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GENERALIZATION OF SCOTT'S PERMANENT IDENTITY

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Abstract. Let $\mathbf{x} = \{x_1, \ldots, x_r\}$, $\mathbf{y} = \{y_1, \ldots, y_n\}$, $\mathbf{z} = \{z_1, \ldots, z_n\}$ be three sets of indeterminates. We give the value of the determinant

$$\left| \prod_{x \in \mathbf{x}} (xy - z)^{-1} \right|_{y \in \mathbf{y}, z \in \mathbf{z}}$$

when specializing $\mathbf{y}$ and $\mathbf{z}$ to the set of roots of $y^n - 1$ and $z^n - \xi^n$, respectively.

In the case where $r = 2$ and $\mathbf{x} = \{1, 1\}$ the determinant $\left|(y-z)^{-2}\right|_{y \in \mathbf{y}, z \in \mathbf{z}}$ factorizes into the determinant of the Cauchy matrix $[(y-z)^{-1}]$ and its permanent (see [1], respectively [11, vol. 2, pp. 173–175]). Scott [13, 10] found the value of this permanent when specializing $\mathbf{y}$ to the roots of $y^n - 1$ and $\mathbf{z}$ to the roots of $z^n + 1$. Han [4] described more generally the case where $\mathbf{z}$ is the set of roots of $z^n + az^k + b$ instead of $z^n + 1$.

Instead of restricting to $r = 2$ and specializing $\mathbf{x}$, we shall consider the determinant

$$\left| \prod_{x \in \mathbf{x}} (xy - z)^{-1} \right|_{y \in \mathbf{y}, z \in \mathbf{z}},$$

and obtain in Theorem 2 its value when specializing $\mathbf{y}$ and $\mathbf{z}$. The remarkable feature is that this value is a product of sums of monomial functions in $\mathbf{x}$ without multiplicities, thus extending the factorized expressions of [13, 4].

We first need a few generalities about symmetric functions [7].

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Given two sets of indeterminates $\mathbf{x}, \mathbf{z}$ (we say alphabets), the complete functions $S_n(\mathbf{x}-\mathbf{z})$ are the coefficients of the generating function

$$\sum_n \gamma^n S_n(\mathbf{x}-\mathbf{z}) = \prod_{z\in\mathbf{z}}(1-\gamma z)\prod_{x\in\mathbf{x}}(1-\gamma x)^{-1}\,.$$

For any r, any $\lambda\in\mathbb{Z}^r$, $S_\lambda(\mathbf{x}-\mathbf{z}) = \det\bigl(S_{\lambda_i+j-i}(\mathbf{x}-\mathbf{z})\bigr)$.

In the case where $\mathbf{z}=0$, and $\mathbf{x}$ of cardinality r, these functions can be obtained by symmetrization over the symmetric group $\mathfrak{S}_r$. Let π_ω be the following operator on functions in $\mathbf{x}$:

$$f \to f\pi_\omega := \sum_{\sigma\in\mathfrak{S}_r}\left(f\prod_{1\le i<j\le r}(1-x_j/x_i)^{-1}\right)^{\sigma}\,.$$

Then, when $\lambda \ge [1-r,\ldots,-1,0]$ (i.e., $\lambda_1\ge 1-r,\ \ldots,\ \lambda_r\ge 0$), the monomial $x^\lambda = x_1^{\lambda_1}\cdots x_r^{\lambda_r}$ is sent to $S_\lambda(\mathbf{x})$ under π_ω. When λ is a partition (i.e., $\lambda_1\ge\cdots\ge\lambda_r\ge 0$), $S_\lambda(\mathbf{x})$ is the usual Schur function of index λ.

Let n be a positive integer, ξ an indeterminate, and $\mathbf{z}$ be the set of roots of $z^n-\xi^n$. Equivalently, $e_i(\mathbf{z})=0$ for $1\le i\le n-1$, $e_n(\mathbf{z})=(-1)^{n-1}\xi^n$. For any integer j, any $\mathbf{x}$, one has

$$S_j(\mathbf{x}-\mathbf{z}) = S_j(\mathbf{x}) - \xi^n S_{j-n}(\mathbf{x})\,,$$

and, more generally, from the determinantal expression for Schur functions, for any $\lambda\in\mathbb{N}^r$,

$$S_\lambda(\mathbf{x}-\mathbf{z}) = \sum_{u\in\{0,n\}^r} S_{\lambda-u}(\mathbf{x})(-1)^{|u|/n}\xi^{|u|}\,,$$

where $|u|$ denotes the sum of the coordinates of u.

In particular, when $\lambda = \underbrace{n-1,\ldots,n-1}_{r-1},p$ (which we shall also denote as $\lambda=(n-1)^r p)$, then the terms with negative last component $-(n-p)$ vanish, and the set $\bigl\{\lambda-u : u\in\{0,n\}^r\bigr\}$ to consider is $\bigl\{[v,p] :\ v\in\{n-1,-1\}^{r-1}\bigr\}$. Reordering indices, putting $q=p-r+1$, one can rewrite the sum as

$$S_{(n-1)^{r-1}p}(\mathbf{x}-\mathbf{z}) = (-1)^{r-1}\sum_{u\in\{0,n\}^{r-1}} S_{q,u}(\mathbf{x})(-1)^{(|\lambda|-|u|-q)/n}\xi^{|\lambda|-|u|-q}\,. \quad (1)$$

Since $x^v\pi_\omega = S_v(\mathbf{x})$, for any $v\ge[1-r,\ldots,-1,0]$, one can rewrite (1) as a symmetrization of monomial functions in $\mathbf{x}-x_1=\{x_2,\ldots,x_r\}$:

$$S_{(n-1)^{r-1}p}(\mathbf{x}-\mathbf{z}) = (-1)^{r-1}\sum_{j=0}^{r-1} x_1^q m_{n^j}(\mathbf{x}-x_1)(-1)^{r-1-j}\xi^{|\lambda|-jn-q}\,\pi_\omega\,. \quad (2)$$

From the identity

$$m_{n^j}(\mathbf{x}-x_1) = m_{n^j}(\mathbf{x}) - x_1^n m_{n^{j-1}}(\mathbf{x}) + x_1^{2n} m_{n^{j-2}}(\mathbf{x}) + \cdots + (-x_1^n)^j ,$$

one sees that $x_1^q m_{n^j}(\mathbf{x}-x_1)\,\pi_\omega$ is equal to

$$S_q(\mathbf{x}) m_{n^j}(\mathbf{x}) - S_{q+n}(\mathbf{x}) m_{n^{j-1}}(\mathbf{x}) + \cdots + (-1)^j S_{q+jn}(\mathbf{x}) . \tag{3}$$

On the other hand, $S_{(n-1)^{r-1}p}(\mathbf{x}-\mathbf{z})$ belongs to the linear span of Schur functions indexed by partitions μ such that $\mu_1 \le n-1$. This implies that $S_{(n-1)^{r-1}p}(\mathbf{x}-\mathbf{z})$ belongs to the linear span of monomial functions indexed by the same set of partitions. Consequently, one can restrict the sum in (3) to the term $(-1)^j S_{q+jn}(\mathbf{x})$.

In summary, one has the following expression for the specialization of the Schur function that we are considering.

Proposition 1. *Let $\mathbf{x}$ be an alphabet of cardinality r, $\mathbf{z}$ be the set of roots $z^n - \xi^n = 0$, $p \le n-1$, $N = (n-1)(r-1)$. Then*

$$S_{(n-1)^{r-1}p}(\mathbf{x}-\mathbf{z}) = \sum_\mu m_\mu(\mathbf{x})\, \xi^{N+p-|\mu|} , \tag{4}$$

where the sum is over all partitions $\mu \in \mathbb{N}^r$, $\mu_1 \le n-1$, $|\mu| \equiv p-r+1 \bmod n$.

For example, for $n=4$, $r=2$, one has

$$S_{30}(\mathbf{x}-\mathbf{z}) = m_3(\mathbf{x}) + m_{21}(\mathbf{x}),\ S_{31}(\mathbf{x}-\mathbf{z}) = m_{31}(\mathbf{x}) + m_{22}(\mathbf{x}) + \xi^4 ,$$

$$S_{32}(\mathbf{x}-\mathbf{z}) = m_{32}(\mathbf{x}) + \xi^4 m_1(\mathbf{x}),\ S_{33}(\mathbf{x}-\mathbf{z}) = m_{33}(\mathbf{x}) + \xi^4 (m_2(\mathbf{x}) + m_{11}(\mathbf{x})) .$$

Let

$$D(\mathbf{x},\mathbf{y},\mathbf{z}) = \Big| \prod_{x\in\mathbf{x}} (xy-z)^{-1} \Big|_{y\in\mathbf{y}, z\in\mathbf{z}} .$$

In the case $r=2$, this determinant has been obtained by Izergin and Korepin [5] as the partition function of the Heisenberg XXZ-antiferromagnetic model (modulo some normalization factor). Gaudin [3] had previously described the partition function of some other model as the determinant $\big|(x-y)^{-1}(x-y+\gamma)^{-1}\big|$ for some parameter γ.

The Izergin–Korepin determinant is used in the enumeration of alternating sign matrices [2]. In that case, one first specializes $\mathbf{x} = \{e^{2i\pi/3}, e^{4i\pi/3}\}$. Kuperberg [6] and Okada [12] evaluate more general partition functions corresponding to similar determinants or Pfaffians, and to other roots of unity (see also [8, Th. 7.2]).

We shall take another point of view, keep $\mathbf{x}$ generic, but specialize instead $\mathbf{y}$ and $\mathbf{z}$. In [9, Formula 4], it is shown that the function

$$G(\mathbf{x},\mathbf{y},\mathbf{z}) = \frac{D(\mathbf{x},\mathbf{y},\mathbf{z})}{\Delta(\mathbf{z})} \prod_{x\in\mathbf{x}}\prod_{y\in\mathbf{y}}\prod_{z\in\mathbf{z}}(xy-z)$$

is equal to the determinant of the matrix

$$\Big[S_{(n-1)^{r-1}j}(y_i\mathbf{x}-\mathbf{z})\Big]_{j=0\ldots n-1,\, i=1\ldots n}, \tag{5}$$

where $\Delta(\mathbf{z}) = \prod_{1\le i<j\le n}(z_i-z_j)$.

For any $k\in\mathbb{N}$, let φ_k be the sum of all monomial functions $m_\mu(\mathbf{x})$ of degree k, with $\mu_1 \le n-1$ (notice that $\varphi_k = 0$ when $k>(n-1)r$). From (4), one sees that $S_{(n-1)^{r-1}j}(y_i\mathbf{x}-\mathbf{z})$ specializes, when $\mathbf{z}$ is the set of roots of $z^n-\xi^n$, to

$$\begin{aligned} &S_{(n-1)^{r-1}j}(y_i\mathbf{x}-\mathbf{z}) \\ &\qquad = y_i^{N+j}\varphi_{N+j} + \xi^n y_i^{N+j-n}\varphi_{N+j-n} + \xi^{2n}y_i^{N+j-2n}\varphi_{N+j-2n} + \cdots, \end{aligned} \tag{6}$$

where, as before, $N=(n-1)(r-1)$. Further specializing $\mathbf{y}$ to the roots of y^n-1, one sees that the matrix (5) factorizes into the product of the matrix $\left[y_i^{(N+j)}\right]$, where the symbol $(N+j)$ stands for the remainder of division of $N+j$ by n, and the diagonal matrix

$$\begin{aligned} \mathrm{diag}\Big((\varphi_N+\xi^n\varphi_{N-n}+\xi^{2n}\varphi_{N-2n}+\cdots),(\varphi_{N+1}+\xi^n\varphi_{N+1-n}+\xi^{2n}\varphi_{N+1-2n}+\cdots),\\ \cdots,\ (\varphi_{N+n-1}+\xi^n\varphi_{N+n-1-n}+\xi^{2n}\varphi_{N+1-n}+\cdots)\Big). \end{aligned}$$

For example, for $n=3$, $r=3$,

$$\begin{aligned} S_{220}(y_i\mathbf{x}-\mathbf{z}) = y_i^4\varphi_4 + y_i\varphi_1\xi^3, S_{221}(y_i\mathbf{x}-\mathbf{z}) = y_i^5\varphi_5 + y_i^2\varphi_2\xi^3,\\ S_{222}(y_i\mathbf{x}-\mathbf{z}) = y_i^6\varphi_6 + y_i^3\varphi_3\xi^3 + \xi^6, \end{aligned}$$

and the matrix factorizes into

$$\begin{bmatrix} y_1 & y_1^2 & 1\\ y_2 & y_2^2 & 1\\ y_3 & y_3^2 & 1\end{bmatrix} \begin{bmatrix} \varphi_4+\varphi_1\xi^3 & 0 & 0\\ 0 & \varphi_5+\varphi_2\xi^3 & 0\\ 0 & 0 & \varphi_6+\varphi_3\xi^3+\xi^6\end{bmatrix}.$$

Taking into account that $\prod_{x,y,z}(xy-z)$ specializes to $\prod_{y,x}(x^ny^n-\xi^n) = \prod_{x\in\mathbf{x}}(x^n-\xi^n)^n$, and that the determinant of powers of the $y\in\mathbf{y}$ is a permutation of the Vandermonde in $\mathbf{y}$, one obtains the following theorem.

Theorem 2. *Let n, r be two positive integers, $N = (n-1)(r-1)$. Let $\mathbf{x}$ be an alphabet of cardinality r, $\mathbf{y}$ be the set of roots of $y^n - 1$, $\mathbf{z}$ be the set of roots of $z^n - \xi^n$. Then*

$$\Delta(\mathbf{y})\Delta(\mathbf{z}) \left| \prod_{k=1}^{r} (x_k y_i - z_j)^{-1} \right|_{i,j=1\ldots n} = \frac{(-1)^{(n-1)(n/2+r-1)}}{\prod_{x\in\mathbf{x}}(x^n - \xi^n)^n} \prod_{i=0}^{n-1} \left(\sum_{j=0}^{\infty} \varphi_{N+i-nj}\, \xi^{nj} \right) .$$

For $\mathbf{x} = \{1,1\}$, this theorem is due to Han[4]. In that case, $\varphi_i = i+1$ and $\varphi_{n-1+i} = n-i$ for $i = 0, \ldots, n-1$, and the product appearing in the theorem is

$$n\,(n-1+\xi^n)\,(n-2+2\xi^n)\cdots(1+(n-1)\xi^n)\,.$$

For $r=5$, $n=3$, as a further example, the theorem yields the expression

$$\prod_{k=1}^{5}(x_k^3 - \xi^3)^{-3} \left(\varphi_8 + \varphi_5\xi^3 + \varphi_2\xi^6\right)\left(\varphi_9 + \varphi_6\xi^3 + \varphi_3\xi^6 + \xi^9\right) \left(\varphi_{10} + \varphi_7\xi^3 + \varphi_4\xi^6 + \varphi_1\xi^9\right) ,$$

which specializes, for $\mathbf{x} = \{1,1,1,1,1\}$, to

$$(1-\xi^3)^{-15}(15 + 51\xi^3 + 15\xi^6)(5 + 45\xi^3 + 30\xi^6 + \xi^9)(1 + 30\xi^3 + 45\xi^6 + 5\xi^9)\,.$$

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