Séminaire Lotharingien de Combinatoire **67** (2012), Article B67e

FLAG WEAK ORDER ON WREATH PRODUCTS

RON M. ADIN, FRANCESCO BRENTI, AND YUVAL ROICHMAN

ABSTRACT. A generating set for the wreath product $\mathbb{Z}_r \wr S_n$ which leads to a nicely behaved weak order is presented. It is shown that the resulting poset has properties analogous to those of the weak order on the symmetric group: it is a self-dual lattice, ranked by the Foata–Han flag inversion number; any two maximal chains are connected via Tits-type pseudo-Coxeter moves; and its intervals have the desired homotopy types. The associated Möbius function and relevant generating functions are computed.

1. Introduction

The weak order on a Coxeter group is a fundamental tool in the study of the combinatorial structure of this group. A natural problem is to give a "correct definition" of a weak order on the wreath product $G(r,n) := \mathbb{Z}_r \wr S_n$. The weak order on a Coxeter group is determined via the generating set of simple reflections and the associated length function. In this paper we address the basic question: which generating set for the wreath product is the counterpart of the set of simple reflections? Unfortunately, the natural analogue — the set of complex reflections — does not lead to a nicely behaved partial order. It will be shown that there is a generating set yielding an order on $G(r,n)$ with properties analogous to those of the weak order on $S_n = G(1,n)$: the resulting poset is ranked by the Foata–Han flag inversion number; it is a self-dual lattice; it has a Tits-type property; and its intervals have the desired homotopy types. Finally, the associated Möbius function and relevant generating functions will be computed.

The rest of the paper is organized as follows. Necessary preliminaries and notation are given in Section 2. For the sake of clarity, results are first stated and proved for the hyperoctahedral group $B_n = G(2,n)$: the generating set and corresponding presentation are described in Section 3, the flag weak order is defined in Section 4, and its properties are studied in Sections 4–6. The corresponding results for general r are discussed in Section 7. Section 8 contains final remarks and open problems.

2. Preliminaries

Let (W,S) be a Coxeter system; thus W is a group with a set of generators $S = \{s_0, s_1, \ldots, s_n\}$ and a presentation of the form

$$W = \langle s_0, s_1, \ldots, s_n \mid (s_i s_j)^{m_{ij}} = e,\ 0 \le i \le j \le n \rangle,$$

where $m_{ij} = m_{ji} \in \{2, 3, \ldots\} \cup \{\infty\}$ and $m_{ii} = 1$.

The (*right*) *weak order* $\le$ on W is the reflexive and transitive closure of the relation

$$w \lessdot ws \quad \text{if and only if} \quad w \in W,\ s \in S,\ \text{and } \ell(w) + 1 = \ell(ws),$$

Work of the first and third authors was supported in part by an Internal Research Grant from the Office of the Rector, Bar-Ilan University.

where $\ell(\cdot)$ is the standard length function with respect to the Coxeter generating set S. The left weak order is defined similarly, with sw instead of ws. For combinatorial and other properties of the weak order the reader is referred to [3].

Let S_n be the symmetric group on the letters $[n] := \{1, \ldots, n\}$. Recall that S_n is a Coxeter group with respect to the set of Coxeter generators $S := \{s_i \mid 1 \le i \le n-1\}$, where s_i may be interpreted as the adjacent transposition $(i, i+1)$.

For $\pi \in S_n$ let the *inversion set* be $\mathrm{Inv}(\pi) := \{(i,j) : \ i < j, \ \pi(i) > \pi(j)\}$, the *inversion number* be $\mathrm{inv}(\pi) := \#\mathrm{Inv}(\pi)$, and the *descent set* be $\mathrm{Des}(\pi) := \{i \in [n-1] : \pi(i) > \pi(i+1)\}$. Recall the classical combinatorial interpretations of the Coxeter length function and of the (right) weak order [3, Cor. 1.5.2, Prop. 3.1.3]:

$$\ell(\pi) = \mathrm{inv}(\pi) = \mathrm{inv}(\pi^{-1}), \qquad \pi \le \sigma \text{ if and only if } \mathrm{Inv}(\pi^{-1}) \subseteq \mathrm{Inv}(\sigma^{-1}). \tag{1}$$

Let B_n be the group of all bijections σ of the set $[\pm n] := \{-n, \ldots, -1, 1, \ldots, n\}$ onto itself such that

$$\sigma(-a) = -\sigma(a), \quad \text{for all } a \in [\pm n],$$

with composition as the group operation. B_n is known as the group of "signed permutations" on $[n]$, or as the *hyperoctahedral group* of rank n. We identify S_n as a subgroup of B_n, and B_n as a subgroup of S_{2n}, in the natural ways.

For $\sigma \in B_n$ let $\mathrm{Neg}(\sigma) := \{i \in [n] : \sigma(i) < 0\}$, $\mathrm{neg}(\sigma) := \#\mathrm{Neg}(\sigma)$ and $|\sigma| = [|\sigma(1)|, \ldots, |\sigma(n)|] \in S_n$.

More generally, consider the wreath product $= \mathbb{Z}_r \wr S_n$, where $\mathbb{Z}_r$ is the (additive) cyclic group of order r:

$$G(r,n) := \{g = ((c_1, \ldots, c_n), \sigma) \mid c_i \in \mathbb{Z}_r \; 1 \le i \le n, \, \sigma \in S_n\},$$

with the group operation

$$((c_1, \ldots, c_n), \sigma) \cdot ((d_1, \ldots, d_n), \tau) := ((c_{\tau(1)} + d_1, \ldots, c_{\tau(n)} + d_n), \sigma\tau).$$

(This definition is slightly non-standard, and it is chosen for compatibility with the case $r = 2$; see below.) The elements of $\mathbb{Z}_r \wr S_n$ may be interpreted as r-colored permutations, i.e., bijections g of the set $\mathbb{Z}_r \times [n]$ onto itself such that

$$g(c,i) = (d,j) \text{ implies } g(c+c', i) = (d+c', j), \quad \text{for all } c, c', d \in \mathbb{Z}_r, \, i, j \in [n].$$

For example, $G(1,n)$ is naturally isomorphic to the symmetric group S_n, and $G(2,n)$ is isomorphic to the hyperoctahedral group B_n, where $((c_1, \ldots, c_n), \sigma) \in G(2,n)$ corresponds to the element $g \in B_n$ such that

$$g(i) = (-1)^{c_i} \sigma(i), \quad \text{for all } i \in [n].$$

Thus $\mathrm{Neg}(g) = \{i : c_i = 1\}$, a basic compatibility (for $r = 2$) that underlies the choice of group operation in $G(r,n)$ above. Informally, this means that the colors (or signs) c_i are attached *before* the permutation σ is applied.

In the special cases $r = 1, 2$, $G(r,n)$ is of course a Coxeter group.

For an r-colored permutation $\pi = ((c_1, \ldots, c_n), \sigma) \in G(r,n)$ let $|\pi| := \sigma$ and $\mathrm{n}(\pi) := \sum_{i=1}^{n} c_i \in \mathbb{Z}$, where elements of $\mathbb{Z}_r$ are interpreted as the corresponding elements of $\{0, \ldots, r-1\} \subseteq \mathbb{Z}$. Note that, for $r = 2$, $\mathrm{n}(\pi) = \mathrm{neg}(\pi)$.

The classical inversion number on permutations has a counterpart for wreath products, the *flag inversion number*. It was introduced by Foata and Han [8, 9] and further investigated in [7, 6].

Definition 2.1. The *flag inversion number* of an r-colored permutation $\pi \in G(r,n)$ is defined by

$$\mathrm{finv}(\pi) := r \cdot \mathrm{inv}(|\pi|) + \mathrm{n}(\pi).$$

For a positive integer m and an indeterminate q, let us write

$$[m]_q := \frac{1-q^m}{1-q}.$$

Proposition 2.2 ([7, Theorem 7.4])**.** *For every r and n, we have*

$$\sum_{\pi \in G(r,n)} q^{\mathrm{finv}(\pi)} = \prod_{i=1}^{n} [ri]_q.$$

3. Generators and Presentations

The *alternating subgroup* of a reflection group is the kernel of the *sign homomorphism* which maps all the Coxeter generators (simple reflections) to -1.

Proposition 3.1. *The alternating subgroup of the hyperoctahedral group $B_n = G(2,n)$ is isomorphic to the abstract group generated by $\{a_i : 1 \le i \le n-1\}$ with defining relations*

$$(A1) \qquad a_i^4 = 1, \quad 1 \le i \le n-1,$$

$$(A2) \qquad a_i a_j = a_j a_i, \quad |i-j| > 1,$$

$$(A3) \qquad a_i a_{i+1} a_i = a_{i+1} a_i a_{i+1}, \quad 1 \le i \le n-1,$$

and

$$(A4) \qquad (a_i a_{i+1})^3 = 1, \quad 1 \le i \le n-1.$$

Proof. Denote by B_n^+ the alternating subgroup of B_n, and let $\widehat{B}_n^+$ be the abstract group with the above presentation. Define a map ϕ from the *free group* generated by $a_1, \ldots, a_{n-1}$ to B_n^+ by

$$\phi(a_i) := [1, \ldots, -(i+1), i, \ldots, n], \quad 1 \le i \le n-1.$$

Since $\phi(a_i) = (i, i+1)(i, -i)$ is a product of two reflections in B_n, it indeed belongs to B_n^+. It is easy to check that relations $(A1)$–$(A4)$ are satisfied when each a_i is replaced by $\phi(a_i)$. Therefore, this defines a group homomorphism, which we again denote by ϕ, from $\widehat{B}_n^+$ to B_n^+. We shall show that it is actually an isomorphism.

Now, B_n^+ is generated by the set $\{(i, i+1)(1, -1) : 1 \le i \le n-1\}$; see, e.g., [10, §5.1, Exercise 1]. Since $\phi(a_i)^2 = (i+1, -(i+1))(i, -i)$, it follows that

$$\phi(a_i)\phi(a_{i-1})^2\phi(a_{i-2})^2 \cdots \phi(a_1)^2 = (i, i+1)(1, -1)$$

for $1 \le i \le n-1$, and therefore $\phi : \widehat{B}_n^+ \to B_n^+$ is surjective.

It remains to show that ϕ is injective. Since it is surjective and $\#B_n^+ = 2^{n-1}n!$, it suffices to show that $\#\widehat{B}_n^+ \le 2^{n-1}n!$.

Let $\widehat{N}_n^+$ be the subgroup of $\widehat{B}_n^+$ generated by $a_1^2, \ldots, a_{n-1}^2$. We shall show that $\widehat{N}_n^+$ is a commutative normal subgroup of $\widehat{B}_n^+$. Indeed, $(A4)$ can be written as

$$a_i a_{i+1} a_i a_{i+1} a_i a_{i+1} = 1,$$

or, using $(A3)$, as

$$a_i a_{i+1} a_i a_i a_{i+1} a_i = 1.$$

Rearrangement gives

$$a_i^2 a_{i+1} = a_{i+1}^{-1} a_i^{-2},$$

or

$$a_{i+1}^{-1} a_i^2 a_{i+1} = a_{i+1}^{-2} a_i^{-2} \in \widehat{N}_n^+. \tag{2}$$

Similarly, $(A4)$ and $(A3)$ for $i-1$ imply

$$a_{i-1}^{-1} a_i^2 a_{i-1} = a_{i-1}^{-2} a_i^{-2} \in \widehat{N}_n^+.$$

Finally, by $(A2)$,

$$a_j^{-1} a_i a_j = a_i, \quad |i-j| > 1,$$

so that

$$a_j^{-1} a_i^2 a_j = a_i^2 \in \widehat{N}_n^+, \quad |i-j| > 1.$$

Thus $\widehat{N}_n^+$ is a normal subgroup of $\widehat{B}_n^+$.

Commutativity of $\widehat{N}_n^+$ is also easy: (2) and $(A1)$ imply that

$$a_{i+1}^{-1} a_i^2 a_{i+1} = a_{i+1}^{-2} a_i^2,$$

or

$$a_{i+1} a_i^2 a_{i+1} = a_i^2,$$

so that also

$$a_{i+1}^2 a_i^2 a_{i+1}^2 = a_{i+1}(a_{i+1} a_i^2 a_{i+1}) a_{i+1} = a_{i+1} a_i^2 a_{i+1} = a_i^2.$$

Thus, again by $(A1)$,

$$a_i^2 a_{i+1}^2 = a_{i+1}^{-2} a_i^2 = a_{i+1}^2 a_i^2,$$

i.e., a_i^2 and a_{i+1}^2 commute. This is certainly also the case for a_i^2 and a_j^2 when $|i-j| > 1$, so $\widehat{N}_n^+$ is commutative.

We can now wrap up the proof: $\widehat{N}_n^+$ is a commutative group generated by the involutions $a_1^2, \ldots, a_{n-1}^2$. Thus each element of $\widehat{N}_n^+$ can be written as a product $a_{i_1}^2 \cdots a_{i_k}^2$ for some $k \geq 0$ and $1 \leq i_1 < \cdots < i_k \leq n-1$. In particular, $\#\widehat{N}_n^+ \leq 2^{n-1}$. Also, $\widehat{N}_n^+$ is a normal subgroup of $\widehat{B}_n^+$. The quotient $\widehat{B}_n^+/\widehat{N}_n^+$ is generated by $\bar{a}_i$, the cosets corresponding to the generators a_i of $\widehat{B}_n^+$, $1 \leq i \leq n-1$. The $\bar{a}_i$ satisfy the same relations $(A2)$–$(A4)$ as the a_i, with $(A1)$ replaced by

$$\bar{a}_i^2 = 1, \quad 1 \leq i \leq n-1.$$

These are exactly the Coxeter relations defining the symmetric group S_n (actually, $(A3)$ is now equivalent to $(A4)$), so that $\widehat{B}_n^+/\widehat{N}_n^+$ is a homomorphic image of S_n, and in particular $\#(\widehat{B}_n^+/\widehat{N}_n^+) \leq n!$. Everything put together, we obtain $\#\widehat{B}_n^+ \leq 2^{n-1} n!$, as required. $\square$

The above presentation may be extended to the whole group $B_n = G(2, n)$.

Proposition 3.2. *The hyperoctahedral group $B_n = G(2, n)$ is isomorphic to the abstract group generated by $S_{2,n} := \{a_i : 1 \le i \le n-1\} \cup \{b_i : 1 \le i \le n\}$ with defining relations*

$$(B1)\qquad b_i^2 = 1, \quad 1 \le i \le n,$$

$$(B2)\qquad b_i b_j = b_j b_i, \quad 1 \le i < j \le n,$$

$$(B3)\qquad a_i^2 = b_i b_{i+1}, \quad 1 \le i \le n-1,$$

$$(B4)\qquad a_i a_j = a_j a_i, \quad |i-j| > 1,$$

$$(B5)\qquad a_i a_{i+1} a_i = a_{i+1} a_i a_{i+1}, \quad 1 \le i \le n-1,$$

$$(B6)\qquad a_i b_j = b_j a_i, \quad j \ne i, i+1,$$

$$(B7)\qquad a_i b_i = b_{i+1} a_i, \quad 1 \le i \le n-1,$$

and

$$(B8)\qquad a_i b_{i+1} = b_i a_i, \quad 1 \le i \le n-1.$$

Remark 3.3. Note that relations $(A1)$–$(A4)$ in Proposition 3.1 follow from relations $(B1)$–$(B8)$ in Proposition 3.2. Relation $(A1)$ follows from relations $(B1)$, $(B2)$ and $(B3)$. Relations $(A2)$–$(A3)$ are relations $(B4)$–$(B5)$. Finally, relation $(A4)$ follows from relations $(B1)$, $(B3)$, $(B5)$–$(B8)$ as follows:

$$\begin{aligned}(a_i a_{i+1})^3 &= (a_i a_{i+1} a_i)(a_{i+1} a_i a_{i+1}) &&= (a_i a_{i+1} a_i)(a_i a_{i+1} a_i)\\ &= a_i a_{i+1} b_i b_{i+1} a_{i+1} a_i &&= b_{i+1} a_i a_{i+1} a_{i+1} a_i b_{i+2}\\ &= b_{i+1} a_i b_{i+1} b_{i+2} a_i b_{i+2} &&= b_{i+1} b_i a_i a_i b_{i+2} b_{i+2}\\ &= b_{i+1} b_i b_i b_{i+1} b_{i+2} b_{i+2} &&= 1.\end{aligned}$$

Proof. This is similar to the proof of Proposition 3.1 (and somewhat simpler).

Let $\widehat{B}_n$ be the abstract group with the presentation described in Proposition 3.2. Define a map ϕ from the free group generated by $a_1, \ldots, a_{n-1}, b_1, \ldots, b_n$ to B_n by

$$\phi(a_i) := [1, \ldots, -(i+1), i, \ldots, n], \quad 1 \le i \le n-1,$$

and

$$\phi(b_i) := [1, \ldots, -i, \ldots, n], \quad 1 \le i \le n.$$

Thus $\phi(a_i) = (i, i+1)(i, -i)$ and $\phi(b_i) = (i, -i)$. It is easy to check that relations $(B1)$–$(B8)$ are satisfied when each a_i (b_i) is replaced by $\phi(a_i)$ ($\phi(b_i)$, respectively). Therefore this defines a group homomorphism, which we again denote by ϕ, from $\widehat{B}_n$ to B_n. We shall show that it is actually an isomorphism.

Clearly, $\{\phi(a_i)\phi(b_i) : 1 \le i \le n-1\}$ is the set of Coxeter generators for the symmetric group S_n, embedded naturally into B_n. A similar assertion holds for $\{\phi(b_i) : 1 \le i \le n\}$ and $\mathbb{Z}_2^n$. Since $B_n = \mathbb{Z}_2^n \rtimes S_n$, it follows that $\{\phi(a_i) : 1 \le i \le n-1\} \cup \{\phi(b_i) : 1 \le i \le n\})$ generates B_n. Thus ϕ is surjective and, in particular,

$$\#\widehat{B}_n \ge \#B_n.$$

It remains to show that ϕ is injective. Since it is surjective and $\#B_n = 2^n n!$, it suffices to show that $\#\widehat{B}_n \le 2^n n!$.

Let $\widehat{N}_n$ be the subgroup of $\widehat{B}_n$ generated by $b_1, \ldots, b_n$. We shall show that $\widehat{N}_n$ is a commutative normal subgroup of $\widehat{B}_n$. Commutativity follows from $(B2)$, while normality follows from $(B6)$–$(B8)$, which may be written as

$$a_i^{-1}b_ja_i = b_j, \quad j \neq i, i+1,$$
$$a_i^{-1}b_{i+1}a_i = b_i, \quad 1 \le i \le n-1,$$

and

$$a_i^{-1}b_ia_i = b_{i+1}, \quad 1 \le i \le n-1.$$

We can now wrap up the proof: $\widehat{N}_n$ is a commutative group generated by the involutions $b_1, \ldots, b_n$. Thus each element of $\widehat{N}_n$ can be written as a product $b_{i_1} \cdots b_{i_k}$ for some $k \ge 0$ and $1 \le i_1 < \cdots < i_k \le n$. In particular, $\#\widehat{N}_n \le 2^n$. Also, $\widehat{N}_n$ is a normal subgroup of $\widehat{B}_n$. The quotient $\widehat{B}_n/\widehat{N}_n$ is generated by $\bar{a}_i$, the cosets corresponding to the generators a_i of $\widehat{B}_n$, $1 \le i \le n-1$. The $\bar{a}_i$ satisfy relations $(B4)$–$(B5)$, with $(B3)$ replaced by

$$\bar{a}_i^2 = 1, \quad 1 \le i \le n-1.$$

These are exactly the Coxeter relations defining the symmetric group S_n, so that $\widehat{B}_n/\widehat{N}_n$ is a homomorphic image of S_n, and in particular $\#(\widehat{B}_n/\widehat{N}_n) \le n!$. Everything put together, we obtain $\#\widehat{B}_n \le 2^n n!$, as required. $\square$

4. Flag Weak Order

From now on we identify the abstract generating set of B_n,

$$S_{2,n} = \{a_i : \ 1 \le i < n\} \cup \{b_i : \ 1 \le i \le n\},$$

with the choice

$$a_i := [1, \ldots, i-1, -(i+1), i, i+2, \ldots, n]$$

and

$$b_i := [1, \ldots, i-1, -i, i+1, \ldots, n],$$

used in the proof of Proposition 3.2.

Following Foata and Han [8, 9], let the *flag inversion number* of $\pi \in B_n$ be

$$\mathrm{finv}(\pi) := 2 \cdot \mathrm{inv}(|\pi|) + \mathrm{neg}(\pi).$$

Definition 4.1. The *flag* (*right*) *weak order* $\preceq$ on B_n is the reflexive and transitive closure of the relation

$$\pi \lessdot \pi s \quad \text{if and only if} \quad \pi \in G(2,n), s \in S_{2,n} \quad \text{and} \quad \mathrm{finv}(\pi) < \mathrm{finv}(\pi s).$$

Note that this order is not isomorphic to the classical weak order on B_n.

Proposition 4.2. *The poset* $(B_n, \preceq)$ *is*

(i) *ranked* (*by flag inversion number*);
(ii) *self-dual* (*by* $\pi \mapsto \pi\mu_0$, *where* $\mu_0 := [\bar{n}, \ldots, \bar{1}]$ *is the unique maximal element in this order*);
(iii) *rank-symmetric and unimodal.*

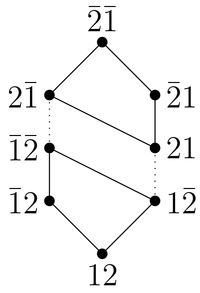

FIGURE 1. The Hasse diagram of the flag weak order on B_2. Edges are drawn in dotted lines for a_i's and as solid lines for b_i's.

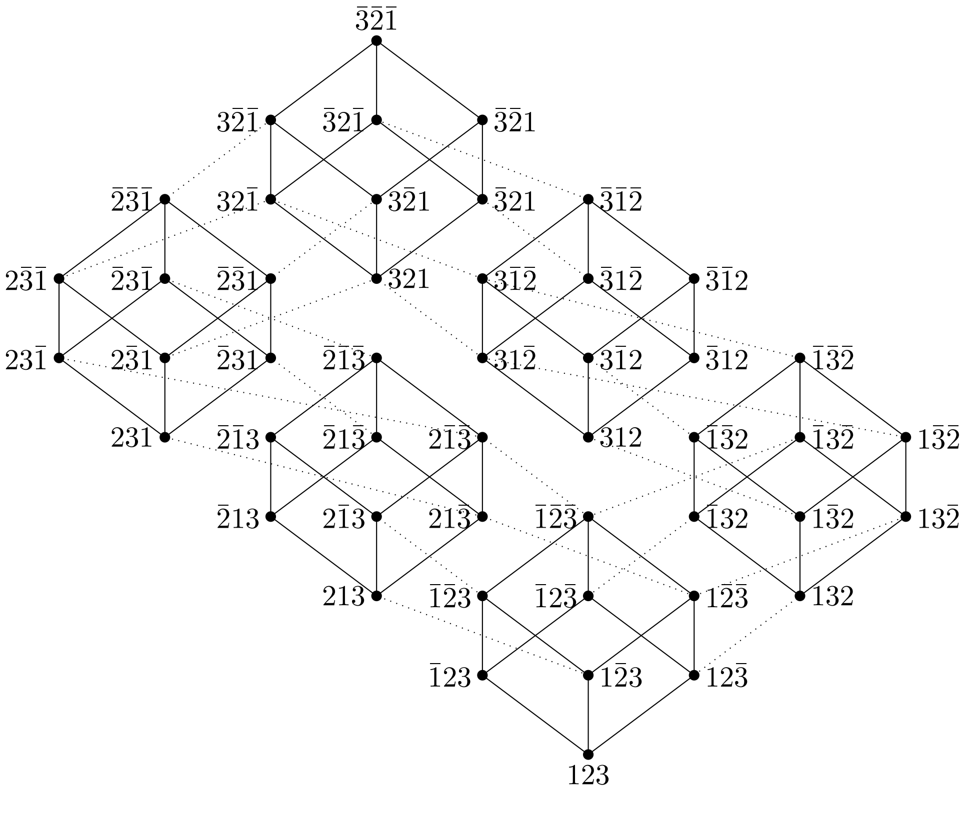

FIGURE 2. The Hasse diagram of the flag weak order on B_3. Edges are drawn as dotted lines for a_i's and as solid lines for b_i's.

Proof. (i) In order to show that all maximal chains between two elements have the same length (the difference between their finv values), it suffices to show that, if $\sigma = \pi s$, with $s \in S_{2,n}$ and $\operatorname{finv}(\pi) < \operatorname{finv}(\sigma)$, then there exists $\pi \prec w \preceq \sigma$ with $\operatorname{finv}(w) = \operatorname{finv}(\pi) + 1$. If $s \in \{b_1, \dots, b_n\}$, then

$$\operatorname{finv}(\pi s) - \operatorname{finv}(\pi) = 2 \cdot (\operatorname{inv}(|\pi s|) - \operatorname{inv}(|\pi|)) + (\operatorname{neg}(\pi s) - \operatorname{neg}(\pi)) = 2 \cdot 0 \pm 1 = 1$$

(positive by the assumption $\operatorname{finv}(\pi) < \operatorname{finv}(\pi s)$). In this case we can take $w := \pi s = \sigma$. Otherwise, $s \in \{a_1, \dots, a_{n-1}\}$. Then

$$\operatorname{finv}(\pi s) - \operatorname{finv}(\pi) = 2 \cdot (\operatorname{inv}(|\pi s|) - \operatorname{inv}(|\pi|)) + (\operatorname{neg}(\pi s) - \operatorname{neg}(\pi)) = 2 \cdot (\pm 1) \pm 1.$$

Being positive by assumption, this number is either 1 or 3. In the first case, which occurs if $s = a_i$, $1 \le i \le n$, $|\pi(i)| < |\pi(i+1)|$ and $\pi(i+1) < 0$, we can again take $w := \pi s$. In the second case, which occurs if $s = a_i$, $|\pi(i)| < |\pi(i+1)|$ and $\pi(i+1) > 0$, we have, by relations (B1) and (B7) in Proposition 3.2, $\sigma = \pi b_{i+1} a_i b_i$, with $\mathrm{finv}(\pi) + 3 = \mathrm{finv}(\pi b_{i+1}) + 2 = \mathrm{finv}(\pi b_{i+1} a_i) + 1 = \mathrm{finv}(\sigma)$, and we can take $w := \pi b_{i+1}$.

(ii) Let $\mu_0 := [\bar{n}, \dots, \bar{1}]$. Then, for every $\pi \in B_n$,

$$\begin{aligned} \mathrm{finv}(\pi\mu_0) &= 2 \cdot \mathrm{inv}\left(|\pi\mu_0|\right) + \mathrm{neg}(\pi\mu_0) \\ &= 2\left[\binom{n}{2} - \mathrm{inv}\left(|\pi|\right)\right] + [n - \mathrm{neg}(\pi)] \\ &= \mathrm{finv}(\mu_0) - \mathrm{finv}(\pi). \end{aligned}$$

If $\sigma = \pi s$ with $s \in S_{2,n}$ and $\mathrm{finv}(\pi) < \mathrm{finv}(\sigma)$, then

$$\mathrm{finv}(\pi\mu_0) - \mathrm{finv}(\sigma\mu_0) = \mathrm{finv}(\sigma) - \mathrm{finv}(\pi) > 0.$$

Moreover, we have $\pi\mu_0 = \sigma\mu_0\tilde{s}$, where

$$\tilde{s} = \mu_0^{-1} s^{-1} \mu_0 = \begin{cases} b_{n+1-i}, & \text{if } s = b_i; \\ a_{n-i}, & \text{if } s = a_i. \end{cases}$$

It follows, by Definition 4.1, that $\pi \preceq \sigma$ if and only if $\sigma\mu_0 \preceq \pi\mu_0$, and, since right multiplication by μ_0 is a bijection on B_n, this proves self-duality.

(iii) Rank-symmetry follows from (ii) (and (i)). Unimodality follows from (i) together with Proposition 2.2. ☐

The proof of Proposition 4.2 implies the following statement.

Corollary 4.3. *σ covers π in $(B_n, \preceq)$ if and only if either*

(i) *there exists an integer i with $1 \le i \le n$ such that*

$$i \notin \mathrm{Neg}(\pi) \quad \text{and} \quad \sigma = \pi b_i;$$

or

(ii) *there exists $1 \le i \le n-1$ such that*

$$i+1 \in \mathrm{Neg}(\pi), \quad |\pi(i)| < |\pi(i+1)| \quad \text{and} \quad \sigma = \pi a_i.$$

5. Properties of the Flag Weak Order

5.1. **Lattice Structure.** For a set of pairs $A \subseteq \{(i,j) : 1 \le i < j \le n\}$ let $\mathrm{M}(A) := \{j : (i,j) \in A\}$. For example, $\mathrm{M}(\{(1,6),(1,4),(2,3),(4,6)\}) = \{3,4,6\}$.

Proposition 5.1. *For every $\pi, \sigma \in B_n$,*

$$\pi \preceq \sigma \quad \textit{if and only if} \quad \begin{aligned} &\mathrm{Inv}(|\pi^{-1}|) \subseteq \mathrm{Inv}(|\sigma^{-1}|) \quad \text{and} \\ &\mathrm{Neg}(\pi^{-1}) \setminus \mathrm{Neg}(\sigma^{-1}) \subseteq \mathrm{M}\left[\mathrm{Inv}(|\sigma^{-1}|) \setminus \mathrm{Inv}(|\pi^{-1}|)\right]. \end{aligned} \tag{3}$$

Proof. $\Longrightarrow$: It suffices to show that the right-hand side of (3) holds whenever σ covers π in $(B_n, \preceq)$. By Corollary 4.3, there are two cases to check:

(i) There exists an integer i with $1 \le i \le n$ such that $i \notin \mathrm{Neg}(\pi)$ and $\sigma = \pi b_i$. Then clearly $\mathrm{Inv}(|\pi^{-1}|) = \mathrm{Inv}(|\sigma^{-1}|)$ and $\mathrm{Neg}(\pi^{-1}) \setminus \mathrm{Neg}(\sigma^{-1}) = \emptyset$.

(ii) There exists an integer i with $1 \le i \le n-1$ such that $i+1 \in \mathrm{Neg}(\pi)$, $|\pi(i)| < |\pi(i+1)|$ and $\sigma = \pi a_i$. Defining $p := |\pi(i)|$ and $q := |\pi(i+1)|$, we have

$$p < q, \qquad \sigma(i+1) = \pi(i) = \pm p, \qquad \sigma(i) = -\pi(i+1) = q.$$

Thus $\mathrm{Inv}(|\sigma^{-1}|) = \mathrm{Inv}(|\pi^{-1}|) \cup \{(p,q)\}$ and $\mathrm{Neg}(\pi^{-1}) \setminus \mathrm{Neg}(\sigma^{-1}) = \{q\}$.

$\Longleftarrow$: Assume that the right-hand side of (3) holds. There are two cases to consider:

(i) $\mathrm{Inv}(|\pi^{-1}|) = \mathrm{Inv}(|\sigma^{-1}|)$. Then $|\pi^{-1}| = |\sigma^{-1}|$ and $\mathrm{Neg}(\pi^{-1}) \setminus \mathrm{Neg}(\sigma^{-1}) = \emptyset$, i.e., $\mathrm{Neg}(\pi^{-1}) \subseteq \mathrm{Neg}(\sigma^{-1})$. It is clear that one can get from π to σ by a sequence of right multiplications by various b_i, each step increasing $\mathrm{finv}(\cdot)$ by 1. Thus $\pi \preceq \sigma$.

(ii) $\mathrm{Inv}(|\pi^{-1}|)$ is strictly contained in $\mathrm{Inv}(|\sigma^{-1}|)$. Thus $|\pi|$ is strictly smaller than $|\sigma|$ in the right weak order on S_n, and one can get from $|\pi|$ to $|\sigma|$ by a sequence of right multiplications by various Coxeter generators s_i of S_n, each step increasing the cardinality of the inversion set by 1. Let $s_{i_1}, \ldots, s_{i_k}$ be such a sequence, so that $|\sigma| = |\pi| s_{i_1} \cdots s_{i_k}$. Let $a_{i_1}, \ldots, a_{i_k}$ be the corresponding sequence of generators of B_n. Define $\pi_0 := \pi$ and, recursively,

$$\pi_j := \pi_{j-1} \tilde{a}_j, \quad 1 \le j \le k,$$

where

$$\tilde{a}_j := \begin{cases} a_{i_j}, & \text{if } i_j + 1 \in \mathrm{Neg}(\pi_{j-1}); \\ b_{i_j+1} a_{i_j}, & \text{otherwise.} \end{cases}$$

It is easy to see that

$$\pi = \pi_0 \preceq \pi_1 \preceq \cdots \preceq \pi_k,$$

with $\mathrm{finv}(\pi_j) - \mathrm{finv}(\pi_{j-1}) \in \{1, 2\}$, $j = 2, \ldots, k$. We shall show that $\pi_k \preceq \sigma$, implying $\pi \preceq \sigma$.

Indeed $|\pi_k| = |\sigma|$, and in particular $\mathrm{Inv}(|\pi_k^{-1}|) = \mathrm{Inv}(|\sigma^{-1}|)$. Moreover, for each $1 \le j \le k$, we have

$$\mathrm{Neg}(\pi_j^{-1}) = \begin{cases} \mathrm{Neg}(\pi_{j-1}^{-1}) \setminus \{|\pi_{j-1}(i_j+1)|\}, & \text{if } i_j + 1 \in \mathrm{Neg}(\pi_{j-1}); \\ \mathrm{Neg}(\pi_{j-1}^{-1}), & \text{otherwise.} \end{cases}$$

Since $\mathrm{M}\left[\mathrm{Inv}(|\pi_j^{-1}|) \setminus \mathrm{Inv}(|\pi_{j-1}^{-1}|)\right] = \{|\pi_{j-1}(i_j+1)|\}$ we conclude that, in both cases,

$$\mathrm{Neg}(\pi_j^{-1}) = \mathrm{Neg}(\pi_{j-1}^{-1}) \setminus \mathrm{M}\left[\mathrm{Inv}(|\pi_j^{-1}|) \setminus \mathrm{Inv}(|\pi_{j-1}^{-1}|)\right].$$

Thus

$$\begin{aligned} \mathrm{Neg}(\pi_k^{-1}) &= \mathrm{Neg}(\pi_0^{-1}) \setminus \bigcup_{j=1}^{k} \mathrm{M}\left[\mathrm{Inv}(|\pi_j^{-1}|) \setminus \mathrm{Inv}(|\pi_{j-1}^{-1}|)\right] \\ &= \mathrm{Neg}(\pi_0^{-1}) \setminus \mathrm{M}\left[\mathrm{Inv}(|\pi_k^{-1}|) \setminus \mathrm{Inv}(|\pi_0^{-1}|)\right], \end{aligned}$$

where we have used the property $\bigcup_j M[A_j] = M\left[\bigcup_j A_j\right]$ and the fact that $\mathrm{Inv}(|\pi_{j-1}^{-1}|) \subseteq \mathrm{Inv}(|\pi_j^{-1}|)$. Since $\pi_0 = \pi$ and $\mathrm{Inv}(|\pi_k^{-1}|) = \mathrm{Inv}(|\sigma^{-1}|)$, we conclude that

$$\mathrm{Neg}(\pi_k^{-1}) = \mathrm{Neg}(\pi^{-1}) \setminus M\left[\mathrm{Inv}(|\sigma^{-1}|) \setminus \mathrm{Inv}(|\pi^{-1}|)\right].$$

Our assumption

$$\mathrm{Neg}(\pi^{-1}) \setminus \mathrm{Neg}(\sigma^{-1}) \subseteq \mathrm{M}\left[\mathrm{Inv}(|\sigma^{-1}|) \setminus \mathrm{Inv}(|\pi^{-1}|)\right]$$

is equivalent to

$$\mathrm{Neg}(\sigma^{-1}) \supseteq \mathrm{Neg}(\pi^{-1}) \setminus \mathrm{M}\left[\mathrm{Inv}(|\sigma^{-1}|) \setminus \mathrm{Inv}(|\pi^{-1}|)\right],$$

namely to

$$\mathrm{Neg}(\sigma^{-1}) \supseteq \mathrm{Neg}(\pi_k^{-1}).$$

Together with $\mathrm{Inv}(|\pi_k^{-1}|) = \mathrm{Inv}(|\sigma^{-1}|)$, this implies, by case (i) above, that $\pi_k \preceq \sigma$. □

Proposition 5.2. *The poset $(B_n, \preceq)$ is a lattice.*

Proof. For simplicity of notation, let

$$\mathrm{M}(|\pi|, |\sigma|) := \mathrm{M}\left[\mathrm{Inv}(|\sigma^{-1}|) \setminus \mathrm{Inv}(|\pi^{-1}|)\right], \quad \pi, \sigma \in B_n. \tag{4}$$

Proposition 5.1 can be stated as:

$$\pi \preceq \sigma \quad \text{if and only if}$$
$$\mathrm{Inv}(|\pi^{-1}|) \subseteq \mathrm{Inv}(|\sigma^{-1}|) \;\text{ and }\; \mathrm{Neg}(\pi^{-1}) \subseteq \mathrm{Neg}(\sigma^{-1}) \cup \mathrm{M}(|\pi|, |\sigma|).$$

Let σ_1, σ_2 be two elements of B_n. It follows that, for any $\pi \in B_n$,

$$\begin{aligned} &\pi \preceq \sigma_1 \;\text{ and }\; \pi \preceq \sigma_2 \quad \text{if and only if} \\ &\mathrm{Inv}(|\pi^{-1}|) \subseteq \mathrm{Inv}(|\sigma_1^{-1}|) \cap \mathrm{Inv}(|\sigma_2^{-1}|) \;\text{ and} \\ &\mathrm{Neg}(\pi^{-1}) \subseteq \left(\mathrm{Neg}(\sigma_1^{-1}) \cup \mathrm{M}(|\pi|, |\sigma_1|)\right) \cap \left(\mathrm{Neg}(\sigma_2^{-1}) \cup \mathrm{M}(|\pi|, |\sigma_2|)\right). \end{aligned} \tag{5}$$

We shall now define a candidate for the meet (in B_n) of σ_1 and σ_2, and prove that it has the required properties. First note that the intersection of inversion sets (of permutations in S_n) is not necessarily an inversion set. Nevertheless, since S_n under right weak order is a lattice, there exists a meet

$$\tau = |\sigma_1| \wedge_{S_n} |\sigma_2| \in S_n$$

which satisfies, by (1),

$$\mathrm{Inv}(\tau^{-1}) \subseteq \mathrm{Inv}(|\sigma_1^{-1}|) \cap \mathrm{Inv}(|\sigma_2^{-1}|)$$

and

(6)

$$\mathrm{Inv}(\gamma^{-1}) \subseteq \mathrm{Inv}(|\sigma_1^{-1}|) \cap \mathrm{Inv}(|\sigma_2^{-1}|) \quad \text{implies} \quad \mathrm{Inv}(\gamma^{-1}) \subseteq \mathrm{Inv}(\tau^{-1}), \quad \text{for all } \gamma \in S_n).$$

Define $\sigma_\wedge \in B_n$ by

$$|\sigma_\wedge| := |\sigma_1| \wedge_{S_n} |\sigma_2| \quad (= \tau) \tag{7}$$

and

$$\mathrm{Neg}(\sigma_\wedge^{-1}) := \left(\mathrm{Neg}(\sigma_1^{-1}) \cup \mathrm{M}(\tau, |\sigma_1|)\right) \cap \left(\mathrm{Neg}(\sigma_2^{-1}) \cup \mathrm{M}(\tau, |\sigma_2|)\right). \tag{8}$$

Then clearly $\sigma_\wedge \preceq \sigma_1$ and $\sigma_\wedge \preceq \sigma_2$. It remains to show that $\pi \preceq \sigma_1$ and $\pi \preceq \sigma_2$ implies $\pi \preceq \sigma_\wedge$. This is straightforward if $\mathrm{Inv}(|\pi^{-1}|) = \mathrm{Inv}(|\sigma_\wedge^{-1}|)$, but more intricate otherwise.

Assume that $\pi \preceq \sigma_1$ and $\pi \preceq \sigma_2$. Then

$$\mathrm{Inv}(|\pi^{-1}|) \subseteq \mathrm{Inv}(|\sigma_1^{-1}|) \cap \mathrm{Inv}(|\sigma_2^{-1}|),$$

so that, by (6),

$$\mathrm{Inv}(|\pi^{-1}|) \subseteq \mathrm{Inv}(\tau^{-1}) = \mathrm{Inv}(|\sigma_\wedge^{-1}|).$$

From

$$\mathrm{Inv}(|\pi^{-1}|) \subseteq \mathrm{Inv}(|\sigma_\wedge^{-1}|) \subseteq \mathrm{Inv}(|\sigma_1^{-1}|)$$

it now follows that

$$\mathrm{Inv}(|\sigma_1^{-1}|) \setminus \mathrm{Inv}(|\pi^{-1}|) = \left(\mathrm{Inv}(|\sigma_1^{-1}|) \setminus \mathrm{Inv}(|\sigma_\wedge^{-1}|)\right) \cup \left(\mathrm{Inv}(|\sigma_\wedge^{-1}|) \setminus \mathrm{Inv}(|\pi^{-1}|)\right),$$

and therefore

$$\mathrm{M}(|\pi|, |\sigma_1|) = \mathrm{M}(|\pi|, |\sigma_\wedge|) \cup \mathrm{M}(|\sigma_\wedge|, |\sigma_1|).$$

A similar assertion holds for σ_2. From (5) it thus follows that

$$\begin{aligned}
\mathrm{Neg}(\pi^{-1}) &\subseteq \left(\mathrm{Neg}(\sigma_1^{-1}) \cup \mathrm{M}(|\pi|, |\sigma_1|)\right) \cap \left(\mathrm{Neg}(\sigma_2^{-1}) \cup \mathrm{M}(|\pi|, |\sigma_2|)\right) \\
&= \left(\mathrm{Neg}(\sigma_1^{-1}) \cup \mathrm{M}(|\pi|, |\sigma_\wedge|) \cup \mathrm{M}(|\sigma_\wedge|, |\sigma_1|)\right) \\
&\qquad\qquad \cap \left(\mathrm{Neg}(\sigma_2^{-1}) \cup \mathrm{M}(|\pi|, |\sigma_\wedge|) \cup \mathrm{M}(|\sigma_\wedge|, |\sigma_2|)\right) \\
&= \mathrm{M}(|\pi|, |\sigma_\wedge|) \\
&\qquad \cup \left[\left(\mathrm{Neg}(\sigma_1^{-1}) \cup \mathrm{M}(|\sigma_\wedge|, |\sigma_1|)\right) \cap \left(\mathrm{Neg}(\sigma_2^{-1}) \cup \mathrm{M}(|\sigma_\wedge|, |\sigma_2|)\right)\right] \\
&= \mathrm{M}(|\pi|, |\sigma_\wedge|) \cup \mathrm{Neg}(\sigma_\wedge^{-1}),
\end{aligned}$$

using definition (8) of $\mathrm{Neg}(\sigma_\wedge^{-1})$. In other words, $\pi \preceq \sigma_\wedge$, as required.

We have shown the existence of meets in $(B_n, \preceq)$. The existence of joins follows by self-duality (Proposition 4.2(ii)):

$$\sigma_1 \vee \sigma_2 = (\sigma_1\mu_0 \wedge \sigma_2\mu_0)\mu_0.$$

□

Note that (7)–(8) in the proof of Proposition 5.2 provide an explicit description of the meet of two elements. One can generalize this description to any number of elements, using the notation $\mathrm{M}(|\pi|, |\sigma|)$ from (4). For the corresponding description of the join it is convenient to use also the notation $\mathrm{Pos}(\sigma) := [n] \setminus \mathrm{Neg}(\sigma)$ for $\sigma \in B_n$.

Lemma 5.3. *Let A be an arbitrary subset of B_n.*

(i) *The meet $A_\wedge$ of A in $(B_n, \preceq)$ is determined by*

$$|A_\wedge| := \bigwedge_{\sigma \in A} |\sigma|,$$

where the meet is taken with respect to the (right) weak order on S_n, and by

$$\mathrm{Neg}(A_\wedge^{-1}) := \bigcap_{\sigma \in A} \left(\mathrm{Neg}(\sigma^{-1}) \cup \mathrm{M}(|A_\wedge|, |\sigma|)\right).$$

(ii) *The join $A_\vee$ of A in $(B_n, \preceq)$ is determined by*

$$|A_\vee| := \bigvee_{\sigma \in A} |\sigma|$$

and by

$$\mathrm{Pos}(A_\vee^{-1}) := \bigcap_{\sigma\in A} \left(\mathrm{Pos}(\sigma^{-1}) \cup \mathrm{M}(|\sigma|, |A_\vee|)\right).$$

Remark 5.4. For $n \geq 3$, the lattice $(B_n, \preceq)$ is not semi-modular. To verify this, notice that $\pi = 2\bar{1}\bar{3}$ and $\sigma = \bar{1}3\bar{2}$ cover their meet $\pi \wedge \sigma = \bar{1}\bar{2}\bar{3}$ but are not covered by their join $\pi \vee \sigma = 32\bar{1}$. Moreover, for $n \geq 2$, the lattice $(B_n, \preceq)$ is not complemented, since $1\bar{2}$ has no complement in $(B_2, \preceq)$.

5.2. **Homotopy Type and Möbius Function.** The following results generalize well-known properties of the classical weak order on a Coxeter group. Recall that an *atom* in an interval $[\pi, \sigma]$ is an element $\tau \in [\pi, \sigma]$ covering π. Recall also the notation $A_\vee$ from Lemma 5.3(ii).

Lemma 5.5. *Suppose that $\pi \prec \sigma$ in B_n. Then, for any two sets A and B of atoms in the interval $[\pi, \sigma]$, we have*

$$A \neq B \implies A_\vee \neq B_\vee.$$

Proof. For a set A of atoms in the interval $[\pi, \sigma]$, define

$$A_1 := A \cap \{\pi a_i \,:\, 1 \leq i \leq n-1\}$$

and

$$A_2 := A \cap \{\pi b_i \,:\, 1 \leq i \leq n\}.$$

Assume now that A and B are sets of atoms in $[\pi, \sigma]$ such that $A_\vee = B_\vee$. We shall prove that $A = B$.

Since $|\pi b_i| = |\pi|$ for all i, it follows from Lemma 5.3(ii) that

$$|A_\vee| = \bigvee_{\tau\in A} |\tau| = \bigvee_{\tau\in A_1} |\tau|,$$

where joins are taken in S_n. Consequently, we have

$$A_\vee = B_\vee \quad \text{implies} \quad |A_\vee| = |B_\vee|, \quad \text{and the latter implies} \quad A_1 = B_1.$$

The last implication holds since joins of sets of atoms uniquely determine the sets in any interval in the usual weak order on S_n; see, e.g., [3, Lemma 3.2.4(i)].

We still need to show that $A_2 = B_2$. If $\sigma = \pi a_i$ covers π then, by definition, $\mathrm{Pos}(\sigma^{-1})$ is the (disjoint) union of $\mathrm{Pos}(\pi^{-1})$ and $\{|\pi(i+1)|\}$, while $\mathrm{M}(|\pi|, |A_\vee|)$ is the (not necessarily disjoint) union of $\mathrm{M}(|\sigma|, |A_\vee|)$ and $\{|\pi(i+1)|\}$. Hence, for every $\sigma \in A_1$, we have

$$\mathrm{Pos}(\sigma^{-1}) \cup \mathrm{M}(|\sigma|, |A_\vee|) = \mathrm{Pos}(\pi^{-1}) \cup \mathrm{M}(|\pi|, |A_\vee|).$$

On the other hand, if $\sigma = \pi b_i \in A_2$, then

$$\mathrm{Pos}(\sigma^{-1}) \cup \mathrm{M}(|\sigma|, |A_\vee|) = (\mathrm{Pos}(\pi^{-1}) \setminus \{\pi(i)\}) \cup \mathrm{M}(|\pi|, |A_\vee|).$$

Thus, by Lemma 5.3(ii), we obtain

$$\begin{aligned}
\mathrm{Pos}(A_\vee^{-1}) &= \bigcap_{\sigma\in A_1\cup A_2} \big(\mathrm{Pos}(\sigma^{-1}) \cup \mathrm{M}(|\sigma|, |A_\vee|)\big) \\
&= \big(\mathrm{Pos}(\pi^{-1}) \cup \mathrm{M}(|\pi|, |A_\vee|)\big) \cap \bigcap_{\sigma\in A_2} \big(\mathrm{Pos}(\sigma^{-1}) \cup \mathrm{M}(|\sigma|, |A_\vee|)\big) \\
&= \big(\mathrm{Pos}(\pi^{-1}) \cup \mathrm{M}(|\pi|, |A_\vee|)\big) \cap \bigcap_{\pi b_i\in A_2} \big((\mathrm{Pos}(\pi^{-1}) \setminus \{\pi(i)\}) \cup \mathrm{M}(|\pi|, |A_\vee|)\big) \\
&= \big(\mathrm{Pos}(\pi^{-1}) \setminus \{\pi(i) \,:\, \pi b_i \in A_2\}\big) \cup \mathrm{M}(|\pi|, |A_\vee|).
\end{aligned}$$

(The intermediate steps, though not the end result, should be slightly rewritten if $A_1 = \emptyset$.) If we show that

$$\pi b_i \in A_2 \quad \text{implies} \quad \pi(i) \notin \mathrm{M}(|\pi|, |A_\vee|), \tag{9}$$

it will then follow that, assuming $|A_\vee| = |B_\vee|$,

$$\mathrm{Pos}(A_\vee^{-1}) = \mathrm{Pos}(B_\vee^{-1}) \quad \text{if and only if} \quad A_2 = B_2. \tag{10}$$

Indeed,

$$\pi b_i \in A_2 \quad \text{implies} \quad \pi(i) > 0, \quad \text{and the latter implies} \quad \pi a_{i-1} \notin A_1.$$

An examination of $|A_\vee|$ as a join of atoms in an interval of S_n shows that

$$\mathrm{M}(|\pi|, |A_\vee|) \subseteq \{|\pi(i)| \,:\, \pi a_{i-1} \in A_1\},$$

which implies (9) and (10) and completes the proof. $\square$

Lemma 5.5 leads to an easy way to determine the homotopy type and Möbius function for open intervals in $(B_n, \preceq)$, generalizing [3, Theorem 3.2.7 and Corollary 3.2.8].

Proposition 5.6. *Suppose that $\pi \prec \sigma$ in B_n and $\mathrm{finv}(\sigma) - \mathrm{finv}(\pi) \geq 2$. Then the order complex of the open interval (π, σ) is homotopy equivalent to the sphere $\mathbf{S}^{k-2}$ if σ is the join of k atoms in the interval $[\pi, \mu_0]$, and is contractible otherwise.*

Corollary 5.7. *For every $\pi, \sigma \in B_n$,*

$$\mu(\pi, \sigma) = \begin{cases} (-1)^k, & \text{if } \sigma \text{ is the join of } k \text{ atoms in } [\pi, \mu_0]; \\ 0, & \text{otherwise.} \end{cases}$$

The proofs of Proposition 5.6 and Corollary 5.7 are along the lines of the analogous proofs for the symmetric group [3, Theorem 3.2.7 and Corollary 3.2.8], and are left to the reader.

5.3. **Tits Property.** In this subsection, it will be shown that maximal chains in $(B_n, \preceq)$ exhibit a Tits-type connectivity property.

Let $\pi, \sigma \in B_n$ such that $\pi \preceq \sigma$. Each maximal chain in the interval $[\pi, \sigma]$ of $(B_n, \preceq)$ corresponds to a unique word $w = s_{i_1} \cdots s_{i_d}$ of length $d = \mathrm{finv}(\sigma) - \mathrm{finv}(\pi)$ with letters s_{i_j} in the alphabet $S_{2,n}$, such that $\mathrm{finv}(\pi s_{i_1} \cdots s_{i_j}) - \mathrm{finv}(\pi) = j$ for all $1 \leq j \leq d$.

Proposition 5.8 (Tits Property)**.** *Any two maximal chains in any interval $[\pi,\sigma]$ of $(B_n,\preceq)$ are connected via the following pseudo-Coxeter moves on the corresponding words:*

$$(T1)\qquad b_ib_j \longleftrightarrow b_jb_i, \quad 1\le i<j\le n,$$
$$(T2)\qquad a_ib_j \longleftrightarrow b_ja_i, \quad j\ne i,i+1,$$
$$(T3)\qquad a_ib_{i+1} \longleftrightarrow b_ia_i, \quad 1\le i\le n-1,$$
$$(T4)\qquad a_ia_j \longleftrightarrow a_ja_i, \quad |i-j|>1,$$

and

$$(T5)\qquad a_ia_{i+1}b_{i+1}a_i \longleftrightarrow a_{i+1}b_{i+1}a_ia_{i+1}, \quad 1\le i\le n-1.$$

In order to prove this proposition, we first classify maximal chains in certain special intervals.

Lemma 5.9. *Let $\pi\in B_n$, $s,s'\in S_{2,n}$, $s\ne s'$, such that both πs and $\pi s'$ cover π. Then the following assertions hold true:*

(i) *The interval $[\pi,\pi s\vee\pi s']$ contains exactly two maximal chains, one described by a word starting with s, and one by a word starting with s'.*
(ii) *The above words are independent of π, as long as both πs and $\pi s'$ cover π. Denote by $\alpha(s,s')$ the word corresponding to the chain starting with s.*
(iii) *The complete list of words corresponding to maximal chains in intervals of the form $[\pi,\pi s\vee\pi s']$ in $(B_n,\preceq)$ is:*

$$\alpha(b_i,b_j)=b_ib_j,\quad i\ne j;$$
$$\alpha(a_i,b_j)=a_ib_j,\quad \alpha(b_j,a_i)=b_ja_i,\quad j\ne i,i+1;$$
$$\alpha(a_i,b_i)=a_ib_{i+1},\quad \alpha(b_i,a_i)=b_ia_i,\quad 1\le i\le n-1;$$
$$\alpha(a_i,a_j)=a_ia_j,\quad |i-j|>1;$$

and

$$\alpha(a_i,a_{i+1})=a_ia_{i+1}b_{i+1}a_i,\quad \alpha(a_{i+1},a_i)=a_{i+1}b_{i+1}a_ia_{i+1},\quad 1\le i\le n-2.$$

The words $\alpha(a_i,b_{i+1})$ and $\alpha(b_{i+1},a_i)$ do not exist, since πa_i and πb_{i+1} cannot cover π simultaneously.

Proof of Lemma 5.9. First of all, πa_i and πb_{i+1} cannot cover π simultaneously since, by Corollary 4.3,

$$\pi a_i \text{ covers } \pi \quad\text{implies}\quad \pi(i+1)<0, \quad\text{and the latter implies}\quad \pi b_{i+1}\prec\pi.$$

We shall deal with the other cases one by one.

If $i\ne j$ then, by Corollary 4.3, πb_i and πb_j cover π if and only if $\pi(i)>0$ and $\pi(j)>0$. Then $\pi b_i\vee\pi b_j=\pi b_ib_j$, and indeed $\alpha(b_i,b_j)=b_ib_j$ is unique and independent of π.

If $j\ne i,i+1$ then, by Corollary 4.3, πa_i and πb_j cover π if and only if $\pi(i+1)<0$, $|\pi(i)|<|\pi(i+1)|$ and $\pi(j)>0$. Then $\pi a_i\vee\pi b_j=\pi a_ib_j=\pi b_ja_i$, so that $\alpha(a_i,b_j)=a_ib_j$ and $\alpha(b_j,a_i)=b_ja_i$ are clearly unique and independent of π.

If πa_i and πb_i cover π (for some $1 \le i \le n-1$) then, by Corollary 4.3, $\pi(i) > 0$, $\pi(i+1) < 0$ and $|\pi(i)| < |\pi(i+1)|$. Then $\pi a_i \vee \pi b_i = \pi b_i a_i = \pi a_i b_{i+1}$, and again $\alpha(a_i, b_i) = a_i b_{j+1}$ and $\alpha(b_i, a_i) = b_i a_i$ are unique and independent of π.

Similarly, if $|i-j| > 1$ then πa_i and πa_j cover π if and only if $\pi(i+1) < 0$, $|\pi(i)| < |\pi(i+1)|$, $\pi(j+1) < 0$ and $|\pi(j)| < |\pi(j+1)|$. Then $\pi a_i \vee \pi a_j = \pi a_i a_j$, so that $\alpha(a_i, a_j) = a_i a_j$ is clearly unique and independent of π.

In all of the above cases, the interval $[\pi, \pi s \vee \pi s']$ is of length 2. It is easy to identify which generators a_k appear in a maximal chain, and the rest readily follows.

Let us now turn to the last case, and assume that πa_i and πa_{i+1} cover π, for some $1 \le i \le n-2$. By Corollary 4.3, $\pi(i+1) < 0$, $\pi(i+2) < 0$ and $|\pi(i)| < |\pi(i+1)| < |\pi(i+2)|$. By Lemma 5.3(ii), $\sigma := \pi a_i \vee \pi a_{i+1}$ satisfies $|\sigma| = |\pi| s_i s_{i+1} s_i$ and $\mathrm{Pos}(\sigma^{-1}) = \mathrm{Pos}(\pi^{-1}) \cup \{|\pi(i+1)|, |\pi(i+2)|\}$. Thus $\mathrm{finv}(\sigma) - \mathrm{finv}(\pi) = 2 \cdot 3 - 2 = 4$. The only maximal chains in S_n from $|\pi|$ to $|\sigma|$ correspond to $s_i s_{i+1} s_i$ and $s_{i+1} s_i s_{i+1}$, and thus each maximal chain from π to σ must correspond to either $a_i a_{i+1} a_i$ or $a_{i+1} a_i a_{i+1}$, with one additional letter of type b_k. It is easy to see that the only possibilities are $\alpha(a_i, a_{i+1}) = a_i a_{i+1} b_{i+1} a_i$ and $\alpha(a_{i+1}, a_i) = a_{i+1} b_{i+1} a_i a_{i+1}$. □

Proof of Proposition 5.8. The proof is similar to the analogous proof for the symmetric group [3, Theorem 3.3.1]. It proceeds by induction on the difference between the ranks of the top and bottom elements.

If the difference is zero then the statement obviously holds.

Assume that the difference is $k > 0$. Consider two maximal chains in the interval $[\pi, \sigma]$, corresponding to the words $s s_2 \cdots s_k$ and $s' s'_2 \cdots s'_k$, all letters being in $S_{2,n}$. Thus

$$\sigma = \pi s s_2 \cdots s_k = \pi s' s'_2 \cdots s'_k.$$

If $s = s'$ then the statement holds by the induction hypothesis for the interval $[\pi s, \sigma]$.

If $s \ne s'$ then $\pi s \preceq \sigma$ and $\pi s' \preceq \sigma$. By the lattice property, $\pi s \vee \pi s' \preceq \sigma$. By Lemma 5.9 there exists a maximal chain in the interval $[\pi, \pi s \vee \pi s']$ corresponding to the word $\alpha(s, s')$ starting with s. It can be extended to a maximal chain in $[\pi, \sigma]$ corresponding to the word $\alpha(s, s')\beta$, where β corresponds to some maximal chain in $[\pi s \vee \pi s', \sigma]$. Both words $s s_2 \cdots s_k$ and $\alpha(s, s')\beta$ start with s. By the induction hypothesis for $[\pi s, \sigma]$, it is possible to transform $s s_2 \cdots s_k$ into $\alpha(s, s')\beta$ using the moves $(T1)$–$(T5)$. By the same argument applied to s', it is possible to transform $\alpha(s', s)\beta$ into $s' s'_2 \cdots s'_k$ using the moves $(T1)$–$(T5)$. Finally, by Lemma 5.9, it is possible to transform $\alpha(s, s')\beta$ into $\alpha(s', s)\beta$ using one of the moves $(T1)$–$(T5)$, thus completing the proof. □

6. Bivariate Distribution

Let

$$E_n(t) := \sum_{\pi \in S_n} t^{\mathrm{des}(\pi)}$$

be the *Eulerian Polynomial.* More generally, let

$$S_n(q, t) := \sum_{\pi \in S_n} q^{\mathrm{inv}(\pi)} t^{\mathrm{des}(\pi)}.$$

Recall that $(B_n, \preceq)$ is graded by finv.

Definition 6.1. For every $\pi \in B_n$ let $\mathrm{wdes}(\pi)$ be the number of elements in B_n which are covered by π in the poset $(B_n, \preceq)$.

Lemma 6.2. *For every $\pi \in B_n$, we have*

$$\mathrm{wdes}(\pi) = \#\left(\mathrm{Des}(|\pi|) \cup \mathrm{Neg}(\pi)\right).$$

Proof. By Corollary 4.3 (with π and σ interchanged), σ is covered by π in $(B_n, \preceq)$ if and only if

(i) there exists $1 \le i \le n$, such that

$$i \in \mathrm{Neg}(\pi) \qquad \text{and} \qquad \sigma = \pi b_i;$$

or

(ii) there exists $1 \le i \le n-1$, such that

$$i \notin \mathrm{Neg}(\pi), \qquad |\pi(i)| > |\pi(i+1)| \qquad \text{and} \qquad \sigma = \pi a_i^{-1}.$$

Hence, the set of elements which are covered by π in $(B_n, \preceq)$ is

$$\{\pi b_i :\ i \in \mathrm{Neg}(\pi)\} \cup \{\pi a_i^{-1} :\ i \in \mathrm{Des}(|\pi|) \setminus \mathrm{Neg}(\pi)\},$$

a disjoint union.

It follows that

$$\mathrm{wdes}(\pi) = \#\mathrm{Neg}(\pi) + \#(\mathrm{Des}(|\pi|) \setminus \mathrm{Neg}(\pi)) = \#(\mathrm{Des}(|\pi|) \cup \mathrm{Neg}(\pi)).$$

□

Proposition 6.3. *For every n, we have*

$$\sum_{\pi \in B_n} t^{\mathrm{wdes}(\pi)} = (1+t)^n \cdot E_n\left(\frac{2t}{1+t}\right)$$

and

$$\sum_{\pi \in B_n} q^{\mathrm{finv}(\pi)} t^{\mathrm{wdes}(\pi)} = (1+qt)^n \cdot S_n\left(q^2, \frac{(1+q)t}{1+qt}\right).$$

Remark 6.4. By a well known result of Stanley [12], $S_n(q,t)$ has an elegant q-exponential generating function; for details and generalizations see [3, §7.2]. It follows that the same is true when the pair $(\mathrm{finv}, \mathrm{wdes})$ is used instead of $(\mathrm{inv}, \mathrm{des})$.

Proof of Proposition 6.3. $\mathbb{Z}_2^n$ and S_n can be viewed as subgroups of B_n, restricting elements $\pi \in B_n$ to have $|\pi| = id$ or $\pi(i) > 0$, $i = 1, \ldots, n$, respectively. Moreover, every $\pi \in B_n$ can be written in the form $\pi = vu$ for some $u \in \mathbb{Z}_2^n$ and $v = |\pi| \in S_n$. Hence,

$$\sum_{\pi \in B_n} t^{\mathrm{wdes}(\pi)} = \sum_{u \in \mathbb{Z}_2^n} \sum_{v \in S_n} t^{\mathrm{wdes}(vu)}.$$

By Lemma 6.2, the right-hand side is equal to

$$\begin{aligned}
\sum_{u\in\mathbb{Z}_2^n}\sum_{v\in S_n} t^{\#(\mathrm{Des}(v)\cup\mathrm{Neg}(u))} &= \sum_{v\in S_n}\sum_{u\in\mathbb{Z}_2^n} t^{\#\mathrm{Des}(v)}t^{\#(\mathrm{Neg}(u)\setminus\mathrm{Des}(v))}\\
&= \sum_{v\in S_n} t^{\#\mathrm{Des}(v)}\sum_{u\in\mathbb{Z}_2^n} t^{\#(\mathrm{Neg}(u)\setminus\mathrm{Des}(v))}\\
&= \sum_{v\in S_n} t^{\#\mathrm{Des}(v)}2^{\#\mathrm{Des}(v)}(1+t)^{n-\#\mathrm{Des}(v)}\\
&= (1+t)^n\sum_{v\in S_n}\left(\frac{2t}{1+t}\right)^{\#\mathrm{Des}(v)}\\
&= (1+t)^n\cdot E_n\left(\frac{2t}{1+t}\right).
\end{aligned}$$

The proof of the second identity is similar:

$$\begin{aligned}
\sum_{\pi\in B_n} q^{\mathrm{finv}(\pi)}t^{\mathrm{wdes}(\pi)} &= \sum_{u\in\mathbb{Z}_2^n}\sum_{v\in S_n} q^{\mathrm{finv}(vu)}t^{\mathrm{wdes}(vu)}\\
&= \sum_{u\in\mathbb{Z}_2^n}\sum_{v\in S_n} q^{2\cdot\mathrm{inv}(v)+\#\mathrm{Neg}(u)}t^{\#(\mathrm{Des}(v)\cup\mathrm{Neg}(u))}\\
&= \sum_{v\in S_n} q^{2\cdot\mathrm{inv}(v)}t^{\#\mathrm{Des}(v)}\sum_{u\in\mathbb{Z}_2^n} q^{\#\mathrm{Neg}(u)}t^{\#(\mathrm{Neg}(u)\setminus\mathrm{Des}(v))}\\
&= \sum_{v\in S_n} q^{2\cdot\mathrm{inv}(v)}t^{\#\mathrm{Des}(v)}(1+q)^{\#\mathrm{Des}(v)}(1+qt)^{n-\#\mathrm{Des}(v)}\\
&= (1+qt)^n\sum_{v\in S_n} q^{2\cdot\mathrm{inv}(v)}\left(\frac{(1+q)t}{1+qt}\right)^{\#\mathrm{Des}(v)}\\
&= (1+qt)^n\cdot S_n\left(q^2,\frac{(1+q)t}{1+qt}\right).
\end{aligned}$$

□

7. Wreath Products

The above results generalize to the group $G(r,n) := \mathbb{Z}_r \wr S_n$, for every positive integer r. The proofs are similar and will be left to the reader.

For $1 \le i \le n$ define the vector $d_i := (\delta_{i1},\ldots,\delta_{in}) \in \mathbb{Z}_r^n$, where

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j;\\ 0, & \text{otherwise.}\end{cases}$$

Let

$$a_i := (d_i, s_i), \quad 1 \le i \le n-1,$$

and

$$b_i := (d_i, id), \quad 1 \le i \le n.$$

Proposition 7.1. *The wreath product $G(r,n) = \mathbb{Z}_r \wr S_n$ is generated by the set $S_{r,n} := \{a_i : 1 \le i \le n-1\} \cup \{b_i : 1 \le i \le n\}$ with defining relations $(B1)$–$(B8)$ of Proposition 3.2, except that relation $(B1)$ is replaced by*

$$(B1_r) \qquad b_i^r = 1, \quad 1 \le i \le n,$$

Recall Definition 2.1 of the flag inversion number.

Definition 7.2. The *flag* (*right*) *weak order* on $G(r,n)$, $\preceq$, is the reflexive and transitive closure of the relation

$$\pi \lessdot \pi s \qquad \text{implies} \qquad \pi \in G(r,n), s \in S_{r,n} \quad \text{and} \quad \operatorname{finv}(\pi) < \operatorname{finv}(\pi s).$$

Proposition 7.3. *The poset $(G(r,n), \preceq)$ is*

(i) *ranked (by flag inversion number);*
(ii) *self-dual (by $\pi \mapsto \bar{\pi}\mu_0$, where $\mu_0 = ((r-1,\ldots,r-1),[n,\ldots,1])$ is the unique maximal element in this order and $\bar{\pi} = ((-c_1,\ldots,-c_n),\tau)$ when $\pi = ((c_1,\ldots,c_n),\tau)$); and*
(iii) *rank-symmetric and unimodal.*

Proposition 7.4. *σ covers π in $(G(r,n), \preceq)$ if and only if either*

(i) *there exists $1 \le i \le n$ such that*

$$c_i(\pi) \ne r-1 \qquad \text{and} \qquad \sigma = \pi b_i;$$

or

(ii) *there exists $1 \le i \le n-1$ such that*

$$c_{i+1}(\pi) = r-1, \qquad |\pi(i)| < |\pi(i+1)| \qquad \text{and} \qquad \sigma = \pi a_i.$$

In the following statement, elements $-c_j(\pi^{-1}) = c_{|\pi^{-1}(j)|}(\pi) \in \mathbb{Z}_r$ are compared using the natural linear order $0 < 1 < \cdots < r-1$ on $\mathbb{Z}_r$.

Proposition 7.5. *For every $\pi, \sigma \in G(r,n)$, we have*

$$\pi \preceq \sigma \quad \text{if and only if} \quad \operatorname{Inv}(|\pi^{-1}|) \subseteq \operatorname{Inv}(|\sigma^{-1}|) \quad \text{and}$$
$$\{j : -c_j(\pi^{-1}) > -c_j(\sigma^{-1})\} \subseteq \mathrm{M}\left[\operatorname{Inv}(|\sigma^{-1}|) \setminus \operatorname{Inv}(|\pi^{-1}|)\right].$$

It follows that all the results of Section 5 can be generalized to $G(r,n)$. In particular, the following assertions hold true.

Proposition 7.6. *The poset $(G(r,n), \preceq)$ is a lattice.*

Lemma 7.7. *Let A be an arbitrary subset of $G(r,n)$.*

(i) *The meet $A_\wedge$ of A in $(G(r,n), \preceq)$ is determined by*

$$|A_\wedge| := \bigwedge_{\sigma \in A} |\sigma|,$$

where the meet is taken with respect to the (right) weak order on S_n, and by

$$-c_j(A_\wedge^{-1}) := \min\{-c_j(\sigma^{-1}) : \sigma \in A,\, j \notin M(|A_\wedge|, |\sigma|)\}, \quad 1 \le j \le n,$$

where $M(|\pi|,|\sigma|) := \mathrm{M}\left[\operatorname{Inv}(|\sigma^{-1}|) \setminus \operatorname{Inv}(|\pi^{-1}|)\right]$, and the minimum is taken with respect to the linear order $0 < 1 < \cdots < r-1$ on $\mathbb{Z}_r$, using the convention $\min \emptyset := \max \mathbb{Z}_r = r-1$.

(ii) *The join* $A_\vee$ *of* A *in* $(G(r,n),\preceq)$ *is determined by*

$$|A_\vee| := \bigvee_{\sigma\in A} |\sigma|$$

and by

$$-c_j(A_\vee^{-1}) := \max\{-c_j(\sigma^{-1}) \,:\, \sigma \in A,\, j \notin M(|\sigma|,|A_\vee|)\}, \quad 1 \le j \le n,$$

using the convention $\max\emptyset := \min \mathbb{Z}_r = 0$.

Proposition 7.8. *Suppose that* $\pi \prec \sigma$ *in* $G(r,n)$ *and* $\operatorname{finv}(\sigma) - \operatorname{finv}(\pi) \ge 2$. *Then the order complex of the open interval* (π,σ) *is homotopy equivalent to the sphere* $\mathbf{S}^{k-2}$ *if* σ *is the join of* k *atoms in the interval* $[\pi,\mu_0]$, *and is contractible otherwise.*

Corollary 7.9. *For every* $\pi,\sigma \in G(r,n)$, *we have*

$$\mu(\pi,\sigma) = \begin{cases} (-1)^k, & \textit{if } \sigma \textit{ is a join of } k \textit{ atoms in } [\pi,\mu_0]; \\ 0, & \textit{otherwise.} \end{cases}$$

Definition 7.10. For every $\pi \in G(r,n)$, we write $\operatorname{wdes}(\pi)$ for the number of elements in $G(r,n)$ which are covered by π in the poset $(G(r,n),\preceq)$.

Clearly, for $r=1$, wdes is the standard descent number. For $r=2$ it coincides with Definition 6.1.

Proposition 7.11. *For every* n *and* r, *we have*

$$\sum_{\pi\in G(r,n)} t^{\operatorname{wdes}(\pi)} = (1+(r-1)t)^n \cdot E_n\left(\frac{rt}{1+(r-1)t}\right)$$

and

$$\sum_{\pi\in G(r,n)} q^{\operatorname{finv}(\pi)} t^{\operatorname{wdes}(\pi)} = (1+[r-1]_q qt)^n \cdot S_n\left(q^r, \frac{[r]_q t}{1+[r-1]_q qt}\right).$$

8. Final Remarks and Open Problems

Recall the pseudo-Coxeter moves $(T1)$–$(T5)$ from Proposition 5.8. Consider the graph Γ_n, whose vertices are all maximal chains in the flag weak order on B_n and whose edges correspond to these moves. By Proposition 5.8, Γ_n is connected.

Problem 8.1. *Find the diameter of* Γ_n.

For a solution of an analogous problem for the classical weak orders of types A and B see [11].

Following comments of an anonymous referee, it should be noted that progress toward a solution of Problem 8.1 may be obtained by explicit calculation of various poset parameters such as order dimension and width. Another approach is a search for symmetries induced by group actions, as well as recursive poset properties such as supersolvability. Such methods were found useful in similar contexts; see, e.g., [1, 11].

It is now natural to look for a definition of a nicely-behaved weak order on other complex reflection groups. A key tool may be the discovery of convenient presentations for kernels of one-dimensional characters.

Finally, a challenging problem is to find a "correct" definition of strong (Bruhat) order on wreath products and other complex reflection groups, having desired properties (such as a nice interval structure and a subword property) which, hopefully, demonstrate an interplay with the flag weak order. Such an order may be useful in developing an appropriate Kazhdan–Lusztig theory.

Acknowledgements. The authors thank the anonymous referees for many helpful comments.

References

[1] R. M. Adin, M. Firer and Y. Roichman, *Triangle-free triangulations*, Adv. Appl. Math. **45** (2010), 77–95.

[2] R. M. Adin, Y. Cherniavsky and Y. Roichman, *Strong order on colored rook monoids and (q,t)-Stirling numbers*, in preparation.

[3] A. Björner and F. Brenti, Combinatorics of Coxeter groups, Graduate Texts in Mathematics, 231, Springer, New York, 2005.

[4] N. Bourbaki, Lie Groups and Lie Algebras, English translation by Andrew Pressley, Springer, New York, 2002.

[5] F. Brenti, V. Reiner and Y. Roichman, *Alternating subgroups of Coxeter groups*, J. Combin. Theory Ser. A **115** (2008), 845–877.

[6] H. L. M. Faliharimalala and A. Randrianarivony, *Flag-major index and flag-inversion number on colored words and wreath product*, Sém. Lothar. Combin. **62** (2010), Art. B62c, 10 pp. (electronic).

[7] M. Fire, *Statistics on wreath products*, preprint; `ar`χ`iv:math/0409421`.

[8] D. Foata and G. -N. Han, *Signed words and permutations, I. A fundamental transformation*, Proc. Amer. Math. Soc. **135** (2007), 31–40.

[9] D. Foata and G.-N. Han, Statistical Distributions on Words and q-Calculus on Permutations, Strasbourg, 2007; available at `http://www-irma.u-strasbg.fr/~foata/paper/pub109.html`.

[10] J. E. Humphreys, Reflection groups and Coxeter groups, Cambridge Studies in Advanced Mathematics, 29, Cambridge University Press, Cambridge, 1990.

[11] V. Reiner and Y. Roichman, *Diameter of reduced words*, Trans. Amer. Math. Soc. (to appear); `ar`χ`iv:0906.4768`.

[12] R. P. Stanley, *Binomial posets, Möbius inversion, and permutation enumeration*, J. Combin. Theory Ser. A **20** (1976), 336–356.

[13] A. Vershik and M. Vsemirnov, *The local stationary presentation of the alternating groups and the normal form*, J. Algebra **319** (2008), 4222–4229.

Department of Mathematics, Bar-Ilan University, 52900 Ramat-Gan, Israel
E-mail address: `radin@math.biu.ac.il`

Dipartimento di Matematica, Universitá di Roma "Tor Vergata", Via della Ricerca Scientifica, 00133 Roma, Italy
E-mail address: `brenti@mat.uniroma2.it`

Department of Mathematics, Bar-Ilan University, 52900 Ramat-Gan, Israel
E-mail address: `yuvalr@math.biu.ac.il`