A NOTE ON THE NUMBER OF k-ROOTS IN S_n

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Abstract. The number of k-roots of an arbitrary permutation is expressed as an alternating sum of μ-unimodal k-roots of the identity permutation.

1. A Combinatorial Identity

1.1. **Outline.** μ-unimodality, which was introduced in computations of Iwahori–Hecke algebra characters [9, 13, 14], was applied most recently to prove conjectures of Regev regarding induced characters [6] and of Shareshian and Wachs regarding Stanley's chromatic symmetric function [5]. In this note it will be shown that the number of k-roots of a permutation of cycle type μ is equal to an alternating sum of μ-unimodal k-roots of the identity permutation.

1.2. **μ-unimodal permutations.** Let $\mu = (\mu_1, \ldots, \mu_t)$ be a partition of n with t nonzero parts. Write

$$\mu_{(0)} := 0$$

$$\mu_{(i)} := \sum_{j=1}^{i} \mu_i \qquad (1 \le i \le t)$$

and

$$S(\mu) := (\mu_{(1)}, \ldots, \mu_{(t)}). \tag{1}$$

A permutation $\pi \in S_n$ is *μ-unimodal* if for every i with $0 \le i < t$ there exists $\imath$ with $0 \le \imath \le \mu_{i+1}$ such that

$$\pi(\mu_{(i)} + 1) > \pi(\mu_{(i)} + 2) > \cdots > \pi(\mu_{(i)} + \imath) < \pi(\mu_{(i)} + \imath + 1) < \cdots < \pi(\mu_{(i+1)}).$$

Denote the set of μ-unimodal permutations in S_n by U_μ.

For example, let $\mu = (\mu_1, \mu_2, \mu_3) = (4, 3, 1)$ then $S(\mu) = (\mu_{(1)}, \mu_{(2)}, \mu_{(3)}) = (4, 7, 8)$. The permutations 53687142 and 35687412 are μ-unimodal, but 53867142 and 53681742 are not.

Note that $U_{(1,\ldots,1)} = S_n$.

1.3. **k-roots in S_n.** For $n \ge 1$ and $k \ge 0$, we write

$$I_n^k := \{\pi \in S_n : \ \pi^k = 1\}$$

for the set of k-roots of the identity permutation in S_n.

Theorem 1.1. *For every $n \ge 1$, $k \ge 0$, partition $\mu \vdash n$, and $\pi \in S_n$ of cycle type μ, we have*

$$\#\{\sigma \in S_n : \ \sigma^k = \pi\} = \sum_{\sigma \in I_n^k \cap U_\mu} (-1)^{|\mathrm{Des}(\sigma) \setminus S(\mu)|}. \tag{2}$$

It follows that the set of k-roots of the identity permutation is a fine set in the sense of [3]. The case $k = 2$ follows from [2, Prop.1.5]. Note that the proof there does not apply to a general k.

2. Proof of Theorem 1.1

2.1. **Induced representations.** For every $n \geq 1$ and $k \geq 0$, let $\theta^{k,n} : S_n \longrightarrow \mathbb{N} \cup \{0\}$ be the enumerator of k-roots of a permutation π in S_n,

$$\theta^{(k,n)}(\pi) := \#\{\sigma \in S_n : \ \sigma^k = \pi\}.$$

Clearly, $\theta^{(k,n)}$ is a class function. By a classical result of Frobenius and Schur, $\theta^{(2,n)}$ is not virtual, see e.g. [11, §4]. It was conjectured by Kerber and proved by Scharf [16] that, for every $k \geq 0$, $\theta^{(k,n)}$ is a non-virtual character.

Let Z_λ be the centralizer of a permutation of cycle type λ in S_n. Z_λ is isomorphic to the direct product $\times_{i=1}^{n} C_i \wr S_{m_i}$, where m_i is the multiplicity of the part i in λ. Denote by ρ_i the one-dimensional representation of $C_i \wr S_{m_i}$ indexed by the i-tuple of partitions $(\emptyset, (m_i), \emptyset, \ldots, \emptyset)$. Let

$$\rho^\lambda := \bigotimes_{i=1}^{n} \rho_i$$

be a one-dimensional representation of Z_λ, and

$$\psi^\lambda = \rho^\lambda \uparrow_{Z_\lambda}^{S_n}$$

the corresponding induced S_n-representation.

Denote by $\phi^{k,n}$ the representation whose character is $\theta^{(k,n)}$. The following theorem implies that $\phi^{k,n}$ is not virtual.

Theorem 2.1 ([16])**.** *For every $n \geq 1$ and $k \geq 0$, we have*

$$\phi^{k,n} = \bigoplus_{\substack{\lambda \vdash n \\ \text{all parts divide } k}} \psi^\lambda.$$

See also [18, Cor. 5.2] and [17, Ex. 7.69(c)]. Note that by letting $k = 2$ one obtains the well known construction of Inglis, Richardson, and Saxl of a Gelfand model for S_n [10].

2.2. **Descents over conjugacy classes.** Let C_λ be the conjugacy class of cycle type λ in S_n and $\mathrm{SYT}(\nu)$ be the set of all standard Young tableaux of shape ν. Denote the multiplicity of the Specht module S^ν in ψ^λ by $m(\nu, \lambda)$. The following is a reformulation of [8, Thm. 2.1], see also [12].

Theorem 2.2. *For every $\lambda \vdash n$, we have*

$$\sum_{\pi \in C_\lambda} \mathbf{x}^{\mathrm{Des}(\pi)} = \sum_{\nu \vdash n} m(\nu, \lambda) \sum_{T \in \mathrm{SYT}(\nu)} \mathbf{x}^{\mathrm{Des}(T)}. \tag{3}$$

Proof. Denote by L_λ the image of ψ^λ under the Frobenius characteristic map. For an explicit description of this symmetric function, see e.g. [17, Ex. 7.89]. For $J \subseteq [n-1]$, let z_J be the skew Schur function which corresponds to the zigzag skew shape with down steps on positions which belong to J. For example, in French notation, $J = \{1,4,5\} \subseteq [7]$ corresponds to the shape

$$\begin{array}{|c|c|c|cc} \hline 1 & \multicolumn{4}{l}{} \\ \hline 2 & 3 & \multicolumn{1}{c|}{4} & & \\ \hline \multicolumn{2}{c|}{} & \multicolumn{1}{c|}{5} & & \\ \cline{3-5} \multicolumn{2}{c|}{} & 6 & \multicolumn{1}{c|}{7} & \multicolumn{1}{c|}{8} \\ \cline{3-5} \end{array} .$$

By [8, Thm. 2.1], the coefficient of $\mathbf{x}^J$ in the left-hand side of Equation (3), which is the number of permutations of cycle type λ and descent set J, is equal to $\langle L_\lambda, z_J\rangle$.

Now,

$$\langle L_\lambda, z_J\rangle = \langle L_\lambda, \sum_{\nu\vdash n}\langle s_\nu, z_J\rangle s_\nu\rangle = \sum_{\nu\vdash n}\langle L_\lambda, s_\nu\rangle\langle s_\nu, z_J\rangle = \sum_{\nu\vdash n} m(\nu,\lambda)\langle s_\nu, z_J\rangle.$$

Since $\langle s_\nu, z_J\rangle$ is equal to the number of standard Young tableaux of shape ν and descent set J [7, Thm. 7] (see also [1, Thm. 4.1]), this is equal to the coefficient of $\mathbf{x}^J$ in the right-hand side of Equation (3). □

Corollary 2.3. *For every partition $\mu \vdash n$, the value of ψ^λ at a permutation of cycle type μ is*

$$\psi^\lambda_\mu = \sum_{\sigma\in C_\lambda\cap U_\mu} (-1)^{|\mathrm{Des}(\sigma)\setminus S(\mu)|}. \tag{4}$$

Proof. For partitions μ and ν of n, let χ^ν_μ be the character value of the Specht module S^ν on a conjugacy class of cycle type μ. A standard Young tableau T of size n is μ-unimodal if $\mathrm{Des}(T)\setminus S(\mu)$ is a disjoint union of intervals of the form $[\mu_{(i)}+1, \mu_{(i)}+\imath]$ for some $0 \le \imath < \mu_{i+1}$. For example, the standard Young tableau

$$T = \begin{array}{|c|c|c|} \hline 5 & \multicolumn{2}{l}{} \\ \hline 2 & 4 & \multicolumn{1}{l}{} \\ \hline 1 & 3 & 6 \\ \hline \end{array}$$

has $\mathrm{Des}(T) = \{1,3,4\}$, and is therefore $(3,3)$-unimodal but not $(4,2)$-unimodal.
By [14, Theorem 4] [13], we have

$$\chi^\nu_\mu = \sum_{T\in\mathrm{SYT}(\nu)\cap\mathrm{SYT}_\mu} (-1)^{|\mathrm{Des}(T)\setminus S(\mu)|},$$

where $\mathrm{SYT}(\nu)$ is the set of all standard Young tableaux of shape ν and SYT_μ is the set of μ-unimodal standard Young tableaux of size n.

Combining this with Theorem 2.2 gives

$$\begin{aligned}\psi^\lambda_\mu = \sum_{\nu\vdash n} m(\nu,\lambda)\chi^\nu_\mu &= \sum_{\nu\vdash n} m(\nu,\lambda) \sum_{T\in\mathrm{SYT}(\nu)\cap\mathrm{SYT}_\mu} (-1)^{|\mathrm{Des}(T)\setminus S(\mu)|} \\ &= \sum_{\sigma\in C_\lambda\cap U_\mu} (-1)^{|\mathrm{Des}(\sigma)\setminus S(\mu)|}.\end{aligned}$$

□

2.3. **Conclusion.** By Theorem 2.1 together with Corollary 2.3, for every $\pi \in S_n$ of cycle type μ, we have

$$\#\{\sigma \in S_n : \ \sigma^k = \pi\} = \theta^{(k,n)}(\pi) = \sum_{\substack{\lambda \vdash n \\ \text{all parts divide } k}} \psi^\lambda_\mu$$
$$= \sum_{\substack{\lambda \vdash n \\ \text{all parts divide } k}} \sum_{\sigma \in C_\lambda \cap U_\mu} (-1)^{|\mathrm{Des}(\sigma) \setminus S(\mu)|} = \sum_{\sigma \in I_n^k \cap U_\mu} (-1)^{|\mathrm{Des}(\sigma)\setminus S(\mu)|},$$

completing the proof of Theorem 1.1. □

3. Remarks and questions

It is desirable to prove Theorem 1.1 via generalizations of the explicit combinatorial construction of Gelfand models described in [2].

Question 3.1. *Find a "simple" S_n-linear action on a basis of $\phi^{k,n}$ indexed by I_n^k, which implies the character formula given on the right-hand side of Equation* (2)*.*

Another desirable approach to prove Theorem 1.1 is purely combinatorial.

Question 3.2. *Define, for any given partition μ of n, an involution on the set of k-roots of the identity permutation, which changes the parity of* $\mathrm{Des}(\cdot) \setminus S(\mu)$ *on non-fixed points, such that the cardinality of the fixed point set is equal to the left-hand side of Equation* (2)*.*

The case $\mu = (n)$ was recently solved by Archer [4].

Question 3.3. *Prove Theorem 2.2 by constructing a map from C_λ to standard Young tableaux of size n, under which for every $\nu \vdash n$ and $T \in \mathrm{SYT}(\nu)$ the cardinality of the preimage of T is exactly $m(\nu, \lambda)$.*

Note that for $\lambda = (2^k, 1^{n-2k})$, $0 \le k \le n/2$, the Robinson–Schensted–Knuth map satisfies this property.

Finally, a natural objective is to extend the setting of the current note to other finite groups. Complex reflection groups are of special interest.

Question 3.4. *Generalize Theorem 1.1 to other Coxeter and complex reflection groups.*

This question is intimately related to the problem of characterizing the finite groups for which the character $\theta^{(k,n)}$ is non-virtual. For wreath products, see [15].

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