Séminaire Lotharingien de Combinatoire **69** (2013), Article B69d

ON THE **c**-VECTORS AND **g**-VECTORS OF THE MARKOV CLUSTER ALGEBRA

ALFREDO NÁJERA CHÁVEZ

Abstract. We describe the **c**-vectors and **g**-vectors of the Markov cluster algebra in terms of a special family of triples of rational numbers, which we call the Farey triples.

Introduction

In papers such as [3, 5, 9, 10, 13] (to mention just a few), the authors present (as examples and propositions) interesting properties of the cluster algebra arising from the quiver of Figure 1, as well as of the potentials, Jacobian algebras and cluster categories associated to it.

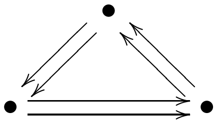


Figure 1. Double cyclic triangle

The cluster algebra arising from this quiver is frequently called the *Markov cluster algebra*. In the theory of cluster algebras associated to triangulated surfaces (see [7]), it can be regarded as the cluster algebra arising from a torus with one puncture. When this algebra has principal coefficients, we denote it by $\mathcal{A}_M$. In this paper, we describe the (extended) exchange matrices of $\mathcal{A}_M$ and thus the set of its **c**-vectors, which can be considered as a generalization of a root system. In [11] and [12, Theorem 1.2], the authors show how the **c**-vectors of a cluster algebra associated to a quiver are related to its **g**-vectors. Thus, in our case, we can also obtain the **g**-vectors. Section 1 is a reminder on cluster algebras with principal coefficients. In Section 2 we introduce the Farey triples, which are triples of rational numbers satisfying a specific arithmetic condition. We define a mutation operation on Farey triples and show that the resulting *exchange graph* is a 3-regular tree $\mathbb{T}_3$. In Section 3 we associate to each Farey triple T an exchange matrix M_T of $\mathcal{A}_M$, and write the entries of M_T in terms of the components of T.

Acknowledgments

I would like to express my sincere thanks to my master advisor, Professor Christof Geiss Hahn, for his support during my studies and for introducing me to the theory of cluster algebras. I am grateful to my current advisor, Professor Bernhard Keller, for helpful comments on previous versions of this article.

1. PRELIMINARIES ON CLUSTER ALGEBRAS

1.1. **Cluster algebras with principal coefficients.** In this section, we recall the construction of cluster algebras with principal coefficients from [8]. For an integer x, we use the notations

$$[x]_+ = \max(x, 0)$$

and

$$\operatorname{sgn}(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

Definition 1.1.1. The *tropical semifield* on a finite family of variables u_j, $j \in J$, is the Abelian group (written multiplicatively) $\operatorname{Trop}(u_j : j \in J)$ freely generated by the u_j, $j \in J$. It is endowed with an auxiliary addition $\oplus$ defined by

$$\prod_j u_j^{a_j} \oplus \prod_j u_j^{b_j} = \prod_j u_j^{\min(a_j, b_j)}. \tag{1.1}$$

Remark 1.1.2. From now on, we let n be a positive integer and $\mathbb{P}$ be the tropical semifield on the indeterminates $x_{n+1}, \ldots, x_{2n}$. Notice that $\mathbb{QP}$, the group algebra on the Abelian group $\mathbb{P}$, is naturally identified with the algebra of Laurent polynomials in the variables $x_{n+1}, \ldots, x_{2n}$ with rational coefficients. We denote by $\mathcal{F}$ the field of fractions of the ring of polynomials with coefficients in $\mathbb{QP}$ in n indeterminates.

Definition 1.1.3. A *seed* in $\mathcal{F}$ is a pair $(\tilde{B}, \mathbf{x})$ formed by

- a $2n \times n$ integer matrix $\tilde{B} = (b_{ij})$, whose principal part B, i.e., the top $n \times n$ submatrix, is skew-symmetric (equivalently, associated to a quiver Γ with no oriented cycles of length ≤ 2 via the formula

  $$b_{ij} = |\text{number of arrows } i \to j \text{ in } \Gamma| - |\text{number of arrows } j \to i \text{ in } \Gamma|$$

  for $1 \leq i, j \leq n$);
- a free generating set $\mathbf{x} = \{x_1, \ldots, x_n\}$ of the field $\mathcal{F}$.

The matrix $\tilde{B}$ is called the *(extended) exchange matrix* and the set $\mathbf{x}$ the *cluster* of the seed $(\tilde{B}, \mathbf{x})$.

Definition 1.1.4. Let $(\tilde{B}, \mathbf{x})$ be a seed in $\mathcal{F}$, and let k be an integer with $1 \leq k \leq n$. The *seed mutation* μ_k in direction k transforms $(\tilde{B}, \mathbf{x})$ into the seed $\mu_k(\tilde{B}, \mathbf{x}) = (\tilde{B}', \mathbf{x}')$, which is defined as follows:

- The entries of $\tilde{B}' = (b'_{ij})$ are given by

  $$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k; \\ b_{ij} + \operatorname{sgn}(b_{ik})[b_{ik}b_{kj}]_+ & \text{otherwise.} \end{cases} \tag{1.2}$$

- The cluster $\mathbf{x}' = \{x'_1, \ldots, x'_n\}$ is given by $x'_j = x_j$ for $j \neq k$, whereas $x'_k \in \mathcal{F}$ is determined by the *exchange relation*

  $$x'_k x_k = \prod_{i=1}^{2n} x_i^{[b_{ik}]_+} + \prod_{i=1}^{2n} x_i^{[-b_{ik}]_+}. \tag{1.3}$$

Remark 1.1.5. Mutation in a fixed direction is an involution, i.e., $\mu_k\mu_k(\tilde{B},\mathbf{x}) = (\tilde{B},\mathbf{x})$.

Definition 1.1.6. Let $\mathbb{T}_n$ be the n-regular tree whose edges are labeled by the numbers $1,\dots,n$ so that the n edges emanating from each vertex carry different labels. A *cluster pattern* is the assignment of a seed $(\tilde{B}_t,\mathbf{x}_t)$ to each vertex t of $\mathbb{T}_n$ such that the seeds assigned to vertices t and t', linked by an edge with label k, are obtained from each other by the seed mutation μ_k. Fix a vertex t_0 of the n-regular tree $\mathbb{T}_n$. Clearly, a cluster pattern is uniquely determined by the initial seed $(\tilde{B}_{t_0}, x_{t_0})$, which can be chosen arbitrarily. The **c**-*vectors* of a seed $(\tilde{B}_t,\mathbf{x}_t)$ are the elements of the set $\{\mathbf{c}_{j;t} = (b^t_{n+1j},\dots,b^t_{2nj}) \in \mathbb{Z}^n : 1 \le j \le n\}$, where $\tilde{B}_t = (b^t_{ij})$.

Definition 1.1.7. Fix a seed $(\tilde{B},\mathbf{x})$ and let $(\tilde{B}_t,\mathbf{x}_t)$, $t \in \mathbb{T}_n$, be the unique cluster pattern with initial seed $(\tilde{B},\mathbf{x})$. The *clusters* associated with $(\tilde{B},\mathbf{x})$ are the sets $\mathbf{x}_t$, $t \in \mathbb{T}_n$. The *cluster variables* are the elements of the clusters. The *cluster algebra* $\mathcal{A}(\tilde{B}) = \mathcal{A}(\tilde{B},\mathbf{x})$ is the $\mathbb{ZP}$-subalgebra of $\mathcal{F}$ generated by the cluster variables. Its *ring of coefficients* is $\mathbb{ZP}$. We say that the cluster algebra $\mathcal{A}(\tilde{B})$ has *principal coefficients* at the vertex t_0 if the complementary part of $\tilde{B}_{t_0}$ (i.e., the bottom $n \times n$ submatrix) is the $n \times n$ identity matrix.

1.2. **The g-vectors.** In this section we recall the natural $\mathbb{Z}^n$-grading in a cluster algebra with principal coefficients. This leads us to the definition of the **g**-vectors.

Definition 1.2.1. Let $\mathcal{A}(\tilde{B},\mathbf{x})$ be the cluster algebra with principal coefficients at a vertex t_0, defined by the initial seed $(\tilde{B},\mathbf{x}) = ((b_{ij}),\{x_1,\dots,x_n\})$. The *(natural) $\mathbb{Z}^n$-grading* of the ring $\mathbb{Z}[x_1^{\pm1},\dots,x_n^{\pm1};x_{n+1},\dots,x_{2n}]$ is given by

$$\deg(x_j) = \begin{cases} \mathbf{e}_j, & \text{if } 1 \le j \le n, \\ -\mathbf{b}_j, & \text{if } n+1 \le j \le 2n, \end{cases} \tag{1.4}$$

where $\mathbf{e}_1,\dots,\mathbf{e}_n$ are the standard basis (column) vectors in $\mathbb{Z}^n$, and $\mathbf{b}_j = \Sigma_i b_{ij}\mathbf{e}_i$.

Theorem 1.2.2 ([8, Corollary 6.3])**.** *Under the $\mathbb{Z}^n$-grading given by* (1.4)*, the cluster algebra $\mathcal{A}(\tilde{B},\mathbf{x})$ is a $\mathbb{Z}^n$-graded subalgebra of* $\mathbb{Z}[x_1^{\pm1},\dots,x_n^{\pm1};x_{n+1},\dots,x_{2n}]$.

Definition 1.2.3. By iterating the exchange relation (1.3), we can express every cluster variable $x_{j;t}$ as a (unique) rational function in $x_1,\dots,x_{2n}$ given by a subtraction-free rational expression; we denote this rational function by $X_{j;t}$. The **g**-vectors of a seed $(\tilde{B}_t,\mathbf{x}_t$ are

$$\mathbf{g}_{j;t} = \begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix} = \deg(X_{j;t}), \tag{1.5}$$

we call a matrix of the form $(\mathbf{g}_{1;t},\dots,\mathbf{g}_{n;t})$ a **g**-matrix.

2. Farey triples

2.1. **The Farey sum.** In this section we introduce some basic properties of the Farey triples, which are also useful in the study of tubular cluster algebras (see [1, 2]). Let $\mathbb{Q}_\infty = \mathbb{Q} \cup \{\infty\}$ be the totally ordered set of extended rational numbers.

Notation 2.1.1. For $q \in \mathbb{Q}_\infty$, we denote by $d(q)$ and $r(q)$ the integers defined by:

- $q = \dfrac{d(q)}{r(q)}$,
- $\gcd(d(q), r(q)) = 1$,
- $r(q) \geq 0$.

It is obvious that $d(q)$ and $r(q)$ are uniquely determined. In particular, $0 = \frac{0}{1}$ and $\infty = \frac{1}{0}$.

Definition 2.1.2. Given $q, q' \in \mathbb{Q}_\infty$ we define

$$\Delta(q, q') := \left| \det \begin{pmatrix} d(q) & d(q') \\ r(q) & r(q') \end{pmatrix} \right| \tag{2.1}$$

In case that $\Delta(q, q') = 1$, we call q and q' *Farey neighbors.*

Definition 2.1.3. Let $q, q' \in \mathbb{Q}_\infty$ be Farey neighbors. The *Farey sum* of q and q' is

$$q \oplus q' := \frac{d(q) + d(q')}{r(q) + r(q')}, \tag{2.2}$$

and their *Farey difference* is

$$q \ominus q' := \frac{d(q) - d(q')}{r(q) - r(q')}. \tag{2.3}$$

Note that both of these operations are commutative.

Definition 2.1.4. We call $(q_1, q_2, q_3) \in \mathbb{Q}^3_\infty$ a *triple of neighbors* provided that $\Delta(q_i, q_j) = 1$ for $1 \leq i < j \leq 3$. A *Farey triple* $[q_1, q_2, q_3]$ is a triple of neighbors considered up to permutation of its components. We use the indices $\{f, s, t\} = \{1, 2, 3\}$ for a Farey triple to indicate that $q_f < q_s < q_t$.

Lemma 2.1.5. *Let $(q_1, q_2, q_3) \in \mathbb{Q}^3_\infty$ and $q_1 < q_2 < q_3$. Then (q_1, q_2, q_3) is a triple of neighbors if and only if $q_2 = q_1 \oplus q_3$, i.e., the Farey triples are essentially of the form $[q_f, q_f \oplus q_t, q_t]$.*

Proof. Fix q_1 and q_3 and consider the following system of equations in $d(q_2)$ and $r(q_2)$:

$$\begin{aligned} d(q_3)r(q_2) - r(q_3)d(q_2) &= 1, \\ -d(q_1)r(q_2) + r(q_1)d(q_2) &= 1. \end{aligned}$$

The determinant of the system is $\Delta(q_1, q_3) = 1$, so it has a unique solution, which is $d(q_2) = d(q_1) + d(q_3)$ and $r(q_2) = r(q_1) + r(q_3)$. □

Lemma and Definition 2.1.6. *For all $q \in \mathbb{Q}$, there exist two unique Farey neighbors $q', q'' \in \mathbb{Q}_\infty$ such that $q = q' \oplus q''$. We call this relation the Farey decomposition of q.*

Proof. Consider the Diophantine equation $d(q)y - r(q)x = 1$. Let $x = a$, $y = b$ be a solution of the equation with $0 \leq b < r(q)$. Put $q' = \frac{a}{b}$ and $q'' = q \ominus q'$, then $q = q' \oplus q''$. To see the uniqueness of q' and q'', suppose that $q = p \oplus p'$ is another decomposition. Without loss of generality we can assume that $p, q' < q < p', q''$. Since $\Delta(q, p) = \Delta(q, q') = 1$ and $p, q' < q$, it follows that $d(q)(r(q') - r(p)) =$

$r(q)(d(q')-d(p))$. In particular, we have $q' \neq p$ if and only if $r(q') \neq r(p)$. Suppose that $q' \neq p$. Then we have

$$\frac{d(q)}{r(q)} = \frac{d(q')-d(p)}{r(q')-r(p)},$$

but $-r(q) < r(q') - r(p) < r(q)$, a contradiction. Hence $q' = p$. We do the same for $q'' = p'$. $\square$

Definition 2.1.7. Let $[q_f, q_s, q_t]$ be a Farey triple (with $q_f < q_s < q_t$; see Definition 2.1.4) and $k \in \{f, s, t\}$. The *mutation* μ_k in direction k of $[q_f, q_s, q_t]$ is the Farey triple $\mu_k[q_f, q_s, q_t]$ defined by

$$\mu_k[q_f, q_s, q_t] = \begin{cases} [q_s, q_s \oplus q_t, q_t], & \text{if } k = f, \\ [q_f, q_f \ominus q_t, q_t], & \text{if } k = s, \\ [q_f, q_f \oplus q_s, q_s], & \text{if } k = t. \end{cases} \tag{2.4}$$

In view of Lemma 2.1.6, we infer that the mutation is an involution, i.e., we have $\mu_k\mu_k[q_f, q_s, q_t] = [q_f, q_s, q_t]$ for each k.

Definition 2.1.8. The *exchange graph* of the Farey triples is the graph whose vertices are the Farey triples, and whose edges connect triples related by a single mutation.

Proposition 2.1.9. *The exchange graph of the Farey triples is a* 3*-regular tree* $\mathbb{T}_3$*.*

Proof. First we show that the graph is connected. It is sufficient to show that any Farey triple is connected with the triple $[\frac{-1}{1}, \frac{0}{1}, \frac{1}{0}]$. Call this triple the *initial triple* and all others *non-initial triples*. For any Farey triple $[q_f, q_s, q_t]$, define its *complexity* as $c[q_f, q_s, q_t] = |d(q_s)|+r(q_s)$. Since $c[q_f, q_s, q_t] = 1$ implies that $[q_f, q_s, q_t] = [\frac{-1}{1}, \frac{0}{1}, \frac{1}{0}]$, it will be sufficient to show that, for any non-initial triple, there is a (unique) direction $k \in \{f, s, t\}$ such that $c(\mu_k[\frac{-1}{1}, \frac{0}{1}, \frac{1}{0}]) < c[q_f, q_s, q_t]$. If $c[q_f, q_s, q_t] = [\frac{-z-1}{1}, \frac{-z}{1}, \frac{1}{0}]$ with $z \geq 1$, then take $k = f$. In any other case we have either $q_i \geq 0$ or $q_i \leq 0$ for all i, so $k = s$ works. It only remains to show that the graph does not contain cycles. Since the complexity of a triple always changes under mutation, the graph does not contain loops. Now, notice that, if the graph contains an n-cycle with $n \geq 2$, then we can find a vertex of the cycle for which the complexity decreases in two different directions, a contradiction. $\square$

Remark 2.1.10. The argument used in the proof of Proposition 2.1.9 is parallel to the one used on page 28 of [4] to prove a classical result on Markov numbers.

2.2. Some isomorphisms of $\mathbb{T}_3 \setminus [\frac{0}{1}, \frac{-1}{1}, \frac{1}{0}]$.

In the rest of this note, we write μ_0, μ_{-1} and μ_∞ to denote the mutations in direction of the fractions of the form $\frac{even}{odd}$, $\frac{odd}{odd}$ and $\frac{odd}{even}$, respectively. We label the edges of $\mathbb{T}_3$ by the same indices, and denote by $\mathbb{T}_3^i$ the connected component of $\mathbb{T}_3 \setminus [\frac{0}{1}, \frac{-1}{1}, \frac{1}{0}]$ having the triple $\mu_i[\frac{0}{1}, \frac{-1}{1}, \frac{1}{0}]$ as a vertex.

Remark 2.2.1. From now on, even if the Farey triples are considered up to permutation of their components, when we write a Farey triple, we assume that it is written in the order $[\frac{even}{odd}, \frac{odd}{odd}, \frac{odd}{even}]$.

Definition 2.2.2. We define a graph isomorphism $\phi : \mathbb{T}_3^{-1} \to \mathbb{T}_3^{\infty}$ as follows:

- The triple $[\frac{0}{1},\frac{1}{1},\frac{1}{0}] = \mu_{-1}[\frac{0}{1},\frac{-1}{1},\frac{1}{0}]$ is mapped to the triple $[\frac{0}{1},\frac{-1}{1},\frac{-1}{2}] = \mu_{\infty}[\frac{0}{1},\frac{-1}{1},\frac{1}{0}]$,
- The edges with label $-1,\ 0,\ \infty$ are mapped to the edges with label $\infty,\ -1,\ 0$, respectively.

Analogously, we define a graph isomorphism $\psi : \mathbb{T}_3^{-1} \to \mathbb{T}_3^{0}$ as follows:

- The triple $[\frac{0}{1},\frac{1}{1},\frac{1}{0}]$ is mapped to the triple $[\frac{-2}{1},\frac{-1}{1},\frac{1}{0}] = \mu_0[\frac{0}{1},\frac{-1}{1},\frac{1}{0}]$,
- The edges with label $-1,\ 0,\ \infty$ are mapped to the edges with label $0,\ \infty,\ -1$, respectively.

Proposition 2.2.3. *The isomorphisms of Definition* 2.2.2 *can be expressed in terms of the following formulas:*

$$\phi[q_0,q_{-1},q_\infty] = \left[\frac{-r(q_\infty)}{r(q_\infty)+d(q_\infty)}, \frac{-r(q_0)}{r(q_0)+d(q_0)}, \frac{-r(q_{-1})}{r(q_{-1})+d(q_{-1})}\right], \tag{2.5}$$

$$\psi[q_0,q_{-1},q_\infty] = \left[\frac{-(r(q_{-1})+d(q_{-1}))}{d(q_{-1})}, \frac{-(r(q_\infty)+d(q_\infty))}{d(q_\infty)}, \frac{-(r(q_0)+d(q_0))}{d(q_0)}\right]. \tag{2.6}$$

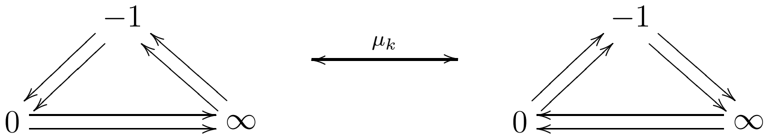
Proof. We only prove the formula for ϕ, using induction on $c[q_0,q_{-1},q_\infty]$. For ψ we can proceed in the same way. We see that the formula in Definition 2.2.2 for $\phi[\frac{0}{1},\frac{1}{1},\frac{1}{0}]$ satisfies the statement, so we take it as base for the induction. Without loss of generality, we assume that $q_0 < q_{-1} < q_\infty$. Thus, the complexity of the triple increases only in directions 0 and ∞. We consider only direction 0, direction ∞ is similar. By induction, we obtain

$$\begin{aligned}\phi(\mu_0[q_0,q_{-1},q_\infty]) &= \mu_{-1}\left[\frac{-r(q_\infty)}{r(q_\infty)+d(q_\infty)}, \frac{-r(q_0)}{r(q_0)+d(q_0)}, \frac{-r(q_{-1})}{r(q_{-1})+d(q_{-1})}\right] \\ &= \left[\frac{-r(q_\infty)}{r(q_\infty)+d(q_\infty)}, \frac{-(r(q_\infty)+r(q_{-1}))}{r(q_\infty)+d(q_\infty)+r(q_{-1})+d(q_{-1})}, \frac{-r(q_{-1})}{r(q_{-1})+d(q_{-1})}\right].\end{aligned}$$

Since $q_{-1}\oplus q_\infty = \frac{d(q_{-1})+d(q_\infty)}{r(q_{-1})+r(q_\infty)}$, it is clear that the result follows for ϕ. $\square$

3. The exchange matrices of $\mathcal{A}_M$

3.1. **Description.** In this section, we use the Farey triples in order to give an explicit description of the extended exchange matrices of the Markov cluster algebra (with principal coefficients) $\mathcal{A}_M$. An important property of the quiver is that the mutation at any vertex only reverses the orientation of the arrows.



(3.1)

We denote by B^+ (respectively B^-) the matrix associated with the left hand (respectively right hand) quiver of (3.1). Then the initial matrix of $\mathcal{A}_M$ is

$$\tilde{B}_0 = \begin{array}{c} \begin{array}{ccc} 0 & -1 & \infty \\ \downarrow & \downarrow & \downarrow \end{array} \\ \left(\begin{array}{ccc} 0 & -2 & 2 \\ 2 & 0 & -2 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right) \end{array}.$$

We consider the following assignment defined recursively. To the initial Farey triple $[\frac{0}{1}, \frac{-1}{1}, \frac{1}{0}]$, assign the initial matrix $\tilde{B}_0$. To any Farey triple $\mu_{a_n} \dots \mu_{a_1}[\frac{0}{1}, \frac{-1}{1}, \frac{1}{0}]$ with $a_i \in \{0, -1, \infty\}$, assign the extended exchange matrix $\mu_{a_n} \dots \mu_{a_1}(\tilde{B}_0)$. Note that this assignment is a surjection onto the set of the exchange matrices of $\mathcal{A}_M$. We use the following notation to denote the assignment $[q_0, q_{-1}, q_\infty] \longrightarrow M$ (see Remark 2.2.1):

$$M = \begin{array}{c} \begin{array}{ccc} q_0 & q_{-1} & q_\infty \end{array} \\ \left(\begin{array}{ccc} & B^{\pm} & \\ a & d & g \\ b & e & h \\ c & f & i \end{array}\right) \end{array} \tag{3.2}$$

Thus we may speak of mutating both, a triple and the associated matrix. Our goal is to describe all the extended exchange matrices using the Farey triples. First we describe the matrices obtained by alternating mutations in directions -1 and 0.

$$\begin{tikzcd}
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \arrow[r, "\mu_{-1}"] & \begin{pmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \arrow[d, "\mu_0"] \\
\begin{pmatrix} 3 & -2 & 0 \\ 4 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \arrow[d, "\mu_0"] & \begin{pmatrix} -1 & 2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \arrow[l, "\mu_{-1}"'] \\
\begin{pmatrix} -3 & 4 & 0 \\ -4 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \arrow[r, "\mu_{-1}"] & \cdots
\end{tikzcd} \tag{3.3}$$

These matrices correspond to triples of the form $[\frac{a}{1}, \frac{a\pm1}{1}, \frac{1}{0}]$. The next result describes all the matrices associated to triples of that form.

Proposition 3.1.1. *The matrices obtained by alternating the mutations μ_{-1} and μ_0 (beginning with μ_{-1}) are of the form*

$$\begin{array}{c} \begin{array}{ccc} \frac{a}{1} & \frac{a-1}{1} & \frac{1}{0} \end{array} \\ \begin{pmatrix} & B^+ & \\ 1-a & a & 0 \\ -a & a+1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array} \quad \textit{or} \quad \begin{array}{c} \begin{array}{ccc} \frac{a}{1} & \frac{a+1}{1} & \frac{1}{0} \end{array} \\ \begin{pmatrix} & B^- & \\ a+1 & -a & 0 \\ a+2 & -(a+1) & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}. \tag{3.4}$$

Proof. We proceed by induction on a. Take as base for the induction the computation made in (3.3). Suppose that $[\frac{a}{1}, \frac{a-1}{1}, \frac{1}{0}] \longrightarrow M$ satisfies the statement. Then, mutating both in direction -1, we obtain

$$M = \begin{pmatrix} & B^+ & \\ -a+1 & a & 0 \\ -a & a+1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{\mu_{-1}} \begin{pmatrix} & B^- & \\ -a+1+2a & -a & 0 \\ -a+2(a+1) & -(a+1) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

In this case we are done. The second case is obtained in the same way. □

Notice that the matrices just described are associated to a triple lying in $\mathbb{T}_3^{-1}$. The next theorem describes the rest of the matrices associated to the triples in $\mathbb{T}_3^{-1}$. All other exchange matrices will be obtained from those described in Proposition 3.1.1 and Theorem 3.1.2 below.

Theorem 3.1.2. *Let M be the exchange matrix associated to a triple $[q_0, q_{-1}, q_\infty] \neq [\frac{a}{1}, \frac{a\pm1}{1}, \frac{1}{0}]$ lying in $\mathbb{T}_3^{-1}$. Consider the following six cases:*

(i) $q_0 < q_\infty < q_{-1}$,
(ii) $q_{-1} < q_\infty < q_0$,
(iii) $q_0 < q_{-1} < q_\infty$,
(iv) $q_\infty < q_{-1} < q_0$,
(v) $q_{-1} < q_0 < q_\infty$,
(vi) $q_\infty < q_0 < q_{-1}$.

Let $q_0 = \frac{a}{b}$, $q_{-1} = \frac{c}{d}$ and $q_\infty = \frac{e}{f}$. Then, in each case, the matrix M is of the form:

$$\text{(i)} \begin{pmatrix} a+1 & -c+1 & c-a-1 \\ a+b+1 & -c-d+1 & c+d-a-b-1 \\ b+1 & -d+1 & d-b-1 \end{pmatrix},$$

$$\text{(ii)} \begin{pmatrix} -a+1 & c+1 & a-c-1 \\ -a-b+1 & c+d+1 & a+b-c-d-1 \\ -b+1 & d+1 & b-d-1 \end{pmatrix},$$

$$\text{(iii)} \begin{pmatrix} a+1 & e-a-1 & -e+1 \\ a+b+1 & e+f-a-b-1 & -e-f+1 \\ b+1 & f-b-1 & -f+1 \end{pmatrix},$$

$$\text{(iv)}\ \begin{pmatrix} -a+1 & a-e-1 & e+1 \\ -a-b+1 & a+b-e-f-1 & e+f+1 \\ -b+1 & b-f-1 & f+1 \end{pmatrix},$$

$$\text{(v)}\ \begin{pmatrix} e-c-1 & c+1 & -e+1 \\ e+f-c-d-1 & c+d+1 & -e-f+1 \\ f-d-1 & d+1 & -f+1 \end{pmatrix},$$

$$\text{(vi)}\ \begin{pmatrix} c-e-1 & -c+1 & e+1 \\ c+d-e-f-1 & -c-d+1 & e+f+1 \\ d-f-1 & -d+1 & f+1 \end{pmatrix}.$$

In cases (i), (iv) and (v), the principal part of M is B^+, in the other cases it is B^-.

Proof. First we show that the matrices obtained by mutating those of Proposition 3.1.1 in direction ∞ satisfy

$$\begin{array}{c} \begin{array}{ccc} \frac{a}{1} & \frac{a-1}{1} & \frac{1}{0} \end{array} \\ \left(\begin{array}{ccc} & B^+ & \\ 1-a & a & 0 \\ -a & a+1 & 0 \\ 0 & 0 & 1 \end{array}\right) \end{array} \xrightarrow{\mu_\infty} \begin{array}{c} \begin{array}{ccc} \frac{a}{1} & \frac{a-1}{1} & \frac{2a-1}{2} \end{array} \\ \left(\begin{array}{ccc} & B^- & \\ 1-a & a & 0 \\ -a & a+1 & 0 \\ 0 & 2 & -1 \end{array}\right) \end{array}$$

For the matrices obtained by the assignment $[\frac{a}{1}, \frac{a+1}{1}, \frac{1}{0}] \longrightarrow M$, we do the same. Suppose that $[q_{-1}, q_0, q_\infty] = \mu_\infty[\frac{a}{1}, \frac{a\pm1}{1}, \frac{1}{0}]$ for some $a > 0$. We proceed by induction on $c[q_{-1}, q_0, q_\infty]$. Take as base for the induction the first part of the proof. Now, notice that the cases (for the triples) are related by mutation in the directions described in the following diagram (we only consider directions which increase the complexity):

$$\begin{tikzcd}
 & (i) \arrow[r, shift right, "0"'] \arrow[dl, shift right, "-1"'] & (vi) \arrow[l, shift right, "\infty"'] \arrow[dr, shift left, "-1"] & \\
(iii) \arrow[ur, shift right, "\infty"'] \arrow[dr, shift left, "0"'] & & & (iv) \arrow[ul, shift left, "0"] \arrow[dl, shift left, "\infty"] \\
 & (v) \arrow[ul, shift left, "-1"'] \arrow[r, shift right, "\infty"'] & (ii) \arrow[l, shift right, "0"'] \arrow[ur, shift left, "-1"'] &
\end{tikzcd}$$

Thus, we will be done if we prove that the matrices of the statement satisfy the same diagram. Once again, we only prove one case and in one direction; it will be obvious that all other cases are similar. Consider case (i), and mutate in direction 0:

$$\begin{tikzcd}
\begin{array}{cccc} & \frac{a}{b} & \frac{c}{d} & \frac{a+c}{b+d} \\ & \multicolumn{3}{c}{\begin{pmatrix} & B^{+} & \\ a+1 & -c+1 & c-a-1 \\ a+b+1 & -c-d+1 & c+d-a-b-1 \\ b+1 & -d+1 & d-b-1 \end{pmatrix}} \end{array} \arrow[d, "\mu_0"] \\
\begin{array}{cccc} & \frac{a+2c}{b+2d} & \frac{c}{d} & \frac{a+c}{b+d} \\ & \multicolumn{3}{c}{\begin{pmatrix} & B^{-} & \\ -1-a & -c+1 & c-a-1+2(a+1) \\ -a-b-1 & -c-d+1 & c+d-a-b-1+2(a+b+1) \\ -b-1 & -d+1 & d-b-1+2(b+1) \end{pmatrix}} \end{array}
\end{tikzcd}$$

□

We have described all the matrices associated to a triple in $\mathbb{T}_3^{-1}$. In order to describe the rest, we consider the action of the alternating group $A_3 = \{id, (123), (132)\}$ on $\mathrm{Mat}(3 \times 3, \mathbb{Z})$ by cyclic permutation of its "diagonals", i.e., if $B = (b_{ij})$ then $\sigma \cdot B = (b_{\sigma(i)\sigma(j)})$.

Proposition 3.1.3. *Let M be the matrix associated to a triple $[q_0, q_{-1}, q_\infty] \in \mathbb{T}_3^{-1}$. Then $\phi[q_0, q_{-1}, q_\infty] \longrightarrow (123) \cdot M$ and $\psi[q_0, q_{-1}, q_\infty] \longrightarrow (132) \cdot M$.*

Proof. Since the matrices $B^{\pm}$ and the 3×3 identity matrix are invariant under the action of (123) and (132), and the functions ϕ and ψ preserve the mutation in the sense of Definition 2.2.2, it is sufficient to verify the statement for the matrices associated to $\phi[\frac{0}{1}, \frac{-1}{1}, \frac{1}{0}]$ and $\psi[\frac{0}{1}, \frac{-1}{1}, \frac{1}{0}]$, but this follows at once. □

Observation 3.1.4. *Each $\mathbf{c}$-vector of $\mathcal{A}_M$ has either only non-negative entries or only non-positive entries. The corresponding fact is known for each skew-symmetric cluster algebra of geometric type, cf. Section 1 of [12].*

In the same way as we did with the exchange matrices, we assign to each Farey Triple a $\mathbf{g}$-matrix.

Theorem 3.1.5. *Let M be the $\mathbf{g}$-matrix associated to a triple $[q_0, q_{-1}, q_\infty] \in \mathbb{T}_3^{-1}$. Consider the following three cases:*

(i) $[q_0, q_{-1}, q_\infty] = [\frac{a}{1}, \frac{a-1}{1}, \frac{1}{0}]$,
(ii) $[q_0, q_{-1}, q_\infty] = [\frac{a}{1}, \frac{a+1}{1}, \frac{1}{0}]$,
(iii) $q_0 = \frac{a}{b}$, $q_{-1} = \frac{c}{d}$ *and* $q_\infty = \frac{e}{f} \neq \frac{1}{0}$.

Then, going through the cases, the matrix M is of the form:

(i) $\begin{pmatrix} a+1 & a & 0 \\ -a & -a+1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$,

(ii) $\begin{pmatrix} a+1 & a+2 & 0 \\ -a & -(a+1) & 0 \\ 0 & 0 & 1 \end{pmatrix}$,

(iii) $\begin{pmatrix} a+1 & c+1 & e+1 \\ b-a-1 & d-c-1 & f-e-1 \\ 1-b & 1-d & 1-f \end{pmatrix}$.

Proof. In view of [12, Theorem 1.2], we may calculate M by inverting and transposing the complementary part of the exchange matrices of Proposition 3.1.1 and Theorem 3.1.2. We may easily see that for the cases (i) and (ii) we are done. Now assume that $[q_0, q_{-1}, q_\infty]$ is as in case (iii). Notice that all the matrices of Theorem 3.1.2 can be expressed in terms of a single matrix, namely

$$\lambda \begin{pmatrix} e-c+cf-de & a-e+bf-af & c-a+ad-bc \\ e+f-c-d+cf-de & a+b-e-f+bf-af & c+d-a-b+ad-bc \\ f-d+cf-de & b-f+bf-af & d-b+ad-bc \end{pmatrix},$$

where $\lambda = (cf-de+be-af+ad-bc)^{-1}$. A straightforward calculation shows that the inverse of this matrix is

$$\begin{pmatrix} a+1 & b-a-1 & 1-b \\ c+1 & d-c-1 & 1-d \\ e+1 & f-e-1 & 1-f \end{pmatrix}.$$

□

Proposition 3.1.6. *Let M be the* **g***-matrix associated to a triple $[q_0, q_{-1}, q_\infty] \in \mathbb{T}_3^{-1}$. Then $\phi[q_0, q_{-1}, q_\infty] \longrightarrow (123) \cdot M$ and $\psi[q_0, q_{-1}, q_\infty] \longrightarrow (132) \cdot M$.*

Proof. The operations of transposing and inverting a matrix commute with the action of A_3. □

We notice that the **g**-vectors of $\mathcal{A}_M$ lie in the plane $x+y+z=1$. The first vectors obtained by iterated mutation of the initial seed are presented in Figure 3.1. Another interpretation of this figure is given in [6].

Figure 2. The **g**-vectors of $\mathcal{A}_M$

References

[1] M. Barot and C. Geiss, *Tubular cluster algebras I: Categorification*, Math. Z. **271** (2012), 1091–1115.

[2] M. Barot, C. Geiss and G. Jasso *Tubular cluster algebras II*, J. Pure Appl. Algebra **217** (2013), 1825–1837.

[3] A. Berenstein, S. Fomin and A. Zelevinsky, *Cluster algebras III: Upper bounds and double Bruhat cells*, Duke Math. J. **126** (2005), 1–52.

[4] J. W. S. Cassels, *An introduction to Diophantine approximation*, Cambridge Tracts in Mathematics and Mathematical Physics **45** (1957), Cambridge University Press.
[5] H. Derksen, J. Weyman and A. Zelevinsky, *Quivers with potentials and their representations I: Mutations*, Selecta Math. **14** (2008), 59–119.
[6] V. V. Fock and A. B. Goncharov, *Cluster $\mathcal{X}$-varieties at infinity*, preprint, *arχiv:1104.0407v1.*
[7] S. Fomin, M. Shapiro and D. Thurston, *Cluster algebras and triangulated surfaces. Part I: Cluster complexes*, Acta Math. **201** (2008), 83–146.
[8] S. Fomin and A. Zelevinsky, *Clusters algebras IV: Coefficients*, Compositio Math. **143** (2007), 112–164.
[9] D. Labardini-Fragoso, *Quivers with potentials associated to triangulated surfaces*, Proc. London Math. Soc. (2009) **98**, 797–839.
[10] D. Labardini-Fragoso, *Quivers with potentials associated to triangulated surfaces, Part II: Arc representations*, `ar`χ`iv:0909.4100v2`.
[11] T. Nakanishi, *Periodicities in cluster algebras and dilogarithm identities*, in Representations of algebras and related topics (A. Skowronski and K. Yamagata, eds.), EMS Series of Congress Reports, European Mathematical Society, 2011, pp. 407–444.
[12] T. Nakanishi and A. Zelevinsky, *On tropical dualities in cluster algebras*, Contemp. Math. **565** (2012), 217–226.
[13] P.-G. Plamondon, *Generic bases for cluster algebras from the cluster category*, Int. Math. Res. Notices 2013, no. 10, 2368–2420.

Université Paris Diderot – Paris 7, Institut de Mathématiques de Jussieu, UMR 7586 du CNRS, Case 7012, Bâtiment Chevaleret, 75205 Paris Cedex 13, France
E-mail address: `najera@math.jussieu.fr`