Séminaire Lotharingien de Combinatoire **60** (2008), Article B60b

A CONTINUED FRACTION EXPANSION FOR A q-TANGENT FUNCTION: AN ELEMENTARY PROOF

HELMUT PRODINGER

Abstract. We prove a continued fraction expansion for a certain q-tangent function that was conjectured by the present writer, then proved by Fulmek, now in a completely elementary way.

1. Introduction

In [3], the present writer defined the following q-trigonometric functions

$$\sin_q(z) = \sum_{n\ge 0} \frac{(-1)^n z^{2n+1}}{[2n+1]_q!} q^{n^2},$$
$$\cos_q(z) = \sum_{n\ge 0} \frac{(-1)^n z^{2n}}{[2n]_q!} q^{n^2}.$$

Here, we use standard q-notation:

$$[n]_q := \frac{1-q^n}{1-q}, \qquad [n]_q! := [1]_q[2]_q \dots [n]_q.$$

These q-functions are variations of Jackson's [2] q-sine and q-cosine functions.

For the q-tangent function $\tan_q(z) = \frac{\sin_q(z)}{\cos_q(z)}$, the following continued fraction expansion was conjectured in [3]:

$$z\tan_q(z) = \cfrac{z^2}{[1]_q q^0 - \cfrac{z^2}{[3]_q q^{-2} - \cfrac{z^2}{[5]_q q^1 - \cfrac{z^2}{[7]_q q^{-9} - \cdots}}}}.$$

Here, the powers of q are of the form $(-1)^{n-1} n(n-1)/2 - n + 1$.

In [1], this statement was proven using heavy machinery from q-analysis.

Happily, after about 8 years, I was now successful to provide a complete *elementary* proof that I will present in the next section.

Key words and phrases. q-tangent, continued fraction.

2. The proof

We write

$$\frac{z\sin_q(z)}{\cos_q(z)}=\frac{z^2}{N_0}=\cfrac{z^2}{C_1-\cfrac{z^2}{N_1}}=\cfrac{z^2}{C_1-\cfrac{z^2}{C_2-\cfrac{z^2}{N_2}}}=\dots,$$

and set

$$N_i=\frac{a_i}{b_i}.$$

This means that

$$N_i=C_{i+1}-\frac{z^2}{N_{i+1}}$$

or

$$\frac{z^2}{N_{i+1}}=C_{i+1}-N_i$$

and

$$\frac{b_{i+1}z^2}{a_{i+1}}=C_{i+1}-\frac{a_i}{b_i}=\frac{C_{i+1}b_i-a_i}{b_i}.$$

Therefore we may identify numerators and denominators, and put $a_i=b_{i-1}$ and

$$b_{i+1}z^2=C_{i+1}b_i-b_{i-1}.$$

The initial conditions are

$$b_{-1}=\cos_q(z)\quad\text{and}\quad b_0=\sum_{n\ge0}\frac{(-1)^nq^{n^2}z^{2n}}{[2n+1]_q!}.$$

The constants C_i guarantee that all the b_i are power series, i.e., they make the constant term in $C_{i+1}b_i-b_{i-1}$ disappear. Our goal is to show that $C_i=[2i-1]_q q^{(-1)^{i-1}i(i-1)/2-i+1}$ are the (unique) numbers that do this. We are proving the claim by proving the following *explicit* formula for b_i:

$$b_i=\sum_{n\ge0}\frac{(-1)^nz^{2n}}{[2n+2i+1]_q!}\biggl(\prod_{j=1}^{i}[2n+2j]_q\biggr)q^{(n+\lfloor\frac{i+1}{2}\rfloor)^2+[i\text{ odd}]\binom{i+1}{2}}.$$

Note that the C_i are uniquely determined by the imposed condition, and since the b_i are power series, we are done once we prove this formula by induction. The first two instances satisfy this, and we do the induction step now:

$$\begin{aligned}
&C_{i+1}b_i-b_{i-1}\\
&\quad=[2i+1]_qq^{(-1)^i\binom{i+1}{2}-i}\sum_{n\ge0}\frac{(-1)^nz^{2n}}{[2n+2i+1]_q!}\biggl(\prod_{j=1}^{i}[2n+2j]_q\biggr)q^{(n+\lfloor\frac{i+1}{2}\rfloor)^2+[i\text{ odd}]\binom{i+1}{2}}\\
&\qquad-\sum_{n\ge0}\frac{(-1)^nz^{2n}}{[2n+2i-1]_q!}\biggl(\prod_{j=1}^{i-1}[2n+2j]_q\biggr)q^{(n+\lfloor\frac{i}{2}\rfloor)^2+[i-1\text{ odd}]\binom{i}{2}}\\
&\quad=\sum_{n\ge0}\frac{(-1)^nz^{2n}}{[2n+2i+1]_q!}\Biggl([2i+1]_q\biggl(\prod_{j=1}^{i}[2n+2j]_q\biggr)q^{(n+\lfloor\frac{i+1}{2}\rfloor)^2+[i\text{ odd}]\binom{i+1}{2}+(-1)^i\binom{i+1}{2}-i}
\end{aligned}$$

$$
\begin{aligned}
&\quad - [2n+2i+1]_q \Bigg(\prod_{j=1}^{i} [2n+2j]_q \Bigg) q^{(n+\lfloor\frac{i}{2}\rfloor)^2 + [i-1 \text{ odd}]\binom{i}{2}} \Bigg) \\
&= \frac{1}{1-q} \sum_{n\ge 0} \frac{(-1)^n z^{2n}}{[2n+2i+1]_q!} \Bigg(\prod_{j=1}^{i} [2n+2j]_q \Bigg) \\
&\quad \times \Bigg((1-q^{2i+1}) q^{(n+\lfloor\frac{i+1}{2}\rfloor)^2 + [i \text{ even}]\binom{i+1}{2} - i} - (1-q^{2n+2i+1}) q^{(n+\lfloor\frac{i}{2}\rfloor)^2 + [i \text{ even}]\binom{i}{2}} \Bigg).
\end{aligned}
$$

The last bracket in this expression can be simplified for i even:

$$
-q^{(n+\frac{i}{2})^2 + \binom{i}{2} + 2i + 1} (1-q^{2n})
$$

and for i odd:

$$
-q^{(n+\frac{i-1}{2})^2} (1-q^{2n}).
$$

Putting everything together, we arrive at

$$
C_{i+1} b_i - b_{i-1} = \sum_{n\ge 0} \frac{(-1)^{n-1} z^{2n}}{[2n+2i+1]_q!} \Bigg(\prod_{j=0}^{i} [2n+2j]_q \Bigg) q^{(n+\lfloor\frac{i}{2}\rfloor)^2 + [i \text{ even}]\binom{i+2}{2}}.
$$

Notice that the constant term vanishes, whence

$$
\begin{aligned}
b_{i+1} &= \sum_{n\ge 1} \frac{(-1)^{n-1} z^{2n-2}}{[2n+2i+1]_q!} \Bigg(\prod_{j=0}^{i} [2n+2j]_q \Bigg) q^{(n+\lfloor\frac{i}{2}\rfloor)^2 + [i \text{ even}]\binom{i+2}{2}} \\
&= \sum_{n\ge 0} \frac{(-1)^{n} z^{2n}}{[2n+2(i+1)+1]_q!} \Bigg(\prod_{j=1}^{i+1} [2n+2j]_q \Bigg) q^{(n+\lfloor\frac{i+2}{2}\rfloor)^2 + [i+1 \text{ odd}]\binom{i+2}{2}},
\end{aligned}
$$

which is the announced formula.

References

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Helmut Prodinger, Department of Mathematics, University of Stellenbosch, 7602 Stellenbosch, South Africa

E-mail address: hproding@sun.ac.za