALS

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Abstract. Multivariate partial Bell polynomials have been defined by E.T. Bell in 1934. These polynomials have numerous applications in Combinatorics, Analysis, Algebra, Probabilities, etc. Many of the formulas on Bell polynomials involve combinatorial objects (set partitions, set partitions into lists, permutations, etc.). So it seems natural to investigate analogous formulas in some combinatorial Hopf algebras with bases indexed by these objects. In this paper we investigate the connections between Bell polynomials and several combinatorial Hopf algebras: the Hopf algebra of symmetric functions, the Faà di Bruno algebra, the Hopf algebra of word symmetric functions, etc. We show that Bell polynomials can be defined in all these algebras, and we give analogs of classical results. To this aim, we construct and study a family of combinatorial Hopf algebras whose bases are indexed by colored set partitions.

1. Introduction

Multivariate partial Bell polynomials (Bell polynomials for short) have been defined by E.T. Bell in [1] in 1934. But their name is due to Riordan [29], who studied the Faà di Bruno formula [11, 12] allowing one to write the nth derivative of a composition $f \circ g$ in terms of the derivatives of f and g [28]. The applications of Bell polynomials in Combinatorics, Analysis, Algebra, Probability Theory, etc. are so numerous that it would take too long to exhaustively list them here. Let us give only a few seminal examples.

- The main applications to Probability Theory are based on the fact that the nth moment of a probability distribution is a complete Bell polynomial of the cumulants.
- Partial Bell polynomials are linked to Lagrange inversion. This follows from the Faà di Bruno formula.
- Many combinatorial formulas for Bell polynomials involve classical combinatorial numbers like Stirling numbers, Lah numbers, etc.

The Faà di Bruno formula and many combinatorial identities can be found in [7]. The Ph.D. thesis of Mihoubi [24] contains a rather complete survey of the applications of these polynomials together with numerous formulas.

Some of the simplest formulas are related to the enumeration of combinatorial objects (set partitions, set partitions into lists, permutations, etc.). So it seems natural to investigate analogous formulas in some combinatorial Hopf algebras with bases indexed by these objects. We recall that combinatorial Hopf algebras are graded bialgebras with

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bases indexed by combinatorial objects such that the product and the coproduct have some compatibilities.

This paper is organized as follows. In Section 2, we investigate the combinatorial properties of colored set partitions. Section 3 is devoted to the study of the Hopf algebras of colored set partitions. After having introduced this family of algebras, we give some special cases which can be found in the literature. The main application explains the connections with Sym, the algebra of symmetric functions. This explains that we can recover some identities for Bell polynomials when the variables are specialized to combinatorial numbers from analogous identities in some combinatorial Hopf algebras. We show that the algebra **WSym** of word symmetric functions has an important role for this construction. In Section 4, we give a few analogs of complete and partial Bell polynomials in **WSym**, ΠQSym $=$ **WSym***, and $\mathbb{C}\langle\mathbb{A}\rangle$ where $\mathbb{A} = \{\mathsf{a}_1, \dots, \mathsf{a}_n, \dots\}$ is an infinite alphabet and investigate their main properties. Finally, in Section 5 we investigate the connection with other noncommutative analogs of Bell polynomials defined by Munthe-Kaas [33].

2. Definition, background and basic properties of colored set partitions

2.1. **Colored set partitions.** Let $a = (a_m)_{m\geq 1}$ be a sequence of nonnegative integers. A *colored set partition* associated with the sequence a is a set of pairs

$$\Pi = \{[\pi_1, i_1], [\pi_2, i_2], \dots, [\pi_k, i_k]\}$$

such that $\pi = \{\pi_1, \dots, \pi_k\}$ is a partition of $\{1, \dots, n\}$ for some $n \in \mathbb{N}$, and $1 \leq i_\ell \leq a_{\#\pi_\ell}$ for $1 \leq \ell \leq k$, where $\#s$ denotes the cardinality of the set s. The integer n is the *size* of Π. We write $|\Pi| = n$, $\Pi \vDash n$, and $\Pi \Rrightarrow \pi$. We denote the set of colored partitions of size n associated with the sequence a by $\mathcal{CP}_n(a)$. Notice that these sets are finite. We also set $\mathcal{CP}(a) = \bigcup_n \mathcal{CP}_n(a)$. We endow $\mathcal{CP}$ with the additional statistic $\#\Pi$, and set $\mathcal{CP}_{n,k}(a) = \{\Pi \in \mathcal{CP}_n(a) : \#\Pi = k\}$.

Example 1. Consider the sequence whose first terms are $a = (1, 2, 3, \dots)$. The colored partitions of size 3 associated with a are

$$\begin{aligned}\mathcal{CP}_3(a) = \{&\{[\{1,2,3\},1]\}, \{[\{1,2,3\},2]\}, \{[\{1,2,3\},3]\}, \{[\{1,2\},1],[\{3\},1]\},\\ &\{[\{1,2\},2],[\{3\},1]\}, \{[\{1,3\},1],[\{2\},1]\}, \{[\{1,3\},2],[\{2\},1]\},\\ &\{[\{2,3\},1],[\{1\},1]\}, \{[\{2,3\},2],[\{1\},1]\}, \{[\{1\},1],[\{2\},1],[\{3\},1]\}\}.\end{aligned}$$

The colored partitions of size 3 and cardinality 2 are

$$\begin{aligned}\mathcal{CP}_{3,2}(a) = \{&\{[\{1,2\},1],[\{3\},1]\}, \{[\{1,2\},2],[\{3\},1]\}, \{[\{1,3\},1],[\{2\},1]\},\\ &\{[\{1,3\},2],[\{2\},1]\}, \{[\{2,3\},1],[\{1\},1]\}, \{[\{2,3\},2],[\{1\},1]\}\}.\end{aligned}$$

It is well-known (see, e.g., [24]) that the number of colored set partitions of size n for a given sequence $a = (a_n)_n$ is equal to the evaluation of the complete Bell polynomial $A_n(a_1, \dots, a_m, \dots)$. It is also known that the number of colored set partitions of size n and cardinality k is given by the evaluation of the partial Bell polynomial $B_{n,k}(a_1, a_2, \dots, a_m, \dots)$. That is,

$$\#\mathcal{CP}_n(a) = A_n(a_1, a_2, \dots) \text{ and } \#\mathcal{CP}_{n,k}(a) = B_{n,k}(a_1, a_2, \dots).$$

Now, let $\Pi = \{[\pi_1, i_1], \dots [\pi_k, i_k]\}$ be a set such that the π_j's are finite sets of nonnegative integers with the property that no integer belongs to more than one π_j, and $1 \leq i_j \leq a_{\#(\pi_j)}$

for any j. Then the *standardization* $\mathrm{std}(\Pi)$ of Π is well-defined as the unique colored set partition obtained by replacing the ith smallest integer in the π_j's by i.

Example 2. For instance, we have

$$\mathrm{std}(\{[\{1,4,7\},1],[\{3,8\},1],[\{5\},3]\}) = \{[\{1,3,5\},1],[\{2,6\},1],[\{4\},3]\} .$$

We define two binary operations, $\uplus : \mathcal{CP}_{n,k}(a) \otimes \mathcal{CP}_{n',k'}(a) \longrightarrow \mathcal{CP}_{n+n',k+k'}(a)$,

$$\Pi \uplus \Pi' = \Pi \cup \Pi'[n],$$

where $\Pi'[n]$ means that we add n to each integer occurring in the sets of Π', and $\Cup$: $\mathcal{CP}_{n,k} \otimes \mathcal{CP}_{n',k'} \longrightarrow \mathcal{P}(\mathcal{CP}_{n+n',k+k'})$,

$$\Pi \Cup \Pi' = \{\hat{\Pi} \cup \hat{\Pi}' \in \mathcal{CP}_{n+n',k+k'}(a) : \mathrm{std}(\hat{\Pi}) = \Pi \text{ and } \mathrm{std}(\hat{\Pi}') = \Pi'\}.$$

Example 3. We have

$$\{[\{1,3\},5],[\{2\},3]\} \uplus \{[\{1\},2],[\{2,3\},4]\} = \{[\{1,3\},5],[\{2\},3],[\{4\},2],[\{5,6\},4]\},$$

and

$$\begin{aligned}\{[\{1\},5],[\{2\},3]\} \Cup \{[\{1,2\},2]\} = \{&\{[\{1\},5],[\{2\},3],[\{3,4\},2]\},\\ &\{[\{1\},5],[\{3\},3],[\{2,4\},2]\},\{[\{1\},5],[\{4\},3],[\{2,3\},2]\},\\ \{[\{2\},5],[\{3\},3],[\{1,4\},2]\},&\{[\{2\},5],[\{4\},3],[\{1,3\},2]\},\{[\{3\},5],[\{4\},3],[\{1,2\},2]\}\}.\end{aligned}$$

The operator $\Cup$ provides an algorithm which computes all colored partitions:

$$\mathcal{CP}_{n,k}(a) = \bigcup_{i_1+\cdots+i_k=n} \bigcup_{j_1=1}^{a_{i_1}} \cdots \bigcup_{j_k=1}^{a_{i_k}} \{[\{1,\dots,i_1\},j_1]\} \Cup \cdots \Cup \{[\{1,\dots,i_k\},j_k]\}. \tag{2.1}$$

Nevertheless, some colored partitions are generated more than once using this process. For a triple (Π,Π',Π''), we denote by $\alpha^{\Pi}_{\Pi',\Pi''}$ the number of pairs of disjoint subsets ($\hat{\Pi}'$, $\hat{\Pi}''$) of Π such that $\hat{\Pi}' \cup \hat{\Pi}'' = \Pi$, $\mathrm{std}(\hat{\Pi}') = \Pi'$, and $\mathrm{std}(\hat{\Pi}'') = \Pi''$.

Remark 4. Notice that, for $a = \mathbf{1} = (1,1,\dots)$ (i.e., the ordinary set partitions), there is an alternative way to construct the set $\mathcal{CP}_n(\mathbf{1})$ efficiently. It suffices to use the induction

$$\begin{aligned}\mathcal{CP}_0(\mathbf{1}) &= \{\emptyset\},\\ \mathcal{CP}_{n+1}(\mathbf{1}) &= \{\pi \cup \{\{n+1\}\} : \pi \in \mathcal{CP}_n(\mathbf{1})\} \cup \{(\pi \setminus \{e\})\\ &\qquad \cup \{e \cup \{n+1\}\} : \pi \in \mathcal{CP}_n(\mathbf{1}),\ e \in \pi\}.\end{aligned} \tag{2.2}$$

By the application of this recurrence, the set partitions of $\mathcal{CP}_{n+1}(\mathbf{1})$ are each obtained exactly once from the set partitions of $\mathcal{CP}_n(\mathbf{1})$.

2.2. **Generating functions.** The generating functions of the colored set partitions $\mathcal{CP}(a)$ is obtained from the cycle generating function for the species of colored set partitions. The construction is rather classical, see, e.g., [3]. Recall first that a species of structures is a rule F which produces for each finite set U, a finite set $F[U]$, and for each bijection $\phi : U \longrightarrow V$, a function $F[\phi] : F[U] \longrightarrow F[V]$ satisfying the following properties:

- for all pairs of bijections $\phi : U \longrightarrow V$ and $\psi : V \longrightarrow W$, we have $F[\psi \circ \phi] = F[\psi] \circ F[\phi]$;
- if Id_U denotes the identity map on U, then $F[Id_U] = Id_{F[U]}$.

An element $s \in F[U]$ is called an F-structure on U. The cycle generating function of a species F is the formal power series in infinitely many independent variables $p_1, p_2, \ldots$ (called power sums) defined by the formula

$$Z_F(p_1, p_2, \ldots) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} |F([n])^\sigma| p^{\mathrm{ct}(\sigma)}, \tag{2.3}$$

where $F([n])^\sigma$ denotes the set of F-structures on $[n] := \{1, \ldots, n\}$ which are fixed by the permutation σ, $\mathrm{ct}(\sigma)$ is the cycle type of σ, that is, the decreasing vector of the cardinalities of the cycles of σ, and $p^\lambda = p_{\lambda_1} \cdots p_{\lambda_k}$ if λ is the vector $[\lambda_1, \ldots, \lambda_k]$. For instance, the trivial species TRIV has only one TRIV-structure for every n. Hence, its cycle generating function is nothing else but the Cauchy function

$$\sigma_1 := \exp\left\{\sum_{n \geq 1} \frac{p_n}{n}\right\} = \sum_{n \geq 0} h_n. \tag{2.4}$$

Here, h_n denotes the complete function $h_n = \sum_{\lambda \vdash n} \frac{1}{z_\lambda} p^\lambda$, where $\lambda \vdash n$ means that λ is a partition of n, and $z_\lambda = \prod_i i^{m_i(\lambda)} m_i(\lambda)!$ if $m_i(\lambda)$ is the multiplicity of the part i in λ.

We consider also the species $\mathrm{NCS}(a)$ of non-empty colored sets having a_n $\mathrm{NCS}(a)$-structures on $[n]$ which are invariant under permutations. Its cycle generating function is

$$Z_{\mathrm{NCS}(a)} = \sum_{n \geq 1} a_n h_n. \tag{2.5}$$

As a species, $\mathcal{CP}(a)$ is the composition $\mathrm{TRIV} \circ \mathrm{NCS}(a)$. Hence, its cycle generating function is obtained by computing the plethysm

$$Z_{\mathrm{NCS}(a)}(p_1, p_2, \ldots) = \sigma_1[Z_{\mathrm{NCS}(a)}] = \exp\left\{\sum_{n>0} \frac{1}{n} \sum_{k>0} a_k p_n[h_k]\right\}. \tag{2.6}$$

The exponential generating function of $\mathcal{CP}(a)$ is obtained by setting $p_1 = t$ and $p_i = 0$ for $i > 1$ in (2.6):

$$\sum_{n \geq 0} A_n(a_1, a_2, \ldots) \frac{t^n}{n!} = \exp\left\{\sum_{i>0} \frac{a_i}{i!} t^i\right\}. \tag{2.7}$$

We deduce easily that the $A_n(a_1, a_2, \ldots)$ are multivariate polynomials in the variables a_i's. These polynomials are called *complete Bell polynomials* [1]. The double generating function of $\#(\mathcal{CP}_{n,k}(a))$ is easily deduced from (2.7) by

$$\sum_{n \geq 0} \sum_{k \geq 0} B_{n,k}(a_1, a_2, \ldots) \frac{x^k t^n}{n!} = \exp\left\{x \sum_{i>0} \frac{a_i}{i!} t^i\right\}. \tag{2.8}$$

Hence,

$$\sum_{n \geq k} B_{n,k}(a_1, a_2, \ldots) \frac{t^n}{n!} = \frac{1}{k!} \left(\sum_{i>0} \frac{a_i}{i!} t^i\right)^n. \tag{2.9}$$

So, we have

$$A_n(a_1, a_2, \dots) = \sum_{k=1}^{n} B_{n,k}(a_1, a_2, \dots), \quad \text{for all } n \geqslant 1 \text{ and } A_0(a_1, a_2, \dots) = 1. \tag{2.10}$$

The multivariate polynomials $B_{n,k}(a_1, a_2, \dots)$ are called *partial Bell polynomials* [1]. Let $S_{n,k}$ denote the Stirling number of the second kind, which counts the number of ways to partition a set of n objects into k nonempty subsets. We have

$$B_{n,k}(1, 1, \dots) = S_{n,k}. \tag{2.11}$$

Note also that $A_n(x, x, \dots) = \sum_{k=0}^{n} S_{n,k} x^k$ is the classical univariate Bell polynomial denoted by $\phi_n(x)$ in [1]. There are several other identities that involve combinatorial numbers, for instance, we have

$$B_{n,k}(1!, 2!, 3!, \dots) = \binom{n-1}{k-1} \frac{n!}{k!}, \text{ (Unsigned Lah numbers A105278 in [30])}, \tag{2.12}$$

$$B_{n,k}(1, 2, 3, \dots) = \binom{n}{k} k^{n-k}, \text{ (Idempotent numbers A059297 in [30])}, \tag{2.13}$$

$$B_{n,k}(0!, 1!, 2!, \dots) = |s_{n,k}|, \text{ (Stirling numbers of the first kind A048994 in [30])}. \tag{2.14}$$

We can also find many other examples in [1, 7, 23, 34, 25].

Remark 5. Without loss of generality, when needed, we will suppose $a_1 = 1$ in the remainder of this paper. Indeed, if $a_1 \neq 0$, then the generating function gives

$$B_{n,k}(a_1, a_2, \dots, a_p, \dots) = a_1^k B_{n,k}\left(1, \frac{a_2}{a_1}, \cdots, \frac{a_p}{a_1}\right) \tag{2.15}$$

and, when $a_1 = 0$,

$$B_{n,k}(0, a_2, \dots, a_p, \dots) = \begin{cases} 0, \ \text{if } n < k, \\ \frac{n!}{(n-k)!} B_{n,k}(a_2, \dots, a_p, \dots), & \text{if } n \geq k. \end{cases} \tag{2.16}$$

Notice that the ordinary series of the isomorphism types of $\mathcal{CP}(a)$ is obtained by setting $p_i = t^i$ in (2.6). Observing that under this specialization we have $p_k[h_n] = t^{nk}$, we obtain, unsurprisingly, the ordinary generating function of colored (integer) partitions

$$\prod_{i>0} \frac{1}{(1-t^i)^{a_i}}. \tag{2.17}$$

2.3. **Bell polynomials and symmetric functions.** The algebra of symmetric functions [22, 20] is isomorphic to its polynomial realization $Sym(\mathbb{X})$ on an infinite set $\mathbb{X} = \{x_1, x_2, \dots\}$ of commuting variables, where the algebra $Sym(\mathbb{X})$ is defined as the set of polynomials invariant under permutation of the variables. As an algebra, $Sym(\mathbb{X})$ is freely generated by the power sum symmetric functions $p_n(\mathbb{X})$, defined by $p_n(\mathbb{X}) = \sum_{i\geqslant 1} x_i^n$, or the complete symmetric functions h_n, where h_n is the sum of all monomials of total degree n in the variables $x_1, x_2, \dots$. The generating function for the h_n, called *Cauchy function*, is

$$\sigma_t(\mathbb{X}) = \sum_{n\geqslant 0} h_n(\mathbb{X}) t^n = \prod_{i\geqslant 1} (1 - x_i t)^{-1}. \tag{2.18}$$

The relationship between the two families $(p_n)_{n\in\mathbb{N}}$ and $(h_n)_{n\in\mathbb{N}}$ is described in terms of generating functions by the *Newton formula*:

$$\sigma_t(\mathbb{X}) = \exp\left\{\sum_{n\geqslant 1} p_n(\mathbb{X})\frac{t^n}{n}\right\}. \tag{2.19}$$

Notice that *Sym* is the free commutative algebra generated by $p_1, p_2 \dots$, i.e., $Sym = \mathbb{C}[p_1, p_2, \dots]$ and $Sym(\mathbb{X}) = \mathbb{C}[p_1(\mathbb{X}), p_2(\mathbb{X}), \dots]$ when $\mathbb{X}$ is an infinite alphabet without relations among the variables. As a consequence of the Newton Formula (2.19), it is also the free commutative algebra generated by $h_1, h_2, \dots$. The freeness of the algebra provides a mechanism of specialization. For any sequence of commuting scalars $u = (u_n)_{n\in\mathbb{N}}$, there is an algebra homomorphism ϕ_u sending p_n to u_n, for $n \in \mathbb{N}$ (respectively sending h_n to a certain v_n which can be deduced from u). These homomorphisms are manipulated *as if* there exists an underlying alphabet (so called *virtual alphabet*) $\mathbb{X}_u$ such that $p_n(\mathbb{X}_u) = u_n$ (respectively $h_n(\mathbb{X}_u) = v_n$). The interest of such a vision is that one defines operations on sequences and symmetric functions by manipulating alphabets.

The bases of *Sym* are indexed by the *partitions* $\lambda \vdash n$ of all the integers n. A partition λ of n is a finite nondecreasing sequence of positive integers ($\lambda_1 \geq \lambda_2 \geq \cdots$) such that $\sum_i \lambda_i = n$.

By specializing either the power sums p_i or the complete functions h_i to the numbers $\frac{a_i}{i!}$, the partial and complete Bell polynomials are identified with well-known bases.

The algebra *Sym* is usually endowed with three coproducts:

- the coproduct Δ such that the power sums are Lie-like ($\Delta(p_n) = p_n \otimes 1 + 1 \otimes p_n$);
- the coproduct Δ' such that the power sums are group-like ($\Delta'(p_n) = p_n \otimes p_n$);
- the coproduct of Faà di Bruno (see, e.g., [9, 18]).

Most of the formulas on Bell polynomials can be stated and proved using specializations and these three coproducts. Since this is not really the purpose of our article, we have deferred a list of examples which are alternative proofs, in terms of symmetric functions, of existing formulas to Appendix A. One of the aims of our paper is to lift some of these identities to other combinatorial Hopf algebras.

3. Hopf algebras of colored set partitions

3.1. The Hopf algebras CWSym(a) and CΠQSym(a)**.** Let **CWSym**(a) (**CWSym** for short when there is no ambiguity) be the algebra defined by its basis $(\Phi_\Pi)_{\Pi\in\mathcal{CP}(a)}$ indexed by colored set partitions associated with the sequence $a = (a_m)_{m\geq 1}$ and the product

$$\Phi_\Pi \Phi_{\Pi'} = \Phi_{\Pi\uplus\Pi'}. \tag{3.1}$$

Example 6. For instance,

$$\Phi_{\{[\{1,3,5\},3],[\{2,4\},1]\}}\Phi_{\{[\{1,2,5\},4],[\{3\},1],[\{4\},2]\}} = \Phi_{\{[\{1,3,5\},3],[\{2,4\},1],[\{6,7,10\},4],[\{8\},1],[\{9\},2]\}}.$$

Let $\mathbf{CWSym}_n$ be the subspace generated by the elements Φ_Π with $\Pi \vDash n$. For any n, we consider an infinite alphabet $\mathbb{A}_n$ of noncommuting variables, and we suppose $\mathbb{A}_n \cap \mathbb{A}_m = \emptyset$ when $n \neq m$.

For any colored set partition $\Pi = \{[\pi_1, i_1], [\pi_2, i_2], \dots, [\pi_k, i_k]\}$, we construct a polynomial $\Phi_\Pi(\mathbb{A}_1, \mathbb{A}_2, \dots) \in \mathbb{C}\langle \bigcup_n \mathbb{A}_n \rangle$,

$$\Phi_\Pi(\mathbb{A}_1, \mathbb{A}_2, \dots) := \sum_{\mathtt{w}=\mathtt{a}_1\dots\mathtt{a}_n} \mathtt{w}, \tag{3.2}$$

where the sum is over the words $\mathtt{w} = \mathtt{a}_1 \dots \mathtt{a}_n$ satisfying

- For $1 \le \ell \le k$, $\mathtt{a}_j \in \mathbb{A}_{i_\ell}$ if and only if $j \in \pi_\ell$.
- If $j_1, j_2 \in \pi_\ell$, then $\mathtt{a}_{j_1} = \mathtt{a}_{j_2}$.

Example 7. We have

$$\Phi_{\{[\{1,3\},3],[\{2\},1],[\{4\},3]\}}(\mathbb{A}_1, \mathbb{A}_2, \dots) = \sum_{\substack{\mathtt{a}_1,\mathtt{a}_2\in\mathbb{A}_3 \\ \mathtt{b}\in\mathbb{A}_1}} \mathtt{a}_1\mathtt{b}\mathtt{a}_1\mathtt{a}_2.$$

Proposition 8. *The family*

$$\Phi(a) := (\Phi_\Pi(\mathbb{A}_1, \mathbb{A}_2, \dots))_{\Pi\in\mathcal{CP}(a)}$$

spans a subalgebra of $\mathbb{C}\langle \bigcup_n \mathbb{A}_n \rangle$ *which is isomorphic to* $\mathbf{CWSym}(a)$*.*

Proof. First, observe that $\mathrm{span}(\Phi(a))$ is stable under concatenation. Indeed,

$$\Phi_\Pi(\mathbb{A}_1, \mathbb{A}_2, \dots)\Phi_{\Pi'}(\mathbb{A}_1, \mathbb{A}_2, \dots) = \Phi_{\Pi\uplus\Pi'}(\mathbb{A}_1, \mathbb{A}_2, \dots).$$

Furthermore, this shows that $\mathrm{span}(\Phi(a))$ is homomorphic to $\mathbf{CWSym}(a)$ and that an explicit (surjective) homomorphism is given by $\Phi_\Pi \longrightarrow \Phi_\Pi(\mathbb{A}_1, \mathbb{A}_2, \dots)$. Observing that the family $\Phi(a)$ is linearly independent, the fact that the algebra $\mathbf{CWSym}(a)$ is graded in finite dimension implies the result. $\square$

We turn **CWSym** into a Hopf algebra by defining the coproduct

$$\Delta(\Phi_\Pi) = \sum_{\substack{\hat\Pi_1\cup\hat\Pi_2=\Pi \\ \hat\Pi_1\cap\hat\Pi_2=\emptyset}} \Phi_{\mathrm{std}(\hat\Pi_1)} \otimes \Phi_{\mathrm{std}(\hat\Pi_2)} = \sum_{\Pi_1,\Pi_2} \alpha^\Pi_{\Pi_1,\Pi_2} \Phi_{\Pi_1} \otimes \Phi_{\Pi_2}. \tag{3.3}$$

Indeed, **CWSym** splits as a direct sum of finite dimension spaces as

$$\mathbf{CWSym} = \bigoplus_n \mathbf{CWSym}_n.$$

This defines a natural graduation on **CWSym**. Hence, since it is a connected algebra, it suffices to verify that it is a bialgebra. More precisely:

$$\begin{aligned}\Delta(\Phi_\Pi\Phi_{\Pi'}) &= \Delta(\Phi_{\Pi\uplus\Pi'}) \\ &= \sum_{\substack{\hat\Pi_1\cup\hat\Pi_2=\Pi, \hat\Pi'_1\cup\hat\Pi'_2=\Pi'[n] \\ \hat\Pi_1\cap\hat\Pi_2=\emptyset, \hat\Pi'_1\cap\hat\Pi'_2=\emptyset}} \Phi_{\mathrm{std}(\hat\Pi_1)\uplus\mathrm{std}(\hat\Pi'_1)} \otimes \Phi_{\mathrm{std}(\hat\Pi_2)\uplus\mathrm{std}(\hat\Pi'_2)} \\ &= \Delta(\Phi_\Pi)\Delta(\Phi_{\Pi'}).\end{aligned}$$

Notice that Δ is cocommutative.

Example 9. For instance,

$$\begin{aligned}\Delta\left(\Phi_{\{[\{1,3\},5],[\{2\},3]\}}\right) = \Phi_{\{[\{1,3\},5],[\{2\},3]\}} \otimes 1 + \Phi_{\{[\{1,2\},5]\}} \otimes \Phi_{\{[\{1\},3]\}} \\ + \Phi_{\{[\{1\},3]\}} \otimes \Phi_{\{[\{1,2\},5]\}} + 1 \otimes \Phi_{\{[\{1,3\},5],[\{2\},3]\}}.\end{aligned}$$

The graded dual CΠQSym(a) (which will be called CΠQSym for short when there is no ambiguity) of **CWSym** is the Hopf algebra generated as a space by the dual basis $(\Psi_\Pi)_{\Pi\in\mathcal{CP}(a)}$ of $(\Phi_\Pi)_{\Pi\in\mathcal{CP}(a)}$. Its product and its coproduct are given by

$$\Psi_{\Pi'}\Psi_{\Pi''} = \sum_{\Pi\in\Pi'\Cup\Pi''} \alpha^{\Pi}_{\Pi',\Pi''}\Psi_\Pi \text{ and } \Delta(\Psi_\Pi) = \sum_{\Pi'\uplus\Pi''=\Pi} \Psi_{\Pi'}\otimes\Psi_{\Pi''}.$$

Example 10. For instance, we have

$$\begin{aligned}\Psi_{\{[\{1,2\},3]\}}\Psi_{\{[\{1\},4],[\{2\},1]\}} = {}& \Psi_{\{[\{1,2\},3],[\{3\},4],[\{4\},1]\}} + \Psi_{\{[\{1,3\},3],[\{2\},4],[\{4\},1]\}}\\ &+ \Psi_{\{[\{1,4\},3],[\{2\},4],[\{3\},1]\}} + \Psi_{\{[\{2,3\},3],[\{1\},4],[\{4\},1]\}}\\ &+ \Psi_{\{[\{2,4\},3],[\{1\},4],[\{3\},1]\}} + \Psi_{\{[\{3,4\},3],[\{1\},4],[\{2\},1]\}}\end{aligned}$$

and

$$\begin{aligned}\Delta(\Psi_{\{[\{1,3\},3],[\{2\},4],[\{4\},1]\}}) = {}& 1\otimes\Psi_{\{[\{1,3\},3],[\{2\},4],[\{4\},1]\}} + \Psi_{\{[\{1,3\},3],[\{2\},4]\}}\otimes\Psi_{\{[\{1\},1]\}}\\ &+ \Psi_{\{[\{1,3\},3],[\{2\},4],[\{4\},1]\}}\otimes 1.\end{aligned}$$

3.2. **Special cases.** In this section, we investigate a few interesting special cases of the construction that we presented in the previous section.

3.2.1. *Word symmetric functions.* The most prominent example follows from the specialization $a_n = 1$ for all n. In this case, the Hopf algebra **CWSym** is isomorphic to **WSym**, the Hopf algebra of word symmetric functions. Let us briefly recall its construction. The algebra of word symmetric functions is a way to construct a noncommutative analog of the algebra *Sym*. Its bases are indexed by set partitions. After the seminal paper [32], this algebra was investigated in [2, 16] as well as an abstract algebra as in its realization with noncommutative variables. Its name comes from its realization as a subalgebra of $\mathbb{C}\langle\mathbb{A}\rangle$ where $\mathbb{A} = \{\mathtt{a}_1,\ldots,\mathtt{a}_n,\ldots\}$ is an infinite alphabet.

Consider the family of functions $\Phi := \{\Phi_\pi\}_\pi$ whose elements are indexed by set partitions of $\{1,\ldots,n\}$. The algebra **WSym** is formally generated by Φ using the shifted concatenation product: $\Phi_\pi\Phi_{\pi'} = \Phi_{\pi\pi'[n]}$, where π and π' are set partitions of $\{1,\ldots,n\}$ and $\{1,\ldots,m\}$, respectively, and $\pi'[n]$ is the partition arising from π' by adding n to each integer occurring in π'. The polynomial realization $\mathbf{WSym}(\mathbb{A})\subset\mathbb{C}\langle\mathbb{A}\rangle$ is defined by $\Phi_\pi(\mathbb{A}) = \sum_{\mathtt{w}}\mathtt{w}$, where the sum is over the words $\mathtt{w} = \mathtt{a}_1\cdots\mathtt{a}_n$, and where $i,j\in\pi_\ell$ implies $\mathtt{a}_i = \mathtt{a}_j$, if $\pi = \{\pi_1,\ldots,\pi_k\}$ is a set partition of $\{1,\ldots,n\}$.

Example 11. For instance, we have $\Phi_{\{\{1,4\},\{2,5,6\},\{3,7\}\}}(\mathbb{A}) = \sum_{\mathtt{a},\mathtt{b},\mathtt{c}\in\mathbb{A}}\mathtt{abcabbc}$.

Although the construction of $\mathbf{WSym}(\mathbb{A})$, the polynomial realization of **WSym**, seems to be close to $Sym(\mathbb{X})$, the structures of the two algebras are quite different since the Hopf algebra **WSym** is not self-dual. The graded dual ΠQSym $:=$ **WSym*** of **WSym** admits a realization in the same subspace $(\mathbf{WSym}(\mathbb{A}))$ of $\mathbb{C}\langle\mathbb{A}\rangle$, but for the shuffle product.

With no surprise, we notice the following fact:

Proposition 12.

- *The algebras* $\mathbf{CWSym}(1,1,\ldots)$*,* **WSym***, and* $\mathbf{WSym}(\mathbb{A})$ *are isomorphic.*
- *The algebras* CΠQSym$(1,1,\ldots)$*,* ΠQSym*, and* $(\mathbf{WSym}(\mathbb{A}),\shuffle)$ *are isomorphic.*

In the rest of the paper, when there is no ambiguity, we will identify the algebras **WSym** and **WSym**($\mathbb{A}$).

The word analog of the basis $(c_\lambda)_\lambda$ of Sym [1] is the dual basis $(\Psi_\pi)_\pi$ of $(\Phi_\pi)_\pi$.

Other bases are known, for example, the word monomial functions defined by $\Phi_\pi = \sum_{\pi\leq\pi'} M_{\pi'}$, where $\pi \leq \pi'$ indicates that π is finer than π', *i.e.*, that each block of π' is a union of blocks of π.

Example 13. For instance,

$$\begin{aligned}\Phi_{\{\{1,4\},\{2,5,6\},\{3,7\}\}} = M_{\{\{1,4\},\{2,5,6\},\{3,7\}\}} + M_{\{\{1,2,4,5,6\},\{3,7\}\}} + M_{\{\{1,3,4,7\},\{2,5,6\}\}}\\ + M_{\{\{1,4\},\{2,3,5,6,7\}\}} + M_{\{\{1,2,3,4,5,6,7\}\}}.\end{aligned}$$

From the definition of the M_π, we deduce that the polynomial representation of the word monomial functions is given by $M_\pi(\mathbb{A}) = \sum_{\mathtt{w}} \mathtt{w}$ where the sum is over the words $\mathtt{w} = \mathtt{a}_1 \cdots \mathtt{a}_n$ where $i, j \in \pi_\ell$ if and only if $\mathtt{a}_i = \mathtt{a}_j$, where $\pi = \{\pi_1, \ldots, \pi_k\}$ is a set partition of $\{1, \ldots, n\}$.

Example 14. $M_{\{\{1,4\},\{2,5,6\},\{3,7\}\}}(\mathbb{A}) = \sum_{\substack{\mathtt{a},\mathtt{b},\mathtt{c}\in\mathbb{A}\\ \mathtt{a}\neq\mathtt{b},\mathtt{a}\neq\mathtt{c},\mathtt{b}\neq\mathtt{c}}} \mathtt{abcabbc}.$

The analog of complete symmetric functions is the basis $(S_\pi)_\pi$ of ΠQSym which is the dual of the basis $(M_\pi)_\pi$ of **WSym**.

The algebra ΠQSym is also realized in the space **WSym**($\mathbb{A}$): it is the subalgebra of $(\mathbb{C}\langle\mathbb{A}\rangle, \sqcup\!\sqcup)$ generated by $\Psi_\pi(\mathbb{A}) = \pi!\,\Phi_\pi(\mathbb{A})$ where $\pi! = \#\pi_1! \cdots \#\pi_k!$ for $\pi = \{\pi_1, \ldots, \pi_k\}$. Indeed, the linear map $\Psi_\pi \longrightarrow \Psi_\pi(\mathbb{A})$ is a bijection sending $\Psi_{\pi_1}\Psi_{\pi_2}$ to

$$\begin{aligned}\sum_{\substack{\pi=\pi_1'\cup\pi_2',\ \pi_1'\cap\pi_2'=\emptyset\\ \pi_1=\mathrm{std}(\pi_1'),\ \pi_2=\mathrm{std}(\pi_2')}} \Psi_\pi(\mathbb{A}) &= \pi_1!\,\pi_2! \sum_{\substack{\pi=\pi_1'\cup\pi_2',\ \pi_1'\cap\pi_2'=\emptyset\\ \pi_1=\mathrm{std}(\pi_1'),\ \pi_2=\mathrm{std}(\pi_2')}} \Phi_\pi(\mathbb{A})\\ &= \pi_1!\,\pi_2!\Phi_{\pi_1}(\mathbb{A}) \sqcup\!\sqcup \Phi_{\pi_2}(\mathbb{A}) = \Psi_{\pi_1}(\mathbb{A}) \sqcup\!\sqcup \Psi_{\pi_2}(\mathbb{A}).\end{aligned}$$

With these notations the image of S_π is $S_\pi(\mathbb{A}) = \sum_{\pi'\leq\pi} \Psi_{\pi'}(\mathbb{A})$. For our realization, the duality bracket $\langle\ |\ \rangle$ implements the scalar product $\langle\ |\ \rangle$ on the space **WSym**($\mathbb{A}$) for which $\langle S_{\pi_1}(\mathbb{A})|M_{\pi_2}(\mathbb{A})\rangle = \langle\Phi_{\pi_1}(\mathbb{A})|\Psi_{\pi_2}(\mathbb{A})\rangle = \delta_{\pi_1,\pi_2}$.

The subalgebra of (**WSym**($\mathbb{A}$), $\sqcup\!\sqcup$) generated by the complete functions $S_{\{\{1,\ldots,n\}\}}(\mathbb{A})$ is isomorphic to Sym. Therefore, we define $\sigma_t^W(\mathbb{A})$ and $\phi_t^W(\mathbb{A})$ by

$$\sigma_t^W(\mathbb{A}) = \sum_{n\geq 0} S_{\{\{1,\ldots,n\}\}}(\mathbb{A})t^n$$

and

$$\phi_t^W(\mathbb{A}) = \sum_{n\geq 1} \Psi_{\{\{1,\ldots,n\}\}}(\mathbb{A})t^{n-1}.$$

These series are linked by the equality

$$\sigma_t^W(\mathbb{A}) = \exp_{\sqcup\!\sqcup}\left(\phi_t^W(\mathbb{A})\right), \tag{3.4}$$

where $\exp_{\sqcup\!\sqcup}$ is the exponential in (**WSym**($\mathbb{A}$), $\sqcup\!\sqcup$). Furthermore, the coproduct of **WSym** consists in identifying the algebra **WSym** $\otimes$ **WSym** with **WSym**($\mathbb{A} + \mathbb{B}$), where $\mathbb{A}$ and $\mathbb{B}$ are two alphabets such that the letters of $\mathbb{A}$ commute with those of $\mathbb{B}$. Hence, we have

[1] The basis $(c_\lambda)_\lambda$, with $c_\lambda = \frac{p_\lambda}{z_\lambda}$, denotes, as usual, the dual basis of the power sum basis $(p_\lambda)_\lambda$.

$\sigma_t^W(\mathbb{A}+\mathbb{B}) = \sigma_t^W(\mathbb{A}) \sqcup\!\sqcup \sigma_t^W(\mathbb{B})$. In particular, we define the multiplication of an alphabet $\mathbb{A}$ by a constant $k \in \mathbb{N}$ by

$$\sigma_t^W(k\mathbb{A}) = \sum_{n\geq 0} S_{\{\{1,\dots,n\}\}}(k\mathbb{A})t^n = \sigma_t^W(\mathbb{A})^k.$$

Unlike in Sym, the knowledge of the complete functions $S_{\{\{1,\dots,n\}\}}(\mathbb{A})$ does not allow us to recover all the polynomials using only the algebraic operations. In [5], we made an attempt to define virtual alphabets by reconstituting the whole algebra using the action of an operad. Although the general mechanism remains to be defined, the case where each complete function $S_{\{\{1,\dots,n\}\}}(\mathbb{A})$ is specialized to a sum of words of length n can be understood via this construction. More precisely, we consider the family of multilinear k-ary operators $\sqcup\!\sqcup_\Pi$ indexed by set compositions (a set composition is a sequence $[\pi_1,\dots,\pi_k]$ of subsets of $\{1,\dots,n\}$ such that $\{\pi_1,\dots,\pi_k\}$ is a set partition of $\{1,\dots,n\}$) acting on words by $\sqcup\!\sqcup_{[\pi_1,\dots,\pi_k]}(a_1^1\cdots a_{n_1}^1,\dots,a_1^k\cdots a_{n_k}^k) = b_1\cdots b_n$ with $b_{i_\ell^p} = a_\ell^p$ if $\pi_p = \{i_1^p < \cdots < i_{n_p}^p\}$ and $\sqcup\!\sqcup_{[\pi_1,\dots,\pi_k]}(a_1^1\cdots a_{n_1}^1,\dots,a_1^k\cdots a_{n_k}^k) = 0$ if $\#\pi_p \neq n_p$ for some $1\leq p\leq k$.

Let $P = (P_n)_{n\geq 1}$ be a family of homogeneous word polynomials such that $\deg(P_n) = n$ for all n. We set $S_{\{\{1,\dots,n\}\}}\left[\mathbb{A}^{(P)}\right] = P_n$ and

$$S_{\{\pi_1,\dots,\pi_k\}}\left[\mathbb{A}^{(P)}\right] = \sqcup\!\sqcup_{[\pi_1,\dots,\pi_k]}(S_{\{\{1,\dots,\#\pi_1\}\}}\left[\mathbb{A}^{(P)}\right],\dots,S_{\{\{1,\dots,\#\pi_k\}\}}\left[\mathbb{A}^{(P)}\right]).$$

The space $\mathbf{WSym}\left[\mathbb{A}^{(P)}\right]$ generated by the polynomials $S_{\{\pi_1,\dots,\pi_k\}}\left[\mathbb{A}^{(P)}\right]$ and endowed with the two products $\cdot$ and $\sqcup\!\sqcup$ is homomorphic to the double algebra $(\mathbf{WSym}(\mathbb{A}),\cdot,\sqcup\!\sqcup)$. Indeed, let $\pi = \{\pi_1,\dots,\pi_k\} \Vdash n$ and $\pi' = \{\pi'_1,\dots,\pi'_{k'}\} \Vdash n'$ be two set partitions. Then we have

$$\begin{aligned}
S_\pi\left[\mathbb{A}^{(P)}\right]\cdot S_{\pi'}\left[\mathbb{A}^{(P)}\right] &= \sqcup\!\sqcup_{[\{1,\dots,n\},\{n+1,\dots,n+n'\}]}\left(S_\pi\left[\mathbb{A}^{(P)}\right], S_{\pi'}\left[\mathbb{A}^{(P)}\right]\right)\\
&= \sqcup\!\sqcup_{[\pi_1,\dots,\pi_k,\pi'_1[n],\dots,\pi'_{k'}[n]]}\Big(S_{\{1,\dots,\#\pi_1\}}\left[\mathbb{A}^{(P)}\right],\dots,S_{\{1,\dots,\#\pi_k\}}\left[\mathbb{A}^{(P)}\right],\\
&\qquad S_{\{1,\dots,\#\pi'_1\}}\left[\mathbb{A}^{(P)}\right],\dots,S_{\{1,\dots,\#\pi'_{k'}\}}\left[\mathbb{A}^{(P)}\right]\Big)\\
&= S_{\pi\Cup\pi'}\left[\mathbb{A}^{(P)}\right]
\end{aligned}$$

and

$$\begin{aligned}
S_\pi\left[\mathbb{A}^{(P)}\right]\sqcup\!\sqcup S_{\pi'}\left[\mathbb{A}^{(P)}\right] &= \sum_{I\cup J=\{1,\dots,n+n'\},\ I\cap J=\emptyset} \sqcup\!\sqcup_{[I,J]}\left(S_\pi\left[\mathbb{A}^{(P)}\right], S_{\pi'}\left[\mathbb{A}^{(P)}\right]\right)\\
&= \sum \sqcup\!\sqcup_{[\pi''_1,\dots,\pi''_{k+k'}]}\Big(S_{\{1,\dots,\#\pi_1\}}\left[\mathbb{A}^{(P)}\right],\dots,S_{\{1,\dots,\#\pi_k\}}\left[\mathbb{A}^{(P)}\right],\\
&\qquad S_{\{1,\dots,\#\pi'_1\}}\left[\mathbb{A}^{(P)}\right],\dots,S_{\{1,\dots,\#\pi'_{k'}\}}\left(\mathbb{A}^{(P)}\right]\Big),
\end{aligned}$$

where the second sum is over the partitions $\{\pi''_1,\dots,\pi''_{k+k'}\} \in \pi \Cup \pi'$ satisfying $\mathrm{std}(\{\pi''_1,\dots,\pi''_k\}) = \pi$, $\mathrm{std}(\{\pi''_{k+1},\dots,\pi''_{k+k'}\}) = \pi'$, $\#\pi''_i = \pi_i$, for $k+1\leq i\leq k+k'$. Hence,

$$S_\pi\left[\mathbb{A}^{(P)}\right]\sqcup\!\sqcup S_{\pi'}\left[\mathbb{A}^{(P)}\right] = \sum_{\pi''\in\pi\Cup\pi'} S_{\pi''}\left[\mathbb{A}^{(P)}\right].$$

In other words, we consider the elements of $\mathbf{WSym}\left[\mathbb{A}^{(P)}\right]$ as word polynomials in the virtual alphabet $\mathbb{A}^{(P)}$ specializing the elements of $\mathbf{WSym}(\mathbb{A})$.

3.2.2. *Biword symmetric functions.* The bi-indexed word algebra **BWSym** was defined in [5]. We recall its definition here: the bases of **BWSym** are indexed by set partitions into lists, which can be constructed from a set partition by ordering each block. We denote the set of the set partitions of $\{1,\dots,n\}$ into lists by $\mathcal{PL}_n$.

Example 15. The sets $\{[1,2,3],[4,5]\}$ and $\{[3,1,2],[5,4]\}$ are two distinct set partitions into lists of the set $\{1,2,3,4,5\}$.

The number of set partitions into lists of an n-element set (or set partitions into lists of size n) is given by Sloane's sequence A000262 [30]. The first values are

$$1, 1, 3, 13, 73, 501, 4051, \dots$$

If $\hat{\Pi}$ is a set partition into lists of $\{1,\dots,n\}$, we write $\Pi \Vdash n$. Set

$$\hat{\Pi} \uplus \hat{\Pi}' = \hat{\Pi} \cup \{[l_1+n,\dots,l_k+n] : [l_1,\dots,l_k] \in \hat{\Pi}'\} \Vdash n+n'.$$

Let $\hat{\Pi}' \subset \hat{\Pi} \Vdash n$. Since the integers appearing in $\hat{\Pi}'$ are all distinct, the standardization $\mathrm{std}(\hat{\Pi}')$ of $\hat{\Pi}'$ is the unique set partition into lists obtained by replacing the ith smallest integer in $\hat{\Pi}$ by i. For example, $\mathrm{std}(\{[5,2],[3,10],[6,8]\}) = \{[3,1],[2,6],[4,5]\}$.

The Hopf algebra **BWSym** is formally defined by its basis $(\Phi_{\hat{\Pi}})$, where the $\hat{\Pi}$'s are set partitions into lists, its product

$$\Phi_{\hat{\Pi}}\Phi_{\hat{\Pi}'} = \Phi_{\hat{\Pi}\uplus\hat{\Pi}'} \tag{3.5}$$

and its coproduct

$$\Delta(\Phi_{\hat{\Pi}}) = \sum \Phi_{\mathrm{std}(\hat{\Pi}')} \otimes \Phi_{\mathrm{std}(\hat{\Pi}'')}, \tag{3.6}$$

where the sum is over the pairs $(\hat{\Pi}', \hat{\Pi}'')$ such that $\hat{\Pi}' \cup \hat{\Pi}'' = \hat{\Pi}$ and $\hat{\Pi}' \cap \hat{\Pi}'' = \emptyset$.

The product of the graded dual $\mathrm{B\Pi QSym}$ of **BWSym** is completely described in the dual basis $(\Psi_{\hat{\Pi}})_{\hat{\Pi}}$ of $(\Phi_{\hat{\Pi}})_{\hat{\Pi}}$ by

$$\Psi_{\hat{\Pi}_1}\Psi_{\hat{\Pi}_2} = \sum \Psi_{\hat{\Pi}}, \tag{3.7}$$

where the sum is over the $\hat{\Pi}$'s such that there exist $\hat{\Pi}'_1$ and $\hat{\Pi}'_2$ satisfying $\hat{\Pi} = \hat{\Pi}'_1 \cup \hat{\Pi}'_2$, $\hat{\Pi}'_1 \cap \hat{\Pi}'_2 = \varnothing$, $\mathrm{std}(\hat{\Pi}'_1) = \hat{\Pi}_1$, and $\mathrm{std}(\hat{\Pi}'_2) = \hat{\Pi}_2$.

Now consider a sequence of bijections ι_n from $\{1,\dots,n!\}$ to the symmetric group $\mathfrak{S}_n$, for all positive integers n. The linear map $\kappa : \mathcal{CP}(1!,2!,3!,\dots) \longrightarrow \mathcal{PL} := \bigcup \mathcal{PL}_n$ sending

$$\{[\{i^1_1,\dots,i^1_{n_1}\},m_1],\dots,[\{i^k_1,\dots,i^k_{n_k}\},m_k]\} \in \mathcal{CP}_n(1!,2!,3!,\dots),$$

with $i^j_1 \le \cdots \le i^j_{n_j}$, to

$$\{[i^1_{(\iota_{n_1}(m_1))_1},\dots,i^1_{(\iota_{n_1}(m_1))_{n_1}}],\dots,[i^k_{(\iota_{n_k}(m_k))_1},\dots,i^k_{(\iota_{n_k}(m_k))_{n_k}}]\}$$

is a bijection. Hence, a simple check shows that the linear map sending Ψ^{Π} to $\Psi^{\kappa(\Pi)}$ is an isomorphism. Thus, we have the following facts.

Proposition 16.

- *The Hopf algebras* $\mathbf{CWSym}(1!,2!,3!,\dots)$ *and* **BWSym** *are isomorphic.*
- *The Hopf algebras* $\mathrm{C\Pi QSym}(1!,2!,3!,\dots)$ *and* $\mathrm{B\Pi QSym}$ *are isomorphic.*

3.2.3. *Word symmetric functions of level* 2. We consider the algebra $\mathbf{WSym}_{(2)}$ which is spanned by the Φ_Π's where Π is a set partition of level 2, that is, a partition of a partition π of $\{1,\dots,n\}$ for some n. More explicitly, a partition of partition of size n is a set $\{\{\pi_{1,1},\dots,\pi_{1,m_1}\},\dots,\{\pi_{k,1},\dots,\pi_{k,m_k}\}\}$ such that the $\pi_{i,j}$'s are pairwise disjoint and $\pi_{1,1}\cup\dots\cup\pi_{1,m_1}\cup\dots\cup\pi_{k,1}\cup\dots\cup\pi_{k,m_k}=\{1,\dots,n\}$.

Example 17. The 12 partitions of partition of size 3 are

$$\begin{array}{l}\{\{\{1\}\},\{\{2\}\},\{\{3\}\}\},\\ \{\{\{1\},\{2\}\},\{\{3\}\}\},\{\{\{1,2\}\},\{\{3\}\}\},\{\{\{1,2\},\{3\}\}\},\\ \{\{\{1\},\{3\}\},\{\{2\}\}\},\{\{\{1,3\}\},\{\{2\}\}\},\{\{\{1,3\},\{2\}\}\},\\ \{\{\{2\},\{3\}\},\{\{1\}\}\},\{\{\{2,3\}\},\{\{1\}\}\},\{\{\{2,3\},\{1\}\}\},\\ \{\{\{1\},\{2\},\{3\}\}\},\{\{\{1,2,3\}\}\}.\end{array}$$

To obtain this set, it suffices to list the set partitions of size 3 and replace each block by the partitions of the block in all the possible ways. For instance, the set partition $\{\{1,3\},\{2\}\}$ yields the 2 partitions of partition $\{\{\{1,3\}\},\{\{2\}\}\}$ and $\{\{\{1\},\{3\}\},\{\{2\}\}\}$.

Notice that partitions of partition are in bijection with pairs of partitions (Π_1,Π_2) such that Π_2 is coarser than Π_1, for instance,

$$\{\{\{1,3,4\},\{5\}\},\{\{2,6\},\{7\}\},\{\{8\}\}\}\\ \sim(\{\{1,3,4\},\{2,6\},\{5\},\{7\},\{8\}\},\{\{1,3,4,5\},\{2,7,6\},\{8\}\})$$

The product of this algebra is given by $\Phi_\Pi\Phi_{\Pi'}=\Phi_{\Pi\cup\Pi'[n]}$, where $\Pi'[n]=\{e[n]:e\in\Pi'\}$. The dimensions of the homogeneous components of this algebra are given by the exponential generating function

$$\sum_i b_i^{(2)}\frac{t^i}{i!}=\exp(\exp(\exp(t)-1)-1).$$

The first values are

$$1,3,12,60,358,2471,19302,167894,1606137,\dots$$

see sequence A000258 of [30].

The coproduct is defined by

$$\Delta(\Phi_\Pi)=\sum_{\substack{\Pi'\cup\Pi''=\Pi\\ \Pi'\cap\Pi''=\emptyset}}\Phi_{\mathrm{std}(\Pi')}\otimes\Phi_{\mathrm{std}(\Pi'')},$$

where, if Π is a partition of a partition of $\{i_1,\dots,i_k\}$, $\mathrm{std}(\Pi)$ denotes the standardization of Π, that is, the partition of partition of $\{1,\dots,k\}$ obtained by replacing each occurrence of i_j by j in Π. The coproduct being co-commutative, the dual algebra $\Pi\mathrm{QSym}_{(2)}:=\mathbf{WSym}^*_{(2)}$ is commutative. The algebra $\Pi\mathrm{QSym}_{(2)}$ is spanned by a basis $(\Psi_\Pi)_\Pi$ satisfying $\Psi_\Pi\Psi_{\Pi'}=\sum_{\Pi''}C^{\Pi''}_{\Pi,\Pi'}\Psi_{\Pi''}$, where $C^{\Pi''}_{\Pi,\Pi'}$ is the number of ways to write $\Pi''=A\cup B$ with $A\cap B=\emptyset$, $\mathrm{std}(A)=\Pi$, and $\mathrm{std}(B)=\Pi'$.

Let b_n be the nth Bell number $A_n(1,1,\dots)$. Considering a bijection from $\{1,\dots,b_n\}$ to the set of the set partitions of $\{1,\dots,n\}$ for all n, we obtain, in the same way as in the previous subsection, the following result.

Proposition 18.

- *The Hopf algebras* $\mathbf{CWSym}(b_1, b_2, b_3, \dots)$ *and* $\mathbf{WSym}_{(2)}$ *are isomorphic.*
- *The Hopf algebras* $\mathrm{C\Pi QSym}(b_1, b_2, b_3, \dots)$ *and* $\mathrm{\Pi QSym}_{(2)}$ *are isomorphic.*

3.2.4. *Cycle word symmetric functions.* We consider the Grossman–Larson Hopf algebra of heap-ordered trees $\mathfrak{S}\mathbf{Sym}$ [15]. The combinatorics of this algebra has been extensively investigated in [16]. This Hopf algebra is spanned by the Φ_σ where σ is a permutation. We identify each permutation with the set of its cycles (for example, the permutation 321 is $\{(13),(2)\}$). The product in this algebra is given by $\Phi_\sigma\Phi_\tau = \Phi_{\sigma\cup\tau[n]}$, where n is the size of the permutation σ and $\tau[n] = \{(i_1+n, i_2+n, \dots, i_k+n) \mid (i_1,\dots,i_k) \in \tau\}$. The coproduct is given by

$$\Delta(\Phi_\sigma) = \sum \Phi_{\mathrm{std}(\sigma|_I)} \otimes \Phi_{\mathrm{std}(\sigma|_J)}, \tag{3.8}$$

where the sum is over the partitions of $\{1,\dots,n\}$ into 2 sets I and J such that the action of σ leaves the sets I and J globally invariant, $\sigma|_I$ denotes the restriction of the permutation σ to the set I and $\mathrm{std}(\sigma|_I)$ is the permutation obtained from $\sigma|_I$ by replacing the ith smallest label by i in $\sigma|_I$.

Example 19. We have

$$\Delta(\Phi_{3241}) = \Phi_{3241}\otimes 1 + \Phi_1\otimes\Phi_{231} + \Phi_{231}\otimes\Phi_1 + 1\otimes\Phi_{3241}.$$

The basis (Φ_σ) and its dual basis (Ψ_σ) are denoted by (S^σ) and (M_σ), respectively, in [16]. The Hopf algebra $\mathfrak{S}\mathbf{Sym}$ is not commutative but it is cocommutative, so it is not self-dual and not isomorphic to the Hopf algebra of free quasi-symmetric functions.

Let ι_n be a bijection from the set of the permutations of $\mathfrak{S}_n$ that are cycles to $\{1,\dots,(n-1)!\}$. We define the bijection $\kappa : \mathfrak{S}_n \leftrightarrow \mathcal{CP}(0!,1!,2!,\dots)$ by

$$\kappa(\sigma) = \left\{[\mathrm{support}(c_1), \iota_{\#\mathrm{support}(c_1)}(\mathrm{std}(c_1))], \dots, [\mathrm{support}(c_k), \iota_{\#\mathrm{support}(c_k)}(\mathrm{std}(c_k))]\right\},$$

if $\sigma = c_1\cdots c_k$ is the decomposition of σ into disjoint cycles and $\mathrm{support}(c)$ denotes the support of the cycle c, i.e., the set of the elements which are permuted by the cycle.

Example 20. For instance, set

$$\iota_1(1) = 1,\ \iota_3(231) = 2,\ \text{and}\ \iota_3(312) = 1.$$

Then we have

$$\kappa(32415867) = \{[\{2\},1],[\{1,3,4\},2],[\{5\},1],[\{6,7,8\},1]\}.$$

The linear map $K : \mathfrak{S}\mathbf{Sym} \longrightarrow \mathbf{CWSym}(0!,1!,2!,\dots)$ sending Φ_σ to $\Phi_{\kappa(\sigma)}$ is an algebra isomorphism. Indeed, it is straightforward to see that it is a bijection, furthermore $\kappa(\sigma\cup\tau[n]) = \kappa(\sigma)\uplus\kappa(\tau)$. Moreover, if $\sigma\in\mathfrak{S}_n$ is a permutation and $\{I,J\}$ is a partition of $\{1,\dots,n\}$ into two subsets such that the action of σ leaves I and J globally invariant, we check that $\kappa(\sigma) = \Pi_1\cup\Pi_2$ with $\Pi_1\cap\Pi_2=\emptyset$, $\mathrm{std}(\Pi_1) = \kappa(\mathrm{std}(\sigma|_I))$, and $\mathrm{std}(\Pi_2) = \kappa(\mathrm{std}(\sigma|_J))$. Conversely, if $\kappa(\sigma) = \Pi_1\cup\Pi_2$ with $\Pi_1\cap\Pi_2=\emptyset$, then there exists a partition $\{I,J\}$ of $\{1,\dots,n\}$ into two subsets such that the action of σ leaves I and J globally invariant, $\mathrm{std}(\Pi_1) = \kappa(\mathrm{std}(\sigma|_I))$, and $\mathrm{std}(\Pi_2) = \kappa(\mathrm{std}(\sigma|_J))$.

In other words,

$$\Delta(\Phi_{\kappa(\sigma)}) = \sum \Phi_{\kappa(\mathrm{std}(\sigma|_I))}\otimes\Phi_{\kappa(\mathrm{std}(\sigma|_J))}, \tag{3.9}$$

where the sum is over the partitions of $\{1,\dots,n\}$ into 2 sets I and J such that the action of σ leaves the sets I and J globally invariant. Hence K is a coalgebra homomorphism and, as with the previous examples, we have the following isomorphism property.

Proposition 21.

- *The Hopf algebras* $\mathbf{CWSym}(0!,1!,2!,\dots)$ *and* $\mathfrak{S}\mathbf{Sym}$ *are isomorphic.*
- *The Hopf algebras* $\mathrm{C\Pi QSym}(0!,1!,2!,\dots)$ *and* $\mathfrak{S}\mathbf{Sym}^*$ *are isomorphic.*

3.2.5. *Miscellaneous subalgebras of the Hopf algebra of endofunctions.* We denote by End the combinatorial class of endofunctions (an endofunction of size $n\in\mathbb{N}$ is a function from $\{1,\dots,n\}$ to itself). Given a function f from a finite subset A of $\mathbb{N}$ to itself, we denote by $\mathrm{std}(f)$ the endofunction $\phi\circ f\circ\phi^{-1}$, where ϕ is the unique increasing bijection from A to $\{1,2,\dots,\#(A)\}$. Given a function g from a finite subset B of $\mathbb{N}$ (disjoint from A) to itself, we denote by $f\cup g$ the function from $A\cup B$ to itself whose f and g are the restrictions to A and B, respectively. Finally, given two endofunctions f and g, of size n and m, respectively, we denote the endofunction $f\cup\tilde{g}$ by $f\bullet g$, where $\tilde{g}$ is the unique function from $\{n+1,n+2,\dots,n+m\}$ to itself such that $\mathrm{std}(\tilde{g})=g$.

Now, let EQSym be the Hopf algebra of endofunctions [16]. This Hopf algebra is defined by its basis (Ψ_f) indexed by endofunctions, the product

$$\Psi_f\Psi_g=\sum_{\mathrm{std}(\tilde{f})=f,\mathrm{std}(\tilde{g})=g,\tilde{f}\cup\tilde{g}\in\mathrm{End}}\Psi_{\tilde{f}\cup\tilde{g}} \tag{3.10}$$

and the coproduct

$$\Delta(\Psi_h)=\sum_{f\bullet g=h}\Psi_f\otimes\Psi_g. \tag{3.11}$$

This algebra is commutative but not cocommutative. We denote its graded dual by $\mathbf{ESym}:=\mathrm{EQSym}^*$ and the basis of $\mathbf{ESym}$ dual to (Ψ_f) by (Φ_f). In [16], the bases (Φ_σ) and (Ψ_σ) are denoted by (S^σ) and (M_σ), respectively. The product and the coproduct in $\mathbf{ESym}$ are given by

$$\Phi_f\Phi_g=\Phi_{f\bullet g} \tag{3.12}$$

and

$$\Delta(\Phi_h)=\sum_{f\cup g=h}\Phi_{\mathrm{std}(f)}\otimes\Phi_{\mathrm{std}(g)}, \tag{3.13}$$

respectively.

Remark 22. The Ψ_f's, where f is a bijective endofunction, span a Hopf subalgebra of EQSym obviously isomorphic to $\mathfrak{S}\mathrm{QSym}:=\mathfrak{S}\mathbf{Sym}^*$, that is, isomorphic to $\mathrm{C\Pi QSym}(0!,1!,2!,\dots)$ from Section 3.2.4.

As suggested by [16], we investigate a few other Hopf subalgebras of EQSym.

- The Hopf algebra of idempotent endofunctions is isomorphic to the Hopf algebra $\mathrm{C\Pi QSym}(1,2,3,\dots)$. The explicit isomorphism sends Ψ_f to $\Psi_{\phi(f)}$, where, for any idempotent endofunction f of size n,

$$\phi(f)=\left\{\left[f^{-1}(i),\#(\{j\in f^{-1}(i)\mid j\le i\})\right]\middle|1\le i\le n,f^{-1}(i)\ne\emptyset\right\}. \tag{3.14}$$

- The Hopf algebra of involutive endofunctions is isomorphic to

$$\mathrm{C\Pi QSym}(1,1,0,\dots,0,\dots) \hookrightarrow \mathrm{\Pi QSym}.$$

Namely, it is a Hopf subalgebra of $\mathfrak{S}$QSym, and the natural isomorphism from $\mathfrak{S}$QSym to CΠQSym$(0!,1!,2!,\dots)$ sends it to the subalgebra CΠQSym$(1,1,0,\dots,0,\dots)$.
- In the same way, the endofunctions such that $f^3=\mathrm{Id}$ generate a Hopf subalgebra of $\mathfrak{S}$QSym $\hookrightarrow$ EQSym isomorphic to the Hopf algebra CΠQSym$(1,0,2,0,\dots,0,\dots)$.
- More generally, the endofunctions such that $f^p=\mathrm{Id}$ generate a Hopf subalgebra of $\mathfrak{S}$QSym $\hookrightarrow$ EQSym isomorphic to CΠQSym$(\tau(p))$, where $\tau(p)_i=(i-1)!$ if $i\mid p$ and $\tau(p)_i=0$ otherwise.

3.3. **Specializations.** The aim of this section is to show how the specialization $c_n \longrightarrow \frac{a_n}{n!}$ factorizes through ΠQSym and CΠQSym.

Firstly, we notice that the algebra Sym is isomorphic to the subalgebra of ΠQSym generated by the family $(\Psi_{\{\{1,\dots,n\}\}})_{n\in\mathbb{N}}$; the explicit isomorphism α sends c_n to $\Psi_{\{\{1,\dots,n\}\}}$. The image of h_n is $S_{\{\{1,\dots,n\}\}}$, and the image of $c_\lambda=\frac{1}{\lambda^!}c_{\lambda_1}\cdots c_{\lambda_k}$ is $\sum_{\pi\Vdash\lambda}\Psi_\pi$, where $\pi\Vdash\lambda$ means that $\pi=\{\pi_1,\dots,\pi_k\}$ is a set partition such that $\#\pi_1=\lambda_1,\ \dots,\#\pi_k=\lambda_k$, and $\lambda^!=\frac{\lambda_1\cdots\lambda_k}{z_\lambda}=\prod_i\mathfrak{m}_i(\lambda)!$, where $\mathfrak{m}_i(\lambda)$ denotes the multiplicity of i in λ. Indeed, c_λ is mapped to $\frac{1}{\lambda^!}\Psi_{\{\{1,\dots,\lambda_1\}\}}\cdots\Psi_{\{\{1,\dots,\lambda_k\}\}}$ and $\Psi_{\{\{1,\dots,\lambda_1\}\}}\cdots\Psi_{\{\{1,\dots,\lambda_k\}\}}=\lambda^!\sum_{\pi\Vdash\lambda}\Psi_\pi$.

Now, the linear map $\beta_a:\Pi\mathrm{QSym}\longrightarrow \mathrm{C\Pi QSym}(a)$ sending Ψ_π to the element $\sum\limits_{\Pi\Rrightarrow\pi}\Psi_\Pi$ for all π is an algebra homomorphism and the subalgebra $\widetilde{\Pi\mathrm{QSym}}:=\beta_a(\Pi\mathrm{QSym})$ is isomorphic to ΠQSym if and only if $a\in(\mathbb{N}\setminus\{0\})^{\mathbb{N}}$.

Let $\gamma_a:\mathrm{C\Pi QSym}(a)\longrightarrow\mathbb{C}$ be the linear map sending Ψ_Π to $\frac{1}{|\Pi|!}$. We have

$$\gamma_a(\Psi_{\Pi_1}\Psi_{\Pi_2})=\sum_{\substack{\Pi=\Pi_1'\cup\Pi_2',\Pi_1'\cap\Pi_2'=\emptyset\\ \mathrm{std}(\Pi_1')=\Pi_1,\,\mathrm{std}(\Pi_2')=\Pi_2}}\gamma_a(\Psi_\Pi). \tag{3.15}$$

We remark that, for each Π occurring on the right-hand side of (3.15), we have $\gamma_a(\Pi)=\frac{1}{(|\Pi_1|+|\Pi_2|)!}$. The number of terms in the sum being $\begin{pmatrix}|\Pi_1|+|\Pi_2|\\|\Pi_1|\end{pmatrix}$, one obtains

$$\gamma_a(\Psi_{\Pi_1}\Psi_{\Pi_2})=\frac{1}{(|\Pi_1|+|\Pi_2|)!}\binom{|\Pi_1|+|\Pi_2|}{|\Pi_1|}=\frac{1}{|\Pi_1|!\,|\Pi_2|!}=\gamma_a(\Psi_{\Pi_1})\gamma_a(\Psi_{\Pi_2}). \tag{3.16}$$

In other words, γ_a is an algebra homomorphism. Furthermore, the restriction $\hat\gamma_a$ of γ_a to $\widetilde{\Pi\mathrm{QSym}}$ is an algebra homomorphism that sends $\beta_a(\Psi_{\{\{1,\dots,n\}\}})$ to $\frac{a_n}{n!}$. It follows that, if $f\in Sym$, then we have

$$f(\mathbb{X}^{(a)})=\hat\gamma_a(\beta_a(\alpha(f))). \tag{3.17}$$

The following theorem summarizes this section.

Theorem 23. *The diagram*

$$\begin{tikzcd}
C\Pi\mathrm{QSym}(a) \arrow[dr, "\gamma_a"'] & \widetilde{\Pi\mathrm{QSym}} \arrow[l, hook'] \arrow[d, two heads, "\hat{\gamma}_a"] & \Pi\mathrm{QSym} \arrow[l, two heads, "\beta_a"'] \\
 & \mathbb{C} = Sym[\mathbb{X}^{(a)}] & Sym \arrow[l, two heads] \arrow[u, hook, "\alpha"']
\end{tikzcd}$$

is commutative.

4. Word Bell polynomials

4.1. **Bell polynomials in ΠQSym.** Since Sym is isomorphic to the subalgebra of ΠQSym generated by the elements $\Psi_{\{\{1,\dots,n\}\}}$, we can compute

$$A_n(\Psi_{\{\{1\}\}}, \Psi_{\{\{1,2\}\}}, \dots, \Psi_{\{\{1,\dots,m\}\}}, \dots).$$

From (3.4), we have

$$A_n(1!\,\Psi_{\{\{1\}\}}, 2!\,\Psi_{\{\{1,2\}\}}, \dots, m!\,\Psi_{\{\{1,\dots,m\}\}}, \dots) = n!\,S_{\{\{1,\dots,n\}\}} = n! \sum_{\pi \Vdash n} \Psi_\pi. \tag{4.1}$$

Notice that, from the previous section, the image of the Bell polynomial

$$A_n(\Psi_{\{\{1\}\}}, \Psi_{\{\{1,2\}\}}, \dots, \Psi_{\{\{1,\dots,m\}\}}, \dots)$$

under the homomorphism γ sending $\Psi_{\{\{1,\dots,n\}\}}$ to $\frac{1}{n!}$ is

$$\gamma(A_n(1!\,\Psi_{\{\{1\}\}}, 2!\,\Psi_{\{\{1,2\}\}}, \dots, m!\,\Psi_{\{\{1,\dots,m\}\}}, \dots)) = b_n = A_n(1,1,\dots).$$

In the same way, we have

$$B_{n,k}(1!\,\Psi_{\{\{1\}\}}, 2!\,\Psi_{\{\{1,2\}\}}, \dots, m!\,\Psi_{\{\{1,\dots,m\}\}}, \dots) = n! \sum_{\substack{\pi \Vdash n \\ \#\pi = k}} \Psi_\pi. \tag{4.2}$$

If $(F_n)_n$ is a homogeneous family of elements of ΠQSym, such that $|F_n| = n$, we define

$$\mathcal{A}_n(F_1, F_2, \dots) = \frac{1}{n!} A_n(1!\,F_1, 2!\,F_2, \dots, m!\,F_m, \dots) \tag{4.3}$$

and

$$\mathcal{B}_{n,k}(F_1, F_2, \dots) = \frac{1}{n!} B_{n,k}(1!\,F_1, 2!\,F_2, \dots, m!\,F_m, \dots). \tag{4.4}$$

By considering the map $\beta_a \circ \alpha$ as a specialization of Sym, we see that the following identities hold in CΠQSym(a):

$$\mathcal{A}_n\left(\sum_{1\le i\le a_1} \Psi_{\{[\{1\},i]\}}, \sum_{1\le i\le a_2} \Psi_{\{[\{1,2\},i]\}}, \dots, \sum_{1\le i\le a_m} \Psi_{\{[\{1,\dots,m\},i]\}}, \dots\right) = \sum_{\Pi \vDash n} \Psi_\Pi$$

and

$$\mathcal{B}_{n,k}\left(\sum_{1\le i\le a_1} \Psi_{\{[\{1\},i]\}}, \sum_{1\le i\le a_2} \Psi_{\{[\{1,2\},i]\}}, \dots, \sum_{1\le i\le a_m} \Psi_{\{[\{1,\dots,m\},i]\}}, \dots\right) = \sum_{\substack{\Pi \vDash n \\ \#\Pi = k}} \Psi_\Pi.$$

Example 24. In BΠQSym $\sim$ CΠQSym$(1!, 2!, \dots)$, we have

$$\mathcal{B}_{n,k}\left(\Psi_{\{[1]\}}, \Psi_{\{[1,2]\}} + \Psi_{\{[2,1]\}}, \dots, \sum_{\sigma\in\mathfrak{S}_m} \Psi_{\{[\sigma]\}}, \dots\right) = \sum_{\substack{\hat{\Pi}\vDash n\\ \#\hat{\Pi}=k}} \Psi_{\hat{\Pi}},$$

where the sum on the right is over the set partitions of $\{1,\dots,n\}$ into k lists. By considering the homomorphism sending $\Psi_{\{[\sigma_1,\dots,\sigma_n]\}}$ to $\frac{1}{n!}$, we see that Theorem 23 allows us to recover $B_{n,k}(1!, 2!, 3!, \dots) = L_{n,k}$, the number of set partitions of $\{1,\dots,n\}$ into k lists.

Example 25. In $\Pi\text{QSym}_{(2)} \sim$ CΠQSym$(b_1, b_2, \dots)$, we have

$$\mathcal{B}_{n,k}\left(\Psi_{\{\{\{1\}\}\}}, \Psi_{\{\{\{1,2\}\}\}} + \Psi_{\{\{\{1\},\{2\}\}\}}, \dots, \sum_{\pi\Vdash m} \Psi_{\{\pi\}}, \dots\right) = \sum_{\substack{\Pi \text{ partition of } \pi\Vdash n\\ \#\Pi=k}} \Psi_{\Pi},$$

where the sum on the right is over the set partitions of $\{1,\dots,n\}$ of level 2 into k blocks. By considering the homomorphism sending $\Psi_{\{\pi\}}$ to $\frac{1}{n!}$ for $\pi \Vdash n$, we see that Theorem 23 allows us to recover $B_{n,k}(b_1, b_2, b_3, \dots) = S^{(2)}_{n,k}$, the number of set partitions into k sets of a partition of $\{1,\dots,n\}$.

Example 26. In $\mathfrak{S}\mathbf{Sym}^* \sim$ CΠQSym$(0!, 1!, 2! \dots)$, we have

$$\mathcal{B}_{n,k}\left(\Psi_{[1]}, \Psi_{[2,1]}, \Psi_{[2,3,1]} + \Psi_{[3,1,2]}, \dots, \sum_{\substack{\sigma\in\mathfrak{S}_n\\ \sigma \text{ is a cycle}}} \Psi_{\{\pi\}}, \dots\right) = \sum_{\substack{\sigma\in\mathfrak{S}_n\\ \sigma \text{ has } k \text{ cycles}}} \Psi_{\sigma},$$

where the sum on the right is over the permutations of size n having k cycles. By considering the homomorphism sending Ψ_σ to $\frac{1}{n!}$ for $\sigma \in \mathfrak{S}_n$, we see that Theorem 23 allows us to recover $B_{n,k}(0!, 1!, 2!, \dots) = s_{n,k}$, the number of permutations of $\mathfrak{S}_n$ having exactly k cycles.

Example 27. In the Hopf algebra of idempotent endofunctions, we have

$$\begin{aligned}&\mathcal{B}_{n,k}\left(\Psi_{f_{1,1}}, \Psi_{f_{2,1}} + \Psi_{f_{2,2}}, \Psi_{f_{3,1}} + \Psi_{f_{3,2}} + \Psi_{f_{3,3}}, \dots, \textstyle\sum_{i=1}^n \Psi_{f_{n,i}}, \dots\right)\\ &= \sum_{|f|=n, \#(f(\{1,\dots,n\}))=k} \Psi_f,\end{aligned}$$

where for $i \geq j \geq 1$, $f_{i,j}$ is the constant endofunction of size i and of image $\{j\}$. Here, the sum on the right is over idempotent endofunctions f of size n such that the cardinality of the image of f is k. By considering the homomorphism sending Ψ_f to $\frac{1}{n!}$ for $|f| = n$, we see that Theorem 23 allows us to recover that $B_{n,k}(1, 2, 3, \dots)$ is the number of these idempotent endofunctions. This number equals the idempotent number $\binom{n}{k}k^{n-k}$ [17, 31].

4.2. **Bell polynomials in WSym.** Bell polynomials can alternatively be defined recursively by using the derivative that sends letter a_i to a_{i+1} for all i. This definition is particularly interesting since noncommutative analogs (Munthe-Kaas polynomials) are defined in the same way (see [10] and Section 5). In this section we describe a word analog of this formula.

We define the operator ∂ acting linearly on the right of **WSym** by

$$1\partial = 0 \text{ and } \Phi_{\{\pi_1,\dots,\pi_k\}}\partial = \sum_{i=1}^{k} \Phi_{(\{\pi_1,\dots,\pi_k\}\setminus\pi_i)\cup\{\pi_i\cup\{n+1\}\}}.$$

In fact, the operator ∂ acts on Φ_π almost as the multiplication of $M_{\{\{1\}\}}$ on M_π. More precisely, we have the following relation.

Proposition 28. *We have:*

$$\partial = \phi^{-1} \circ \mu \circ \phi - \mu,$$

where ϕ is the linear operator satisfying $M_\pi \phi = \Phi_\pi$, and μ is the multiplication by $\Phi_{\{\{1\}\}}$.

Example 29. For instance, we have

$$\begin{aligned}
\Phi_{\{\{1,3\},\{2,4\}\}}\partial &= \Phi_{\{\{1,3\},\{2,4\}\}}(\phi^{-1}\mu\phi - \mu)\\
&= M_{\{\{1,3\},\{2,4\}\}}\mu\phi - \Phi_{\{\{1,3\},\{2,4\},\{5\}\}}\\
&= (M_{\{\{1,3,5\},\{2,4\}\}} + M_{\{\{1,3\},\{2,4,5\}\}} + M_{\{\{1,3\},\{2,4\},\{5\}\}})\phi - \Phi_{\{\{1,3\},\{2,4\},\{5\}\}}\\
&= \Phi_{\{\{1,3,5\},\{2,4\}\}} + \Phi_{\{\{1,3\},\{2,4,5\}\}} + \Phi_{\{\{1,3\},\{2,4\},\{5\}\}} - \Phi_{\{\{1,3\},\{2,4\},\{5\}\}}\\
&= \Phi_{\{\{1,3,5\},\{2,4\}\}} + \Phi_{\{\{1,3\},\{2,4,5\}\}}.
\end{aligned}$$

Following Remark 4, we define the elements $\mathfrak{A}_n$ of **WSym** recursively as

$$\mathfrak{A}_0 = 1,\ \mathfrak{A}_{n+1} = \mathfrak{A}_n(\Phi_{\{\{1\}\}} + \partial). \tag{4.5}$$

So we have

$$\mathfrak{A}_n = 1(\Phi_{\{\{1\}\}} + \partial)^n. \tag{4.6}$$

Easily, one shows that $\mathfrak{A}_n$ provides an analog of complete Bell polynomials in **WSym**.

Proposition 30.

$$\mathfrak{A}_n = \sum_{\pi \Vdash n} \Phi_\pi.$$

Noticing that the multiplication by $\Phi_{\{\{1\}\}}$ adds one block to each partition, we give the following analog for partial Bell polynomials.

Proposition 31. *If we set*

$$\mathfrak{B}_{n,k} = [t^k]1(t\Phi_{\{\{1\}\}} + \partial)^n, \tag{4.7}$$

then we have $\mathfrak{B}_{n,k} = \sum_{\substack{\pi \Vdash n\\ \#\pi = k}} \Phi_\pi.$

Example 32. We have

$$\begin{aligned}
1(t\Phi_{\{\{1\}\}} + \partial)^4 = t^4\Phi_{\{\{1\},\{2\},\{3\},\{4\}\}} &+ t^3(\Phi_{\{\{1,2\},\{3\},\{4\}\}} + \Phi_{\{\{1,3\},\{2\},\{4\}\}}\\
&+ \Phi_{\{\{1\},\{2,3\},\{4\}\}} + \Phi_{\{\{1,4\},\{2\},\{3\}\}} + \Phi_{\{\{1\},\{2,4\},\{3\}\}} + \Phi_{\{\{1\},\{2\},\{3,4\}\}})\\
+ t^2(\Phi_{\{\{1,3,4\},\{2\}\}} &+ \Phi_{\{\{1,2,3\},\{4\}\}} + \Phi_{\{\{1,2,4\},\{3\}\}} + \Phi_{\{\{1,2\},\{3,4\}\}} + \Phi_{\{\{1,3\},\{2,4\}\}}\\
&+ \Phi_{\{\{1,4\},\{2,3\}\}} + \Phi_{\{\{1\},\{2,3,4\}\}}) + t\Phi_{\{\{1\},\{3\},\{4\},\{2\}\}}.
\end{aligned}$$

Hence,

$$\mathfrak{B}_{4,2} = \Phi_{\{\{1,3,4\},\{2\}\}} + \Phi_{\{\{1,2,3\},\{4\}\}} + \Phi_{\{\{1,2,4\},\{3\}\}} + \Phi_{\{\{1,2\},\{3,4\}\}} \\ + \Phi_{\{\{1,3\},\{2,4\}\}} + \Phi_{\{\{1,4\},\{2,3\}\}} + \Phi_{\{\{1\},\{2,3,4\}\}}.$$

4.3. **Bell polynomials in $\mathbb{C}\langle\mathbb{A}\rangle$.** Both **WSym** and ΠQSym admit word polynomial realizations in a subspace **WSym**$(\mathbb{A})$ of the free associative algebra $\mathbb{C}\langle\mathbb{A}\rangle$ over an infinite alphabet $\mathbb{A}$. When endowed with the concatenation product, **WSym**$(\mathbb{A})$ is isomorphic to **WSym**, and, when endowed with the shuffle product, it is isomorphic to ΠQSym. Alternatively to the definitions of partial Bell polynomials in ΠQSym (4.4) and in **WSym**, we set, for a sequence of polynomials $(F_i)_{i\in\mathbb{N}}$ in $\mathbb{C}\langle\mathbb{A}\rangle$,

$$\sum_{n\geq 0} \mathtt{B}_{n,k}(F_1,\dots,F_m,\dots)t^n = \frac{1}{k!}\left(\sum_i F_i t^i\right)^{\sqcup\!\sqcup k} \tag{4.8}$$

and

$$\mathtt{A}_n(F_1,\dots,F_m,\dots) = \sum_{k\geq 1} \mathtt{B}_{n,k}(F_1,\dots,F_m,\dots). \tag{4.9}$$

This definition generalizes (4.4) and (4.7) in the following sense.

Proposition 33. *We have*

$$\mathtt{B}_{n,k}(\Psi_{\{\{1\}\}}(\mathbb{A}),\dots,\Psi_{\{\{1,\dots,m\}\}}(\mathbb{A}),\dots) = \mathcal{B}_{n,k}(\Psi_{\{\{1\}\}}(\mathbb{A}),\dots,\Psi_{\{\{1,\dots,m\}\}}(\mathbb{A}),\dots)$$

and

$$\mathtt{B}_{n,k}(\Phi_{\{\{1\}\}}(\mathbb{A}),\dots,\Phi_{\{\{1,\dots,m\}\}}(\mathbb{A}),\dots) = \mathfrak{B}_{n,k}(\mathbb{A}).$$

Proof. The two identities follow from

$$\Psi_{\pi_1}(\mathbb{A})\sqcup\!\sqcup\Psi_{\pi_2}(\mathbb{A}) = \sum_{\substack{\pi=\pi_1'\cup\pi_2',\ \pi_1'\cap\pi_2'=\emptyset \\ \mathrm{std}(\pi_1')=\pi_1,\ \mathrm{std}(\pi_2')=\pi_2}} \Psi_\pi(\mathbb{A}). \qquad \square$$

Equality (4.8) allows us to show more general properties. For instance, let $\mathbb{A}'$ and $\mathbb{A}''$ be two disjoint subalphabets of $\mathbb{A}$, and set

$$S_n^{\mathbb{A}'}(\mathbb{A}'') = S_{\{\{1\}\}}(\mathbb{A}')\sqcup\!\sqcup S_{\{\{1,\dots,n-1\}\}}(\mathbb{A}'').$$

Observing that

$$\sum_n \mathtt{B}_{n,k}(S_1^{\mathbb{A}'}(\mathbb{A}''),\dots,S_m^{\mathbb{A}'}(\mathbb{A}''),\dots)t^n \\ = t^k S_{\{\{1\},\dots,\{k\}\}}(\mathbb{A}')\sqcup\!\sqcup(\sum_{n\geq 0} S_{\{\{1,\dots,n\}\}}(\mathbb{A}'')t^n)^{\sqcup\!\sqcup k} = t^k S_{\{\{1\},\dots,\{k\}\}}(\mathbb{A}')\sqcup\!\sqcup\sigma_t^W(k\mathbb{A}''),$$

we obtain a word analog of the formula allowing one to write a Bell polynomial as a symmetric function (see Eq. (A.8) in Appendix A).

Proposition 34. *We have*

$$\mathtt{B}_{n,k}(S_1^{\mathbb{A}'}(\mathbb{A}''),\dots,S_m^{\mathbb{A}'}(\mathbb{A}''),\dots) = S_{\{\{1\},\dots,\{k\}\}}(\mathbb{A}')\sqcup\!\sqcup S_{\{\{1,\dots,n-k\}\}}(k\mathbb{A}'').$$

For simplicity, let us write $\mathtt{B}^{\mathbb{A}'}_{n,k}(\mathbb{A}'') := \mathtt{B}_{n,k}(S^{\mathbb{A}'}_1(\mathbb{A}''), \ldots, S^{\mathbb{A}'}_m(\mathbb{A}''), \ldots)$.

Let $k = k_1 + k_2$. From

$$S_{\{\{1\},\ldots,\{k_1\}\}}(\mathbb{A}') ⧢ S_{\{\{1\},\ldots,\{k_2\}\}}(\mathbb{A}') = \binom{k}{k_1} S_{\{\{1\},\ldots,\{k\}\}}(\mathbb{A}')$$

and

$$S_{\{\{1,\ldots,n-k\}\}}(k\mathbb{A}'') = \sum_{i+j=n-k} S_{\{\{1,\ldots,i\}\}}(k_1\mathbb{A}'') ⧢ S_{\{\{1,\ldots,i\}\}}(k_2\mathbb{A}''),$$

we deduce an analog of the binomiality of the partial Bell polynomials (see Eq. (A.9) in Appendix A).

Corollary 35. *Let $k = k_1 + k_2$ be three nonnegative integers. Then we have*

$$\binom{k}{k_1} \mathtt{B}^{\mathbb{A}'}_{n,k}(\mathbb{A}'') = \sum_{i=0}^{n} \mathtt{B}^{\mathbb{A}'}_{i,k_1}(\mathbb{A}'') ⧢ \mathtt{B}^{\mathbb{A}'}_{n-i,k_2}(\mathbb{A}''). \tag{4.10}$$

Example 36. Consider a family of functions $(f_k)_k$ such that $f_k : \mathbb{N} \longrightarrow \mathbb{C}\langle\mathbb{A}\rangle$ and

$$f_0 = 1 \text{ and } f_n(\alpha + \beta) = \sum_{n=i+j} f_i(\alpha) ⧢ f_j(\beta). \tag{4.11}$$

From (4.8), we obtain

$$\mathtt{B}_{n,k}(f_0(a), \ldots, f_{m-1}(a), \ldots)t^n = \frac{1}{k!} \sum_{i_1+\cdots+i_k=n-k} f_{i_1}(a) ⧢ \cdots ⧢ f_{i_k}(a).$$

Hence, iterating (4.11), we deduce

$$\mathtt{B}_{n,k}(f_0(a), \ldots, f_{m-1}(a), \ldots) = \frac{1}{k!} f_{n-k}(ka).$$

Set $f_n(k) = k!\,\mathtt{B}^{\mathbb{A}'}_{n,k}(\mathbb{A}'')$ and $f_0(k) = 1$. By (4.10), the family $(f_n)_{n\in\mathbb{N}}$ satisfies (4.11). Hence we obtain an analog of the composition formula (see Eq. (A.13) in Appendix A):

$$k_1!\,\mathtt{B}_{n,k_1}(1, \ldots, k_2!\,\mathtt{B}^{\mathbb{A}'}_{m-1,k_2}(\mathbb{A}''), \ldots) = (k_1k_2)!\,\mathtt{B}^{\mathbb{A}'}_{n-k_1,k_1k_2}(\mathbb{A}'').$$

Suppose now that $\mathbb{A}'' = \mathbb{A}''_1 + \mathbb{A}''_2$. By

$$S_{\{\{1,\ldots,n\}\}}(\mathbb{A}'') = \sum_{i=0}^{n} S_{\{\{1,\ldots,i\}\}}(\mathbb{A}''_1) ⧢ S_{\{\{1,\ldots,n-i\}\}}(\mathbb{A}''_2),$$

Proposition 34 allows us to write a word analog of the convolution formula for Bell polynomials (see formula (A.10) in Appendix A).

Corollary 37. *We have*

$$S_{\{\{1\},\ldots,\{k\}\}}(\mathbb{A}') ⧢ \mathtt{B}^{\mathbb{A}'}_{n,k}(\mathbb{A}'') = \sum_{i=0}^{n} \mathtt{B}^{\mathbb{A}'}_{i,k}(\mathbb{A}''_1) ⧢ \mathtt{B}^{\mathbb{A}'}_{n-i,k}(\mathbb{A}''_2). \tag{4.12}$$

Let k_1 and k_2 be two positive integers. We have

$$\sum_n \mathtt{B}_{n,k_1}(\mathtt{B}^{\mathbb{A}'}_{k_2,k_2}(\mathbb{A}''),\dots,\mathtt{B}^{\mathbb{A}'}_{k_2+m-1,k_2}(\mathbb{A}''),\dots)t^n = \frac{1}{k_1!}\left(\sum_{m\geq 1}\mathtt{B}^{\mathbb{A}'}_{k_2+m-1,k_2}(\mathbb{A}'')t^m\right)^{\sqcup\!\sqcup k_1}$$
$$= t^{k_1}S_{\{\{1\},\dots,\{k_2k_1\}\}}(\mathbb{A}')\sqcup\!\sqcup\,\sigma^W_t(k_1k_2\mathbb{A}'').$$

This implies the following identity.

Proposition 38. *We have*

$$\mathtt{B}_{n,k_1}(\mathtt{B}^{\mathbb{A}'}_{k_2,k_2}(\mathbb{A}''),\dots,\mathtt{B}^{\mathbb{A}'}_{k_2+m-1,k_2}(\mathbb{A}''),\dots) = \mathtt{B}^{\mathbb{A}'}_{n-k_1+k_1k_2,k_1k_2}(\mathbb{A}''). \tag{4.13}$$

4.4. **Specialization again.** In [5], we have shown that one can construct a double algebra which is homomorphic to $(\mathbf{WSym}(\mathbb{A}),.,\sqcup\!\sqcup)$. This is a general construction which is an attempt to properly define the concept of a virtual alphabet for **WSym**. In our context, the construction is simpler and can be described as follows.

Let $F=(F_\pi(\mathbb{A}))_\pi$ be a basis of $\mathbf{WSym}(\mathbb{A})$. We say that F is *shuffle-compatible* if

$$F_{\{\pi_1,\dots,\pi_k\}}(\mathbb{A}) = \sqcup\!\sqcup_{[\pi_1,\dots,\pi_k]}\left(F_{\{\{1,\dots,\#\pi_1\}\}}(\mathbb{A}),\dots,F_{\{\{1,\dots,\#\pi_k\}\}}(\mathbb{A})\right).$$

Then we have

$$F_{\pi_1}(\mathbb{A})\sqcup\!\sqcup F_{\pi_2}(\mathbb{A}) = \sum_{\substack{\pi=\pi_1'\cup\pi_2',\pi_1'\cap\pi_2'=\emptyset\\ \mathrm{std}(\pi_1')=\pi_1,\mathrm{std}(\pi_2')=\pi_2}} F_\pi \text{ and } F_{\pi_1}(\mathbb{A}).F_{\pi_2}(\mathbb{A}) = F_{\pi_1\uplus\pi_2}(\mathbb{A}).$$

Example 39. The bases $(S_\pi(\mathbb{A}))_\pi$, $(\Phi_\pi(\mathbb{A}))_\pi$, and $(\Psi_\pi(\mathbb{A}))_\pi$ are shuffle-compatible but not the basis $(M_\pi(\mathbb{A}))_\pi$.

Straightforwardly, we have the following fact.

Claim 40. *Let $(F_\pi(\mathbb{A}))_\pi$ be a shuffle-compatible basis of* $\mathbf{WSym}(\mathbb{A})$*. Let $\mathbb{B}$ be another alphabet and let $P=(P_k)_{k>0}$ be a family of noncommutative polynomials of $\mathbb{C}\langle\mathbb{B}\rangle$ such that* $\deg P_k = k$*. Then the space spanned by the polynomials*

$$F_{\{\pi_1,\dots,\pi_k\}}[\mathbb{A}^{(P)}_F] := \sqcup\!\sqcup_{[\pi_1,\dots,\pi_k]}(P_{\#\pi_1},\dots,P_{\#\pi_k})$$

is stable under concatenation and shuffle product in $\mathbb{C}\langle\mathbb{B}\rangle$. So it is a double algebra which is homomorphic to $(\mathbf{WSym}(\mathbb{A}),.,\sqcup\!\sqcup)$. We denote this double algebra by $\mathbf{WSym}[\mathbb{A}^{(P)}_F]$ and the image of an element $f\in\mathbf{WSym}(\mathbb{A})$ under the homomorphism $\mathbf{WSym}(\mathbb{A})\longrightarrow\mathbf{WSym}[\mathbb{A}^{(P)}_F]$ sending $F_{\{\pi_1,\dots,\pi_k\}}$ to $F_{\{\pi_1,\dots,\pi_k\}}[\mathbb{A}^{(P)}_F]$ by $f[\mathbb{A}^{(P)}_F]$.

With these notations, we have

$$\mathtt{B}_{n,k}(P_1,\dots,P_m,\dots) = \mathfrak{B}_{n,k}[\mathbb{A}^{(P)}_\Phi].$$

Example 41. We define a specialization by setting

$$\Phi_{\{\{1,\dots,n\}\}}[\mathfrak{S}] = \sum_{\substack{\sigma\in\mathfrak{S}_n\\ \sigma_1=1}} \mathtt{b}_{\sigma[1]}\dots\mathtt{b}_{\sigma[n]},$$

where the letters $\mathtt{b}_i$ belong to an alphabet $\mathbb{B}$. Let $\sigma\in\mathfrak{S}_n$ be a permutation and $\sigma=c_1\circ\cdots\circ c_k$ its decomposition into disjoint cycles. Each cycle $c^{(i)}$ is denoted by a sequence of integers

$(n_1^{(i)}, \dots, n_{\ell_i}^{(i)})$ such that $n_1^{(i)} = \min\{n_1^{(i)}, \dots, n_{\ell_i}^{(i)}\}$. Let $\widetilde{c^{(i)}} \in \mathfrak{S}_{\ell_i}$ be the permutation which is the standardization of the sequence $n_1^{(i)} \dots n_{\ell_i}^{(i)}$. The cycle support of σ is the partition

$$\mathrm{support}(\sigma) = \{\{n_1^{(1)}, \dots, n_{\ell(i)}^{(1)}\}, \dots, \{n_1^{(k)}, \dots, n_{\ell(k)}^{(k)}\}\}.$$

We define $w[c^{(i)}] = \mathtt{b}_{\widetilde{c^{(i)}}[1]} \cdots \mathtt{b}_{\widetilde{c^{(i)}}[\ell^{(i)}]}$ and $w[\sigma] = ш_{[\pi_1,\dots,\pi_k]}(w[c^{(1)}], \dots, w[c^{(k)}])$, where $\pi_i = \{n_1^{(i)}, \dots, n_{\ell(i)}^{(i)}\}$ for $1 \le i \le k$.

For instance, if $\sigma = 312654 = (132)(46)(5)$, we have $w[(132)] = \mathtt{b}_1\mathtt{b}_3\mathtt{b}_2$, $w[(46)] = \mathtt{b}_1\mathtt{b}_2$, $w[(5)] = \mathtt{b}_1$, and $w[\sigma] = \mathtt{b}_1\mathtt{b}_3\mathtt{b}_2\mathtt{b}_1\mathtt{b}_1\mathtt{b}_2$.

So, we have

$$\mathtt{B}_{n,k}(\Phi_{\{\{1\}\}}[\mathfrak{S}], \dots, \Phi_{\{\{1,\dots,m\}\}}[\mathfrak{S}], \dots) = \mathfrak{B}_{n,k}[\mathfrak{S}] = \sum_{\substack{\pi \Vdash n \\ \#\pi = k}} \Phi_\pi[\mathfrak{S}] = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \#\mathrm{support}(\sigma) = k}} w[\sigma].$$

For instance,

$$\begin{aligned}
&\mathtt{B}_{4,2}(\mathtt{b}_1, \mathtt{b}_1\mathtt{b}_2, \mathtt{b}_1\mathtt{b}_2\mathtt{b}_3 + \mathtt{b}_1\mathtt{b}_3\mathtt{b}_2, \mathtt{b}_1\mathtt{b}_2\mathtt{b}_3\mathtt{b}_4 + \mathtt{b}_1\mathtt{b}_3\mathtt{b}_2\mathtt{b}_4 + \mathtt{b}_1\mathtt{b}_2\mathtt{b}_4\mathtt{b}_3 + \mathtt{b}_1\mathtt{b}_3\mathtt{b}_4\mathtt{b}_2 \\
&\qquad + \mathtt{b}_1\mathtt{b}_4\mathtt{b}_2\mathtt{b}_3 + \mathtt{b}_1\mathtt{b}_4\mathtt{b}_3\mathtt{b}_2, \dots) \\
&\quad = \Phi_{\{\{1\},\{2,3,4\}\}}[\mathfrak{S}] + \Phi_{\{\{2\},\{1,3,4\}\}}[\mathfrak{S}] + \Phi_{\{\{3\},\{1,2,4\}\}}[\mathfrak{S}] + \Phi_{\{\{4\},\{1,2,3\}\}}[\mathfrak{S}] \\
&\qquad + \Phi_{\{\{1,2\},\{3,4\}\}}[\mathfrak{S}] + \Phi_{\{\{1,3\},\{2,4\}\}}[\mathfrak{S}] + \Phi_{\{\{1,4\},\{2,3\}\}}[\mathfrak{S}] \\
&\quad = 2\mathtt{b}_1\mathtt{b}_1\mathtt{b}_2\mathtt{b}_3 + 2\mathtt{b}_1\mathtt{b}_1\mathtt{b}_3\mathtt{b}_2 + \mathtt{b}_1\mathtt{b}_2\mathtt{b}_1\mathtt{b}_3 + \mathtt{b}_1\mathtt{b}_3\mathtt{b}_2\mathtt{b}_1 + \mathtt{b}_1\mathtt{b}_2\mathtt{b}_3\mathtt{b}_1 + \mathtt{b}_1\mathtt{b}_3\mathtt{b}_2\mathtt{b}_1 \\
&\qquad + \mathtt{b}_1\mathtt{b}_2\mathtt{b}_1\mathtt{b}_2 + 2\mathtt{b}_1\mathtt{b}_1\mathtt{b}_2\mathtt{b}_2.
\end{aligned}$$

Notice that the sum of the coefficients of the words occurring in the expansion of $\mathfrak{B}_{n,k}[\mathfrak{S}]$ is equal to the Stirling number $s_{n,k}$. Hence, this specialization gives another word analog of formula (2.14).

5. Munthe-Kaas polynomials

5.1. **Munthe-Kaas polynomials from WSym.** In order to generalize the Runge–Kutta method to integration on manifolds, Munthe-Kaas [33] introduced a noncommutative version of Bell polynomials. We recall here the construction in a slightly different variant adapted to our notation, the operators acting on the right of the algebra. Consider an alphabet $\mathbb{D} = \{d_1, d_2, \dots\}$. The algebra $\mathbb{C}\langle\mathbb{D}\rangle$ is equipped with the derivative defined by $d_i\partial = d_{i+1}$. The noncommutative Munthe-Kaas Bell polynomials are defined by setting $t = 1$ in $\mathtt{MB}_n(t) = 1.(td_1 + \partial)^n$. The partial noncommutative Bell polynomial $\mathtt{MB}_{n,k}$ is the coefficient of t^k in $\mathtt{MB}_n(t)$.

Example 42. We have

- $\mathtt{MB}_1(t) = d_1 t$,
- $\mathtt{MB}_2(t) = d_1^2 t^2 + d_2 t$,
- $\mathtt{MB}_3(t) = d_1^3 t^3 + (2d_2d_1 + d_1d_2)t^2 + d_3 t$,
- $\mathtt{MB}_4(t) = d_1^4 t^4 + (3d_2d_1^2 + 2d_1d_2d_1 + d_1^2d_2)t^3 + (3d_3d_1 + 3d_2^2 + d_1d_3)t^2 + d_4 t$.

We consider the map χ which sends a set partition $\pi = \{\pi_1, \dots, \pi_k\}$, where $\min(\pi_i) < \min(\pi_{i+1})$ for $0 < i < k$, to the integer composition $[\#(\pi_1), \dots, \#(\pi_k)]$. The linear map Ξ sending Φ_π to $d_{\chi(\pi)[1]} \cdots d_{\chi(\pi)[k]}$ is an algebra homomorphism. Hence, we deduce the following proposition.

Proposition 43. *We have*

$$\Xi\left(\mathfrak{B}_{n,k}\right) = \mathtt{MB}_{n,k}. \tag{5.1}$$

Example 44. We have

$$\Xi\left(\mathfrak{B}_{3,2}\right) = \Xi\left(\Phi_{\{\{1\},\{2,3\}\}} + \Phi_{\{\{1,3\},\{2\}\}} + \Phi_{\{\{1,2\},\{3\}\}}\right) = d_1d_2 + 2d_2d_1 = \mathtt{MB}_{3,2}.$$

We recover a result due to Ebrahimi-Fard et al. [10].

Theorem 45. *If $j_1+\cdots+j_k = n$, the coefficient of $d_{j_1}\cdots d_{j_k}$ in $\mathtt{MB}_{n,k}$ is equal to the number of partitions of $\{1,2,\dots,n\}$ into blocks $\pi_1,\dots,\pi_k$ such that $\#(\pi_\ell) = j_\ell$ for $1\le \ell\le k$ and $\min(\pi_1) < \cdots < \min(\pi_k)$.*

5.2. **Dendriform structure and quasideterminant formula.** The algebra ΠQSym is equipped with a Zinbiel structure. The notion of Zinbiel algebra is due to Loday [21]. This is an algebra equipped with two nonassociative products $\prec$ and $\succ$ satisfying

- $(u \prec v) \prec w = u \prec (v \prec w) + u \prec (v \succ w)$,
- $(u \succ v) \prec w = u \succ (v \prec w)$,
- $u \succ (v \succ w) = (u \prec v) \succ w + (u \succ v) \succ w$,
- $u \prec v = v \succ u$.

The Zinbiel structure on ΠQSym is defined for any $\pi \Vdash n$ and $\pi' \Vdash m$ by $\Phi_\pi \prec \Phi_{\pi'} = \sum' \Phi_{\pi[I]\cup\pi'[J]}$ (respectively $\Phi_\pi \succ \Phi_{\pi'} = \sum'' \Phi_{\pi[I]\cup\pi'[J]}$), where $\sum'$ (respectively $\sum''$) means that the sum is over the partitions $I\cup J = \{1,\dots,n+m\}$ ($I\cap J = \emptyset$) such that $\#(I) = n$, $\#(J) = m$, and $1\in I$ (respectively $1 \in J$), and $\pi[I]$ is obtained from π by replacing ℓ by i_ℓ for all ℓ if $I = \{i_1,\dots,i_n\}$ and $i_1<\cdots<i_n$. Refer to [26, 27, 13] for other combinatorial Hopf algebras with a dendriform structure.

We notice that

$$\sum_n \mathcal{B}_{n,k}(\Phi_{\{\{1\}\}}, \Phi_{\{\{1,2\}\}},\dots)t^n = \left(\sum_i \Phi_{\{\{1,\dots,i\}\}}t^i\right)^{\overrightarrow{\prec} k}, \tag{5.2}$$

where $u^{\overrightarrow{\prec} k} = u^{\overrightarrow{\prec} k-1} \prec u$ and $u^{\overrightarrow{\prec} 0} = 1$.

Definition 46. Let $A_n = (a_{ij})_{1\le i\le j\le n}$ be an upper triangular matrix whose entries are in a Zinbiel algebra. We define the polynomial

$$\mathrm{P}(A_n;t) = t\sum_{k=1}^{n} \mathrm{P}(A_{k-1}) \prec a_{k,n} \text{ and } \mathrm{P}(A_0) = 1. \tag{5.3}$$

Example 47. We have

$$\begin{aligned}\mathrm{P}(A_4,t) &= t\mathrm{P}(A_3) \prec a_{44} + t\mathrm{P}(A_2) \prec a_{34} + t\mathrm{P}(A_1) \prec a_{24} + t\mathrm{P}(A_0) \prec a_{14}\\ &= t^4((a_{11} \prec a_{22}) \prec a_{33}) \prec a_{44} + t^3(a_{11} \prec a_{23}) \prec a_{44}\\ &\quad + t^3(a_{12} \prec a_{33}) \prec a_{44} + t^3(a_{11} \prec a_{22}) \prec a_{34} + t^2a_{12} \prec a_{34} + at_{11}^2 \prec a_{24} + ta_{14}.\end{aligned}$$

By induction, we find

$$\mathrm{P}(A_n;t) = ta_{1n} + \sum_{\substack{1\leq j_1<j_2\\<\cdots<j_k}} t^k(\cdots(a_{1j_1} \prec (a_{j_1+1,j_2}) \prec (a_{j_2+1,j_3}) \prec \cdots \prec a_{j_{k-1}+1,j_k}) \prec a_{j_k+1,n}). \tag{5.4}$$

Setting $M_n := (\Phi_{\{\{1,\dots,j-i+1\}\}})_{1\leq i\leq j\leq n}$, we get the following proposition.

Proposition 48. *We have*

$$\mathcal{B}_{n,k}(\Phi_{\{\{1\}\}}, \Phi_{\{\{1,2\}\}}, \dots) = [t^k]\mathrm{P}(M_n;t). \tag{5.5}$$

Example 49. We have

$$\mathrm{P}(A_3;t) = t^3(a_{11} \prec a_{22}) \prec a_{33} + t^2(a_{11} \prec a_{23} + a_{12} \prec a_{33}) + ta_{13}.$$

Hence

$$\begin{aligned}\mathrm{P}(M_3;t) &= t^3(\Phi_{\{\{1\}\}} \prec \Phi_{\{\{1\}\}}) \prec \Phi_{\{\{1\}\}} + t^2(\Phi_{\{\{1\}\}} \prec \Phi_{\{\{1,2\}\}} + \Phi_{\{\{1,2\}\}} \prec \Phi_{\{\{1\}\}})\\ &\quad + t\Phi_{\{\{1,2,3\}\}}\\ &= t^3\Phi_{\{\{1\},\{2\},\{3\}\}} + t^2(\Phi_{\{\{1\},\{2,3\}\}} + \Phi_{\{\{1,2\},\{3\}\}} + \Phi_{\{\{1,3\},\{2\}\}}) + t\Phi_{\{\{1,2,3\}\}}\\ &= t^3\mathcal{B}_{3,3} + t^2\mathcal{B}_{3,2} + t\mathcal{B}_{3,1}.\end{aligned}$$

Formula (5.4) is reminiscent of a well-known result on quasideterminants.

Proposition 50 (Gelfand et al. [14])**.** *We have*

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & \boxed{a_{1n}} \\ -1 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & -1 & a_{33} & \cdots & a_{3n} \\ & \cdots & & & \\ 0 & \cdots & 0 & -1 & a_{nn} \end{vmatrix} = a_{1n} + \sum_{1\leq j_1<j_2<\cdots<j_k<n} a_{1j_1}a_{j_1+1,j_2}\cdots a_{j_k+1,n}. \tag{5.6}$$

Furthermore, formula (5.5) is an analog of the following result of Ebrahimi-Fard et al.

Theorem 51 (Ebrahimi-Fard et al. [10])**.** *We have*

$$\mathrm{MB}_n(1) = \begin{vmatrix} \binom{n-1}{0}d_1 & \binom{n-1}{1}d_2 & \binom{n-1}{2}d_3 & \cdots & \boxed{\binom{n-1}{n-1}d_n} \\ -1 & \binom{n-2}{0}d_1 & \binom{n-2}{1}d_2 & \cdots & \binom{n-2}{n-2}d_{n-1} \\ 0 & -1 & \binom{n-3}{0}d_1 & \cdots & \binom{n-3}{n-3}d_{n-3} \\ & \cdots & & & \\ 0 & \cdots & 0 & -1 & \binom{0}{0}d_1 \end{vmatrix}.$$

The connection between all these results remains to be investigated.

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Appendix A. Bell polynomials and coproducts in Sym

In fact, most of the identities on Bell polynomials can be obtained by manipulating generating functions and they are closely related to some other identities occurring in the literature. Typically, the relation between the complete Bell polynomials $A_n(a_1, a_2, \dots)$ and the variables $a_1, a_2, \dots$ is very closely related to the Newton Formula which links the generating function of complete symmetric functions h_n (Cauchy series) to that of the power sums p_n. The symmetric functions form a commutative algebra Sym freely generated by the complete functions h_n or the power sum functions p_n. Hence, specializing the variable a_n to some numbers is equivalent to specializing the power sum functions p_n. More soundly, the algebra Sym can be endowed with coproducts conferring to it the structure of a Hopf algebra. For instance, the coproduct for which the power sums are primitive turns Sym into a self-dual Hopf algebra. The coproduct can be translated in terms of generating functions by a product of two Cauchy series. This kind of manipulations appears also in the context of Bell polynomials, for instance when computing the complete Bell polynomials of the sum of two sequences of variables $a_1 + b_1,\ a_2 + b_2, \dots$. Another coproduct turns Sym into a non-cocommutative Hopf algebra called the Faà di Bruno algebra which is related to Lagrange inversion. Finally, the coproduct such that the power sums are group-like can be related also to a few other formulas on Bell polynomials. The aim of this section is to investigate these connections and in particular to restate some known results in terms of symmetric functions and virtual alphabets. We also give a few new results that are difficult to prove without the help of symmetric functions.

A.1. **Bell polynomials as symmetric functions.**

First, let us recall some operations on alphabets. Given two alphabets $\mathbb{X}$ and $\mathbb{Y}$, we define (see, e.g., [20]) the alphabet $\mathbb{X}+\mathbb{Y}$ by:

$$p_n(\mathbb{X}+\mathbb{Y}) = p_n(\mathbb{X}) + p_n(\mathbb{Y}) \tag{A.1}$$

and the alphabet $\alpha\mathbb{X}$ (respectively $\mathbb{X}\mathbb{Y}$), for $\alpha \in \mathbb{C}$ by:

$$p_n(\alpha\mathbb{X}) = \alpha p_n(\mathbb{X}) \text{ (respectively } p_n(\mathbb{X}\mathbb{Y}) = p_n(\mathbb{X})p_n(\mathbb{Y})). \tag{A.2}$$

In terms of Cauchy functions, these transforms imply

$$\sigma_t(\mathbb{X}+\mathbb{Y}) = \sigma_t(\mathbb{X})\sigma_t(\mathbb{Y}) \tag{A.3}$$

and

$$\sigma_t(\mathbb{X}\mathbb{Y}) = \sum_\lambda \frac{1}{z_\lambda} p^\lambda(\mathbb{X}) p^\lambda(\mathbb{Y}) t^{|\lambda|}. \tag{A.4}$$

In fact $\sigma_t(\mathbb{X}\mathbb{Y})$ encodes the kernel of the scalar product defined by $\langle p^\lambda, c_\mu\rangle = \delta_{\lambda,\mu}$ with $c_\lambda = \frac{p^\lambda}{z_\lambda}$. Notice that $c_n = \frac{p_n}{n}$ and

$$Sym = \mathbb{C}[c_1, c_2, \ldots]. \tag{A.5}$$

From (2.7) and (2.19), we obtain the following identity.

Proposition 52. *We have* $h_n = \frac{1}{n!}A_n(1!\,c_1, 2!\,c_2, \ldots)$.

Conversely, Equality (A.5) implies that the homomorphism ϕ_a sending c_i to $\frac{a_i}{i!}$ for all i is well defined for any sequence of numbers $a = (a_i)_{i\in\mathbb{N}\setminus\{0\}}$, and $\phi_a(h_n) = \frac{1}{n!}A_n(a_1, a_2, \ldots)$.

Let us also define $h_n^{(k)}(\mathbb{X}) = [\alpha^k]h_n(\alpha\mathbb{X})$. From (2.19) and (A.2), we have

$$h_n^{(k)} = \sum_{\lambda=[\lambda_1,\ldots,\lambda_k]\vdash n} c_\lambda = [t^n]\frac{1}{k!}\left(\sum_{i\geq 1} c_i t^i\right)^k,$$

and thus everything works as if we use a special (virtual) alphabet $\mathbb{X}^{(a)}$ satisfying $c_n(\mathbb{X}^{(a)}) = n!\,a_n$. More precisely, the following identity holds true.

Proposition 53. *We have*

$$\phi_a(h_n^{(k)}) = h_n^{(k)}(\mathbb{X}^{(a)}) = \frac{1}{n!}B_{n,k}(a_1, \ldots, a_k, \ldots). \tag{A.6}$$

Example 54. Let $\mathbf{1}$ be the virtual alphabet defined by $c_n(\mathbf{1}) = \frac{1}{n}$ for all $n \in \mathbb{N}$. In this case, the Newton Formula yields $h_n(\mathbf{1}) = 1$. Hence $A_n(0!, 1!, 2!, \ldots, (m-1)!, \ldots) = n!$ and

$$B_{n,k}(0!, 1!, 2!, \ldots, (m-1)!, \ldots) = n!\,[\alpha^k][t^n]\left(\frac{1}{1-t}\right)^\alpha = s_{n,k},$$

the Stirling number of the first kind.

Example 55. A more complicated example is treated in [4, 19], where $a_i = i^{i-1}$. In this case, the specialization gives

$$\sigma_t(\alpha\mathbb{X}^{(a)}) = \exp\{-\alpha W(-t)\},$$

where $W(t) = \sum_{n=1}^\infty (-n)^{n-1}\frac{t^n}{n!}$ is the Lambert W function satisfying $W(t)\exp\{W(t)\} = t$ (see, e.g., [6]). Hence,

$$\sigma_t(\alpha\mathbb{X}^{(a)}) = \left(\frac{W(-t)}{-t}\right)^\alpha.$$

However, the expansion of the series $\left(\frac{W(t)}{t}\right)^\alpha$ is known to be

$$\left(\frac{W(t)}{t}\right)^\alpha = 1 + \sum_{n=1}^\infty \frac{1}{n!}\alpha(\alpha+n)^{n-1}(-t)^n. \tag{A.7}$$

Hence, we obtain $B_{n,k}(1, 2, 3^2, \ldots, m^{m-1}, \ldots) = \binom{n-1}{k-1}n^{n-k}$. Note that the expansion of $W(t)$ and (A.7) are usually obtained by the use of Lagrange inversion.

Example 56. With these notations, we have $B_{n,k}(a_1+b_1,\dots)=\frac{1}{n!}h_n^{(k)}(\mathbb{X}^{(a)}+\mathbb{X}^{(b)})$, and classical properties of Bell polynomials can be deduced from symmetric functions through this formalism. For instance, the equalities $c_n(\mathbb{X}^{(a)}+\mathbb{X}^{(b)})=c_n(\mathbb{X}^{(a)})+c_n(\mathbb{X}^{(b)})$ and $h_n(\mathbb{X}^{(a)}+\mathbb{X}^{(b)})=\sum_{i+j=n}h_i(\mathbb{X}^{(a)})h_j(\mathbb{X}^{(b)})$ give

$$A_n(a_1+b_1,\dots)=\sum_{i+j=n}\binom{n}{i}A_i(a_1,a_2,\dots)A_j(b_1,b_2,\dots)$$

and

$$B_{n,k}(a_1+b_1,\dots)=\sum_{r+s=k}\sum_{i+j=n}\binom{n}{i}B_{i,r}(a_1,a_2,\dots)B_{j,s}(b_1,b_2,\dots).$$

Example 57. Another example is given by

$$A_n(1a_1b_1,2a_2b_2,\dots,ma_mb_m,\dots)=n!\sum_{\lambda\vdash n}\det\left(\frac{A_{\lambda_i-i+j}(a_1,a_2,\dots)}{(\lambda_i-i+j)!}\right)\times\det\left(\frac{A_{\lambda_i-i+j}(b_1,b_2,\dots)}{(\lambda_i-i+j)!}\right),$$

where we used the convention $A_{-n}=0$ for $n>0$. This formula is a consequence of the Jacobi–Trudi formula and is derived from the Cauchy kernel (A.4), observing that $c_n(\mathbb{X}^{(a)}\mathbb{X}^{(b)})=nc_n(\mathbb{X}^{(a)})c_n(\mathbb{X}^{(b)})$ and

$$h_n(\mathbb{X}^{(a)}\mathbb{X}^{(b)})=\sum_{\lambda\vdash n}s_\lambda(\mathbb{X}^{(a)})s_\lambda(\mathbb{X}^{(b)})=\sum_{\lambda\vdash n}\det\left(h^{(a)}_{\lambda_i-i+j}(\mathbb{X}^a)\right)\det\left(h^{(b)}_{\lambda_i-i+j}(\mathbb{X}^b)\right),$$

where $s_\lambda=\det\left(h^{(a)}_{\lambda_i-i+j}\right)$ is a Schur function (see, e.g., [22]).

A.2. **Other interpretations.** First we focus on Identity (2.9), and we interpret it as the Cauchy function $\sigma_t(k\hat{\mathbb{X}}^{(a)})$, where $\hat{\mathbb{X}}^{(a)}$ is the virtual alphabet such that $h_{i-1}(\hat{\mathbb{X}}^{(a)})=\frac{a_i}{i!}$. This means that we consider the homomorphism $\hat{\phi}_a:Sym\longrightarrow\mathbb{C}$ sending h_i to $\frac{a_{i+1}}{(i+1)!}$. We suppose that $a_1=1$, otherwise we use (2.15) and (2.16). With these notations, the following relation holds true.

Proposition 58. *We have*

$$B_{n,k}(a_1,a_2,\dots)=\frac{n!}{k!}h_{n-k}(k\hat{\mathbb{X}}^{(a)}).\tag{A.8}$$

Example 59. If $a_i=i$, we have $h_i(\hat{\mathbb{X}}^{(a)})=\frac{1}{i!}$, and so $\sigma_t(k\hat{\mathbb{X}}^{(a)})=\exp(kt)$. Hence, we recover the classical result

$$B_{n,k}(1,2,\dots,m,\dots)=\binom{n}{k}k^{n-k}.$$

From $h_n(\mathbb{X}+\mathbb{Y})=\sum_{i+j=n}h_i(\mathbb{X})h_j(\mathbb{Y})$, we deduce two classical identities, namely

$$\binom{k_1+k_2}{k_1}B_{n,k_1+k_2}(a_1,a_2,\dots)=\sum_{i=0}^{n}\binom{n}{i}B_{i,k_1}(a_1,a_2,\dots)B_{n-i,k_2}(a_1,a_2,\dots)\tag{A.9}$$

and

$$\binom{n}{k} B_{n-k,k}\left(a_1 b_1, \ldots, \frac{1}{m+1} \sum_{i=1}^{m} \binom{m+1}{i} a_i b_{m+1-i}, \ldots\right)$$
$$= \sum_{i=k}^{n-k} \binom{n}{i} B_{i,k}(a_1, a_2, \ldots) B_{i,k}(b_1, b_2, \ldots). \quad \text{(A.10)}$$

Indeed, formula (A.9) is obtained by setting $\mathbb{X} = k_1 \hat{\mathbb{X}}^{(a)}$ and $\mathbb{Y} = k_2 \hat{\mathbb{X}}^{(a)}$. Formula (A.10) is called the convolution formula for Bell polynomials (see, e.g., [24]), and is obtained by setting $\mathbb{X} = \hat{\mathbb{X}}^{(a)}$ and $\mathbb{Y} = \hat{\mathbb{X}}^{(b)}$ in the left-hand side and $\mathbb{X} = k\hat{\mathbb{X}}^{(a)}$ and $\mathbb{Y} = k\hat{\mathbb{X}}^{(b)}$ in the right-hand side.

Example 60. The partial Bell polynomials are known to be involved in interesting identities for binomial functions. Let us first recall that a binomial sequence is a family of functions $(f_n)_{n\in\mathbb{N}}$ satisfying $f_0(x) = 1$ and

$$f_n(a+b) = \sum_{k=0}^{n} \binom{n}{k} f_k(a) f_{n-k}(b), \quad \text{(A.11)}$$

for all $a, b \in \mathbb{C}$ and $n \in \mathbb{N}$. Setting $h_n(\mathbb{A}) := \frac{f_n(a)}{n!}$ and $h_n(\mathbb{B}) := \frac{f_n(b)}{n!}$, with these notations we have $f_n(ka) = n!\, h_n(k\mathbb{A})$. Hence,

$$B_{n,k}(1, \ldots, i f_{i-1}(a), \ldots) = \frac{n!}{k!} h_{n-k}(k\mathbb{A}) = \binom{n}{k} f_{n-k}(ka). \quad \text{(A.12)}$$

Notice that from (A.9), we see that

$$f_n(k) = \begin{cases} \binom{n}{k}^{-1} B_{n,k}(a_1, a_2, \ldots), & \text{if } n > 0, \\ 1, & \text{if } n = 0, \end{cases}$$

is binomial, and we obtain

$$\binom{n}{k_1 k_2}^{-1} B_{n,k_1}\left(1, \ldots, i \binom{i-1}{k}^{-1} B_{i-1,k_2}(a_1, a_2, \ldots), \ldots\right)$$
$$= \binom{n-k_1}{k_1 k_2}^{-1} B_{n-k_1, k_1 k_2}(a_1, a_2, \ldots). \quad \text{(A.13)}$$

Several related identities are compiled in [24].

Example 61. Extracting the coefficient of t^{n-k-1} on the left-hand side and the right-hand side of the equality $\frac{d}{dt}\sigma_t((k+1)\mathbb{X}) = (k+1)\left(\frac{d}{dt}\sigma_t(\mathbb{X})\right)\sigma_t(k\mathbb{X})$, we obtain

$$(n-k) h_{n-k}((k+1)\mathbb{X}) = (k+1) \sum_{i=1}^{n-k} i h_i(\mathbb{X}) h_{n-i-k}(\mathbb{X}),$$

and we recover the identity (see, e.g., [8])

$$B_{n,k}(a_1, a_2, \ldots) = \frac{1}{n-k} \sum_{i=1}^{n-k} \binom{n}{i} \left[(k+1) - \frac{n+1}{i+1}\right] (i+1) a_i B_{n-i,k}(a_1, a_2, \ldots). \quad \text{(A.14)}$$

Example 62. Let $(a_n)_{n>0}$ and $(b_n)_{n>0}$ be two sequences of numbers such that $a_1 = b_1 = 1$ and

$$d_n = n! \sum_{\lambda \vdash n-1} \det\left(\frac{a_{\lambda_i - i + j + 1}}{(\lambda_i + j + 1)!}\right) \det\left(\frac{b_{\lambda_i - i + j + 1}}{(\lambda_i + j + 1)!}\right)$$

with the convention $a_{-n} = b_{-n} = 0$ if $n \geq 0$. The Cauchy kernel and the orthogonality of Schur functions give

$$B_{n,k}(d_1, d_2, \dots) = \frac{n!}{k!} \sum_{\lambda \vdash n-k} (k_1!\, k_2!)^{\ell(\lambda)} \det\left(\frac{B_{\lambda_i - i + j + k_1, k_1}(a_1, a_2, \dots)}{(\lambda_i - i + j + k_1)!}\right) \times \det\left(\frac{B_{\lambda_i - i + j + k_1, k_1}(b_1, b_2, \dots)}{(\lambda_i - i + j + k_1)!}\right),$$

for $k_1 k_2 = k$. Indeed, it suffices to use the fact that

$$h_n(k\mathbb{X}^{(a)}\mathbb{X}^{(b)}) = \sum_{\lambda \vdash n} s_\lambda(k_1\mathbb{X}^{(a)}) s_\lambda(k_2\mathbb{X}^{(b)}).$$

The sum $\mathbb{X} + \mathbb{Y}$ and the product $\mathbb{X}\mathbb{Y}$ of alphabets are two examples of coproducts endowing Sym with the structure of a Hopf algebra. The sum of alphabets encodes the coproduct Δ for which the power sums are of Lie type (i.e., $\Delta(p_n) = p_n \otimes 1 + 1 \otimes p_n \sim p_n(\mathbb{X} + \mathbb{Y}) = p_n(\mathbb{X}) + p_n(\mathbb{Y})$ by identifying $f \otimes g$ with $f(\mathbb{X})g(\mathbb{Y})$), whilst the product of alphabets encodes the coproduct Δ' for which the power sums are group-like (i.e., $\Delta'(p_n) = p_n \otimes p_n \sim p_n(\mathbb{X}\mathbb{Y}) = p_n(\mathbb{X})p_n(\mathbb{Y})$).

The algebra of symmetric functions can be endowed with another coproduct that confers the structure of a Hopf algebra: this is the Faà di Bruno algebra [9, 18]. This algebra is rather important since it is related to the Lagrange–Bürmann formula. The Bell polynomials also appear in this context. As a consequence, one can define a new operation on alphabets corresponding to the composition of Cauchy generating functions. Let $\mathbb{X}$ and $\mathbb{Y}$ be two alphabets and set $f(t) = t\sigma_t(\mathbb{X})$ and $g(t) = t\sigma_t(\mathbb{Y})$. The composition $\mathbb{X} \circ \mathbb{Y}$ is defined by $\sigma_t(\mathbb{X} \circ \mathbb{Y}) = \frac{1}{t}(f \circ g)(t)$. The relationship with Bell polynomials can be established by observing that we have

$$\frac{1}{t} f \circ g = \sum_{n \geq 0} \left(\sum_{k=1}^{n+1} \frac{k!}{(n+1)!} h_{k-1}(\mathbb{X}) B_{n+1,k}(1, 2h_1(\mathbb{Y}), 3!\, h_2(\mathbb{Y}), \dots) \right) t^n.$$

Equivalently,

$$h_n(\mathbb{X} \circ \mathbb{Y}) = \sum_{k=0}^{n} \frac{(k+1)!}{(n+1)!} h_k(\mathbb{X}) B_{n+1;k+1}(1, 2!\, h_1(\mathbb{Y}), 3!\, h_2(\mathbb{Y}), \dots).$$

The antipode of the Faà di Bruno algebra is also described in terms of alphabets as the operation which associates to an alphabet $\mathbb{X}$ the alphabet $\mathbb{X}^{\langle -1 \rangle}$ satisfying $\sigma_t(\mathbb{X} \circ \mathbb{X}^{\langle -1 \rangle}) = 1$. More explicitly, we have

$$h_n(\mathbb{X}^{\langle -1 \rangle}) = \frac{n!}{(2n+1)!\,(n+1)} B_{2n+1,n}(1, -2!\, e_1(\mathbb{X}), 3!\, e_2(\mathbb{X}), \dots), \tag{A.15}$$

where the $e_n(\mathbb{X})$'s are the elementary symmetric functions defined by $\sum_n e_n(\mathbb{X}) t^n = \frac{1}{\sigma_{-t}(\mathbb{X})}$.

Example 63. Let $\omega(t) = t\sigma_t(\mathbb{X})$. The Lagrange inversion consists in finding an alphabet $\mathbb{X}'$ such that $\phi(t) = \sigma_t(\mathbb{X}')$. According to (A.15), it suffices to set $\mathbb{X}' = -\mathbb{X}^{\langle -1\rangle}$.
Let $F(t) = \sigma_t(\mathbb{Y})$. When stated in terms of alphabets, the Lagrange–Bürmann formula reads

$$F(\omega(t)) = 1 + \sum_{n\geq 1} \frac{d^{n-1}}{du^{n-1}}[\sigma'_u(\mathbb{Y})\sigma_u(-n\mathbb{X}^{\langle -1\rangle})]\Big|_{u=0}\frac{t^n}{n!}.$$

In other words, we have $h_{n-k}(-n\mathbb{X}^{\langle -1\rangle}) = \frac{(k-1)!}{(n-1)!}B_{n,k}(1, 2!\,h_1(\mathbb{X}), 3!\,h_3(\mathbb{X}), \dots)$. So we recover a result due to Sadek Bouroubi and Moncef Abbas [4]:

$$B_{n,k}(1, h_1(2\mathbb{X}), \dots, (m-1)!\,h_{m-1}(m\mathbb{X}), \dots) = \frac{(n-1)!}{(k-1)!}h_{n-k}(n\mathbb{X}).$$

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