Sminaire Lotharingien de Combinatoire **72** (2015), Article B72d

INCIDENCE HOPF ALGEBRA OF THE HYPERTREE POSETS

BÉRÉNICE DELCROIX-OGER

ABSTRACT. We adapt the computation of characters on incidence Hopf algebras introduced by Schmitt in the 1990s for families of bounded posets to a family mixing bounded and unbounded finite posets. This computation relies on the introduction of an auxiliary bialgebra: the coproduct in this bialgebra enables us to compute the convolution of some characters on the incidence Hopf algebra. After establishing a general result on the link between the bialgebra and the incidence Hopf algebra, we apply it to the family of hypertree posets and partition posets. This link for hypertree posets enables us to recover the Möbius numbers of these posets due to the coproduct in the associated bialgebra. This coproduct is computed using the number of hypertrees with fixed valency set and fixed edge sizes set.

CONTENTS

INTRODUCTION

In 1994, Schmitt [13] defined the notion of incidence Hopf algebra associated with a given family of intervals satisfying some closure properties. Using the Hopf algebra structure, one can define a convolution on characters on this algebra. The Möbius number for posets of the family can then be computed using characters on the incidence Hopf algebra.

However, the incidence Hopf algebras of Schmitt are only defined for bounded posets. In the present article, we introduce a way to compute some characters for another type of finite posets, called the triangle and diamond posets. The diamond posets are bounded posets whereas the triangle posets have a least element but no greatest one. If we consider the hereditary family generated by the diamond posets and the augmented triangle posets, i.e., the triangle posets with an added greatest element, we can build the associated incidence Hopf algebra $\mathcal{H}$. The coproduct in the non-counital bialgebra $\mathcal{B}$ generated by isomorphism classes in the hereditary family obtained from diamond and triangle posets can be linked with the coproduct of the incidence Hopf algebra $\mathcal{H}$: this

Key words and phrases. poset, incidence Hopf algebra, hypertree, Möbius number.

relation enables us to identify a computation involving maps from the bialgebra $\mathcal{B}$ to $\mathbb{Q}$ with the convolution of characters on the incidence Hopf algebra $\mathcal{H}$. The advantage of this method is that the computation of such maps on the bialgebra $\mathcal{B}$ is, in most cases, easier. In the rest of the article, we will apply this method to hypertree posets. This method could also be applied to other families mixing bounded and unbounded posets, such as, for instance, to planar hypertree posets or pointed hypertree posets (cf. [10]), which are Cohen–Macaulay but for which there is still no known closed formula for their Möbius numbers.

In the first section of the article, we present generalities on posets and incidence Hopf algebras, before defining the bialgebra $\mathcal{B}$ in the second section.

In the third section of the article, we recall the notion of hypertrees. Hypergraphs have been introduced in 1989 by Berge [3] as a generalization of graphs. Hypertrees are hypergraphs satisfying a kind of connectedness and acyclicity. The set of hypertrees on a vertex set I can be endowed with a partial order given by union of edges. We provide a criterion on sequences (α_i) and (π_j) for the existence of a hypertree with α_i vertices of valency i and π_j edges of size j, which also appears in [1]. In the fourth section, we also present a formula for the number of hypertrees with a given edge-size and valency sets, which was already computed by Bacher [1] and by Bousquet-Mélou and Chapuy in [4] (see also [16]). We include the proof for self-containment, and refer the reader to the beginning of this section for more bibliographic details.

The results of the third and fourth section are then used in the fifth section to compute the coproduct in the bialgebra $\mathcal{B}_{HT}$ associated with the hypertree posets. The Möbius number of the poset of hypertrees on n vertices has been computed by McCammond and Meier in [8]. The hypertree poset has also been studied in [6]. Chapoton has computed its characteristic polynomial in [5], and he conjectured a formula for the action of the symmetric group on the homology of the hypertree poset. This conjecture has been proven by the author in [11]. In the last section, we propose a new way to compute the Möbius number of the hypertree posets. The advantage of this method is that it is algebraic, that it does not need the Cohen–Macaulayness of the posets, and that it uses the usual framework of incidence Hopf algebra. This leads to the enumerative formula (6.4) relating an alternating sum of quotients of factorials to the number of rooted trees on $n-1$ vertices.

We thank the referees for their advice, which helped me to improve this paper.

1. Generalities on posets and incidence Hopf algebras

In this section, we review some general notions on posets and incidence Hopf algebras which will be needed in this article.

1.1. **Generalities on posets.** A *poset* is a set endowed with a partial order $\leq$. All posets considered here will be finite. The *trivial* poset is the poset which has only one element. If P is a poset in which $x \leq y$, then the *interval* $[x, y]$ is the set $\{z \in P : x \leq z \leq y\}$, and the *half-open interval* $[x, y)$ is the set $\{z \in P : x \leq z < y\}$. If P is an interval, P is called a *bounded* poset. In this case, its least and greatest elements will be denoted by $\hat{0}_P$ and $\hat{1}_P$, respectively, or by $\hat{0}$ and $\hat{1}$ if there is no ambiguity.

Given a locally finite poset P, its *Möbius function* μ is recursively defined by

$$\mu(x,x) = 1, \quad \text{for all } x \in P, \tag{1.1}$$

$$\mu(x,y) = -\sum_{x \leq z < y} \mu(x,z), \quad \text{for all } x < y \in P. \tag{1.2}$$

The *Möbius invariant*, or *Möbius number*, of a finite bounded poset P is defined as

$$\mu(P) := \mu(\hat{0}_P, \hat{1}_P). \tag{1.3}$$

Conceptually, the Möbius number of a poset can be defined as the inverse of the zeta function with respect to convolution, as explained in Section 6.1.

Example 1.1. The Möbius number of the poset B_n of subsets of $[\![1,n]\!]$, ordered by inclusion, is $(-1)^n$.

1.2. **Generalities on incidence Hopf algebra.** All the definitions recalled here are extracted from the article [13] of Schmitt.

A family of intervals $\mathcal{P}$ is *interval closed*, if it is non-empty and, for all $P \in \mathcal{P}$ and $x \leq y \in P$, the interval $[x,y]$ belongs to $\mathcal{P}$. An *order compatible relation* on an interval closed family $\mathcal{P}$ is an equivalence relation $\sim$ such that $P \sim Q$ if and only if there exists a bijection $\phi : P \to Q$ such that $[0_P, x] \sim [0_Q, \phi(x)]$ and $[x, 1_P] \sim [\phi(x), 1_Q]$, for all $x \in P$. Isomorphism of posets is an example of an order compatible relation.

Given an order compatible relation $\sim$ on an interval closed family $\mathcal{P}$, we consider the quotient set $\mathcal{P}/\sim$ and denote by $[P]$ the $\sim$-equivalence class of a poset $P \in \mathcal{P}$. We define a $\mathbb{Q}$-coalgebra $C(\mathcal{P})$ as follows.

Proposition 1.2 ([13, Theorem 3.1])**.** *Let $C(\mathcal{P})$ denote the free $\mathbb{Q}$-module generated by $\mathcal{P}/\sim$. We define linear maps $\Delta : C(\mathcal{P}) \to C(\mathcal{P}) \otimes C(\mathcal{P})$ and $\epsilon : C(\mathcal{P}) \to \mathbb{Q}$ by*

$$\Delta[P] = \sum_{x \in P} [0_P, x] \otimes [x, 1_P] \tag{1.4}$$

and

$$\epsilon[P] = \delta_{|P|,1}, \tag{1.5}$$

where $\delta_{i,j}$ is the Kronecker symbol. Then $C(\mathcal{P})$ is a coalgebra with comultiplication Δ and counit ϵ.

The *direct product* of posets P_1 and P_2 is the Cartesian product $P_1 \times P_2$ partially ordered by the relation $(x_1, x_2) \leq (y_1, y_2)$ if and only if $x_i \leq y_i$ in P_i, for $i = 1, 2$. A *hereditary family* is an interval closed family which is also closed under formation of direct products. Let $\sim$ be an order compatible relation on $\mathcal{P}$ which is also a semigroup congruence, i.e., whenever $P \sim Q$ in $\mathcal{P}$, then $P \times R \sim Q \times R$ and $R \times P \sim R \times Q$, for all $R \in \mathcal{P}$. This relation is *reduced* if, whenever $|R| = 1$, then $P \times R \sim R \times P \sim P$: all trivial intervals are then equivalent and give a unit element for the product on the quotient. These hypotheses ensure that the product will be well defined on the quotient. The obtained unit is denoted by ν. An order compatible relation on a hereditary family $\mathcal{P}$ which is also a reduced congruence is called a *Hopf relation* on $\mathcal{P}$. Isomorphism of posets is a Hopf relation.

Proposition 1.3 ([12]). *Let $\sim$ be a Hopf relation on a hereditary family $\mathcal{P}$. Then $H(\mathcal{P}) = (C(\mathcal{P}), \times, \Delta, \nu, \epsilon, S)$ is a Hopf algebra over $\mathbb{Q}$, with the antipode S on an equivalence class of posets $[P]$ in $C(\mathcal{P})$ given by*

$$S[P] = \sum_{k \geq 0} \sum_{\substack{x_0 < \cdots < x_k \\ x_0 = 0_P \\ x_k = 1_P}} (-1)^k \prod_{i=1}^{k} [x_{i-1}, x_i]. \tag{1.6}$$

Example 1.4. The incidence Hopf algebra generated by the family of posets of subsets of $[\![1, n]\!]$ is the polynomial algebra $\mathbb{Q}[x]$, endowed with the coproduct

$$\Delta(x^n) = \sum_{k=0}^{n} \binom{n}{k} x^k \otimes x^{n-k}.$$

We may consider the set of $\mathbb{Q}$-linear algebra homomorphisms between $H(\mathcal{P})$ and $\mathbb{Q}$, which send the trivial poset to the unit of $\mathbb{Q}$. These homomorphisms are called *characters*. The set of characters has a canonical group structure as follows. Given two characters ϕ and ψ and an element P of $H(\mathcal{P})$, the convolution of ϕ and ψ is defined by

$$(\phi * \psi)(P) = \sum \phi(P_{(1)})\psi(P_{(2)}), \tag{1.7}$$

where $\Delta(P) = \sum P_{(1)} \otimes P_{(2)}$, using Sweedler's convention. The unit of this group is the counit of the Hopf algebra $H(\mathcal{P})$.

2. Incidence Hopf algebra of triangle and diamond posets

In this section, we first present the incidence Hopf algebra of triangle and diamond posets. This incidence Hopf algebra is bigger than desired, as some open intervals of triangle posets are products of triangle posets but cannot be decomposed as products anymore once bounded by the addition of a greatest element. We thus present in the second subsection a non-counital bialgebra for the family of triangle and diamond posets, before linking characters on the incidence Hopf algebra of triangle and diamond posets with a computation in the bialgebra associated with triangle and diamond posets. This link is used in Section 6 to compute some characters on the hypertree posets.

2.1. **Presentation of the triangle and diamond posets and their incidence Hopf algebra.** Let us consider the family $\mathcal{F}_0$ consisting of posets $\{(d_i)_{i\geq 1}, (t_j)_{j\geq 3}\}$, such that d_1 is the trivial poset, d_i is a finite interval for all $i \geq 2$, and t_j is a finite poset with a least element but without a greatest element. The elements of the collection $(d_i)_{i\geq 1}$ will be called *diamond posets*, and the elements of $(t_j)_{j\geq 2}$ will be called *triangle posets*. We denote by $\widehat{t_j}$ the augmented triangle poset, bounded by the addition of a greatest element $\hat{1}$ to t_j. We moreover assume that:

> *Decomposition Property 1.* Any closed interval in a diamond poset can be written as a direct product of diamond posets.
>
> *Decomposition Property 2 a.* Any closed interval in a triangle poset can be written as a direct product of diamond posets.

Decomposition Property 2 b. Any half-open interval $[t, \hat{1})$ in an augmented triangle poset $\widehat{t_j}$ is trivial or can be written as a direct product of triangle posets (see Figure 2.1).

We denote by $\mathcal{F}_1$ the smallest hereditary family containing $\mathcal{F}_0$. Due to the decomposition properties, this family is constituted by direct products of diamond and triangle posets. As diamond and triangle posets admit a least element, all posets in $\mathcal{F}_1$ have a least element, some of these are intervals but others are not. We construct a hereditary family of intervals from this family.

To apply Schmitt's construction, we now consider the smallest hereditary family $\mathcal{F}_2$ containing the elements of the family $\mathcal{F}_1$ augmented with a maximal element when they are not intervals. The family $\mathcal{F}_2$ then contains all the augmented elements of $\mathcal{F}_1$, but also upper intervals in augmented triangle posets which cannot be rewritten as a direct product anymore, and direct products of these posets. The family $\mathcal{F}_2$ is then a hereditary family of intervals. We apply Schmitt's construction to this family taking isomorphism of posets as a reduced order compatible relation to obtain the incidence Hopf algebra $\mathcal{H}_{\diamond,\triangledown}$. We show that, under some assumptions, the calculus of some characters on $\mathcal{H}_{\diamond,\triangledown}$ can be reduced to a calculus in a smaller algebra.

2.2. **A smaller bialgebra constructed on triangle and diamond posets.** The family $\mathcal{F}_1$ is closed under direct product and under taking intervals, in the sense that any closed or half-open interval of a poset of the family belongs to the family. We construct a bialgebra from this family in the same way as Schmitt constructs an incidence Hopf algebra from a hereditary family of intervals, adapting the construction to the fact that some posets of the family have no greatest element. The bialgebra obtained is not counital on the right due to the absence of a greatest element in some poset. It would be interesting to generalize this construction to a family $\mathcal{F}$ of finite not necessarily bounded posets, closed under direct product and by taking down- and up-sets, i.e., such that, for any poset P and any x in P, the sets $D(x) = \{y \in P \mid y \leq x\}$ and $U(x) = \{y \in P \mid y \geq x\}$ belong to the family $\mathcal{F}$. The obtained bialgebra would be counital on the right if and only if all posets in $\mathcal{F}$ have a greatest element and counital on the left if and only if all posets in $\mathcal{F}$ have a least element. However, as the posets considered in the rest of this paper have a least element, we will restrict ourselves to families of posets of this type.

Taking isomorphism of posets as an order compatible relation which is also a reduced congruence $\sim$, the set $\tilde{\mathcal{F}}_1 = \mathcal{F}_1 / \sim$ is a monoid, with product induced by direct product of posets and identity element 1 equal to the class of any trivial poset. Let us denote by $V(\mathcal{F}_1)$ the free $\mathbb{Q}$-module generated by $\tilde{\mathcal{F}}_1$. The monoid structure on $\tilde{\mathcal{F}}_1$ induces an algebra structure on $V(\mathcal{F}_1)$, isomorphic to the monoid algebra of $\tilde{\mathcal{F}}_1$ over $\mathbb{Q}$. As $\mathcal{F}_1$ is the set of monomials on triangle and diamond posets of $\mathcal{F}_0$, the algebra $V(\mathcal{F}_1)$ is generated by isomorphism classes of triangle and diamond posets of $\mathcal{F}_0$. Let us remark that all elements of $\mathcal{F}_1$ have a least element, but some of them do not have a greatest element. We define the coproduct on $P \in \mathcal{F}_1$ by

$$\Delta(P) = \sum_{x \in P} D(x) \otimes U(x), \tag{2.1}$$

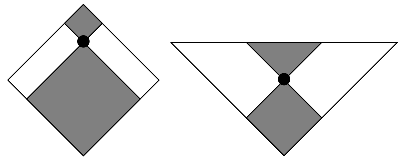


FIGURE 2.1. Intervals in diamond and triangle posets: all of these are products of diamond posets, except half-open upper intervals in triangle posets, which are trivial or products of triangle posets.

where $D(x) = \{y \in P | y \leq x\}$ is called a down-set and $U(x) = \{y \in P | y \geq x\}$ is called an up-set, as above.

Let us remark that diamond posets are intervals, whereas triangle posets are not: these two types of posets thus cannot belong to the same isomorphism class.

Proposition 2.1. *The coproduct Δ is an algebra homomorphism. The algebra $V(\mathcal{F}_1)$ together with this coproduct is then a bialgebra, not counital on the right, and denoted by $\mathcal{B}_{\diamond,\triangledown}$.*

The subalgebra $\mathcal{D}_\diamond$ generated by the diamond posets is a subbialgebra of $\mathcal{B}_{\diamond,\triangledown}$ isomorphic to the incidence Hopf algebra of diamond posets. The subalgebra $\mathcal{T}_\triangledown$ of $\mathcal{B}_{\diamond,\triangledown}$ generated by the triangle posets is a right comodule over $\mathcal{D}_\diamond$.

Proof. An interval I in a product of posets $P_1 \times \cdots \times P_k$ can be seen as a product of intervals $I_1 \times \cdots \times I_k$, where each I_k is an interval in P_k.

Due to this property, we can rewrite the coproduct of a product of two posets P and Q in the form

$$\begin{aligned}
\Delta(P \times Q) &= \sum_{x \in P, y \in Q} D(x,y) \otimes U(x,y) \\
&= \sum_{x \in P} \sum_{y \in Q} (D(x) \times D(y)) \otimes (U(x) \times U(y)) \\
&= \sum_{x \in P} D(x) \otimes U(x) \times \sum_{y \in Q} D(y) \otimes U(y).
\end{aligned}$$

This coproduct is thus an algebra homomorphism. The algebra $V(\mathcal{F}_1)$ is then endowed with a coproduct which is an algebra homomorphism: it is a bialgebra. Let us remark that for a triangle poset P, the coproduct $\Delta(P)$ never contains the term $P \otimes 1$: the bialgebra is therefore not counital on the right.

The subalgebra $\mathcal{D}_\diamond$ generated by the diamond posets is also a subcoalgebra according to the first decomposition property. Indeed, every interval in a diamond poset is a product of diamond posets, therefore the coproduct of a diamond poset is a sum of tensor products of products of diamond posets. The subalgebra $\mathcal{D}_\diamond$ is then a subbialgebra of $\mathcal{B}_{\diamond,\triangledown}$. By definition, this subbialgebra is isomorphic as a bialgebra to the incidence Hopf algebra of diamond posets.

According to the second decomposition property, the coproduct of a triangle poset is a sum of tensor products. The terms on the left of this product are diamond posets and the terms on the right are triangle posets: the subalgebra $\mathcal{T}_\triangledown$ of $\mathcal{B}_{\diamond,\triangledown}$ generated by the triangle posets is then a right comodule over $\mathcal{D}_\diamond$. The subalgebra $\mathcal{T}_\triangledown$ is not unital. $\square$

We next show that the computation of some characters on the incidence Hopf algebra $\mathcal{H}_{\diamond,\triangledown}$ can be reduced to some calculus in the bialgebra $\mathcal{B}_{\diamond,\triangledown}$.

2.3. **Characters on the incidence Hopf algebra of triangle and diamond posets.** We define the linear map

$$\lambda : \mathcal{B}_{\diamond,\triangledown} \to \mathcal{H}_{\diamond,\triangledown} \tag{2.2}$$

which sends an isomorphism class c_{d_i} of a diamond poset d_i in $\mathcal{B}_{\diamond,\triangledown}$ to the isomorphism class c_{d_i} of d_i in $\mathcal{H}_{\diamond,\triangledown}$ and which sends an isomorphism class c_{t_j} of a triangle poset t_j in $\mathcal{B}_{\diamond,\triangledown}$ to the isomorphism class $c_{\widehat{t_j}}$ of the augmented triangle poset $\widehat{t_j}$ in $\mathcal{H}_{\diamond,\triangledown}$. We stress the fact that this map is neither an algebra homomorphism nor a coalgebra homomorphism.

This linear map is well defined. Indeed, if two diamond posets are in the same isomorphism class in $\mathcal{B}_{\diamond,\triangledown}$, then they are also in the same isomorphism class in $\mathcal{H}_{\diamond,\triangledown}$. If two triangle posets are in the same isomorphism class in $\mathcal{B}_{\diamond,\triangledown}$, then these posets augmented with a greatest element are also isomorphic, and thus in the same isomorphism class in $\mathcal{H}_{\diamond,\triangledown}$.

We would like to compute some characters on the isomorphism classes of the diamond posets d_i of $\mathcal{F}_0$ and augmented triangle posets $\widehat{t_j}$ coming from triangle posets t_j of $\mathcal{F}_0$ in $\mathcal{H}_{\diamond,\triangledown}$. As $\mathcal{F}_0$ is a subfamily of $\mathcal{F}_1$, to any element of $\mathcal{F}_0$ corresponds an isomorphism class in $\mathcal{B}_{\diamond,\triangledown}$ which is sent to the isomorphism class of the corresponding element in $\mathcal{H}_{\diamond,\triangledown}$: the elements on which we want to compute characters belong to the image of λ.

We observe that the fibre of any isomorphism class in the image of λ contains at most one isomorphism class of triangle posets and at most one isomorphism class of diamond poset. Indeed, if two isomorphism classes of triangle posets, or two isomorphism classes of diamond posets, are sent by λ to the same isomorphism class, then these isomorphism classes are equal.

A character γ on $\mathcal{H}_{\diamond,\triangledown}$ is said to satisfy the *HtoB*-condition if there exists a rational number ω_γ and a map $\widetilde{\gamma}$ from $\mathcal{B}_{\diamond,\triangledown}$ to $\mathbb{Q}$ which satisfy

$$\gamma(\lambda(c_{d_i})) = \widetilde{\gamma}(c_{d_i}) \quad \text{and} \quad \gamma(\lambda(c_{t_j})) = \omega_\gamma \widetilde{\gamma}(c_{t_j}), \tag{2.3}$$

The number ω will be chosen as -1 in the example of hypertree posets developed in the following sections.

The convolution of two characters on $\mathcal{H}_{\diamond,\triangledown}$ can be computed by means of the following theorem.

Theorem 2.2. *The convolution of the characters α and β on $\mathcal{H}_{\diamond,\triangledown}$ satisfying the HtoB-condition* (2.3) *is given by*

$$(\alpha * \beta)(\lambda(c_{d_i})) = \sum \widetilde{\alpha}(c_{d_i}^{(1)})\widetilde{\beta}(c_{d_i}^{(2)}),$$

and

$$(\alpha * \beta)(\lambda(c_{t_j})) = \omega_\beta \sum \widetilde{\alpha}(c_{t_j}^{(1)})\widetilde{\beta}(c_{t_j}^{(2)}) + \omega_\alpha \widetilde{\alpha}(c_{t_j}),$$

where $\Delta(c_{d_i}) = \sum c_{d_i}^{(1)} \otimes c_{d_i}^{(2)}$ and $\Delta(c_{t_j}) = \sum c_{t_j}^{(1)} \otimes c_{t_j}^{(2)}$ in $\mathcal{B}_{\diamond,\triangledown}$.

Proof. The isomorphism class $\lambda(c_{d_i})$ is the isomorphism class of the diamond poset d_i in $\mathcal{H}_{\diamond,\triangledown}$ by definition of λ. Moreover, the coproduct of $\lambda(c_{d_i})$ in $\mathcal{H}_{\diamond,\triangledown}$ and of c_{d_i} in $\mathcal{B}_{\diamond,\triangledown}$

are the same by definition of the coproduct. As α and $\widetilde{\alpha}$ on the one hand, and β and $\widetilde{\beta}$ on the other hand, are equal on $\lambda(c_{d_i})$ and c_{d_i}, respectively, the first equality follows.

To obtain the second equality, we observe that the isomorphism class $\lambda(c_{t_j})$ corresponds to the isomorphism class of the augmented triangle poset $\widehat{t_j}$ by definition of λ. Hence the coproduct of $\lambda(c_{t_j})$ in $\mathcal{H}_{\diamond,\triangledown}$ has one more term than the coproduct of c_{t_j} in $\mathcal{B}_{\diamond,\triangledown}$, due to the fact that the poset $\widehat{t_j}$ has one more element than the poset t_j. This term is $\lambda(c_{t_j}) \otimes 1$. All the other terms can be matched by associating $\lambda(c_{t_j})$ with the unique isomorphism class of triangle posets of its fibre c_{t_j}. Moreover, the posets on the left part of the coproduct of $\lambda(c_{t_j})$ are of diamond type, except for the term that does not belong to the coproduct of c_{t_j}, and α and $\widetilde{\alpha}$ coincide on diamond posets. Therefore we have

$$(\alpha * \beta)(\lambda(c_{t_j})) = \sum \widetilde{\alpha}(c_{t_j}^{(1)})\omega_\beta \widetilde{\beta}(c_{t_j}^{(2)}) + \omega_\alpha \widetilde{\alpha}(c_{t_j}).$$

This yields the claimed result. □

3. Hypertree posets

A *hypergraph* is a pair (V, E), where the elements of V are called *vertices* and the elements of E, called *edges*, are sets of at least two vertices. We will only consider hypertrees on at least two vertices: each vertex thus belongs to at least one edge. The *size* of an edge e is the number of vertices in the edge e. The *valency* of a vertex v is the number of edges to which v belongs. A *walk* on a hypergraph $H = (V, E)$ from a vertex s of H to a vertex f of H is an alternating sequence of vertices and edges in H, $s = v_0, e_0, v_1, e_1, \ldots, e_{n-1}, v_n = f$, such that e_i is an edge containing the vertices v_i and v_{i+1}, for $i \in [\![0, n-1]\!]$. A *hypertree* is a hypergraph such that, given any pair (s, f) of vertices, there exists exactly one walk from s to f without repeated edges. Given a hypertree T with vertex set V of cardinality n, we say that T is a hypertree on n vertices.

We may define the following order on hypertrees: a hypertree T is smaller than a hypertree T' whenever the edges of T are unions of some edges of T'. The set of hypertrees on n vertices endowed with this partial order is a poset denoted by h_n. This poset has a least element $\hat{0}$, given by the hypertree the only edge of which being the one which is formed by the complete vertex set V. The poset obtained by adding a greatest element $\hat{1}$ is called the (n-)augmented hypertree poset and is denoted by $\widehat{h_n}$.

We now want to characterize tuples $\alpha = (\alpha_1, \ldots \alpha_k)$ and $(\pi_2, \ldots, \pi_\ell)$ such that there exists a hypertree with α_i vertices of valency i and π_j edges of size j, for $i \geq 1$ and $j \geq 2$. For that purpose, we will consider hypertrees as $[\![1, n]\!]$-labelled bipartite trees as in [9]. A *$[\![1, n]\!]$-labelled bipartite tree* is a tree T together with a bijection from $[\![1, n]\!]$ to a subset of its vertex set such that the image of $[\![1, n]\!]$ includes all of the vertices of valency 1, and for every edge in T exactly one of its endpoints lies in the image of $[\![1, n]\!]$. The labelled vertices of a bipartite tree correspond to the vertices of the associated hypertree, and the other vertices correspond to the edges of the hypertree. An example of a hypertree and the associated labelled bipartite tree is presented in Figure 3.2. The number of labelled vertices of valency i is then denoted by α_i and the number of unlabelled vertices of valency j (or of edges of size j in the hypertree) by π_j.

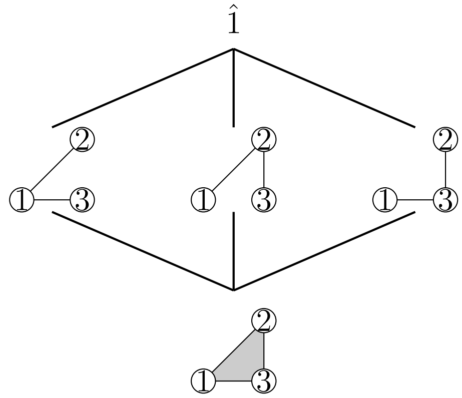


FIGURE 3.1. The poset $\widehat{h_3}$

FIGURE 3.2. A hypertree and its associated labelled bipartite tree.

We obtain the following criterion for the existence of such an hypertree, which was first given in an equivalent formulation by Bacher [1]. Here, we provide a direct proof of his result.

Proposition 3.1. *Given two tuples $\alpha = (\alpha_1, \ldots, \alpha_k)$ and $\pi = (\pi_2, \ldots, \pi_\ell)$, there exists a hypertree on n vertices with α_i vertices of valency i and π_j edges of size j if and only if*

$$\sum_{i=1}^{k} \alpha_i = n, \tag{3.1}$$

$$\sum_{j=2}^{\ell} (j-1)\pi_j = n-1, \tag{3.2}$$

$$\sum_{i=1}^{k} i\alpha_i = n + \sum_{j=2}^{\ell} \pi_j - 1. \tag{3.3}$$

We postpone the proof of this proposition in order to illustrate it first by an example.

Example 3.2. For $n = 4$, the second equation of (3.1) implies that $\sum_{j=2}^{\ell} \pi_j \leq 3$, i.e., $\sum_{i=1}^{k} i\alpha_i \leq 6$. The possible α's are:

- $\alpha = (4)$; in that case, we obtain the condition $\sum_{j=2}^{\ell} \pi_j = 1$, so the only possible π is $\pi = (0, 0, 1)$;
- $\alpha = (3, 1)$; in that case, we obtain the condition $\sum_{j=2}^{\ell} \pi_j = 2$, so the only possible π is $\pi = (1, 1)$;
- $\alpha = (2, 2)$; in that case, we obtain the condition $\sum_{j=2}^{\ell} \pi_j = 3$, so the only possible π is $\pi = (3)$;

- $\alpha = (3,0,1)$; in that case, we obtain the condition $\sum_{j=2}^{\ell} \pi_j = 3$, so the only possible π is $\pi = (3)$.

We now give a new direct proof of the criterion in Proposition 3.1.

Proof of Proposition 3.1. Suppose that there exists such a hypertree. Let T be such a hypertree. Every vertex in T has a fixed valency. Therefore, counting vertices, we obtain the first equation

$$\sum_{i=1}^{k} \alpha_i = n.$$

By construction of the labelled bipartite tree B associated with T, every unlabelled vertex is linked with a labelled vertex. This leads to the following equality by counting edges around labelled and unlabelled vertices of B:

$$\sum_{i=1}^{k} i\alpha_i = \sum_{j=2}^{\ell} j\pi_j. \tag{3.4}$$

Moreover, with a bipartite tree we may associate a simplicial complex with faces of dimension at most 1. This simplicial complex is connected without cycles, therefore its Euler characteristic is equal to 1 and can be expressed as

$$\chi = 1 = \sum_{j\geq 2} \pi_j - \sum_{j\geq 2} j\pi_j + \sum_{i\geq 1} \alpha_i. \tag{3.5}$$

These equations are equivalent to (3.1)–(3.3).

Let us now prove that this condition is also sufficient. We consider a set of α_i labelled vertices with i half-edges and π_j unlabelled vertices with j half-edges, with $i \geq 1$ and $j \geq 2$, such that (3.1)–(3.3) are satisfied. As this implies that (3.4) is satisfied, we can then choose a way to associate the vertices to obtain a $[\![1,n]\!]$-labelled bipartite graph T, i.e., a graph together with a chosen bijection from $[\![1,n]\!]$ to a subset of its vertex set such that the image of $[\![1,n]\!]$ includes all of the vertices of valency 1 and for every edge in T exactly one of its endpoints lies in the image of $[\![1,n]\!]$.

As our assumption also implies that (3.5) is satisfied, the Euler characteristic, i.e., the difference between the number of connected components and the number of cycles, is equal to 1. If the graph T is connected, then it has no cycles: it is a tree and we have constructed a $[\![1,n]\!]$-labelled bipartite tree. The associated hypertree has fixed set of valencies and fixed set of edge sizes.

If the graph T is not connected, then there is a cycle in one of the connected components. Therefore, there is an edge in this connected component that can be removed without increasing the number of connected components. This edge is between an unlabelled vertex u_1 and a labelled vertex l_1. Let us cut an edge in one of the other connected components between two vertices u_2 and l_2. We then obtain a graph with each element of the set $\{u_1, l_1, u_2, l_2\}$ having an unlinked half-edge. Linking u_2 with l_1 and u_1 with l_2, we obtain a $[\![1,n]\!]$-labelled graph satisfying the conditions with one less connected component. Indeed, we may have disconnected the connected component of u_2 and l_2 by deleting the edge but when linking the vertices we create a path from u_2 to l_2 by using the one existing between u_1 and l_1. As this operation decreases the number of connected components, we can repeat it until we find a hypertree matching the required conditions. □

From the second equation of the proposition, we may deduce the following expression for π_2 in terms of π_j for $j \geq 3$:

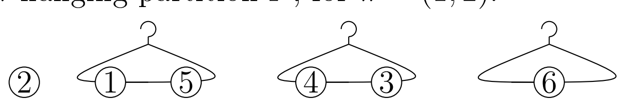
$$\pi_2 = n - 1 - \sum_{j \geq 3} (j-1)\pi_j.$$

4. Computation of the number of hypertrees with fixed parameters

In this section, we compute the number of hypertrees with a given edge size and valency set. After having written the proof, the author was informed that this result was already obtained by Bacher [1] and by Bousquet-Mélou and Chapuy [4] in terms of bicoloured trees (see also [16]). Although the proof of this section is really close to the reasoning of Bacher, we still present it here for self-containment. The Prüfer code presented here is different from the one presented by Bacher, although it was already introduced by Selivanov in [14] according to Bacher [2]. We thank the different mathematicians who have reported to us the already existing literature and helped us improve the accuracy of the references.

We want to compute the number of hypertrees with α_i vertices of valency i and π_j edges of size j. We do this using bijections. Given a tuple π, we call a partition a π-*hanging partition* if it has one block consisting of a vertex, π_j other blocks consisting of $j-1$ vertices, and in each of these π_j other blocks a "fastener", which by definition is a mark in the block indicating that this block will be linked to the vertex of another block, for $j \geq 2$.

Example 4.1. A π-hanging partition P, for $\pi = (1, 2)$:



.

For convenience, we write X for the fastener and represent the π-hanging partition as

$$P = (2) \quad (X|1\ 5) \quad (X|4\ 3) \quad (X|6).$$

Then the assembly of elements of a π-hanging partition of a hypertree can be seen as an assembly of coat-hangers and coat racks. In Figure 4.1, we represent the assembly of the π-hanging partition

$$(2) \quad (X|1\ 5) \quad (X|4\ 3) \quad (X|6) \quad (X|7\ 8\ 9)$$

and the corresponding hypertree.

Let us fix π and α and denote the set of π-hanging partitions by Π_{HP}, and the set of rooted hypertrees with α_i vertices of valency i and π_j edges of size j by $\mathcal{H}^p_{\alpha,\pi}$. We define a map $\varphi : \mathcal{H}^p_{\alpha,\pi} \to \Pi_{\mathrm{HP}}$ in the following way. Given an edge e, we map it to the set of all vertices of e, except the closest to the root, with a fastener added to this set. If we add the singleton made of the root to this set of hanging sets, we obtain a π-hanging partition. Indeed, all sets but one of cardinality one have a fastener, and the

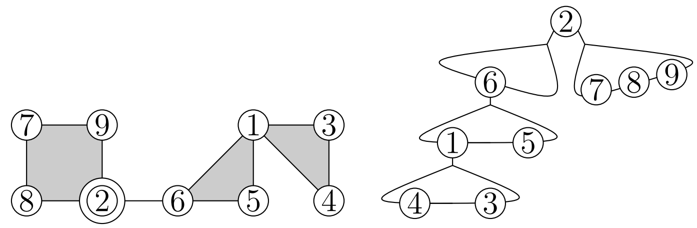


FIGURE 4.1. A hypertree and the corresponding assembly of a π-hanging partition (2) $(X|1\ 5)$ $(X|4\ 3)$ $(X|6)$ $(X|7\ 8\ 9)$.

size of each hanging set is one less than the size of the associated edge. An example of the image of a hypertrees under the map φ is presented in Example 4.3.

Given P in Π_{HP}, we denote the fibre $\varphi^{-1}(P)$ by F_P. The cardinality of $\mathcal{H}^p_{\alpha,\pi}$ is the sum of the cardinalities of the disjoint fibres. As we will see in the proof, the cardinality of a fibre is independent of the considered π-hanging partitions. We denote this cardinality by d^n_α. We will say that we can *construct* a hypertree H from a π-hanging partition P if $\varphi(H) = P$.

Let us now link hypertrees to hanging partitions.

Lemma 4.2. *The number of rooted hypertrees with α_i vertices of valency i and π_j edges of size j is given by*

$$|\mathcal{H}^p_{\alpha,\pi}| = \frac{1}{n} \times \frac{n!}{\prod_{j\geq 2}(j-1)!^{\pi_j}\pi_j!} \times d^n_\alpha. \tag{4.1}$$

Proof. We want to compute the cardinality of $\mathcal{H}^p_{\alpha,\pi}$. Let us consider the action of the symmetric group $\mathfrak{S}_n$ on $\mathcal{H}^p_{\alpha,\pi}$ by permutation of vertices. By definition of the map φ, which is $\mathfrak{S}_n$-equivariant, this action induces an action of the symmetric group on the set Π_{HP}. The action of the symmetric group $\mathfrak{S}_n$ on the set of all hanging partitions of type π is transitive, as it does not change the sizes of the blocks of the partitions. Let $(\mathcal{O}_j)_{1\leq j\leq p}$ be the orbits for the action of $\mathfrak{S}_n$ on the set $\mathcal{H}^p_{\alpha,\pi}$. The fibre F_P has a component f^P_j in every orbit $\mathcal{O}_j$. We summarize all these notations in the diagram of Figure 4.2.

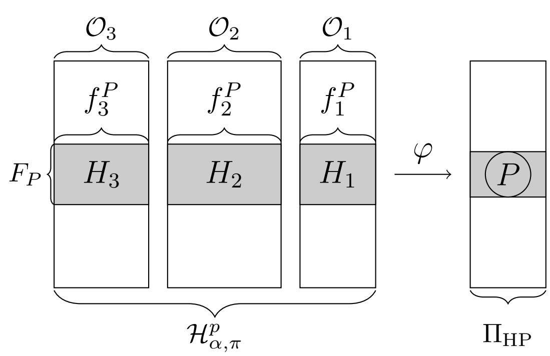


FIGURE 4.2. The map φ.

We consider a hypertree H_j in each f_j^P. The orbit-stabilizer theorem applied to $\mathcal{O}_j$ yields

$$n! = |\mathcal{O}_j| \times |\operatorname{Aut}_{H_j}|,$$

where $|\operatorname{Aut}_{H_j}|$ is the cardinality of the automorphism group of the rooted hypertree H_j. As $\mathcal{H}^p_{\alpha,\pi} = \bigsqcup_{j=1}^p \mathcal{O}_j$, we obtain the relation

$$|\mathcal{H}^p_{\alpha,\pi}| = n! \times \sum_{j=1}^{p} \frac{1}{|\operatorname{Aut}_{H_j}|}. \tag{4.2}$$

Let us consider the group G_P of permutations of $[\![1,n]\!]$ fixing P. There are exactly $\prod_{j\geq 2}(j-1)!^{\pi_j}\pi_j!$ such permutations. The group G_P acts on the fibre F_P transitively on each f_j^P. Indeed, if $\sigma \in \mathfrak{S}_n$ sends a hypertree H of f_k^P to a hypertree H' of f_k^P, then σ stabilizes P since $\varphi(H) = \varphi(\sigma(H)) = P$. Subsequently, the orbit-stabilizer theorem applied to f_j^P gives

$$|f_j^P| \times |\operatorname{Stab}_{G_P} H_j| = |G_P| = \prod_{j\geq 2}(j-1)!^{\pi_j}\pi_j!, \tag{4.3}$$

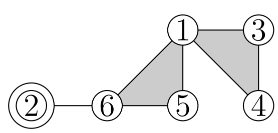
where $\operatorname{Stab}_{G_P} H_j = \{\sigma \in G_P | \sigma(H_j) = H_j\}$.

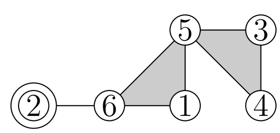
We show that $\operatorname{Stab}_{G_P} H_j = \operatorname{Aut}_{H_j}$. As $G_P \subseteq \mathfrak{S}_n$, it is easily shown that $\operatorname{Stab}_{G_P} H_j \subseteq \operatorname{Aut}_{H_j}$. For a permutation σ in Aut_{H_j}, we have $\varphi(\sigma(H_j)) = \varphi(H_j) = P$ and $\varphi(\sigma(H_j)) = \sigma(P)$. In other words, σ stabilizes P. Therefore, we obtain the relation $\operatorname{Stab}_{G_P} H_j = \operatorname{Aut}_{H_j}$. Combined with (4.2) and (4.3), we get the result, since $d_\alpha^n = \sum_{j=1}^p |f_j^P|$. ☐

Example 4.3. We consider the π-hanging partition P

$$P = (2) \quad (X|1\ 5) \quad (X|4\ 3) \quad (X|6),$$

with $\pi = (1,2)$, where X represents the fastener of the block. For $\alpha = (4,2)$, we may construct the following rooted hypertrees (and many others):

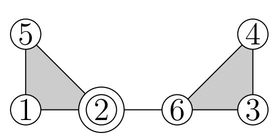
$T =$ 

and $T' =$ 

.

Let us consider the group G_P of permutations of $[\![1,n]\!]$ fixing P, introduced in the proof of Lemma 4.2. We describe an example of the action of the group G_P on the fibre of P. Considering T and T' in the fibre of P ($\varphi(T) = \varphi(T') = P$), the permutation $(3\ 4)$ fixes T and T', but the permutation $(1\ 5)$ sends T to T'. Hence T and T' are in the same orbit.

The following hypertree T'' is not in the orbit of T and T':



.

We now want to compute the number d_α^n of constructions of a hypertree of valency set α from a π-hanging partition P_π. This is also the cardinality of the fibre $\varphi^{-1}(P_\pi)$. This construction is given by a bijection introduced by Bacher in [1], which we recall for self-containment of this article.

Lemma 4.4. *Given a pair (α, π) in $\mathcal{P}_n$ and a π-hanging partition P_π, there is a bijection between the set of constructions of a rooted hypertree of valency set α from P_π and the set of words on $[\![1, n]\!]$ of length $\sum_{j\geq 2} \pi_j - 1$ with $\sum_{i\geq 2} \alpha_i$ different letters, where α_i letters appear $i-1$ times for $i \geq 2$.*

Proof. We prove this lemma using a Prüfer code type proof. We want to count the number of different rooted hypertrees which can be constructed from a π-hanging partition P_π and which have α_i vertices of valency i for $i \geq 1$. Given such a rooted hypertree, we recursively construct a variant of Prüfer code.

If the hypertree has only one edge of size n, then we can separate the root from the edge and put a fastener instead: we obtain two blocks, the one of the root and another hanging one of size $n-1$. Given a π-hanging partition, we assemble the two blocks of the partition into one edge and it gives back the hypertree. The associated word is the empty word, which is of length 0.

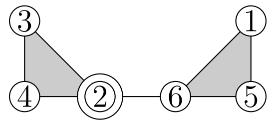
If the rooted hypertree H has more than one edge, we consider the set of leaves of the hypertree, i.e., the set of edges whose vertices but the closest from the root, called the *petiole*, are of valency 1. We order the set of leaves according to their minimal unshared element. The petiole of the minimal leaf will be the first letter w_1 of the word w associated with H. We suppose that this vertex has a valency v. We denote the size of the minimal leaf by s_m. Then, by deleting the minimal leaf and its $s_m - 1$ vertices different from the petiole, we obtain a rooted hypertree H' on $n - s_m + 1$ vertices in which the valency of the petiole w_1 has decreased by one, the number of vertices of valency 1 has decreased by $s_m - 1$, and all other vertices have the same valency. As vertices of valency 1 do not appear in the word associated with the hypertree, the deletion of these vertices only decreases the number of occurrences of w_1 in the word associated with H' by one compared with the word associated with H. If w is the word associated with H and w' the one associated with H', then $w = w_1 w'$.

Moreover, the hanging partition associated with H' can be obtained from P_π by deleting the hanging block of P_π containing the vertices of valency 1 of the minimal leaf. We then construct the word w' associated with H': it is a word of length $\sum_{j\geq 2} \pi_j - 2$ letters, with $\sum_{i\geq 2} \alpha_i$ different letters, where α_i letters appear $i-1$ times for $i \neq v, v-1$, $\alpha_v - 1$ letters appear $v-1$ times, and $\alpha_{v-1} + 1$ letters appear $v-2$ times. We note that the vertex w_1 is of valency $v-1$ in H', so it appears $v-2$ times in w'. Hence, the letter w_1 appears $v-1$ times in the word $w = w_1 w'$, and the word $w = w_1 w'$ satisfies the required conditions.

If we have a π-hanging partition and a word w satisfying the required conditions, we can build the associated rooted hypertree by ordering the blocks with a fastener whose elements are not letters of w according to their minimal element. Then we attach the least element of these blocks to the last letter of the word, which is an element of another block and delete this last letter. We repeat these operations until the word is empty. We finally obtain a rooted hypertree, and this operation is the inverse of the construction above. Hence, this gives a bijection between the constructions of rooted hypertrees from hanging partitions and the set of words of the lemma. $\square$

Example 4.5. Considering the hanging partition P and the hypertrees T, T', and T'' of Example 4.3, the words associated with the construction of T, T' and T'' from P are $1\,6$, $5\,6$, and $2\,6$, respectively.

The hypertree whose construction from P is associated with the word 6 2 is



There are 36 words associated with the π-hanging partition: 6 corresponding to hypertrees with a vertex of valency 3 and the others of valency 1, and 30 corresponding to hypertrees with two vertices of valency 2 and the others of valency 1.

Lemma 4.6. *For a tuple $(\alpha_1, \alpha_2, \dots)$, put $k = \sum_{i\geq 1}(i-1)\alpha_i$ and $n = \sum_{i\geq 1}\alpha_i$. Then the number of length k words on the alphabet $[1, n]$, with α_i letters repeated $i-1$ times, is*

$$d_\alpha^n = \frac{k! \times n!}{\prod_{i\geq 1}(i-1)!^{\alpha_i}\alpha_i!}. \tag{4.4}$$

Proof. This lemma is a classical result. □

Using this lemma, we obtain the following proposition.

Proposition 4.7. *If the tuples $\alpha = (\alpha_1, \dots)$ and $\pi = (\pi_2, \dots)$ satisfy (3.1)–(3.3), the number of hypertrees with α_i vertices of valency i and π_j edges of size j, for $i \geq 1$ and $j \geq 2$, is given by*

$$c_{\alpha,\pi}^n = \frac{1}{n} \times \frac{n!}{\prod_{j\geq 2}(j-1)!^{\pi_j}\pi_j!} \times \frac{(K-1)! \times n!}{\prod_{i\geq 1}(i-1)!^{\alpha_i}\alpha_i!}, \tag{4.5}$$

with $K = \sum_{j\geq 2}\pi_j = \sum_{i\geq 1}(i-1)\alpha_i + 1$ and $n = \sum_{i\geq 1}\alpha_i$.

Proof. This proposition follows from Lemmas 4.2, 4.4, and 4.6. □

5. Coproduct in the bialgebra

Some intervals in the hypertree posets will be described in terms of another type of posets: partition posets. A *partition poset* is a poset on the set of all partitions of a set V. A partition a_1 is smaller than a partition a_2 if each block of a_1 is the union of some blocks of a_2. The partition poset on n vertices p_n is based on the set of partitions of a set of cardinality n. The poset p_4 is presented in Figure 5.1.

We need the following result of McCammond and Meier on intervals in the hypertree poset.

Lemma 5.1 ([8, Lemma 2.5])**.** *Let T be a hypertree on n vertices, for $n \geq 2$. Then the following hold:*

(a) *The interval $[\hat{0}, T]$ is a direct product of partition posets, with one factor p_j for each vertex in T with valency j.*
(b) *The half-open interval $[T, \hat{1})$ is a direct product of hypertree posets, with one factor h_j for each edge in T with size j.*

Let us consider the incidence Hopf algebra $\mathcal{H}_{\widehat{HT}} = (\mathcal{H}_{\widehat{HT}}, \times, \epsilon, \eta, \Delta, S)$ obtained from the construction of Section 2.1 by taking the set of partition posets $(p_i)_{i\geq 1}$ for the set of diamond posets and the set of hypertree posets $(h_n)_{n\geq 3}$ for the set of triangle

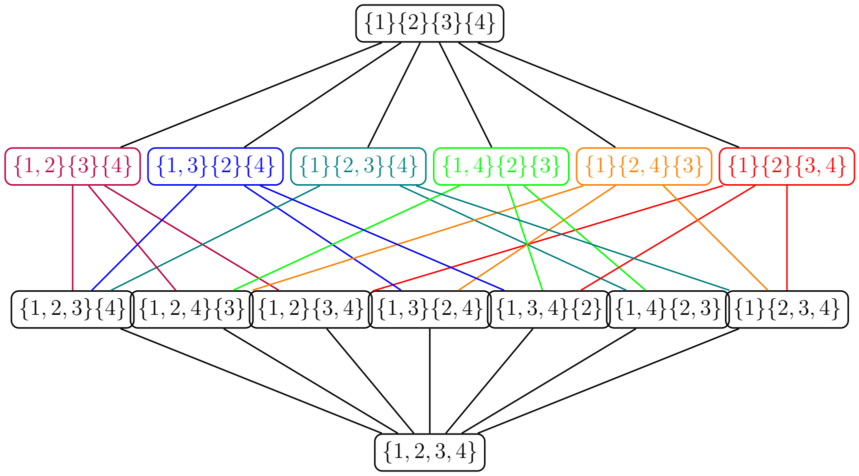


FIGURE 5.1. The poset p_4

posets. Indeed, the partition poset p_1 on one element and the hypertree poset h_2 on two elements are both isomorphic to the trivial poset, partition posets are intervals, and hypertree posets have a least element but no greatest one. Moreover, it is a classical result that every interval in a partition poset is isomorphic to a product of partition posets (see [16]). This fact combined with Lemma 5.1 implies that this family satisfies the decomposition property and therefore all the requirements of Section 2.1. We denote the augmented hypertree poset by $\widehat{h_n}$.

We consider $\mathcal{H}^*_{\widehat{HT}}$, the group of characters $\chi : \mathcal{H}_{\widehat{HT}} \to \mathbb{Q}$. We aim at calculating the Möbius numbers of the augmented hypertree posets using the classical techniques of characters. A good reference for such a computation of characters for the partition posets, and Möbius numbers, is the article [15] of Speicher. To compute the character which associates with a poset of $\mathcal{H}_{\widehat{HT}}$ its Möbius number, we use Theorem 2.2.

We write $\mathcal{B}_{HT}$ for the bialgebra defined in Section 2.2, where partition posets play the role of diamond posets and hypertree posets play the role of triangle posets. Due to Lemma 5.1(b), we see that this bialgebra is not only generated as an algebra by isomorphism classes of partition posets and isomorphism classes of intervals $[\tau, \hat{1})$, where τ is a hypertree, but also by a smaller set: the isomorphism classes of partition posets p_n and the isomorphism classes of hypertree posets h_n. Moreover, partition posets and hypertree posets are both graded, therefore two partition posets or hypertree posets on n and m elements, respectively, are isomorphic if and only if m and n are equal. As every p_i and h_j are pairwise not in the same isomorphism classes, due to gradings, and as we focus on these classes, we will use the same notation for the isomorphism classes of posets and posets themselves.

Let us remark that the subalgebra of $\mathcal{B}_{HT}$ generated by partition posets, as described in Proposition 2.1, is the *Faà di Bruno Hopf algebra*. This is also a subalgebra of $\mathcal{H}_{\widehat{HT}}$. Moreover, the subalgebra of $\mathcal{B}_{HT}$ generated by hypertree posets is a right comodule over the Faà di Bruno Hopf algebra.

Example 5.2. We consider the poset $\widehat{h_3}$ shown in Figure 3.1. The computation of the coproducts gives

$$\Delta(\widehat{h_3}) = 1 \otimes \widehat{h_3} + 3\ p_2 \otimes \widehat{h_2} + \widehat{h_3} \otimes 1, \text{ in } \mathcal{H}_{\widehat{HT}},$$
$$\Delta(h_3) = 1 \otimes h_3 + 3\ p_2 \otimes h_2, \text{ in } \mathcal{B}_{HT}.$$

Hence, the convolution of characters α and β on $\mathcal{H}_{\widehat{HT}}$ can be computed using the bialgebra $\mathcal{B}_{HT}$.

Proposition 5.3. *The convolution of characters α and β on $\mathcal{H}_{\widehat{HT}}$ can be computed using maps $\widetilde{\alpha}$ and $\widetilde{\beta}$ from $\mathcal{B}_{HT}$ to $\mathbb{Q}$, provided α and β satisfy the HtoB-condition* (2.3)*. This computation is given by*

$$(\alpha * \beta)(p_i) = \sum \widetilde{\alpha}(p_i^{(1)})\widetilde{\beta}(p_i^{(2)}),$$

and

$$(\alpha * \beta)(\widehat{h_j}) = \omega_\beta \sum \widetilde{\alpha}(h_j^{(1)})\widetilde{\beta}(h_j^{(2)}) + \omega_\alpha \widetilde{\alpha}(h_j),$$

where $\Delta(p_i) = \sum p_i^{(1)} \otimes p_i^{(2)}$ and $\Delta(h_j) = \sum h_j^{(1)} \otimes h_j^{(2)}$ in $\mathcal{B}_{HT}$.

Proof. This is a corollary of Theorem 2.2 for p_i and h_j. □

We now compute the coproduct Δ in the algebra $\mathcal{B}_{HT}$. We denote the neutral element of $\mathcal{B}_{HT}$ for the product, i.e., the trivial poset, by 1.

The coproduct of isomorphism classes of partition posets p_n has already been computed. It can be found for instance in the article [13] of Schmitt.

Proposition 5.4 ([13, Example 14.1])**.** *The coproduct on the isomorphism classes of partition posets is given by*

$$\Delta\left(\frac{p_n}{n!}\right) = \sum_{k=1}^{n} \sum_{\substack{(j_1,\dots,j_n)\in\mathbb{N} \\ \sum_{i=1}^n j_i = k,\ \sum_{i=1}^n i j_i = n}} \binom{k}{j_1,\dots,j_n} \prod_{i=1}^{n} \left(\frac{p_i}{i!}\right)^{j_i} \otimes \frac{p_k}{k!},$$

where p_1 is the trivial poset.

We now compute the coproduct for h_n. According to the structure of the hypertree posets and Lemma 5.1, the left part of the coproduct of isomorphism classes of the hypertree poset h_n is a product of isomorphism classes of partition posets, and the right part is a product of isomorphism classes of hypertree posets h_k. The coefficient of a term $p_\alpha \otimes h_\pi$ is the number of hypertrees with α_i vertices of valency i and π_j edges of size j, as computed in Proposition 4.7, for tuples α and π satisfying (3.1)–(3.3).

Theorem 5.5. *If the set $\mathcal{P}(n)$ is the set of tuples $\alpha = (\alpha_1, \dots, \alpha_k)$ and $\pi = (\pi_2, \dots, \pi_\ell)$ satisfying* (3.1)–(3.3)*, the coproduct of h_n in $\mathcal{B}_{HT}$ is given by*

$$\Delta(h_n) = \frac{1}{n} \times \sum_{(\alpha,\pi)\in\mathcal{P}(n)} \frac{n!}{\prod_{j\geq 2}(j-1)!^{\pi_j}\pi_j!} \times \frac{(K-1)! \times n!}{\prod_{i\geq 1}(i-1)!^{\alpha_i}\alpha_i!} \prod_{i=2}^{k} p_i^{\alpha_i} \otimes \prod_{j=2}^{\ell} h_j^{\pi_j},$$

with $K = \sum_{j\geq 2}\pi_j = \sum_{i\geq 1}(i-1)\alpha_i + 1$ and $n = \sum_{i\geq 1}\alpha_i$.

Example 5.6. We are now able to compute the coproduct of some h_n. Using the values of (α, π) on which $c^n_{\alpha,\pi}$ does not vanish, computed in Example 3.2, we obtain for h_4 that

$$\begin{aligned}\Delta h_4 &= \frac{1}{4} \times \frac{4!}{3!} \times \frac{0!4!}{4!} \times p_1^4 \otimes h_4 + \frac{1}{4} \times \frac{4!}{2!} \times \frac{1!4!}{3!} \times p_1^3 p_2 \otimes h_2 h_3 \\ &\quad + \frac{1}{4} \times \frac{4!}{3!} \times \frac{2!4!}{2!2!} \times p_1^2 p_2^2 \otimes h_2^3 + \frac{1}{4} \times \frac{4!}{3!} \times \frac{2!4!}{3!2!} \times p_1^3 p_3 \otimes h_2^3, \\ \Delta h_4 &= 1 \otimes h_4 + 12\ p_2 \otimes h_2 h_3 + 12\ p_2^2 \otimes h_2^3 + 4\ p_3 \otimes h_2^3.\end{aligned}$$

When summing the coefficients in this coproduct, we obtain the total number of hypertrees on 4 vertices, which is 29, as expected.

6. Computation of the Möbius number of augmented hypertree posets

On an incidence Hopf algebra $\mathcal{H}$ with algebraic generators g_n (which are posets), we define the characters ζ and μ for $n \geq 1$ by

$$\zeta : g_n \mapsto 1$$

and

$$\mu : g_n \mapsto \mu(g_n),$$

where $\mu(g_n)$ is the Möbius number of the poset g_n.

These characters are inverses of each other. This means that, if ϵ is the counit of $\mathcal{H}$ and $*$ is the convolution on characters, we have

$$\zeta * \mu = \mu * \zeta = \epsilon.$$

Indeed, these equations come from the definitions of the convolution and the Möbius function:

$$(\mu * \zeta)([h, h]) = \mu([h, h]) = 1$$

and

$$(\mu * \zeta)([h, h']) = \sum_{h \leq x \leq h'} \mu([h, x]) \times 1 = \mu(h, h') + \sum_{h \leq x < h'} \mu(h, x),$$

for intervals $[h, h']$ with $h < h'$ in $\mathcal{H}$.

According to Definition 1.1 of the Möbius function, $\mu * \zeta$ and $\zeta * \mu$ vanish on any non-trivial interval.

We want to compute the Möbius number of augmented hypertree posets. We use Proposition 5.3. To prove that the characters satisfy the assumptions of the proposition, we need the notion of a sum function. If P is a finite poset with a unique least element, then we define the *sum function* by $s(P) = \sum_{x \in P} \mu(\hat{0}, x)$.

If $\hat{P}$ is the poset obtained from P by the addition of a greatest element $\hat{1}$, then $\mu(\hat{P}) = -s(P)$.

Lemma 6.1 ([8, Lemma 4.4])**.** *If P_i, $i \in [k]$, is a list of finite posets, each with a unique minimal element, and $Q = \prod_{i=1}^k P_i$, then $s(Q) = \prod_{i=1}^k s(P_i)$.*

We define the following maps from $\mathcal{B}_{HT}$ to $\mathbb{Q}$, for a poset p of $\mathcal{B}_{HT}$ with both a least and a greatest element and a poset h with a least but no greatest element:

$$\widetilde{\zeta}(p) = \zeta(p) = 1,\ \widetilde{\zeta}(h) = \zeta(h) = 1,$$

and
$$\widetilde{\mu}(p) = \mu(p),\ \widetilde{\mu}(h) = s(h).$$
These maps satisfy the following property due to their definitions and Lemma 6.1, for $i \geq 2$ and $j \geq 3$:
$$\widetilde{\zeta}(\prod_{i=1}^{k} p_i) = \prod_{i=1}^{k} \widetilde{\zeta}(p_i), \quad \widetilde{\zeta}(\prod_{j=1}^{\ell} h_j) = \prod_{j=1}^{\ell} \widetilde{\zeta}(h_j),$$
and
$$\widetilde{\mu}(\prod_{i=1}^{k} p_i) = \prod_{i=1}^{k} \widetilde{\mu}(p_i), \quad \widetilde{\mu}(\prod_{j=1}^{\ell} h_j) = \prod_{j=1}^{\ell} \widetilde{\mu}(h_j).$$
As these maps satisfy the conditions of Proposition 5.3, we apply it in the following subsections. Since partition and hypertree posets are not mixed in the coproduct of a hypertree poset, the computation of the convolution of μ and ζ will be given by a computation using only the values of $\widetilde{\zeta}$ and $\widetilde{\mu}$ on partition and hypertree posets. The first part of this section will be devoted to the equation $\zeta * \mu = \epsilon$, and the second part will be devoted to the equation $\mu * \zeta = \epsilon$.

6.1. **Right-sided computation.** In this subsection, we give a simplified proof of the result of McCammond and Meier on the computation of the Möbius number of the augmented hypertree poset. The proof of McCammond and Meier presented in [8] used generating function calculus. The introduction of incidence Hopf algebras enables us to give a shorter proof, only using the hypothesis of Theorem 2.2.

Applying the Möbius function on the right side of the coproduct, we obtain
$$(\zeta * \mu)(\widehat{h_n}) = 0,$$
for $n \geq 2$. Hence, applying the computation of the coproduct of Theorem 5.5 and Proposition 5.3, for $n \geq 2$ we obtain the equality
$$0 = -\sum \widetilde{\mu}(h_n^{(2)}) + 1,$$
where $\Delta(h_n) = \sum h_n^{(1)} \otimes h_n^{(2)}$. Using Lemma 5.1, the definition of the coproduct on $\mathcal{B}_{HT}$ computed in Theorem 5.5, and the multiplicativity of $\widetilde{\mu}$, we thus obtain
$$0 = -\sum_{h \in h_n} \prod_{i \in ES(h)} \widetilde{\mu}(\widehat{h_i}) + 1, \tag{6.1}$$
where $ES(h)$ is the multiset of the edge sizes of h.

The first terms are given by
$$\mu(\widehat{h_2}) = -1,$$
$$\mu(\widehat{h_3}) = 3 \times (-\mu(\widehat{h_2}))^2 - 1 = 2,$$
and
$$\mu(\widehat{h_4}) = 13 \times (-\mu(\widehat{h_3})) + 16 \times (-\mu(\widehat{h_2}))^3 + 1 = -26 + 16 + 1 = -9.$$
To obtain a closed formula, we consider the exponential generating function of hypertrees with a weight $\widetilde{\mu}(\widehat{h_i}) = -\mu(\widehat{h_i})$ for each edge of size i,
$$T(x) = x + \sum_{n \geq 2} \sum_{h \in h_n} \prod_{i \in ES(h)} \left(-\mu(\widehat{h_i})\right) \frac{x^n}{n!},$$

where, again, $ES(h)$ is the multiset of edge sizes of the hypertree h. Using (6.1), we obtain

$$T(x) = x + \sum_{n\geq 2} \frac{x^n}{n!} = e^x - 1.$$

Moreover, it has been proven by Kalikow [7] that the derivative of T satisfies the following functional equation.

Theorem 6.2 ([7]). *The generating function $T(x)$ satisfies the equation*

$$xT'(x) = x \times \exp(y(x)) \text{ where } y(x) = \sum_{j\geq 1} -\mu(\widehat{h_{j+1}}) \frac{x^j T'(x)^j}{j!}. \tag{6.2}$$

Hence, we obtain

$$x = \sum_{j\geq 1} -\mu(\widehat{h_{j+1}}) \frac{x^j e^{jx}}{j!}.$$

At this point, we recall that the compositional inverse of xe^x is the *Lambert W function*, given by

$$W(x) = \sum_{n\geq 1} (-n)^{n-1} \frac{x^n}{n!}. \tag{6.3}$$

This establishes the following theorem by McCammond and Meier.

Theorem 6.3 ([8, Theorem 5.1]). *The Möbius number of the augmented hypertree poset on n vertices is given by*

$$\mu(\widehat{h_n}) = (-1)^{n-1}(n-1)^{n-2}.$$

As the homology of the augmented hypertree poset is concentrated in top degree, this Möbius number is also the dimension of the only homology group of the hypertree poset. The action of the symmetric group on this homology group has been computed by the author in [11].

Remark 6.4. To apply this method to pointed hypertree posets and to other posets satisfying the conditions of Theorem 2.2, it would be sufficient to have a theorem analogous to Theorem 6.2.

Moreover, using the expression for the coproduct computed in Section 5 and the Möbius numbers of the hypertree posets, we obtain the following proposition.

Proposition 6.5. *We have*

$$1 = \sum_{K=1}^{n-1} \sum_{\substack{(\pi_2,\dots) \\ \sum_{j\geq 2}\pi_j = K}} (-1)^{K+n-1}(n-1)! n^{K-1} \prod_{j\geq 2} \frac{1}{\pi_j!} \left(\frac{(j-1)^{j-2}}{(j-1)!} \right)^{\pi_j},$$

with $K = \sum_{j\geq 2} \pi_j$ and $n-1 = \sum_{j\geq 2}(j-1)\pi_j$.

Proof. We use (6.1) together with the expression for the coproduct recalled in Section 5 and the expression for Möbius numbers of hypertree posets. As the left part of the coproduct does not interfere here, we can sum over the possible tuples α. This leads to a sum of multinomial coefficients which can be easily computed. ☐

6.2. **Left-sided computation.** Applying the Möbius function on the left side of the coproduct, we obtain

$$(\mu * \zeta)(\widehat{h_n}) = 0,$$

for $n \geq 2$. By Proposition 5.3, for $n \geq 2$ this can be rewritten as

$$0 = \sum \widetilde{\mu}(h_n^{(1)})\widetilde{\zeta}(h_n^{(2)}) - \widetilde{\mu}(h_n).$$

The formula for the coproduct is

$$\mu(\widehat{h_n}) = -\sum_{(\alpha,\pi)\in\mathcal{P}_n} c^n_{\alpha,\pi} \prod (-1)^{(i-1)\alpha_i}(i-1)!^{\alpha_i}$$

Using Theorems 6.3 and 5.5, we obtain the following proposition.

Proposition 6.6. *We have*

$$(n-1)^{n-2} = \sum_{(\alpha,\pi)\in\mathcal{P}(n)} \frac{(-1)^{i\alpha_i - 1}}{n} \times \frac{n!}{\prod_{j\geq 2}(j-1)!^{\pi_j}\pi_j!} \times \frac{k! \times n!}{\prod_{i\geq 1}\alpha_i!}, \tag{6.4}$$

where $\mathcal{P}(n)$ *is the set of pairs of tuples* (α,π) *with* $\alpha = (\alpha_1,\dots,\alpha_k)$ *and* $\pi = (\pi_2,\dots,\pi_\ell)$ *satisfying* (3.1)–(3.3).

This equation is quite surprising as it links the number of rooted trees on n vertices with an alternating sum of quotient of factorials.

Proof of Proposition 6.6. This equation comes from the computation of the coproduct, combined with the Möbius numbers of the augmented hypertree posets and of the partition posets. Indeed, the Möbius number of the partition poset on n elements is given by $(-1)^{n-1}(n-1)!$. □

Example 6.7. The first terms are

$$\mu(\widehat{h_4}) = -1 + 12 - 12 - 8 = -9$$

and

$$\mu(\widehat{h_5}) = -1 + 20 + 12 - 120 - 60 + 60 + 120 + 30 = 64.$$

References

[1] R. Bacher, *On the enumeration of labelled hypertrees and of labelled bipartite trees*, `arχiv:1102.2708`, 2011.

[2] R. Bacher, *Prüfer codes for acyclic hypertrees*, Australasian Journal of Combinatorics, vol. 60, 109–127, 2014.

[3] C. Berge, *Hypergraphs.* North-Holland Mathematical Library, vol. 45, 1989.

[4] M. Bousquet-Mélou and G. Chapuy, *The vertical profile of embedded trees.* Electronic Journal of Combinatorics, vol. 19, no. 3, Paper 46, 61 pp., 2012.

[5] F. Chapoton, *Hyperarbres, arbres enracinés et partitions pointées*, Homology, Homotopy and Applications, vol. 9, no. 1, 193–212, 2007.

[6] C. Jensen, J. McCammond and J. Meier, *The Euler characteristic of the Whitehead automorphism group of a free product*, Trans. Amer. Math. Soc., vol. 359, no. 6, 2577–2595, 2007.

[7] L. H. Kalikow, *Enumeration of parking functions, allowable permutation pairs, and labeled trees*, Ph.D. thesis, Brandeis University, 1999.

[8] J. McCammond and J. Meier, *The hypertree poset and the l^2-Betti numbers of the motion group of the trivial link*, Mathematische Annalen, vol. 328, no. 4, 633–652, 2004.

[9] D. McCullough and A. Miller, *Symmetric automorphisms of free products*, Memoirs of the American Mathematical Society, vol. 122, no. 582, viii+97 pp., 1996.

[10] B. Delcroix-Oger *Hypertrees and semi-pointed partitions: algebraic, combinatorial and topological aspects* (french), Ph.D. thesis, Université Claude Bernard Lyon 1, 2014.

[11] B. Oger, *Action of the symmetric groups on the homology of the hypertree posets*, Journal of Algebraic Combinatorics, vol. 38, no. 4, 915–945, 2013.

[12] W. R. Schmitt, *Antipodes and incidence coalgebras*, Journal of Combinatorial Theory. Series A, vol. 46, no. 2, 264–290, 1987.

[13] W. R. Schmitt, *Incidence Hopf algebras.* Journal of Pure and Applied Algebra, vol. 96, no. 3, 299–330, 1994.

[14] B. I. Selivanov, *Enumeration of homogeneous hypergraphs with a simple cycle structure*, (Russian) Kombinatornyĭ Analiz Vyp., vol. 2, 60–67, 1972.

[15] R. Speicher, *Free probability theory and non-crossing partitions*, Séminaire Lotharingien de Combinatoire, vol. 39, Art. B39c, 38 pp., 1997.

[16] R. Stanley, *Enumerative Combinatorics*, vol. 2, Cambridge Studies in Advanced Mathematics, Cambridge, 2001.

Institut Camille Jordan, UMR 5208, Université Claude Bernard Lyon 1, Bât. Jean Braconnier nb 101, 43 Bd du 11 novembre 1918, 69622 Villeurbanne Cedex, France

E-mail address: oger@math.univ-lyon1.fr