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ON THE REGULARIZATION PROCESS FOR ARIKI–KOIKE ALGEBRAS

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Abstract. The aim of this note is to study a generalization of theorems by James and Fayers on the modular representations of the symmetric group and its Hecke algebra to the case of the complex reflection groups of type $G(l,1,n)$ and the associated Ariki–Koike algebra.

1. Introduction

One of the main and still open problems in the representation theory of finite groups is the explicit determination of the irreducible representations for the symmetric groups $\mathfrak{S}_n$, for $n \in \mathbb{N}$, over a field of characteristic $p > 0$. The main information on these representations is contained in a fundamental object: the decomposition matrix. By the work of James, the problem of computing this matrix may be attacked using the representation theory of Hecke algebras. Indeed, James' conjecture predicts that, in the case where $p^2 > n$, the decomposition matrix of the symmetric group corresponds to the decomposition matrix of a non-semisimple deformation of the group algebra $\mathbb{C}\mathfrak{S}_n$: the Hecke algebra. Both matrices have their rows labelled by the partitions of rank n (which also label the simple modules of $\mathbb{C}\mathfrak{S}_n$) whereas their columns are labelled by a certain subset of partitions, namely the p-regular partitions (which also label the simple modules of $\mathbb{F}\mathfrak{S}_n$ where $\mathrm{char}(\mathbb{F}) = p$).

An algorithm for the computation of the decomposition matrices for Hecke algebras over $\mathbb{C}$ is available. This algorithm comes from a conjecture given by Lascoux, Leclerc and Thibon [11] and proved by Ariki [1]. It asserts that the decomposition matrix of the Hecke algebra over $\mathbb{C}$ is given by the evaluation at $v = 1$ of the matrix of the canonical basis for the basic representation of the quantum algebra $\mathcal{U}_v(\widehat{\mathfrak{sl}_e})$. Recently, several authors have shown that the matrix of the canonical basis itself (and not only its specialization at $v = 1$) admits an interpretation in terms of "graded representation theory" of Hecke algebras (see [10] and the references therein). This matrix can thus be called the v-decomposition matrix of the Hecke algebra.

A nice general property of these matrices has been revealed by James [9] (for the decomposition matrix of the symmetric group) and Fayers [4] (for the v-decomposition matrix of the Hecke algebra). One can explicitly associate to each partition λ of n a certain p-regular partition, the *regularization* of λ, and computes the associated decomposition number. This result uses a very important partial order on the set of partitions which appears in many ways in the representation theory of the symmetric group: the dominance order.

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This note is concerned with a generalization of the Hecke algebra of $\mathfrak{S}_n$: the Ariki–Koike algebra. We present a result analogous to the ones by James and Fayers for the decomposition matrices of these algebras and for the matrices of the canonical bases for the irreducible highest weight $\mathcal{U}_v(\mathfrak{sl}_\infty)$-modules. The concept of partitions is here replaced by the concept of "multipartitions", the set of p-regular partitions by the set of so called cylindric multipartitions and the dominance order on partitions by the dominance order on multipartitions. The main results, Theorem 5.1 and Theorem 6.3, give the desired analogues of James and Fayers' Theorems. The paper will be organized as follows. In the first section, we introduce the main combinatorial objects we will use in this paper, namely multipartitions and symbols, and present some useful properties which they satisfy. Then we define the notion of regularization of multipartitions. The third part is devoted to a brief exposition of the representation theory of $\mathcal{U}_v(\mathfrak{sl}_\infty)$ using the theory of Fock spaces. All these notions are then used in the two last parts to obtain our main results.

2. Multipartitions and symbols

In this part, we give the combinatorial definitions which are needed to present our main results.

2.1. Let $l \in \mathbb{N}_{>0}$. We write

$$\mathcal{S}^l := \left\{(s_0, \dots, s_{l-1}) \in \mathbb{Z}^l \mid s_{i+1} \geq s_i \text{ for } i = 0, \dots, l-2\right\}.$$

Recall that a *partition* $\lambda = (\lambda_1, \dots, \lambda_r)$ of $n \in \mathbb{N}$ is an ordered sequence of weakly decreasing nonnegative integers such that $|\lambda| := \sum_{1 \leq i \leq r} \lambda_i = n$. The integer $h_\lambda := \max(i \geq 0 \mid \lambda_i \neq 0)$ is called the *height* of λ with the convention that $h_\lambda = 0$ if λ is the empty partition, denoted by $\emptyset$. A *multipartition* or *l-partition* of n is an l-tuple of partitions $\boldsymbol{\lambda} = (\lambda^0, \dots, \lambda^{l-1})$ such that, for each $i \in \{0, 1, \dots, l-1\}$, λ^i is a partition of rank $n_i \in \mathbb{N}$ and $\sum_{0 \leq i \leq l-1} n_i = n$. If $\boldsymbol{\lambda}$ is a multipartition of n, we write $\boldsymbol{\lambda} \vdash_l n$. The height of $\boldsymbol{\lambda}$ is the nonnegative integer

$$h_{\boldsymbol{\lambda}} := \max(h_{\lambda^0}, \dots, h_{\lambda^{l-1}}),$$

and we have $h_{\boldsymbol{\lambda}} = 0$ if and only if $\boldsymbol{\lambda}$ is the empty multipartition, which is denoted by $\boldsymbol{\emptyset}$. The dominance order on multipartitions is defined as follows. Let $\boldsymbol{\lambda} := (\lambda^0, \dots, \lambda^{l-1})$ and $\boldsymbol{\mu} := (\mu^0, \dots, \mu^{l-1})$ be two multipartitions of n then we write

$$\boldsymbol{\lambda} \trianglerighteq \boldsymbol{\mu} \text{ if and only if } \sum_{0 \leq i < c} |\lambda^i| + \sum_{1 \leq j \leq k} \lambda_j^c \geq \sum_{0 \leq i < c} |\mu^i| + \sum_{1 \leq j \leq k} \mu_j^c$$
$$\text{for } c = 0, \dots, l-1 \text{ and } k \in \mathbb{N},$$

where the partitions are considered with an infinite number of empty parts.

2.2. Let $\mathbf{s} = (s_0, \dots, s_{l-1}) \in \mathcal{S}^l$. We now define the notion of shifted symbol. Following [5, § 5.5.5], let $\beta = (\beta_1, \dots, \beta_k)$ be a sequence of integers where $k \in \mathbb{N}$ and let t be a positive integer. We set

$$\beta(t) := (0, 1, \dots, t-1, \beta_1 + t, \dots, \beta_k + t).$$

It is a sequence of exactly $k + t$ integers. Let $\boldsymbol{\lambda} := (\lambda^0, \dots, \lambda^{l-1}) \vdash_l n$. For $i = 0, 1, \dots, l-1$, let h^i be the height of the partitions λ^i. We consider the following sequence of strictly increasing integers

$$\beta^i = (\lambda^i_{h^i} - h^i + h^i, \dots, \lambda^i_j - j + h^i, \dots, \lambda^i_1 - 1 + h^i).$$

For $i = 0, 1, \dots, l-1$, we put

$$hc^i := h^i - s_i \text{ and } hc^{\boldsymbol{\lambda}} := \max(hc^0, \dots, hc^{l-1}).$$

Let h be an integer such that $h \geq hc^{\boldsymbol{\lambda}} + 1$. The *shifted* $\mathbf{s}$*-symbol* of $\boldsymbol{\lambda}$ of size h is the family of sequences

$$\mathfrak{B}_{(\mathbf{s},h)}(\boldsymbol{\lambda}) = (\mathfrak{B}^0, \dots, \mathfrak{B}^{l-1}),$$

such that for $j = 0, \dots, l-1$, we have

$$\mathfrak{B}^j = (\beta^j(h - hc^j)).$$

Each sequence $\mathfrak{B}^j$ contains exactly $h + s_j$ elements $(\mathfrak{B}^j_{h+s_j}, \dots, \mathfrak{B}^j_1)$.

2.3. A shifted symbol is usually represented as (and identified with) an l-row tableau where the c-th row (starting from bottom) is $\mathfrak{B}^c$ (see [5, § 5.5.5]). It is written as follows:

$$\mathfrak{B}_{(\mathbf{s},h)}(\boldsymbol{\lambda}) = \begin{pmatrix} \mathfrak{B}^{l-1}_{h+s_{l-1}} & \dots & \dots & \mathfrak{B}^{l-1}_2 & \mathfrak{B}^{l-1}_1 \\ \mathfrak{B}^{l-2}_{h+s_{l-2}} & \dots & \dots & \mathfrak{B}^{l-2}_1 & \\ \dots & \dots & \dots & & \\ \mathfrak{B}^0_{h+s_0} & \dots & \mathfrak{B}^0_1 & & \end{pmatrix}.$$

With our convention, note that the leftmost column of the symbol contains only entries 0. Also, note that the i^{th} column (starting from the right) of $\mathfrak{B}_{(\mathbf{s},h)}(\boldsymbol{\lambda})$ contains exactly $l - c(i)$ elements where $c(i) = \min(k \in \{0, 1, \dots, l-1\} \mid i + s_k - s_{l-1} > 0)$ and this column is given by

$$\begin{pmatrix} \mathfrak{B}^{l-1}_i \\ \mathfrak{B}^{l-2}_{i+s_{l-2}-s_{l-1}} \\ \dots \\ \mathfrak{B}^{c(i)}_{i+s_{c(i)}-s_{l-1}} \end{pmatrix}.$$

It is easy to recover the multipartition $\boldsymbol{\lambda}$ from a given shifted symbol. Similarly, one can also easily recover $\mathbf{s} \in \mathbb{Z}^l$ modulo a translation by an element $(x, \dots, x) \in \mathbb{Z}^l$.

Example 2.4. With $\boldsymbol{\lambda} = (3, 2.2.2, 2.1)$ and $\mathbf{s} = (0, 0, 2)$, if we take $h = 5$ we obtain

$$\mathfrak{B}_{((0,0,2),5)}(\boldsymbol{\lambda}) = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 6 & 8 \\ 0 & 1 & 4 & 5 & 6 & & \\ 0 & 1 & 2 & 3 & 7 & & \end{pmatrix}.$$

With $\mathbf{s} = (0, 1, 2, 3)$ and $\boldsymbol{\lambda} = (3.1, 1.1, 2.1.1, 3)$ and $h = 3$, the shifted symbol is

$$\mathfrak{B}_{((0,1,2,3),3)}(\boldsymbol{\lambda}) = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 8 \\ 0 & 1 & 3 & 4 & 6 & \\ 0 & 1 & 3 & 4 & & \\ 0 & 2 & 5 & & & \end{pmatrix}.$$

3. Regularization of multipartitions

In this section, after having fixed an element in $\mathcal{S}^l$, we associate to each multipartition another one which belongs to a certain class of multipartitions: the cylindric multipartitions.

Definition 3.1. Assume that $\mathbf{s}\in\mathcal{S}^l$. A shifted symbol $\mathfrak{B}_{(\mathbf{s},h)}(\boldsymbol{\lambda})$ is called *semistandard*[1] if and only if in each column of $\mathfrak{B}_{(\mathbf{s},h)}(\boldsymbol{\lambda})$, the numbers weakly decrease from top to bottom.

Example 3.2. The first symbol in Example 2.4 is not semistandard whereas the second is. Let now $\mathbf{s}=(0,1,2)$ and $\boldsymbol{\lambda}=(3.1,2.2.1,2.1)$. If we take $h=5$ we obtain

$$\mathfrak{B}_{((0,1,2),5)}(\boldsymbol{\lambda})=\left(\begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 6 & 8\\ 0 & 1 & 2 & 4 & 6 & 7 & \\ 0 & 1 & 2 & 4 & 7 & & \end{array}\right).$$

This symbol is semistandard.

Definition 3.3. Assume that $\boldsymbol{s}=(s_0,\dots,s_{l-1})\in\mathcal{S}^l$ then the $\boldsymbol{\lambda}=(\lambda^0,\dots,\lambda^{l-1})\vdash_l n$ is called *cylindric* if for every $c=0,\dots,l-2$ and $i\geq 1$, we have $\lambda_i^c\geq\lambda^{c+1}_{i+s_{c+1}-s_c}$ (the partitions are taken with an infinite number of empty parts). We denote by $\Phi_{\mathbf{s}}$ the set of *cylindric multipartitions* associated to $\boldsymbol{s}\in\mathcal{S}^l$ and by $\Phi_{\mathbf{s}}(n)$ the set of cylindric multipartitions of rank n.

Proposition 3.4. *Let $\mathbf{s}\in\mathcal{S}^l$, $\boldsymbol{\lambda}\vdash_l n$ and let $\mathfrak{B}_{(\mathbf{s},h)}(\boldsymbol{\lambda})$ be an associated shifted symbol. Then $\boldsymbol{\lambda}$ is cylindric if and only if $\mathfrak{B}_{(\mathbf{s},h)}(\boldsymbol{\lambda})$ is semistandard.*

Proof. For all relevant $i\in\mathbb{N}$ and $c\in\{0,1,\dots,l-2\}$, we have

$$\lambda_i^c-i+s_c\geq\lambda^{c+1}_{i+s_{c+1}-s_c}-(i+s_{c+1}-s_c)+s_{c+1}\text{ if and only if }\lambda_i^c\geq\lambda^{c+1}_{i+s_{c+1}-s_c},$$

whence

$$\mathfrak{B}_i^c\geq\mathfrak{B}^{c+1}_{i+s_{c+1}-s_c}\text{ if and only if }\lambda_i^c\geq\lambda^{c+1}_{i+s_{c+1}-s_c},$$

which is exactly what is needed to prove the assertion. □

3.5. We now explain the process of regularization for multipartitions. Let $\boldsymbol{\lambda}\vdash_l n$ and let $\boldsymbol{s}=(s_0,\dots,s_{l-1})\in\mathcal{S}^l$. Let $\mathfrak{B}_{(\mathbf{s},h)}(\boldsymbol{\lambda})=(\mathfrak{B}^0,\dots,\mathfrak{B}^{l-1})$ be a shifted symbol associated to $\boldsymbol{\lambda}$. Then for each column of the symbol, we reorder the elements so that it is weakly decreasing from top to bottom and we denote by $\mathfrak{B}_{(\mathbf{s},h)}(\boldsymbol{\lambda})^R$ this new object. For example, the symbol in Example 2.4

$$\mathfrak{B}_{((0,1,1),4)}(\boldsymbol{\lambda})=\left(\begin{array}{cccc} 0 & 1 & 3 & 9\\ 0 & 4 & 5 & 7\\ 0 & 2 & 3 & \end{array}\right)$$

becomes

$$\left(\begin{array}{cccc} 0 & 1 & 3 & 7\\ 0 & 2 & 3 & 9\\ 0 & 4 & 5 & \end{array}\right).$$

[1]Such symbols are called standard in [12].

We claim that the resulting object is a well-defined semistandard symbol of a multipartition $\mathfrak{B}_{(\mathbf{s},h)}(\boldsymbol{\mu})$. Indeed, denote the symbol by $\mathcal{B}=(\mathcal{B}^0,\dots,\mathcal{B}^{l-1})$ (in an obvious way) and assume that there exist $c\in\{0,1,\dots,l-1\}$ and $j\in\mathbb{N}$ such that

$$\mathcal{B}^c_{j+1}\geq\mathcal{B}^c_j.$$

Note that the elements in each column of $\mathfrak{B}_{(\mathbf{s},h)}(\boldsymbol{\lambda})$ are the same as the elements in the corresponding columns in $\mathcal{B}$. Let $\mathcal{A}_1$ be the multiset of elements in the column containing the entry $\mathcal{B}^c_j$ which are less or equal than $\mathcal{B}^c_j$ in $\mathfrak{B}_{(\mathbf{s},h)}(\boldsymbol{\lambda})$. Let $\mathcal{A}_2$ be the multiset of elements in the column containing the entry $\mathcal{B}^c_{j+1}$ which are greater or equal than $\mathcal{B}^c_j$ in $\mathfrak{B}_{(\mathbf{s},h)}(\boldsymbol{\lambda})$. Assume that the column containing $\mathcal{B}^c_{j+1}$ contains m elements. We know that in the columns of $\mathcal{B}$, the numbers are weakly decreasing from top to bottom. Thus, by the construction of $\mathcal{B}$ and the above assumption, we have $\sharp\mathcal{A}_1\geq l-c$ and $\sharp\mathcal{A}_2\geq m-l+c+1$. Now $\mathfrak{B}_{(\mathbf{s},h)}(\boldsymbol{\lambda})$ is a well-defined symbol so each row contains elements which are strictly increasing from left to right. This implies that the rows containing the elements of $\mathcal{A}_1$ in the column of $\mathfrak{B}^c_j$ and the rows containing the elements of $\mathcal{A}_2$ in the column of $\mathfrak{B}^c_{j+1}$ in $\mathfrak{B}_{(\mathbf{s},h)}(\boldsymbol{\lambda})$ must be disjoint. So the sum $\sharp\mathcal{A}_1+\sharp\mathcal{A}_2$ must be less or equal to m. This is not the case, so the result follows. In addition, $\boldsymbol{\mu}$ is an l-partition of n.

3.6. Let $\boldsymbol{\lambda}\vdash_l n$ and let $\mathfrak{B}_{(\mathbf{s},h)}(\boldsymbol{\lambda})=(\mathfrak{B}^0,\dots,\mathfrak{B}^{l-1})$ be an associated shifted symbol. For all $c_1\in\{0,1,\dots,l-1\}$ and $j_1\in\{1,2,\dots,h+s_{c_1}\}$, we set

$$R(\boldsymbol{\lambda})_{(j_1,c_1)}=\sharp\left\{c\in\{0,1,\dots,l-1\}\mid c>c_1,\ \mathfrak{B}^{c_1}_{j_1}<\mathfrak{B}^c_{j_1+s_c-s_{c_1}},\ \mathfrak{B}^{c_1}_{j_1}\notin\mathfrak{B}^c\right\},$$

and

$$R(\boldsymbol{\lambda})=\sum_{0\leq c\leq l-1}\ \sum_{1\leq j_1\leq h+s_c}R(\boldsymbol{\lambda})_{(j_1,c_1)}.$$

Clearly, this number does not depend of the choice of h (and thus on the choice of the shifted symbol).

Definition 3.7. Let $\boldsymbol{\lambda}$ be a multipartition of n and let $\boldsymbol{s}=(s_0,\dots,s_{l-1})\in\mathcal{S}^l$. Let $\mathfrak{B}_{(\mathbf{s},h)}(\boldsymbol{\lambda})$ be the associated shifted symbol. By the discussion in §3.5, there exists a cylindric multipartition $\boldsymbol{\lambda}^R$ such that $\mathfrak{B}_{(\mathbf{s},h)}(\boldsymbol{\lambda}^R)=\mathfrak{B}_{(\mathbf{s},h)}(\boldsymbol{\lambda})^R$. This multipartition is called the *regularization* of $\boldsymbol{\lambda}$.

By construction, the regularization of a multipartition is a cylindric multipartition by Proposition 3.4. It is also clear that the regularization stabilizes the set of cylindric multipartitions.

Example 3.8. Consider the 3-partition $\boldsymbol{\lambda}:=(\emptyset,\emptyset,6.2)$. We set $\mathbf{s}=(0,1,1)$, $h=3$. Then we have

$$\mathfrak{B}_{((0,1,1),3)}(\boldsymbol{\lambda})=\begin{pmatrix}0&1&4&9\\0&1&2&3\\0&1&2&\end{pmatrix}.$$

We obtain

$$\mathfrak{B}_{((0,1,1),3)}(\boldsymbol{\lambda})=\begin{pmatrix}0&1&2&3\\0&1&2&9\\0&1&4&\end{pmatrix}.$$

We have

$$R(\boldsymbol{\lambda})_{(j,c)}=\left\{\begin{array}{ll}0 & \text{if } (j,c)\notin\{(1,0),(2,1),(1,1)\}\\ 1 & \text{if } (j,c)\in\{(1,0),(2,1),(1,1)\}\end{array}\right.$$

and so we have $R(\boldsymbol{\lambda})=3$. We can check that $\boldsymbol{\lambda}^R=(2,6,\emptyset)$.

Example 3.9. Consider the 3-partition $\boldsymbol{\lambda}:=(5,\emptyset,2.1)$. We set $\mathbf{s}=(0,1,1)$, $h=3$. Then we have

$$\mathfrak{B}_{((0,1,1),3)}(\boldsymbol{\lambda})=\left(\begin{array}{cccc}0&1&3&5\\0&1&2&3\\0&1&7&\end{array}\right).$$

We obtain

$$\mathfrak{B}_{((0,1,1),3)}(\boldsymbol{\lambda})=\left(\begin{array}{cccc}0&1&2&3\\0&1&3&5\\0&1&7&\end{array}\right).$$

and we have $R(\boldsymbol{\lambda})=R(\boldsymbol{\lambda})_{(2,1)}=1$. We can check that $\boldsymbol{\lambda}^R=(5,2.1,\emptyset)$.

4. Action of $\mathcal{U}_v(\mathfrak{sl}_\infty)$ on the Fock space

4.1. Let v be an indeterminate and let $\mathcal{U}_v(\mathfrak{sl}_\infty)$ be the quantized enveloping algebra of $\mathfrak{sl}_\infty$ with Chevalley generators e_j, f_j and t_j ($j\in\mathbb{Z}$), see for example [5, § 6.1]. The simple roots and fundamental weights are denoted by α_k and Λ_k for $k\in\mathbb{Z}$ respectively. Let now $\mathbf{s}\in\mathcal{S}^l$ and let $\mathcal{F}^{\mathbf{s}}$ be the associated Fock space. This is the $\mathbb{Q}(v)$-vector space defined as follows:

$$\mathcal{F}_{\mathbf{s}}=\bigoplus_{n\in\mathbb{Z}_{\geq 0}}\bigoplus_{\boldsymbol{\lambda}\vdash_l n}\mathbb{Q}(v)\boldsymbol{\lambda}.$$

One can define an action of $\mathcal{U}_v(\mathfrak{sl}_\infty)$ which turns $\mathcal{F}_{\mathbf{s}}$ into an integrable $\mathcal{U}_v(\mathfrak{sl}_\infty)$-module.

4.2. Let $\beta:=(\beta_1,\ldots,\beta_m)$ be a sequence of strictly increasing positive numbers. We write $j\in\beta$ if the number j appears in the sequence β and $j\notin\beta$ otherwise. In this case, we write $\beta\cup\{j\}$ for the sequence of strictly positive numbers obtained by inserting the number j in β. Similarly, if $j\in\beta$ then $\beta\setminus\{j\}$ is defined to be the sequence obtained from β by removing j.

Let $\boldsymbol{\lambda}\vdash_l n$ and let $\mathfrak{B}_{(\mathbf{s},h)}(\boldsymbol{\lambda})$ be an associated shifted symbol. Let $\boldsymbol{\mu}\vdash_l n$ and let $\mathfrak{B}_{(\mathbf{s},h)}(\boldsymbol{\mu})$ be an associated shifted symbol (so we assume that $h\geq\max(hc^{\boldsymbol{\lambda}},hc^{\boldsymbol{\mu}})+1$.) Then we write

$$\boldsymbol{\lambda}\xrightarrow[c]{j}\boldsymbol{\mu},$$

if for all $d\in\{0,1,\ldots,l-1\}$, we have

$$\mathfrak{B}_{(\mathbf{s},h)}(\boldsymbol{\mu})^d=\left\{\begin{array}{ll}\mathfrak{B}_{(\mathbf{s},h)}(\boldsymbol{\lambda})^d & \text{if } d\neq c,\\ (\mathfrak{B}_{(\mathbf{s},h)}(\boldsymbol{\lambda})^d\setminus\{j+h-1\}))\cup\{j+h\} & \text{if } d=c.\end{array}\right.$$

For $k\in\mathbb{N}$, we write

$$\boldsymbol{\lambda}\xrightarrow[(c_1,\ldots,c_k)]{j:k}\boldsymbol{\mu},$$

if there exists a sequence of multipartitions

$$\boldsymbol{\lambda}:=\boldsymbol{\lambda}[1],\ldots,\boldsymbol{\lambda}[k],\boldsymbol{\lambda}[k+1]:=\boldsymbol{\mu},$$

such that for all $i = 1, \dots, k$, we have

$$\boldsymbol{\lambda}[i] \xrightarrow[c_i]{j} \boldsymbol{\lambda}[i+1].$$

Hence, we have

$$\mathfrak{B}_{(\mathbf{s},h)}(\boldsymbol{\mu})^d = \begin{cases} \mathfrak{B}_{(\mathbf{s},h)}(\boldsymbol{\lambda})^d & \text{if } d \neq c_i \text{ for all } i = 1, \dots, k \\ (\mathfrak{B}_{(\mathbf{s},h)}(\boldsymbol{\lambda})^d \setminus \{j+h-1\})) \cup \{j+h\} & \text{if } d = c_i \text{ for some } i \in \{1 \dots k\}. \end{cases}$$

We also write $\boldsymbol{\lambda} \xrightarrow{j:k} \boldsymbol{\mu}$ if there exists a sequence $(c_1, \dots, c_k) \in \{0, 1, \dots, l-1\}^k$ such that $\boldsymbol{\lambda} \xrightarrow[(c_1,\dots,c_k)]{j:k} \boldsymbol{\mu}$. Note that in this case, the sequence $(c_1, \dots, c_k) \in \{0, 1, \dots, l-1\}^k$ is uniquely determined. Finally, given $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$ and $j \in \mathbb{Z}$ such that $\boldsymbol{\lambda} \xrightarrow[(c_1,\dots,c_k)]{j:k} \boldsymbol{\mu}$, we define the following number:

$$N_j(\boldsymbol{\lambda}, \boldsymbol{\mu}) = \sum_{1 \leq i \leq k} \big(\sharp\{\text{integers equal to } j+h-1 \text{ in } \mathfrak{B}_{(\mathbf{s},h)}(\boldsymbol{\mu})^c \text{ with } c \leq c_i\} \\ -\sharp\{\text{integers equal to } j+h \text{ in } \mathfrak{B}_{(\mathbf{s},h)}(\boldsymbol{\lambda})^c \text{ with } c \leq c_i\}\big)\,.$$

It is clear that all these notions do not depend on the choice of h.

Example 4.3. Consider the 4-partition $\boldsymbol{\lambda} = (3.1, \emptyset, \emptyset, 6.2)$ and $\mathbf{s} = (0, 0, 1, 1)$. For $h = 3$, we get the symbol

$$\mathfrak{B}_{((0,0,1,1),3)}(\boldsymbol{\lambda}) = \begin{pmatrix} 0 & 1 & 4 & 9 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & \\ 0 & 2 & 5 & \end{pmatrix}.$$

We have

$$(3.1, \emptyset, \emptyset, 6.2) \xrightarrow[(0,1)]{0:2} (3.2, 1, \emptyset, 6.2),$$

where the symbol of $\boldsymbol{\mu}$ is

$$\mathfrak{B}_{((0,0,1,1),3)}(\boldsymbol{\mu}) = \begin{pmatrix} 0 & 1 & 4 & 9 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 3 & \\ 0 & 3 & 5 & \end{pmatrix}.$$

and we have $N_0(\boldsymbol{\lambda}, \boldsymbol{\mu}) = 0$.

4.4. For our purpose, we only need to describe the action of the Chevalley generators f_j for $j \in \mathbb{Z}$ and their divided power $f_j^{(r)} := \dfrac{f_j^r}{[r]_v!}$, for $r \in \mathbb{N}$. This is given as follows. Let $\boldsymbol{\lambda} \vdash_l n$, let $r \in \mathbb{N}$ and let $j \in \mathbb{Z}$, then we have

$$f_j^{(r)}.\boldsymbol{\lambda} = \sum_{\boldsymbol{\lambda} \xrightarrow{j:r} \boldsymbol{\mu}} v^{N_j(\boldsymbol{\lambda}, \boldsymbol{\mu})} \boldsymbol{\mu}.$$

It is known that the $\mathcal{U}_v(\mathfrak{sl}_\infty)$-submodule $V(\mathbf{s})$ of $\mathcal{F}^{\mathbf{s}}$ generated by the empty multipartition is an irreducible highest weight module for $\mathcal{U}_v(\mathfrak{sl}_\infty)$.

4.5. Let $\mathcal{U}_v(\mathfrak{sl}_\infty)^-$ be the subalgebra of $\mathcal{U}_v(\mathfrak{sl}_\infty)$ generated by the f_i's. We have a ring automorphism $x \mapsto \overline{x}$ of $\mathcal{U}_v(\mathfrak{sl}_\infty)^-$ such that

$$\overline{f}_j = f_j \ (j \in \mathbb{N}), \qquad \text{and} \qquad \overline{v} = v^{-1},$$

which induces a $\mathbb{C}$-linear map $v \mapsto \overline{v}$ on $V(\mathbf{s})$ defined by

$$\overline{v.\emptyset} = \overline{v}.\emptyset.$$

Using this, one can define the canonical basis elements of $V(\mathbf{s})$. They are elements of $V(\mathbf{s})$ which are parametrized by the set of cylindric multipartitions

$$\{b_{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda} \in \Phi_{\mathbf{s}}\},$$

and characterized by the following property:

$$\text{for all } \boldsymbol{\lambda} \in \Phi_{\mathbf{s}}, \text{ we have } \overline{b_{\boldsymbol{\lambda}}} = b_{\boldsymbol{\lambda}}, \text{ and } b_{\boldsymbol{\lambda}} = \boldsymbol{\lambda} + \sum_{\boldsymbol{\mu} \vdash_l n} d_{\boldsymbol{\mu},\boldsymbol{\lambda}}(v)\boldsymbol{\mu},$$

for elements $d_{\boldsymbol{\mu},\boldsymbol{\lambda}}(v) \in v\mathbb{Q}[v]$ with $\boldsymbol{\mu} \vdash_l n$ such that $\boldsymbol{\lambda} \neq \boldsymbol{\mu}$.

4.6. We now briefly explain an algorithm for computing the canonical bases of irreducible highest weight $\mathcal{U}_v(\mathfrak{sl}_\infty)$-modules. An analogue of this algorithm has already been described in [7] (and the proofs can be found therein) in a more general setting but it can be simplified here. One can also use the algorithm described in [13] using different conventions. Let $\boldsymbol{\lambda} = (\lambda^0, \ldots, \lambda^{l-1}) \vdash_l n$ be a nonempty cylindric multipartition and let $\mathfrak{B}_{(\mathbf{s},h)} := \mathfrak{B}_{(\mathbf{s},h)}(\boldsymbol{\lambda})$ be an associated shifted symbol. Let $\mathfrak{B}_{(\mathbf{s},h)} = (\mathfrak{B}^0, \ldots, \mathfrak{B}^{l-1})$. From this symbol, we define a new multipartition $\boldsymbol{\lambda}^- \vdash_l n' < n$ together with two integers $j(\boldsymbol{\lambda})$ and $r(\boldsymbol{\lambda})$.

(1) Let $c(\boldsymbol{\lambda})$ be the minimal integer such that $\lambda^{c(\boldsymbol{\lambda})}$ is nonempty. This means that there exists $i(\boldsymbol{\lambda}) \in \mathbb{N}$ such that

$$\mathfrak{B}^{c(\boldsymbol{\lambda})}_{i(\boldsymbol{\lambda})} > \mathfrak{B}^{c(\boldsymbol{\lambda})}_{i(\boldsymbol{\lambda})+1} + 1.$$

Assume that $i(\boldsymbol{\lambda})$ is minimal with this property and set

$$j(\boldsymbol{\lambda}) := \mathfrak{B}^{c(\boldsymbol{\lambda})}_{i(\boldsymbol{\lambda})}.$$

(2) Let $r(\boldsymbol{\lambda}) \in \mathbb{N}$ be maximal such that

$$\mathfrak{B}^{c(\boldsymbol{\lambda})}_{i(\boldsymbol{\lambda})} = \mathfrak{B}^{c(\boldsymbol{\lambda})+1}_{i(\boldsymbol{\lambda})+s_{c(\boldsymbol{\lambda})+1}-s_{c(\boldsymbol{\lambda})}} = \ldots = \mathfrak{B}^{c(\boldsymbol{\lambda})+r(\boldsymbol{\lambda})-1}_{i(\boldsymbol{\lambda})+s_{c(\boldsymbol{\lambda})+r(\boldsymbol{\lambda})-1}-s_{c(\boldsymbol{\lambda})}} = j(\boldsymbol{\lambda}),$$

then we have $r(\boldsymbol{\lambda}) \geq 1$ and because $\boldsymbol{\lambda}$ is cylindric, by Proposition 3.4, we deduce that

$$j(\boldsymbol{\lambda}) > \mathfrak{B}^{c(\boldsymbol{\lambda})+k-1}_{i(\boldsymbol{\lambda})+s_{c(\boldsymbol{\lambda})+k-1}-s_{c(\boldsymbol{\lambda})}+1} + 1$$

for all $k = 1, \ldots, r(\boldsymbol{\lambda})$.

(3) Take the symbol obtained by replacing in $\mathfrak{B}_{(\mathbf{s},h)}$ all the elements

$$\mathfrak{B}^{c(\boldsymbol{\lambda})}_{i(\boldsymbol{\lambda})} = \mathfrak{B}^{c(\boldsymbol{\lambda})+1}_{i(\boldsymbol{\lambda})+s_{c(\boldsymbol{\lambda})+1}-s_{c(\boldsymbol{\lambda})}} = \ldots = \mathfrak{B}^{c(\boldsymbol{\lambda})+r(\boldsymbol{\lambda})-1}_{i(\boldsymbol{\lambda})+s_{c(\boldsymbol{\lambda})+r(\boldsymbol{\lambda})-1}-s_{c(\boldsymbol{\lambda})}} = j(\boldsymbol{\lambda})$$

by $j(\boldsymbol{\lambda}) - 1$. This is a well-defined shifted symbol $\mathfrak{B}'_{(\mathbf{s},h)}$. Thus, there exists a unique l-partition $\boldsymbol{\lambda}^- \vdash_l n - r(\boldsymbol{\lambda})$ such that $\mathfrak{B}_{(\mathbf{s},h)}(\boldsymbol{\lambda}^-) = \mathfrak{B}'_{(\mathbf{s},h)}$.

(4) By construction, $\mathfrak{B}'_{(\mathbf{s},h)}$ is semistandard so $\boldsymbol{\lambda}^-$ is cylindric.

Using this, we can produce a sequence of cylindric multipartitions $\boldsymbol{\lambda}[k]$ with $k = 1, \dots, m+1$ such that $\boldsymbol{\lambda}[1] = \boldsymbol{\lambda}$, $\boldsymbol{\lambda}[m+1] = \boldsymbol{\emptyset}$ and $\boldsymbol{\lambda}[k+1] = \boldsymbol{\lambda}[k]^-$ for $k = 1, \dots, m$. Set $j_k := j(\boldsymbol{\lambda}[k]) - h$ and $a_k := r(\boldsymbol{\lambda}[k])$ for $k = 1, \dots, m$. Then note that we have

$$\boldsymbol{\lambda}[m+1] \overset{j_m:a_m}{\longrightarrow} \boldsymbol{\lambda}[m] \overset{j_{m-1}:a_{m-1}}{\longrightarrow} \dots \overset{j_1:a_1}{\longrightarrow} \boldsymbol{\lambda}[1].$$

We have thus defined two sequences of integers $(j_m, \dots, j_1)$ and $(a_m, \dots, a_1)$. Then we can define

$$a_{\boldsymbol{\lambda}} := f_{j_1}^{(a_1)} f_{j_2}^{(a_2)} \dots f_{j_m}^{(a_m)}.\boldsymbol{\emptyset}.$$

By induction, our construction implies that we have

$$a_{\boldsymbol{\lambda}} = \boldsymbol{\lambda} + \sum_{\boldsymbol{\lambda} \rhd \boldsymbol{\mu}} b_{\boldsymbol{\mu},\boldsymbol{\lambda}}(v) \boldsymbol{\mu}$$

for elements $b_{\boldsymbol{\mu},\boldsymbol{\lambda}}(v) \in \mathbb{N}[v, v^{-1}]$ (the proof is exactly the same as in [6, Prop. 4.6].) Note that if $b_{\boldsymbol{\mu},\boldsymbol{\lambda}}(v) \neq 0$ then the multiset of elements appearing in $\mathfrak{B}_{(\mathbf{s},h)}(\boldsymbol{\mu})$ is the same as the one of $\mathfrak{B}_{(\mathbf{s},h)}(\boldsymbol{\lambda})$ (for h large enough). When $\boldsymbol{\lambda}$ runs over the set of all cylindric multipartitions, these elements provide a basis for $V(\boldsymbol{s})$ and an algorithm for the computation of the canonical basis (see below or [5, Ch. 6]). This, in turn, implies that if $d_{\boldsymbol{\mu},\boldsymbol{\lambda}}(v) \neq 0$ then the multiset of elements appearing in $\mathfrak{B}_{(\mathbf{s},h)}(\boldsymbol{\mu})$ is the same as the one of $\mathfrak{B}_{(\mathbf{s},h)}(\boldsymbol{\lambda})$ (for h large enough) and we have

$$b_{\boldsymbol{\lambda}} = \boldsymbol{\lambda} + \sum_{\boldsymbol{\lambda} \rhd \boldsymbol{\mu}} d_{\boldsymbol{\mu},\boldsymbol{\lambda}}(v) \boldsymbol{\mu},$$

and all the multipartitions appearing in this decomposition have the same rank. Alternatively, these properties are a consequence of the fact that both the $a_{\boldsymbol{\lambda}}$'s and the $b_{\boldsymbol{\lambda}}$'s are weight vectors, and that weight spaces of the Fock spaces are parametrized by symbols with a fixed distribution of entries.

Example 4.7. Let $\mathbf{s} = (0, 0, 1)$ and $\boldsymbol{\lambda} = (2.2, 2.2, 2.2.1)$, for $h = 3$, we get the following symbol:

$$\mathfrak{B}_{((0,0,1),3)}(\boldsymbol{\lambda}) = \begin{pmatrix} 0 & 2 & 4 & 5 \\ 0 & 3 & 4 & \\ 0 & 3 & 4 & \end{pmatrix}$$

which is semistandard. We begin with $c(\boldsymbol{\lambda}) = 0$ and $j(\boldsymbol{\lambda}) = 3$. Then we have $r(\boldsymbol{\lambda}) = 2$. The new symbol is

$$\begin{pmatrix} 0 & 2 & 4 & 5 \\ 0 & 2 & 4 & \\ 0 & 2 & 4 & \end{pmatrix}$$

which is the symbol $\mathfrak{B}_{((0,0,1),3)}(\boldsymbol{\lambda}[2])$ with $\boldsymbol{\lambda}[2] = (2.1, 2.1, 2.2.1)$. Then we have $c(\boldsymbol{\lambda}[2]) = 0$, $j(\boldsymbol{\lambda}[2]) = 4$, $r(\boldsymbol{\lambda}[2]) = 3$ and we get the symbol

$$\begin{pmatrix} 0 & 2 & 3 & 5 \\ 0 & 2 & 3 & \\ 0 & 2 & 3 & \end{pmatrix}$$

which is the symbol $\mathfrak{B}_{((0,0,1),3)}(\boldsymbol{\lambda}[3])$ with $\boldsymbol{\lambda}[3] = (1.1, 1.1, 2.1.1)$. Continuing in this way, keeping the notation of the above remark, we obtain

$$\boldsymbol{\lambda}[4] = (1.1, 1.1, 1.1.1), \qquad \boldsymbol{\lambda}[5] = (1, 1, 1.1), \qquad \boldsymbol{\lambda}[6] = (\emptyset, \emptyset, 1), \qquad \boldsymbol{\lambda}[7] = \boldsymbol{\emptyset}$$

and we have

$$\emptyset \xrightarrow[(2)]{1:1} (\emptyset,\emptyset,1) \xrightarrow[(0,1,2)]{0:3} (1,1,1.1) \xrightarrow[(0,1,2)]{-1:3} (1.1,1.1,1.1.1) \xrightarrow[(2)]{2:1} (1.1,1.1,2.1.1)$$

$$(1.1,1.1,2.1.1) \xrightarrow[(0,1,2)]{1:3} (2.1,2.1,2.2.1) \xrightarrow[(0,1)]{0:2} (2.2,2.2,2.2.1).$$

4.8. Following [14, § 6.25], the above construction provides an algorithm for the computation of the canonical basis of $V(\boldsymbol{s})$: The (modified) LLT algorithm. This is done recursively as follows. Let $n \in \mathbb{N}_{>0}$. For all $k < n$, assume that we have constructed all the canonical basis elements

$$\{b_{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda} \in \Phi_{\boldsymbol{s}}(k)\}.$$

Now, let $\boldsymbol{\lambda} \in \Phi_{\boldsymbol{s}}(n)$. We want to compute $b_{\boldsymbol{\lambda}}$.

(1) We set $c_{\boldsymbol{\lambda}} = f_{j(\boldsymbol{\lambda})-h}^{(r(\boldsymbol{\lambda}))} b_{\boldsymbol{\lambda}^-}$ and we have $\overline{c_{\boldsymbol{\lambda}}} = c_{\boldsymbol{\lambda}}$.

(2) The element $b_{\boldsymbol{\lambda}^-}$ is known by induction. Again, by the above construction, we have

$$c_{\boldsymbol{\lambda}} = \boldsymbol{\lambda} + \sum_{\boldsymbol{\lambda} \rhd \boldsymbol{\nu}} \widehat{d}_{\boldsymbol{\nu},\boldsymbol{\lambda}}(v)\boldsymbol{\nu},$$

for Laurent polynomials $\widehat{d}_{\boldsymbol{\nu},\boldsymbol{\lambda}}(v)$ with $\widehat{d}_{\boldsymbol{\lambda},\boldsymbol{\lambda}}(v) = 1$ and we obtain

$$b_{\boldsymbol{\lambda}} = c_{\boldsymbol{\lambda}} - \sum_{\boldsymbol{\lambda} \rhd \boldsymbol{\nu}} \alpha_{\boldsymbol{\nu},\boldsymbol{\lambda}}(v) b_{\boldsymbol{\nu}},$$

for some Laurent polynomials $\alpha_{\boldsymbol{\nu},\boldsymbol{\lambda}}(v)$ such that $\alpha_{\boldsymbol{\nu},\boldsymbol{\lambda}}(v) = \alpha_{\boldsymbol{\nu},\boldsymbol{\lambda}}(v^{-1})$.

(3) We find the greatest multipartition $\boldsymbol{\nu} \neq \boldsymbol{\lambda}$ with respect to $\rhd$ such that $\widehat{d}_{\boldsymbol{\nu},\boldsymbol{\lambda}}(v) \neq 0$. If no such multipartition exists, then we have $c_{\boldsymbol{\lambda}} = b_{\boldsymbol{\lambda}}$. Otherwise, $\alpha_{\boldsymbol{\nu},\boldsymbol{\lambda}}(v)$ is the unique bar-invariant Laurent polynomial such that the coefficients of $\alpha_{\boldsymbol{\nu},\boldsymbol{\lambda}}(v)$ and $\widehat{d}_{\boldsymbol{\nu},\boldsymbol{\lambda}}(v)$ associated to the v^i with $i \leq 0$ are the same. We then replace $c_{\boldsymbol{\lambda}}$ with the bar-invariant element $c_{\boldsymbol{\lambda}} - \alpha_{\boldsymbol{\nu},\boldsymbol{\lambda}}(v) b_{\boldsymbol{\nu}}$ and we repeat the last step until all of the coefficients $\widehat{d}_{\boldsymbol{\nu},\boldsymbol{\lambda}}(v)$ belong to $v\mathbb{Z}[v]$ for $\boldsymbol{\lambda} \neq \boldsymbol{\nu}$.

5. The main results

The first main result of this paper is the following theorem concerning the computation of the coefficients of the canonical basis elements for irreducible highest weight $\mathcal{U}_v(\mathfrak{sl}_\infty)$-modules.

Theorem 5.1. *Assume that $\boldsymbol{\mu}$ is a fixed cylindric l-partition of rank n. Then for any l-partition $\boldsymbol{\lambda}$ of rank n, we have $d_{\boldsymbol{\lambda},\boldsymbol{\mu}}(v) = 0$ unless $\boldsymbol{\mu} \trianglerighteq \boldsymbol{\lambda}^R$ while $d_{\boldsymbol{\lambda},\boldsymbol{\lambda}^R}(v) = v^{R(\boldsymbol{\lambda})}$.*

The strategy for the proof of this theorem is modeled on the one presented in [4]. However, the proofs of the preparatory results we need here are of course different. The first lemma is the analogue of [4, Lemma 2.3].

Lemma 5.2. *Let $\boldsymbol{\lambda} = (\lambda^0, \ldots, \lambda^{l-1}) \vdash_l n$ and $\boldsymbol{\mu} = (\mu^0, \ldots, \mu^{l-1}) \vdash_l n$. Assume that $\boldsymbol{\mu}$ is cylindric and that $\boldsymbol{\mu}^- \xrightarrow{j:k} \boldsymbol{\mu}$. Let $\boldsymbol{\nu}$ be a partition of $n-k$ such that $\boldsymbol{\nu} \xrightarrow{j:k} \boldsymbol{\lambda}$ and $\boldsymbol{\mu}^- \trianglerighteq \boldsymbol{\nu}^R$. Then $\boldsymbol{\mu} \trianglerighteq \boldsymbol{\lambda}^R$ with equality only if $\boldsymbol{\mu}^- = \boldsymbol{\nu}^R$.*

Proof. Assume that we have

$$\mu^0 = \ldots = \mu^{c(\boldsymbol{\mu})-1} = \emptyset \qquad \text{and} \qquad \mu^{c(\boldsymbol{\mu})} \neq \emptyset.$$

Then, by construction, we have

$$\mathfrak{B}_{(\mathbf{s},h)}(\boldsymbol{\mu})^d = (\mathfrak{B}_{(\mathbf{s},h)}(\boldsymbol{\mu}^-)^d \setminus \{j+h-1\})) \cup \{j+h\}$$

for $d = c(\boldsymbol{\mu}), \ldots, c(\boldsymbol{\mu})+k-1$. Similarly, by the definition of the regularization, we have

$$\mathfrak{B}_{(\mathbf{s},h)}(\boldsymbol{\lambda}^R)^d = (\mathfrak{B}_{(\mathbf{s},h)}(\boldsymbol{\nu}^R)^d \setminus \{j+h-1\})) \cup \{j+h\}$$

for elements $d \in \{r_1, \ldots, r_k\} \subset \{0, \ldots, l-1\}$. Since we have $\boldsymbol{\mu}^- \trianglerighteq \boldsymbol{\nu}^R$, we deduce that

$$(\nu^R)^0 = \ldots = (\nu^R)^{c(\boldsymbol{\mu})-1} = \emptyset.$$

We have to check that $r_i \geq c(\boldsymbol{\mu})$ for all $i \in \{1, \ldots, k\}$. So let us assume that we have $r_1 < c(\boldsymbol{\mu})$ (without loss of generality). Thus, we have

$$\mathfrak{B}_{(\mathbf{s},h)}(\boldsymbol{\nu}^R)_1^{r_1} = 0 - 1 + s_{r_1} + h = j + h - 1$$

which implies that $j = s_{r_1}$. Thus, we deduce that $j + h = s_{r_1} + h \in \mathfrak{B}_{(\mathbf{s},h)}(\boldsymbol{\mu})^{c(\boldsymbol{\mu})}$. Since $j - 1 \notin \mathfrak{B}_{(\mathbf{s},h)}(\boldsymbol{\mu})^{c(\boldsymbol{\mu})}$ by the construction of $\boldsymbol{\mu}$, we deduce that $i(\boldsymbol{\mu}) > s_{c(\boldsymbol{\mu})} - s_{r_1} + 1$. However, this implies that $\mathfrak{B}_{(\mathbf{s},h)}(\boldsymbol{\mu})$ is not semistandard. Indeed, in these case we have $\mathfrak{B}_{(\mathbf{s},h)}(\boldsymbol{\mu})_{i(\boldsymbol{\mu})}^{c(\boldsymbol{\mu})} > \mathfrak{B}_{(\mathbf{s},h)}(\boldsymbol{\mu})_{i(\boldsymbol{\mu})+s_{r_1}-s_{c(\boldsymbol{\mu})}}^{r_1}$. This is a contradiction because $\boldsymbol{\mu}$ is cylindric.

Assume now that we have the equality $\boldsymbol{\mu} = \boldsymbol{\lambda}^R$, this implies that we have

$$\{r_1, \ldots, r_k\} = \{c(\boldsymbol{\mu}), \ldots, c(\boldsymbol{\mu}) + k - 1\}.$$

Thus, we also have $\boldsymbol{\mu}^- = \boldsymbol{\nu}^R$ which concludes the proof. □

The second lemma is the analogue of [4, Prop. 2.5].

Lemma 5.3. *Let $\boldsymbol{\lambda}$ be a cylindric multipartition and let $\boldsymbol{\mu}$ be a multipartition such that $\boldsymbol{\mu}^R = \boldsymbol{\lambda}$. Let $\boldsymbol{\mu}^-$ be a multipartition of $n-k$ such that $\boldsymbol{\mu}^- \xrightarrow{j:k} \boldsymbol{\mu}$ for some $j \in \mathbb{Z}$ and $k \in \mathbb{N}$ and such that $(\boldsymbol{\mu}^-)^R = \boldsymbol{\lambda}^-$ then the coefficient of $\boldsymbol{\mu}$ in $f_j^{(k)} \boldsymbol{\mu}^-$ is $v^{R(\boldsymbol{\mu})-R(\boldsymbol{\mu}^-)}$.*

Proof. Consider a shifted symbol $\mathfrak{B}_{(\mathbf{s},h)}(\boldsymbol{\mu}^-)$. Let (i_1, c_1), $\ldots$, (i_m, c_m) be all the elements of $\mathbb{N} \times \{0, 1, \ldots, l-1\}$ such that

$$\mathfrak{B}_{(\mathbf{s},h)}(\boldsymbol{\mu}^-)_{i_1}^{c_1} = \ldots = \mathfrak{B}_{(\mathbf{s},h)}(\boldsymbol{\mu}^-)_{i_m}^{c_m} = j + h - 1.$$

By hypothesis, we have $m \geq k$ and one can assume without loss of generality that

$$\mathfrak{B}_{(\mathbf{s},h)}(\boldsymbol{\mu})_i^c = \begin{cases} j+h & \text{if } (i_t, c_t) = (i, c) \text{ for one } t \in \{1, \ldots, k\}, \\ \mathfrak{B}_{(\mathbf{s},h)}(\boldsymbol{\mu}^-)_i^c & \text{otherwise.} \end{cases}$$

First, we easily see that for all (i, c) such that $\mathfrak{B}_{(\mathbf{s},h)}(\boldsymbol{\mu}^-)_i^c \notin \{j+h-1, j+h\}$, we have

$$R(\boldsymbol{\mu}^-)_{(i,c)} = R(\boldsymbol{\mu})_{(i,c)}.$$

Let now (i, c) be such that

$$\mathfrak{B}_{(\mathbf{s},h)}(\boldsymbol{\mu}^-)_i^c = j + h - 1.$$

- Assume that $c\neq c_t$ for all $t\in\{1,\dots,k\}$. Then we have $\mathfrak{B}_{(\mathbf{s},h)}(\boldsymbol{\mu})^c_i=j+h-1$. We claim that we have
$$R(\boldsymbol{\mu})_{(i,c)}-R(\boldsymbol{\mu}^-)_{(i,c)}=\sharp\{\text{integers } t\in\{1,\dots,k\} \text{ such that } c<c_t\}.$$
We need to show that if $c<c_t$ then we have
$$\mathfrak{B}(\boldsymbol{\mu}^-)^{c_t}_{i+s_{c_t}-s_c}>\mathfrak{B}(\boldsymbol{\mu}^-)^c_i.$$
Assume to the contrary that $\mathfrak{B}(\boldsymbol{\mu}^-)^{c_t}_{i+s_{c_t}-s_c}\leq\mathfrak{B}(\boldsymbol{\mu}^-)^c_i$, then by the process of regularization, in the construction of $\mathfrak{B}_{(\mathbf{s},h)}(\boldsymbol{\lambda}^-)$, the number $\mathfrak{B}_{(\mathbf{s},h)}(\boldsymbol{\mu}^-)^{c_t}_{i_t}=j+h-1$ is sent to a row c' of the symbol which is greater than the row containing $\mathfrak{B}_{(\mathbf{s},h)}(\boldsymbol{\mu}^-)^c_i=j+h-1$. This is impossible by the construction of $\boldsymbol{\lambda}^-$.
- Assume that $c=c_t$ for $t\in\{1,\dots,k\}$. Then we have $\mathfrak{B}_{(\mathbf{s},h)}(\boldsymbol{\mu})^c_i=j+h$. We claim that we have
$$R(\boldsymbol{\mu})_{(i,c)}=R(\boldsymbol{\mu}^-)_{(i,c)}.$$
This comes from the fact that if there exists $c'>c$ such that $c'\neq c_s$ for $s\in\{1,\dots,k\}$ and
$$\mathfrak{B}(\boldsymbol{\mu}^-)^{c'}_{i+s_{c'}-s_{c'}}>\mathfrak{B}(\boldsymbol{\mu}^-)^c_i,$$
then $j+h-1\notin\mathfrak{B}_{(\mathbf{s},h)}(\boldsymbol{\mu}^-)^{c'}$ which follows from the construction of $\boldsymbol{\lambda}^-$.

Now assume that (i,c) is such that
$$\mathfrak{B}_{(\mathbf{s},h)}(\boldsymbol{\mu}^-)^c_i=j+h.$$
We follow the same kind of reasoning as above by showing that if $c_t>c$ then we have
$$\mathfrak{B}(\boldsymbol{\mu}^-)^{c_t}_{i+s_{c_t}-s_{c_t}}>\mathfrak{B}(\boldsymbol{\mu}^-)^c_i.$$
We deduce that
$$R(\boldsymbol{\mu}^-)_{(i,c)}-R(\boldsymbol{\mu})_{(i,c)}=\sharp\{\text{integers } t\in\{1,\dots,k\} \text{ such that } c<c_t\}$$
Now we use the formula in §4.2 to deduce the result. Namely, we have

$$\begin{aligned}
N_j(\boldsymbol{\mu}^-,\boldsymbol{\mu}) &= \sum_{1\leq i\leq k}\big(\sharp\{\text{integers equal to } j+h-1 \text{ in } \mathfrak{B}_{(\mathbf{s},h)}(\boldsymbol{\mu})^c \text{ with } c\leq c_i\}\\
&\qquad\qquad -\sharp\{\text{integers equal to } j+h \text{ in } \mathfrak{B}_{(\mathbf{s},h)}(\boldsymbol{\mu}^-)^c \text{ with } c\leq c_i\}\big)\\
&= \sum_{c\in\{0,\dots,l-1\}}\sharp\{\text{integers } t\in\{1,\dots,k\} \text{ s.t. } c<c_t \text{ and } j+h-1\in\mathfrak{B}_{(\mathbf{s},h)}(\boldsymbol{\mu})^c\ \}\\
&\quad -\sum_{c\in\{0,\dots,l-1\}}\sharp\{\text{integers } t\in\{1,\dots,k\} \text{ s.t. } c<c_t \text{ and } j+h\in\mathfrak{B}_{(\mathbf{s},h)}(\boldsymbol{\mu}^-)^c\ \}\\
&= R(\boldsymbol{\mu})-R(\boldsymbol{\mu}^-).
\end{aligned}$$

□

Remark 5.4. In [12], B. Leclerc and H. Miyachi have given an explicit closed formula for the elements of the canonical bases for irreducible highest weight modules of level $l=2$. This formula also uses the notion of symbols. When $l>2$, these elements are more difficult to compute because they are non-monomial in terms of the Chevalley operators (contrary to the case $l=2$). When $l=2$, it is easy to check that our formulas are consistent with the ones of Leclerc-Miyachi in this particular case.

5.5. Proof of Theorem 5.1. The proof is exactly the same as in [4, Thm. 2.2]. We give it for the convenience of the reader. We argue by induction on $n \in \mathbb{N}$ and on the dominance order. When $n = 0$ or when $\boldsymbol{\mu}$ is minimal with respect to $\trianglerighteq$, there is nothing to do. So let us assume that $n > 0$ and that we have a cylindric multipartition $\boldsymbol{\mu}$ of rank n. By induction, the result holds when $\boldsymbol{\mu}$ is replaced with $\boldsymbol{\mu}^-$. Assume that we have $\boldsymbol{\mu}^- \xrightarrow{j:k} \boldsymbol{\mu}$ for $j \in \mathbb{Z}$ and $k \in \mathbb{N}$. Then we have

$$f_j^{(k)} b_{\boldsymbol{\mu}^-} = \boldsymbol{\mu} + \sum_{\boldsymbol{\mu} \triangleright \boldsymbol{\lambda}} \widehat{d}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(v) \boldsymbol{\lambda}.$$

We will prove the property of the theorem for the numbers $\widehat{d}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(v)$ and then deduce the result for the numbers $d_{\boldsymbol{\lambda},\boldsymbol{\mu}}(v)$. Assume that $\widehat{d}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(v) \neq 0$. Then there exists $\boldsymbol{\nu} \vdash_l n - k$ such that $d_{\boldsymbol{\nu},\boldsymbol{\mu}^-}(v) \neq 0$ and such that $\boldsymbol{\nu} \xrightarrow{j:k} \boldsymbol{\lambda}$. By induction, this implies that $\boldsymbol{\mu}^- \trianglerighteq \boldsymbol{\nu}^R$. We can then apply Lemma 5.2 which implies that $\boldsymbol{\mu} \trianglerighteq \boldsymbol{\lambda}^R$.

Now let us assume that $\boldsymbol{\lambda}^R = \boldsymbol{\mu}$. If $\boldsymbol{\nu} \vdash_l n$ is such that $d_{\boldsymbol{\nu},\boldsymbol{\mu}^-}(v) \neq 0$ and $\boldsymbol{\nu} \xrightarrow{j:k} \boldsymbol{\lambda}$ then by induction, we have $\boldsymbol{\nu}^R \trianglerighteq \boldsymbol{\mu}^-$. Then we obtain $\boldsymbol{\nu}^R = \boldsymbol{\mu}^-$. Thus, the coefficient $\widehat{d}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(v)$ equals $\widehat{d}_{\boldsymbol{\nu},\boldsymbol{\nu}^R}(v)$ times the coefficient of $\boldsymbol{\lambda}$ in $f_j^{(m)} \boldsymbol{\nu}$. By Lemma 5.3 and by induction, this coefficient is $v^{R(\boldsymbol{\lambda})}$. Hence, we have shown that $\widehat{d}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(v)$ is zero unless $\boldsymbol{\mu} \trianglerighteq \boldsymbol{\lambda}^R$ and $\widehat{d}_{\boldsymbol{\lambda},\boldsymbol{\lambda}^R}(v) = v^{R(\boldsymbol{\lambda})}$. The LLT algorithm in §4.8 implies that

$$d_{\boldsymbol{\lambda},\boldsymbol{\mu}}(v) = \widehat{d}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(v) + \sum_{\xi \triangleleft \boldsymbol{\mu}} \alpha_{\xi,\boldsymbol{\mu}}(v) d_{\boldsymbol{\lambda},\xi}(v).$$

Now assume that $\boldsymbol{\mu}$ is not greater than $\boldsymbol{\lambda}^R$ with respect to the dominance order. This implies that for all $\xi \vdash_l n$ such that $\xi \triangleleft \boldsymbol{\mu}$, ξ is not greater than $\boldsymbol{\lambda}$. By induction, we have $d_{\lambda,\xi}(v) = 0$. Thus, we obtain $d_{\boldsymbol{\lambda},\boldsymbol{\mu}}(v) = \widehat{d}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(v)$ and the result follows. This concludes the proof.

6. Regularization for Ariki–Koike algebras

We now present consequences for the representation theory of Ariki–Koike algebras.

6.1. Let $\mathbf{s} = (s_0, \ldots, s_{l-1}) \in \mathcal{S}^l$ and let $\eta \in \mathbb{C}$. We consider the associative $\mathbb{C}$-algebra $\mathcal{H}(\boldsymbol{s})$ generated by $T_0, \ldots, T_{n-1}$ subject to the relations $(T_0 - \eta^{s_0}) \ldots (T_0 - \eta^{s_{l-1}}) = 0$, $(T_i - \eta)(T_i + 1) = 0$, for $1 \leq i \leq n$ and the type B braid relations

$$(T_0T_1)^2 = (T_1T_0)^2, \quad T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1} \quad \text{for } 1 \leq i < n,$$
$$T_iT_j = T_jT_i \quad \text{for } j \geq i + 2.$$

$\mathcal{H}(\boldsymbol{s})$ is called *the Ariki–Koike algebra.* Set $e = +\infty$ if η is not a root of 1. Otherwise, e is the order of η as a root of 1. We assume that $e \neq 1$. $\mathcal{H}(\boldsymbol{s})$ is non-semisimple in general and its representation theory is usually studied through its decomposition matrix which we now define. Following Dipper, James and Mathas [2], one can construct a certain set of finite dimensional $\mathcal{H}(\boldsymbol{s})$-modules called *Specht modules.* We denote them by

$$\left\{ S^{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda} \vdash_l n \right\}.$$

The simple $\mathcal{H}(\boldsymbol{s})$-modules are indexed by a certain set of multipartitions called FLOTW multipartitions $\Phi_{\boldsymbol{s},e}(n)$ (see [5, § 5.7])

$$\left\{ D^{\boldsymbol{\mu}} \mid \boldsymbol{\mu} \in \Phi_{\boldsymbol{s},e}(n) \right\}.$$

If $e=\infty$, we have $\Phi_{\boldsymbol{s},e}(n)=\Phi_{\boldsymbol{s}}(n)$, the set of cylindric multipartitions.

6.2. If $\boldsymbol{\lambda} \vdash_l n$ then one can consider the composition multiplicities $[S^{\boldsymbol{\lambda}} : D^{\boldsymbol{\mu}}]$, with $\boldsymbol{\mu} \in \Phi_{\boldsymbol{s},e}(n)$. The resulting matrix,

$$\mathcal{D} := ([S^{\boldsymbol{\lambda}} : D^{\boldsymbol{\mu}}])_{\boldsymbol{\lambda} \vdash_l n, \boldsymbol{\mu} \in \Phi_{\boldsymbol{s},e}(n)},$$

is called the decomposition matrix of $\mathcal{H}(\boldsymbol{s})$. The problem of computing the decomposition matrix has been solved by Ariki [1] by proving a generalization of a conjecture by Lascoux, Leclerc and Thibon. The theorem asserts that the decomposition numbers $[S^{\boldsymbol{\lambda}} : D^{\boldsymbol{\mu}}]$ with $\boldsymbol{\mu} \in \Phi_{\boldsymbol{s},e}(n)$ corresponds to the coefficients of the canonical bases for an irreducible highest weight $\mathcal{U}_v(\widehat{\mathfrak{sl}}_e)$-module (realized as a submodule of the Fock space) evaluated at $v=1$.

In the case $e=+\infty$, this decomposition numbers are thus the polynomials $d_{\boldsymbol{\lambda},\boldsymbol{\mu}}(v)$ evaluated at $v=1$. We obtain the following result which is deduced from the above discussion and Theorem 5.1.

Theorem 6.3. *Assume that $e=+\infty$ and that $\boldsymbol{\mu}$ is a fixed cylindric l-partition of rank n. Then, for any l-partition $\boldsymbol{\lambda}$ of rank n, we have $[S^{\boldsymbol{\lambda}} : D^{\boldsymbol{\mu}}]=0$ unless $\boldsymbol{\mu} \trianglerighteq \boldsymbol{\lambda}^R$ while $[S^{\boldsymbol{\lambda}} : D^{\boldsymbol{\lambda}^R}]=1$.*

By results of several authors, the polynomials $d_{\boldsymbol{\lambda},\boldsymbol{\mu}}(v)$ also have an interpretation in terms of the representation theory of Ariki–Koike algebras. They correspond to graded decomposition numbers (see [10]). Thus, Theorem 5.1 can be interpreted as a graded analogue of the above regularization theorem.

6.4. It is natural to ask what happens in the case where $e \in \mathbb{N}$. Here, the main problem is to find a natural order on the set of multipartitions which is the analogue of the dominance order on partitions. A natural choice for it is the one used in [7]. Now, the decomposition matrices for Ariki–Koike algebras can be computed using the algorithm described in this paper and implemented in [8]. Let $e=2$, $l=2$, $\mathbf{s}=(0,1)$ and $n=6$ then for the 2-partitions $\boldsymbol{\lambda}=(3,3)$, $\boldsymbol{\mu}^1=(4,2)$ and $\boldsymbol{\mu}^2=(2,4)$, we have $[S^{\boldsymbol{\lambda}} : D^{\boldsymbol{\mu}^1}]=[S^{\boldsymbol{\lambda}} : D^{\boldsymbol{\mu}^2}]=1$ and there are no 2-partition $\boldsymbol{\nu}$ such that $[S^{\boldsymbol{\lambda}} : D^{\boldsymbol{\nu}}] \neq 0$ which are less than $\boldsymbol{\mu}^1$ and $\boldsymbol{\mu}^2$. Thus, an analogue of Theorem 6.3 is not available for these choices.

References

[1] S. Ariki, *Representations of quantum algebras and combinatorics of Young tableaux,* University Lecture Series, vol. 26, American Mathematical Society, Providence, RI, 2002. Translated from the 2000 Japanese edition and revised by the author.

[2] R. Dipper, G. James, A. Mathas, *Cyclotomic q-Schur Algebras*, Mathematische Zeitschrift **229** (1998), 385–416.

[3] M. Fayers, *Regularisation and the Mullineux map*, Electron. J. Combin. **15**(1) (2008), Research Paper 142, 15 pp.

[4] M. Fayers, *q-analogues of regularisation theorems for linear and projective representations of the symmetric group*, J. Algebra **316** (2007), 346–367.

[5] M. Geck and N. Jacon, *Representations of Hecke algebras at roots of unity,* Algebra and Applications, vol. 15, Springer-Verlag, London, 2011.

[6] N. Jacon, *On the parametrization of the simple modules for Ariki–Koike algebras at roots of unity,* J. Math. Kyoto Univ. **44** (2004), 729–767.

[7] N. Jacon, *An algorithm for the computation of the decomposition matrices for Ariki–Koike algebras*, J. Algebra **292** (2005), 100–109.
[8] N. Jacon, *GAP program for the computation of the decomposition matrices for Ariki–Koike algebras*, https://lmb.univ-fcomte.fr/IMG/zip/jacon_arikikoike.g.zip
[9] G. James, *On the decomposition matrices of the symmetric groups II*, J. Algebra **43** (1976), 45–54.
[10] A. Kleshchev, *Representation theory of symmetric groups and related Hecke algebras*, Bull. Amer. Math. Soc. **47** (2010), 419–481
[11] A. Lascoux, B. Leclerc and J-Y Thibon, *Hecke algebras at roots of unity and crystal bases of quantum affine algebras*, Comm. Math. Phys. **181** (1996), 205–263.
[12] B. Leclerc and H. Miyachi, *Constructible characters and canonical bases*, J. Algebra **277** (2004), 298–317.
[13] B. Leclerc and P Toffin, *A simple algorithm for computing the global crystal basis of an irreducible $U_q(\mathrm{sl}_n)$-module*, Internat. J. Algebra Comput. **10** (2000), 191–208.
[14] A. Mathas, *Iwahori-Hecke algebras and Schur algebras of the symmetric group*, University lecture series, 15, American Mathematical Society (Providence, R.I.), 1999.

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