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POLAR ROOT POLYTOPES THAT ARE ZONOTOPES

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Abstract. Let $\mathcal{P}_\Phi$ be the root polytope of a finite irreducible crystallographic root system Φ, i.e., the convex hull of all roots in Φ. The polar of $\mathcal{P}_\Phi$, denoted $\mathcal{P}_\Phi^*$, coincides with the union of the orbit of the fundamental alcove under the action of the Weyl group. In this paper, we determine which polytopes $\mathcal{P}_\Phi^*$ are zonotopes and which are not. The proof is constructive.

1. Introduction

Let Φ be a finite irreducible crystallographic root system in a Euclidean space V with scalar product $(\ , \ )$, W the Weyl group of Φ, and $\mathfrak{g}_\Phi$ a simple Lie algebra having Φ as root system. Let $\mathcal{P}_\Phi$ be the *root polytope* associated with Φ, i.e., the convex hull of all roots in Φ.

Motivated by the connections of the root polytope $\mathcal{P}_\Phi$ with $\mathfrak{g}_\Phi$ (more precisely, with the Borel subalgebras of $\mathfrak{g}_\Phi$ and their abelian ideals), in [3] we study $\mathcal{P}_\Phi$ for a general Φ. Among other things, we give a presentation of $\mathcal{P}_\Phi$ as an intersection of half-spaces, and describe its faces as special subposets of the root poset, up to the action of W. In [4], we develop these general results, obtaining several special results for the root types A_n and C_n. One of the special properties of these two root types is that the cones on the facets of $\mathcal{P}_\Phi$ are the closures of the regions of a hyperplane arrangement. This means that $\mathcal{P}_\Phi$ is combinatorially dual to a zonotope (see [8, §7.3], or [5, §2.3.1]). More precisely, if $\mathcal{H}_{\mathcal{P}_\Phi}$ is the arrangement of all the hyperplanes through the origin containing some $(n-2)$-dimensional faces of $\mathcal{P}_\Phi$, then the complete fan associated with $\mathcal{H}_{\mathcal{P}}$ is equal to the face fan associated with $\mathcal{P}_\Phi$. This property is satisfied by the polytopes whose polar polytopes are zonotopes (see [8, §7.3]). One of the referees of [4] asked if the polars of the types A_n and C_n root polytopes are actually zonotopes, and if so which.

In this paper, we answer this question. We denote by $\mathcal{P}_\Phi^*$ the polar polytope of $\mathcal{P}_\Phi$: we explicitly describe $\mathcal{P}_\Phi^*$ as a zonotope for the types A_n and C_n, as well as for B_3 and G_2. Moreover, we prove by a direct check that, for all other root types, the set of cones on the facets of $\mathcal{P}_\Phi$ is not equal to the set of closures of the regions of $\mathcal{H}_{\mathcal{P}_\Phi}$. Hence, for all other root types, $\mathcal{P}_\Phi^*$ is not a zonotope.

We point out that $\mathcal{P}_\Phi^*$ is a natural object for the crystallographic root system Φ that can be more familiarly described in terms of alcoves and Weyl groups. Indeed, $\mathcal{P}_\Phi^*$ is the union of the orbit of a fundamental alcove of Φ under the Weyl group, i.e., if we fix any basis of Φ, and denote by $\mathcal{A}$ the corresponding fundamental alcove of the affine Weyl group associated with Φ, then $\mathcal{P}_\Phi^* = \bigcup_{w\in W} w\mathcal{A}$. Thus, $\mathcal{P}_\Phi^*$ is a fundamental domain for the group of translations by elements in the coroot lattice of Φ.

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2. Statement of results

Let Φ^+ be a positive system for Φ, Π the corresponding root basis of Φ, θ the highest root, and $\Omega^\vee$ the dual basis of Π in the dual space V^* of V, i.e., the set of fundamental coweights of Φ.

We set $\Pi = \{\alpha_1, \dots, \alpha_n\}$, $\theta = \sum_{i=1}^n m_i \alpha_i$, $\Omega^\vee = \{\omega_1^\vee, \dots, \omega_n^\vee\}$, so that $\langle \omega_i^\vee, \alpha_j \rangle = \delta_{ij}$, where $\langle\ ,\ \rangle : V^* \times V \mapsto \mathbb{R}$ is the natural pairing of V^* and V, and define

$$o_i = \frac{\omega_i^\vee}{m_i}, \text{ for } i = 1, \dots, n.$$

Consider the fundamental alcove of the affine Weyl group of Φ (see [2, VI, 2.1–2.2], or [7, 4.2–4.3]):

$$\mathcal{A} = \{x \in V^* \mid \langle x, \alpha \rangle \geq 0 \text{ for all } \alpha \in \Pi,\ \langle x, \theta \rangle \leq 1\}.$$

If x is any element or subset of V or V^*, we denote by $W \cdot x$ the orbit of x by the action of W. It is well-known, and easy to see, that $\mathcal{A}$ is the n-simplex with vertices the null vector and $o_1, \dots, o_n$ [2, VI, Corollaire in 2.2], and that

$$\bigcup_{w \in W} w\mathcal{A} = \{x \in V^* \mid \langle x, \beta \rangle \leq 1 \text{ for all } \beta \in \Phi\}. \tag{2.1}$$

We denote by $\mathcal{P}_\Phi$ the *root polytope* associated with Φ, i.e., the convex hull of all roots in Φ. For short, we write $\mathcal{P}$ for $\mathcal{P}_\Phi$ when the root system is clear from the context. It is easy to see that $\mathcal{P}$ is the convex hull of the long roots in Φ. Indeed, this is directly checked if the rank of Φ is 2. In the general case, we observe that, since Φ is irreducible, the set of all long roots and the set of all short roots cannot be mutually orthogonal, hence there exist a short root β and a long root β' such that $(\beta, \beta') \neq 0$. Then β belongs to the convex hull of the long roots in the irreducible dihedral root system generated by β and β', and since W is transitive on the short roots, any short root belongs to the convex hull of some long roots.

We denote by $\mathcal{P}_\Phi^*$ the polar of the root polytope associated with Φ:

$$\mathcal{P}_\Phi^* := \{x \in V^* \mid \langle x, v \rangle \leq 1 \text{ for all } v \in \mathcal{P}_\Phi\},$$

and we call it the *polar root polytope*. Again, for short, we write $\mathcal{P}^*$ for $\mathcal{P}_\Phi^*$. By definition of $\mathcal{P}^*$, we obtain

$$\mathcal{P}^* = \{x \in V^* \mid \langle x, \beta \rangle \leq 1 \text{ for all } \beta \in \Phi\} = \{x \in V^* \mid \langle x, \beta \rangle \leq 1 \text{ for all } \beta \in \Phi_\ell\}, \tag{2.2}$$

where Φ_ℓ is the set of long roots in Φ. Hence, the polar root polytope satisfies

$$\mathcal{P}^* = \bigcup_{w \in W} w\mathcal{A} = \mathrm{Conv}\left(\bigcup_{i=1}^n W \cdot o_i\right). \tag{2.3}$$

Our first result is that, for Φ of type A_n, C_n (hence also for type $\mathrm{B}_2 = \mathrm{C}_2$), B_3, and G_2, the polar root polytope $\mathcal{P}^*$ is a zonotope.

A zonotope is by definition the image of a cube under an affine projection. Let U be a vector space, $v_1, \dots, v_k, p \in U$, and $S = \{v_1, \dots, v_k\}$. We set

$$\mathrm{Zon}_p(S) = \left\{p + \sum_{i=1}^k t_i v_i \mid -\frac{1}{2} \leq t_i \leq \frac{1}{2}\right\}$$

and call $\mathrm{Zon}_p(S)$ the *zonotope generated by S with center p*. Thus a zonotope in U is a polytope of the form $\mathrm{Zon}_p(S)$, for some finite subset S and some vector p in U.

We prove that, for A_n and C_n, the polar root polytope $\mathcal{P}^*$ is the zonotope generated by the orbit of a single o_i (with center the null vector $\underline{0}$), and that a similar result holds for B_3 and G_2. This is done in Section 4, where we find case free conditions for the general inclusions $\mathrm{Zon}_{\underline{0}}(W \cdot c\,\omega_i^\vee) \subseteq \mathcal{P}^*$ ($c \in \mathbb{R}$, $i \in \{1, \dots, n\}$), and check directly the reverse inclusions that we need in the special cases $\mathrm{A}_n, \mathrm{C}_n, \mathrm{B}_3, \mathrm{G}_2$.

The following is a well-known property of zonotopes (see [8, Corollary 7.18]).

Proposition 2.1. *Let U be a vector space, U^* its dual, $S = \{v_1, \dots, v_r\} \subseteq U \setminus \{\underline{0}\}$, $H_i = \{v \in U^* \mid \langle v, v_i \rangle = 0\}$ for $i = 1, \dots, r$, and $\mathcal{A}_S$ be the arrangement of the hyperplanes $H_1, \dots, H_r$. Then the cones on the faces of $\mathrm{Zon}_0(S)^*$ coincide with the faces of $\mathcal{A}_S$.*

We denote by $\mathcal{H}_{\mathcal{P}}$ the central hyperplane arrangement determined by the $(n-2)$-faces of $\mathcal{P}$, i.e., $H \in \mathcal{H}_{\mathcal{P}}$ if and only if H is a hyperplane containing the null vector $\underline{0}$ and some $(n-2)$-face of $\mathcal{P}$. Since the null vector $\underline{0}$ lies in interior of $\mathcal{P}$, the polytopes $\mathcal{P}$ and $\mathcal{P}^*$ are combinatorially dual to each other, and $\mathcal{P} = (\mathcal{P}^*)^*$. By Proposition 2.1, if $\mathcal{P}^*$ is a zonotope, then the cones on the proper faces of $\mathcal{P}$ should coincide with the faces of the hyperplane arrangement $\mathcal{H}_{\mathcal{P}}$. Therefore, $\mathcal{P}^*$ cannot be a zonotope if some hyperplane in $\mathcal{H}_{\mathcal{P}}$ meets the interior of some facet of $\mathcal{P}$. We will see in Section 5 that this happens for all root types other than A_n, C_n, B_3, G_2.

We summarize our results in the following theorem, where we number the simple roots, and hence the fundamental coweights, as in Bourbaki's tables [2]. For any finite subset $S = \{v_1, \dots, v_k\}$, we denote by ZT the zonotope

$$\mathrm{ZT}(S) = \left\{ \sum_{i=1}^{k} t_i v_i \mid 0 \le t_i \le 1 \right\}.$$

If the barycenter of S is the null vector $\underline{0}$, then $\mathrm{ZT}(S)$ coincides with $\mathrm{Zon}_{\underline{0}}(S)$ (see Section 4).

Theorem 2.2. (1) *For Φ of type A_n or C_n,*

$$\mathcal{P}^* = \mathrm{ZT}(W \cdot o_1).$$

(2) *For Φ of type B_3, $\mathcal{P}^* = \mathrm{ZT}\left(W \cdot \frac{o_3}{2}\right)$.*
(3) *For Φ of type G_2, $\mathcal{P}^* = \mathrm{ZT}\left(W \cdot \frac{o_1}{2}\right)$.*
(4) *For all other root types, $\mathcal{P}^*$ is not a zonotope.*

3. Preliminaries

In this section, we fix our further notation and collect some basic results on root systems and Weyl groups. Some of these results are well-known (see [1], [2], or [7]), while other results are more unusual and their proofs will be sketched.

Let Φ be a finite irreducible (reduced) crystallographic root system in the real vector space V endowed with the positive definite bilinear form $(\ ,\ )$. From now on, for notational convenience, we identify V^* with V through the form $(\ ,\ )$.

We summarize our notation on the root system and its Weyl group in the following list:

n	the rank of Φ,
$\Pi = \{\alpha_1, \dots, \alpha_n\}$	the set of simple roots,
$\Omega^\vee = \{\omega_1^\vee, \dots, \omega_n^\vee\}$	the set of fundamental coweights (the dual basis of Π),
Φ^+	the set of positive roots w.r.t. Π,
$\Phi(\Gamma)$	the root subsystem generated by Γ in Φ, for $\Gamma \subseteq \Phi$,
θ	the highest root in Φ,
m_i	the i-th coordinate of θ w.r.t. Π, i.e., $\theta = \sum_{i=1}^n m_i \alpha_i$,
o_i	$= \frac{\omega_i^\vee}{m_i}$, for $i = 1, \dots, n$,
$\Phi^\vee$	$= \{\beta^\vee = \frac{2\beta}{(\beta,\beta)} \mid \beta \in \Phi\}$, the dual root system of Φ,
W	the Weyl group of Φ,
s_β	the reflection with respect to the root β.

For each specific type of the root system, we number the simple roots and hence the fundamental coweights as in Bourbaki's tables [2].

3.1. **Root, coroots and partial orderings.** We denote by $\leq$ the usual partial ordering of V determined by the positive system Φ^+: $x \leq y$ if and only if $y - x = \sum_{\alpha \in \Pi} c_\alpha \alpha$ with c_α a nonnegative integer for all $\alpha \in \Pi$. We denote by $\leq^\vee$ the analogous ordering determined by the dual root system $\Phi^\vee$. For $S \subseteq \Phi$, we let $S^\vee = \{\beta^\vee \mid \beta \in S\}$. Then $\Pi^\vee$ is the basis of $\Phi^\vee$ corresponding to the positive system $(\Phi^+)^\vee$. We have:

$$x \leq^\vee y \quad \text{if and only if} \quad y - x = \sum_{\alpha \in \Pi} c_\alpha \alpha^\vee \quad \text{with } c_\alpha \in \mathbb{Z},\ c_\alpha \geq 0 \quad \text{for all } \alpha \in \Pi.$$

3.2. **Reflection products.** In the following lemma, we provide a result on the product of reflections by general roots (possibly not simple).

Lemma 3.1. *Let $\beta_1, \dots, \beta_k \in \Phi$ and $w = s_{\beta_1} \cdots s_{\beta_k}$. Then, for any $x \in V$,*

$$w(x) = x - \sum_{i=1}^{k} (x, \beta_i^\vee) \nu_i = x - \sum_{i=1}^{k} (x, \beta_i) \nu_i^\vee, \quad \textit{where} \quad \nu_i = s_{\beta_1} \cdots s_{\beta_{i-1}}(\beta_i); \tag{3.1}$$

$$w(x) = x - \sum_{i=1}^{k} (x, \eta_i^\vee) \beta_i = x - \sum_{i=1}^{k} (x, \eta_i) \beta_i^\vee, \quad \textit{where} \quad \eta_i = s_{\beta_k} \cdots s_{\beta_{i+1}}(\beta_i). \tag{3.2}$$

Proof. The first equality in Formula (3.1) is easily proved by induction computing $s_{\beta_1}(s_{\beta_2} \cdots s_{\beta_k}(x))$; the second one is clear since β_i and ν_i have the same length, for all i. Formula (3.2) is an application of (3.1) since, by definition of η_i and the fact that $s_{s_\beta(\beta')} = s_\beta s_{\beta'} s_\beta$ for all roots β and β', we have

$$s_{\beta_h} \cdots s_{\beta_k} = s_{\eta_k} \cdots s_{\eta_h}, \quad \text{for } h = 1, \dots, k,$$

hence $w = s_{\eta_k} \cdots s_{\eta_1}$ and $\beta_i = s_{\eta_k} \cdots s_{\eta_{i+1}}(\eta_i)$. □

For $h = 1, \dots, k$, if we define $w_h = s_{\beta_h} \cdots s_{\beta_k} = s_{\eta_k} \cdots s_{\eta_h}$, Formula (3.2) yields

$$w_h(x) = x - \sum_{i=h}^{k} (x, \eta_i^\vee) \beta_i = x - \sum_{i=h}^{k} (x, \eta_i) \beta_i^\vee. \tag{3.3}$$

3.3. **Reduced expressions.** For any $w \in W$, we set

$$N(w) = \{\gamma \in \Phi^+ \mid w^{-1}(\gamma) < \underline{0}\}.$$

If $w = s_{\beta_1} \cdots s_{\beta_k}$ is a reduced expression of w (so $\beta_i \in \Pi$, for all $i = 1, \ldots, k$), and if we define $\nu_i = s_{\beta_1} \cdots s_{\beta_{i-1}}(\beta_i)$ and $\eta_i = s_{\beta_k} \cdots s_{\beta_{i+1}}(\beta_i)$ for $i = 1, \ldots, k$ as in Lemma 3.1, then we have the following result (see [2, VI, 1.6, Corollaire 2]):

$$N(w) = \{\nu_1, \ldots, \nu_k\} \quad \text{and} \quad N(w^{-1}) = \{\eta_1, \ldots, \eta_k\}. \tag{3.4}$$

3.4. **Stabilizers.** We denote by $\mathcal{C}$ the fundamental chamber of W:

$$\mathcal{C} = \{x \in V \mid (x, \alpha) \geq 0 \text{ for all } \alpha \in \Pi\}.$$

Formula (3.1) implies directly the well-known fact that

$$\mathrm{Stab}_W(x) = \langle s_\alpha \mid \alpha \in \Pi,\ (x, \alpha) = 0\rangle, \quad \text{for all } x \in \mathcal{C}, \tag{3.5}$$

and, as an easy consequence, the equally well-known fact that

$$\mathrm{Stab}_W(x) = \langle s_\beta \mid \beta \in \Phi,\ (x, \beta) = 0\rangle, \quad \text{for all } x \in V. \tag{3.6}$$

For all $j \in \{1, \ldots, n\}$, we set

$$W^j = \langle s_{\alpha_i} \mid i \in [n] \setminus \{j\}\rangle,$$

so that

$$W^j = \mathrm{Stab}_W(\omega_j^\vee).$$

3.5. **Images of fundamental coweights.** Let $w = s_{\beta_1} \cdots s_{\beta_k}$ be a reduced expression of w and $\eta_i = s_{\beta_k} \cdots s_{\beta_{i+1}}(\beta_i)$, for $i = 1, \ldots, k$. Recall that, by definition, the left descents of w are the simple roots in $N(w)$, and the right descents the simple roots in $N(w^{-1})$. For any $j \in \{1, \ldots, n\}$, if w is a minimal length representative in the left coset wW^j, then α_j is the unique right descent of w. Hence, every reduced expression of w ends with s_{α_j} and every reduced expression of w^{-1} starts with s_{α_j}, i.e., in our notation,

$$\beta_k = \eta_k = \alpha_j. \tag{3.7}$$

For $\gamma \in \Phi$, let

$$\mathrm{Supp}(\gamma) = \{\alpha_i \in \Pi \mid (\gamma, \omega_i^\vee) \neq 0\},$$

and, for all $\alpha \in \Pi$, let

$$M_\alpha = \{\gamma \in \Phi^+ \mid \alpha \in \mathrm{Supp}(\gamma)\}.$$

It is clear that if $\mathrm{Supp}(\gamma) \cap N(w^{-1}) = \emptyset$, then $\gamma \notin N(w^{-1})$, by the linearity of w. Hence, if w is the minimal length representative in wW^j, we have

$$\{\eta_1, \ldots, \eta_k\} \subseteq M_{\alpha_j}. \tag{3.8}$$

Equivalently, $(\omega_j^\vee, \eta_i) \geq 1$ for $i = 1, \ldots, k$. Hence, by (3.2) and (3.3), we have

$$w_h(\omega_j^\vee) \leq^\vee \omega_j^\vee - \beta_k^\vee \cdots - \beta_h^\vee. \tag{3.9}$$

In particular, by (3.7), for each $w \notin \mathrm{Stab}_W(\omega_j^\vee)$

$$w(\omega_j^\vee) \leq^\vee \omega_j^\vee - \alpha_j^\vee. \tag{3.10}$$

4. Polar root polytopes that are zonotopes

In this section, we prove items (1), (2), and (3) of Theorem 2.2.

By Proposition 2.1, if $\mathcal{P}^*$ is a zonotope, then the cones on the proper faces of $\mathcal{P}$ coincide with the faces of the hyperplane arrangement $\mathcal{H}_{\mathcal{P}}$, and $\mathcal{P}^* = \mathrm{Zon}_{\underline{0}}(S)$, where S is a complete set of orthogonal vectors to the hyperplanes of $\mathcal{H}_{\mathcal{P}}$. Recall from [4, Proposition 3.2] that the hyperplanes in the arrangement $\mathcal{H}_{\mathcal{P}}$ are of a very special form: there exists a subset $H_\Phi \subseteq \{1, \dots, n\}$ (depending on Φ) such that $\mathcal{H}_{\mathcal{P}} = \{w(\omega_k^\vee)^\perp \mid w \in W, k \in H_\Phi\}$. The sets H_Φ are given in [4, Table 2]. We will see that, when $\mathcal{P}^*$ is a zonotope, the set S generating it is the W-orbit of a multiple of a single coweight.

From the definitions (see Section 2), it is clear that $\mathrm{ZT}(S) = \mathrm{Zon}_p(S)$, with $p = \frac{1}{2}\sum_{s\in S} s$. For any $\lambda \in V$, we get

$$\mathrm{ZT}(W \cdot \lambda) = \mathrm{Zon}_{\underline{0}}(W \cdot \lambda),$$

since $\frac{1}{2}\sum_{w\in W} w(\lambda)$ is fixed by all elements in W and so must be the null vector $\underline{0}$.

To prove items (1), (2), and (3) of Theorem 2.2, we need the following lemmas.

Lemma 4.1. *Let S be a W-stable finite subset of V.*

1. $\mathrm{ZT}(S) \subseteq \mathcal{P}^*$ *if and only if, for each $X \subseteq S$,* $\left(\sum_{x\in X} x, \theta\right) \leq 1$.
2. *If for each $i \in \{1, \dots, n\}$ there exists $X \subseteq S$ such that $\sum_{x\in X} x = o_i$, then* $\mathcal{P}^* \subseteq \mathrm{ZT}(S)$.

Proof. (1) It is easy to see that the set of vertices of $\mathrm{ZT}(S)$ is a subset of $\left\{\sum_{x\in X} x \mid X \subset S\right\}$ (see for example [5, §2.3]). Hence, the claim follows from (2.2) and the stability of S under W, since all long roots are in the same W-orbit.

(2) By (2.3) and the stability of S under W, the assumption that for each $i \in \{1, \dots, n\}$, $o_i = \sum_{x\in X} x$, with $X \subseteq S$, implies that the set of vertices of $\mathcal{P}^*$ is contained in the set of vertices of $\mathrm{ZT}(S)$, hence the claim. □

For each $j \in \{1, \dots, n\}$, we set

$$r_j = \frac{\|\theta\|^2}{\|\alpha_j\|^2}.$$

Lemma 4.2. *Let $j \in \{1, \dots, n\}$. For each long root β, we have*

$$(\beta, \omega_j^\vee) \equiv m_j \mod r_j.$$

Proof. The claim is obvious if α_j is long, in which case $r_j = 1$. We recall that, if α_j is short, then $(\gamma, \alpha_j^\vee) \in \{-r_j, 0, r_j\}$ for each long root γ. Hence the claim follows by induction on the length of $w \in W$ such that $w(\theta) = \beta$. □

For each subset S of Π, we denote by $\Phi(S)$ the standard parabolic subsystem of Φ generated by S, and by $W(S)$ the Weyl group of $\Phi(S)$. We set

$$\Phi_0 = \Phi(\Pi \cap \theta^\perp), \qquad W_0 = W(\Phi_0).$$

For each $\alpha \in \Pi$, $\alpha \perp \theta$ if and only if α is not connected to α_0, the extra root added to Π in the extended Dynkin diagram of Φ (see [7, §4.7], or [2, Chapter VI, n° 4.3]). Since θ is in the fundamental chamber of W, we have

$$W_0 = \mathrm{Stab}_W(\theta).$$

For each $j \in \{1, \ldots, n\}$, we set

$$\Phi_0^j = \Phi((\Pi \cap \theta^\perp) \setminus \{\alpha_j\}), \qquad W_0^j = W(\Phi_0^j), \qquad q_j = [W_0 : W_0^j].$$

It is clear that

$$W(\Phi_0^j) = \mathrm{Stab}_W(\omega_j^\vee) \cap \mathrm{Stab}_W(\theta) = \mathrm{Stab}_{W_0}(\omega_j^\vee).$$

Lemma 4.3. *Let $j \in \{1, \ldots, n\}$ and $x \in W \cdot \omega_j^\vee$. Then, $(x, \theta) = m_j$ if and only if $x \in W_0 \cdot \omega_j^\vee$. In particular,*

$$\left|\left\{x \in W \cdot \omega_j^\vee \mid (x, \theta) = m_j\right\}\right| = q_j.$$

Proof. It is obvious that, if $w \in W_0$, then $\big(w(\omega_j^\vee), \theta\big) = m_j$, since $\big(w(\omega_j^\vee), \theta\big) = \big(\omega_j^\vee, w^{-1}(\theta)\big) = \big(\omega_j^\vee, \theta\big)$. Conversely, assume $x \in W \cdot \omega_j^\vee$ and $(x, \theta) = m_j$. Let w be the minimal length element in W such that $x = w(\omega_j^\vee)$, and $w = s_{\beta_1} \cdots s_{\beta_k}$ be a reduced expression of w. Then, by (3.9), we have $x \leq^\vee \omega_j^\vee - \sum_{i=1}^{k} \beta_i^\vee$, and hence, since θ is in the fundamental chamber and $(\omega_i^\vee, \theta) = m_i$, we obtain $(\theta, \beta_i^\vee) = 0$ for $i = 1, \ldots, k$, i.e., $w \in W_0$. □

Proposition 4.4. *Let $j \in \{1, \ldots, n\}$. If $r_j = m_j$, then for all $c \in \mathbb{R}$,*

$$\mathrm{ZT}\left(W \cdot c\,\omega_j^\vee\right) \subseteq \mathcal{P}^* \text{ if and only if } c \leq \frac{1}{q_j m_j}.$$

Equivalently, if $r_j = m_j$, then the hyperplane $\{(x, \theta) = q_j m_j\}$ is a supporting hyperplane for $\mathrm{ZT}\left(W \cdot \omega_j^\vee\right)$.

Proof. By Lemma 4.3, for all $w \in W \backslash W_0$, we have $(w(\omega_j^\vee), \theta) < m_j$. Hence, by Lemma 4.2, we infer $(w(\omega_j^\vee), \theta) \leq 0$. It follows that

$$\left(\sum_{x \in W_0 \cdot \omega_j^\vee} x, \theta\right) = q_j m_j,$$

and that, for any other $z \in \mathrm{ZT}\left(W \cdot \omega_j^\vee\right)$, we have $(z, \theta) \leq q_j m_j$. This proves the claim. □

For both types A_n and C_n, we have $\mathcal{H}_{\mathcal{P}} = \{w(\omega_1^\vee)^\perp \mid w \in W\}$, and the values of Φ_0, Φ_0^1, q_1, m_1, and r_1 are the following:

$$\mathrm{A}_n: \quad \Phi_0 = \Phi_0^1 = \Phi(\Pi \setminus \{\alpha_1, \alpha_n\}), \quad q_1 = m_1 = r_1 = 1,$$

$$\mathrm{C}_n: \quad \Phi_0 = \Phi_0^1 = \Phi(\Pi \setminus \{\alpha_1\}), \quad q_1 = 1, \quad m_1 = r_1 = 2.$$

By Proposition 4.4, from the previous computations, in types A_n and C_n, we have $\mathrm{ZT}\,(W \cdot o_1) \subseteq \mathcal{P}^*$. In both cases, also the other inclusion holds.

Theorem 4.5. *Let Φ be of type A_n or C_n. Then*

$$\mathrm{ZT}\,(W \cdot o_1) = \mathcal{P}^*.$$

Proof. To prove that $\mathrm{ZT}\,(W\cdot o_1)\supseteq \mathcal{P}^*$, we show that, for each $k\in\{1,\dots,n\}$, o_k is a sum of distinct elements in $W\cdot o_1$ (Lemma 4.1, (2)).

Let $w_i := s_i\cdots s_1$, for $i\in\{1,\dots,n\}$, and $w_0:=e$, the identity element. We show by induction on k, $0\le k<n$, that $\sum\limits_{i=0}^{k} w_i(o_1)=o_{k+1}$. This is trivially true for $k=0$. If $k>0$ and $\sum\limits_{i=0}^{h} w_i(o_1)=o_{h+1}$, $h<k<n$, then $w_h(o_1)=-o_h+o_{h+1}$ (where $o_0:=\underline{0}$). We have

$$\sum_{i=0}^{k} w_i(o_1)=o_k+w_k(o_1)=o_k+s_kw_{k-1}(o_1)=o_k+s_k(-o_{k-1}+o_k)=o_k-o_{k-1}+s_ko_k.$$

Since $(o_i,\alpha_j)=\frac{1}{m_i}\delta_{i,j}$, we have $(s_k(o_k),\alpha_j)=(o_k,s_k(\alpha_j))=\frac{-(\alpha_k^\vee,\alpha_j)}{m_k}$, and hence

$$\left(\sum_{i=0}^{k} w_i(o_1),\alpha_j\right)=0$$

for $j=k$ and for all j with $|j-k|>1$. Only the two cases $j=k-1$ and $j=k+1$ are left out. We have

$$\left(\sum_{i=0}^{k} w_i(o_1),\alpha_{k-1}\right)=-\frac{1}{m_{k-1}}-\frac{(\alpha_k^\vee,\alpha_{k-1})}{m_k}=0,$$

(note that, for type C_n, we need $k<n$), and

$$\left(\sum_{i=0}^{k} w_i(o_1),\alpha_{k+1}\right)=-\frac{(\alpha_k^\vee,\alpha_{k+1})}{m_k}=1.$$

So we get the assertion. □

While the property $\mathrm{ZT}\,(W\cdot o_1)=\mathcal{P}^*$ is a property of the root system Φ, the property that $\mathcal{P}^*$ is a zonotope is a property of the root polytope $\mathcal{P}$. Hence, in view of $\mathcal{P}^*_{\mathrm{B}_3}\cong\mathcal{P}^*_{\mathrm{A}_3}$ and $\mathcal{P}^*_{\mathrm{G}_2}\cong\mathcal{P}^*_{\mathrm{A}_2}$, we deduce that the polar root polytopes of types B_3 and G_2 are also zonotopes. It turns out that, also in these two cases, the polar root polytope is the zonotope generated by the orbit of a single vector, proportional to a coweight. More precisely, the following result holds.

Proposition 4.6. *In types* B_3 *and* G_2*, we have*

$$\mathcal{P}^*_{\mathrm{B}_3}=\mathrm{ZT}\left(W\cdot\frac{o_3}{2}\right),$$
$$\mathcal{P}^*_{\mathrm{G}_2}=\mathrm{ZT}\left(W\cdot\frac{o_1}{2}\right).$$

Proof. One inclusion follows by Proposition 4.4 since we have

$$\begin{aligned}\mathrm{B}_3:\ &\Phi_0=\Phi(\Pi\setminus\{\alpha_2\})\cong \mathrm{A}_1\times\mathrm{A}_1,\\ &\Phi_0^3=\Phi(\Pi\setminus\{\alpha_2,\alpha_3\})\cong\mathrm{A}_1,\quad q_3=2,\quad m_3=r_3=2.\\ \mathrm{G}_2:\ &\Phi_0=\Phi(\Pi\setminus\{\alpha_2\})\cong\mathrm{A}_1,\quad \Phi_0^1=\Phi(\Pi\setminus\{\alpha_1,\alpha_2\})=\emptyset,\quad q_1=2,\quad m_1=r_1=3.\end{aligned}$$

The other inclusion can be directly proved using Lemma 4.1, (2). □

5. Polar root polytopes that are not zonotopes

In this section, we prove item (4) of Theorem 2.2, i.e., that, for all root systems Φ other than those of types A_n, C_n, B_3 and G_2, the polar root polytope $\mathcal{P}^*$ is not a zonotope.

In fact, we show that, for all such root systems, the set of cones on the facets of the root polytope $\mathcal{P}$ is not equal to the set of closures of the regions of the hyperplane arrangement $\mathcal{H}_{\mathcal{P}}$. This is enough to show that $\mathcal{P}^*$ cannot be a zonotope by Proposition 2.1, noting that:

(1) being the convex hull of the long roots in Φ, the polytope $\mathcal{P}$ is centrally symmetric with respect to the null vector $\underline{0}$;
(2) the polar of a polytope which is centrally symmetric with respect to $\underline{0}$ is centrally symmetric with respect to the null vector in the dual space;
(3) every zonotope which is centrally symmetric with respect to the null vector $\underline{0}$ is of the form $\mathrm{Zon}_{\underline{0}}(S)$, for an appropriate set S.

Recall that $\mathcal{H}_{\mathcal{P}}$ is the central hyperplane arrangement determined by the $(n-2)$-faces of $\mathcal{P}$, i.e., $H \in \mathcal{H}_{\mathcal{P}}$ if and only if H is a hyperplane containing $\underline{0}$ and some $(n-2)$-face of $\mathcal{P}$. We show that some hyperplane in $\mathcal{H}_{\mathcal{P}}$ meets the interior of some facet of $\mathcal{P}$, for all root types other than A_n, C_n, B_3, G_2. More precisely, for each of the root systems Φ we are considering, we identify a hyperplane of $\mathcal{H}_{\mathcal{P}}$ containing the barycenter of a facet of $\mathcal{P}$ and hence cutting that facet.

For the reader's convenience, we recall from [3] that $\mathcal{P}$ has certain distinguished faces, which we call *standard parabolic*, which give a complete set of representatives of the W-orbits of faces of $\mathcal{P}$ (see [3, Corollary 4.3 and Theorem 5.11]). For each $I \subseteq \{1, \ldots, n\}$, we let

$$F_I := \mathrm{Conv}\{\alpha \in \Phi^+ \mid (\omega_i^\vee, \alpha) = m_i, \text{ for all } i \in I\}$$

be the standard parabolic face associated with I. Here we need the standard parabolic facets, which are all of the form $F_i := F_{\{i\}}$, for $i \in \{1, \ldots, n\}$ (not all standard parabolic faces indexed by singletons are facets but all standard parabolic faces which are facets are indexed by singletons). The numbers i such that F_i are facets are those such that the extended Dynkin diagram is still connected after removing α_i (see [3, Section 5]). Moreover, we will use the fact that the barycenter of a standard parabolic facet F_i is a multiple of the corresponding fundamental weight ω_i (see [3, Lemma 4.2]).

As we already recalled in Section 4, there exists a subset $H_\Phi \subseteq \{1, \ldots, n\}$ such that $\mathcal{H}_{\mathcal{P}} = \{w(\omega_k^\vee)^\perp \mid w \in W, k \in H_\Phi\}$. We call the hyperplanes $(\omega_k^\vee)^\perp$, $k \in H_\Phi$, *standard hyperplanes.*

For each irreducible root type other than A_n, C_n, B_3, G_2, we will exhibit a standard hyperplane containing the barycenter of a facet. Since each facet is of the form wF_i, and the barycenter of wF_i is a scalar multiple of $w(\omega_i)$, it suffices to find $i, k \in \{1, \ldots, n\}$ and $w \in W$ such that: F_i is a facet, $(\omega_k^\vee)^\perp$ is a standard hyperplane, and $w(\omega_i) \perp \omega_k^\vee$. In the following table, beside each root type other than A_n, C_n, B_3 and G_2, in the first row we list all the standard parabolic facets and all the standard hyperplanes; in the further rows we write down explicitly a particular triple i, k, w such that $i, k \in \{1, \ldots, n\}$, $w \in W$ and $w(\omega_i) \perp \omega_k^\vee$.

B_n $n \geq 4$	$F_1,\, F_n,\qquad (\omega_1^\vee)^\perp,\, (\omega_n^\vee)^\perp$ $\omega_1 = \alpha_1 + \cdots + \alpha_n,\qquad s_1(\omega_1) = (\omega_1 - \alpha_1) \perp \omega_1^\vee$
D_n $n \geq 4$	$F_1,\, F_{n-1},\, F_n,\qquad (\omega_1^\vee)^\perp,\, (\omega_{n-1}^\vee)^\perp,\, (\omega_n^\vee)^\perp$ $\omega_1 = \alpha_1 + \cdots + \alpha_n,\qquad s_1(\omega_1) = (\omega_1 - \alpha_1) \perp \omega_1^\vee$
E_6	$F_1,\, F_6,\qquad (\omega_1^\vee)^\perp,\, (\omega_6^\vee)^\perp$ $\omega_1 = \frac{4}{3}\alpha_1 + \alpha_2 + \frac{5}{3}\alpha_3 + 2\alpha_4 + \frac{4}{3}\alpha_5 + \frac{2}{3}\alpha_6$ $s_2s_4s_3s_1(\omega_1) = \omega_1 - \alpha_1 - \alpha_3 - \alpha_4 - \alpha_2 \perp \omega_2^\vee$
E_7	$F_2,\, F_7,\qquad (\omega_1^\vee)^\perp,\, (\omega_2^\vee)^\perp$ $\omega_7 = \alpha_1 + \frac{3}{2}\alpha_2 + 2\alpha_3 + 3\alpha_4 + \frac{5}{2}\alpha_5 + 2\alpha_6 + \frac{3}{2}\alpha_7$ $s_1s_3s_4s_5s_6s_7(\omega_7) = \omega_7 - \alpha_7 - \alpha_6 - \alpha_5 - \alpha_4 - \alpha_3 - \alpha_1 \perp \omega_1^\vee$
E_8	$F_1,\, F_2,\qquad (\omega_2^\vee)^\perp,\, (\omega_8^\vee)^\perp$ $\omega_1 = 4\alpha_1 + 5\alpha_2 + 7\alpha_3 + 10\alpha_4 + 8\alpha_5 + 6\alpha_6 + 4\alpha_7 + 2\alpha_8$ $s_8s_7s_6s_5s_4s_3s_1(s_2s_4s_3s_5s_4s_2s_6s_5s_4s_3)s_1(\omega_1) =$ $s_8s_7s_6s_5s_4s_3s_1(s_2s_4s_3s_5s_4s_2s_6s_5s_4s_3)(\omega_1 - \alpha_1) = s_8s_7s_6s_5s_4s_3s_1(\omega_1 - \theta_6) =$ $(\omega_1 - \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8) - (\theta_6 - \alpha_7 - \alpha_8) \perp \omega_8^\vee$ [*here* θ_6 *is the highest root of the type* E_6 *root system generated by* $\alpha_1 \ldots, \alpha_6$]
F_4	$F_4,\qquad (\omega_4^\vee)^\perp$ $\omega_4 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$ $s_4s_3(s_2s_3s_4)(\omega_4) = s_4s_3(\omega_4 - \alpha_4 - \alpha_3 - \alpha_2) = \omega_4 - 2\alpha_4 - 2\alpha_3 - \alpha_2 \perp \omega_4^\vee$

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