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KEYS AND ALTERNATING SIGN MATRICES

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Abstract. In [*Invariant Theory and Tableaux*, I.M.A. Vol. Math. Appl. **19**, Springer–Verlag, New York, 1990, pp. 125–144], Lascoux and Schützenberger introduced a notion of key associated to any Young tableau. More recently, Lascoux defined the key of an alternating sign matrix by recursively removing all -1's in such matrices. But alternating sign matrices are in bijection with monotone triangles, which form a subclass of Young tableaux. We show that in this case these two notions of keys coincide. Moreover we obtain an elegant and direct way to compute the key of any Young tableau, and discuss consequences of our result.

1. Introduction

A key is by definition a Young tableau whose columns are comparable for the inclusion order. They were introduced by A. Lascoux and M.-P. Schützenberger [12, 13] to study Demazure characters. For type A, irreducible characters (i.e., Schur functions) are associated to all Young tableaux of a given shape. Demazure characters correspond to subsets of tableaux which can be described using keys (cf. [8, 14, 1, 15] for recent works).

Alternating sign matrices (ASM's for short) are combinatorial objects that were extensively studied in the last two decades [3, 16], with (at least) the great achievement of their enumeration [20, 7]. They may be seen as a representation of square-ice configurations [10], and surprisingly their numbers appeared in the context of the dense $O(n = 1)$ or Temperley–Lieb loop model on the square grid [17, 18, 4, 2].

In [10], A. Lascoux defines an operation which consists in iteratively removing the -1's of a given ASM to obtain a permutation matrix, called the key of the ASM. But ASM's are in bijection with a certain class of Young tableaux, called monotone triangles (or Gog triangles in [20]). Since a permutation (matrix) may be seen as a monotone triangle which is a key as a Young tableau, we may ask whether these two notions coincide. The answer, as stated without proof in [10], is affirmative (cf. Corollary 9).

More generally we extend Lascoux's operation of removing the -1's to matrices associated to (unrestricted) Young tableaux and obtain (Theorem 3) a very simple way to compute the left key of a Young tableau, by far easier and quicker than the original definition [12], and even than the recent method presented in [15].

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This article is organized as follows. In Section 2, we recall the definitions of the left key of a Young tableau, introduce and prove our new way to compute it. We also explain how this method can be used for the computation of the right key. In Section 3, we examine consequences of our result for ASM's, including simple formulas for the number of ASM's with exactly one or two -1 entries.

2. Keys of Young tableaux

In this paper, the French notation for tableau is used. A tableau may be seen as a product of columns: $T = C_1\, C_2\, \cdots\, C_l$, where a column is a strictly decreasing word (often identified with the set of its entries). For example, the tableau $T = \begin{array}{|c|cc} \hline 4 & & \\ \hline 2 & \multicolumn{1}{|c|}{5} & \\ \hline 1 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{5} \\ \hline \end{array}$ is the product of the 3 columns $C_1\, C_2\, C_3$ with $C_1 = 4\,2\,1$, $C_2 = 5\,2$ and $C_3 = 5$. Moreover, this tableau T is a *Young tableau* because the entries are non-decreasing along its rows. The *shape* of a tableau is then the list $(H_1, \ldots, H_l)$ of the heights of its columns. The shape of our tableau T given above is $(3, 2, 1)$.

The *word* of a tableau is simply the result of its reading (column by column). The tableau T reads: 421 52 5 (the gaps are here to mark the changes of columns, and are of course unnecessary). We will now not distinguish a tableau and its word.

A *key* is a tableau such that for any $k \in \{1, \ldots, l-1\}$, the column C_{k+1} is a subword of the column C_k.

We recall that Schensted's insertion gives a bijection from words to pairs of Young tableaux (P, Q) of same shape, with the second one standard. Two words are equivalent in the sense of the plactic congruence, characterized by Knuth, if they give the same tableau P by insertion. The following lemma (cf. [13]) is the crucial step to associate a key to any Young tableau.

Lemma 1. *Let T be a Young tableau of shape $H = (H_1, \ldots, H_l)$. Then for any permutation $I = (I_1, \ldots, I_l)$ of H, there exists in the congruence class of T exactly one word $V = V_1 \cdots V_l$, which is a product of columns of respective degrees $I_1, \ldots, I_l$. If J is another permutation of I and $W = W_1 \cdots W_l$ is the associated word, then $J_k \geq I_k$ implies that V_k is a subword of W_k.*

The words V introduced in this lemma are called the *frank words* of T. The keys are characterized by the fact that their columns (as words) commute (in the plactic monoid). For a tableau T, we shall say that V_1 is the *left factor* of the couple (I, T), and V_l the *right factor*. Thus, for each height H_k, we obtain this way a unique left (respectively right) factor of degree H_k. These columns are all ordered by inclusion. The *left (respectively right) key of the Young tableau* T is then defined as the product of its left (respectively right) factors of respective sizes $H_1, H_2, \ldots, H_l$.

In our example, the tableau T is of shape $(3, 2, 1)$, thus there are six permutations of its shape, which correspond to the six frank words:

$$\begin{array}{ccc}4&&\\2&5&\\1&2&5\end{array} \equiv \begin{array}{ccc}4&5&\\2&2&\\&1&5\end{array} \equiv \begin{array}{ccc}4&&\\2&&\\1&5&5\\&&2\end{array} \equiv \begin{array}{ccc}4&5&\\&2&5\\&1&2\end{array} \equiv \begin{array}{ccc}4&5&5\\&2&2\\&&1\end{array} \equiv \begin{array}{ccc}4&&\\2&5&5\\&&2\\&&1\end{array}$$

We chose a planar presentation of words (i.e., in form of tableaux) to insist on the fact that the frank words may be computed using the "jeu de taquin." We will use the "jeu de taquin" and refer to classical references [19] for definitions, and to [6], Appendix A.5 for its use in the computation of keys. The left and right keys of our tableau T are respectively

$$\begin{array}{ccc}4&&\\2&4&\\1&2&4\end{array} \text{ and } \begin{array}{ccc}5&&\\2&5&\\1&2&5\end{array}.$$

We observe that to compute the left key of a tableau T, it is sufficient to compute its left factors; thus we may restrict the computation to frank words relative to permutations $(I_k, I_1, I_2, \ldots, I_{k-1}, I_{k+1}, \ldots, I_l)$.

Now we associate to any tableau T a sign matrix $M(T)$.

Definition 2. *A* sign matrix *is a matrix $M = (M_{i,j})$ such that:*

- *for all i, j, $M_{i,j} \in \{-1, 0, 1\}$;*
- *for all i, j, $\sum_{r=1}^{i} M_{r,j} \in \{0, 1\}$;*
- *for all i, j, $\sum_{s=1}^{j} M_{i,s} \geq 0$.*

The bijection between Young tableaux and sign matrices is a generalization of the well-known bijection between ASM's and monotone triangles (cf. [20]). We observe the appearances and disappearances of the entries in the columns (from right to left), and we translate this into a matrix:

$$M(T)_{ij} = \begin{cases} 1 & \text{if } j \in C_{l-i+1} \text{ and } j \notin C_{l-i+2}, \\ -1 & \text{if } j \notin C_{l-i+1} \text{ and } j \in C_{l-i+2}, \\ 0 & \text{if not,} \end{cases}$$

with the convention that C_{l+1} is empty.

Proposition 1. *The map that sends a Young tableau T to a matrix $M(T)$ is a bijection from the set of Young tableaux with m columns and entries in $\{1, \ldots, n\}$ to sign matrices with m rows and n columns.*

Proof. First, if we start with a Young tableau T, the matrix obtained is a sign matrix because:

- since T is a Young tableau, the elements of a column of T are all distinct, thus in a given column of $M(T)$, the non-zero entries start with a 1, and then alternate, whence the second condition of sign matrices;

- the elements of T are weakly increasing along the rows, which translates into the third condition of the sign matrices definition.

Conversely, if we start from a sign matrix, we construct a tableau, which is a Young tableau for the same reason. ☐

For example, the tableau $T = \begin{array}{|c|c|c|c|} \cline{1-2} 5 & 5 \\ \cline{1-3} 2 & 4 & 5 \\ \hline 1 & 2 & 4 & 6 \\ \hline \end{array}$ is associated to the sign matrix

$$M(T) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \end{bmatrix}$$

A simple observation is that a Young tableau is a key if and only if its sign matrix does not contain any -1. Since our goal is to associate to any Young tableau its (left) key, we introduce a way to remove the -1's. This is done through an *elimination process*. This process, as we shall see in the next section, is an extension of the *removing process* defined in [10] for monotone triangles, to general Young tableaux. It should be observed here that this process is more than just the restriction of Lascoux's process to the sub-quadrant of an ASM, as may be seen in the previous example, where the first two rows are clearly not a part of an ASM.

A -1 entry in a matrix M is said to be *removable* if there is no -1 in the rows above it, nor in its row and on its right, i.e., for the -1 in position (a, b) (we use matrix coordinates) if:

$$\text{for all } i < a \text{ and for all } j,\ M_{i,j} \neq -1, \quad \text{and for all } j > b,\ M_{a,j} \neq -1.$$

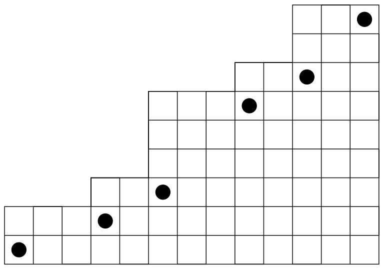
For such a -1 in a given matrix, its *neighbours* are the entries $M_{i,j}$ equal to 1 such that

$$i \leq a,\ \ j \leq b,\ \text{ and for all } k, l \text{ with } i \leq k \leq a \text{ and } j \leq l \leq b,\ M_{k,l} \neq 1,$$

i.e., the rectangle of South-East corner (a, b) and of North-West corner (i, j) contains no other entry equal to 1 than the neighbour itself. For a given removable -1, the union of these rectangles is a Ferrers diagram:

$$\begin{array}{cccccccc} . & . & . & . & . & . & . & 1 \\ . & . & . & . & . & 1 & 0 & 0 \\ . & . & . & . & . & 0 & 0 & 0 \\ . & . & . & 1 & 0 & 0 & 0 & 0 \\ . & . & . & 0 & 0 & 0 & 0 & 0 \\ . & . & . & 0 & 0 & 0 & 0 & 0 \\ 1 & \theta & \theta & \theta & \theta & \theta & \theta & -1 \end{array}$$

where the θ entries are either 0's or -1's. To *remove* a (removable) -1 consists in replacing it by a 0, and to replace its n neighbours by 0 and to place $n-1$ entries equal to 1 such that they form a new Ferrers diagram whose inner corners are the former n neighbours. To make this definition unambiguous, we stress that in our context the *inner corners* of a Ferrers diagram are the dots in the following picture.



In the example, this means to replace the sub-matrix given above by:

$$\begin{array}{cccccccc}
. & . & . & . & . & \mathbf{1} & . & \mathbf{0} \\
. & . & . & \mathbf{1} & . & \mathbf{0} & 0 & 0 \\
. & . & . & . & . & 0 & 0 & 0 \\
\mathbf{1} & . & . & \mathbf{0} & 0 & 0 & 0 & 0 \\
. & . & . & 0 & 0 & 0 & 0 & 0 \\
. & . & . & 0 & 0 & 0 & 0 & 0 \\
\mathbf{0} & \theta & \theta & \theta & \theta & \theta & \theta & \mathbf{0}
\end{array}$$

Our main result is the following.

Theorem 3. *Let T be a Young tableau, and $M(T)$ the sign matrix associated to T. By removing all the -1's of $M(T)$ by the process described, we obtain a sign matrix associated to a tableau U which is the left key of T.*

Let us deal with our example $T = \begin{array}{|c|c|c|c|c|} \hline 5 & 5 \\ \cline{1-4} 2 & 4 & 5 & 6 \\ \hline 1 & 2 & 4 & 4 & 6 \\ \hline \end{array}$. Its sign matrix is

$$M(T) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \end{bmatrix}$$

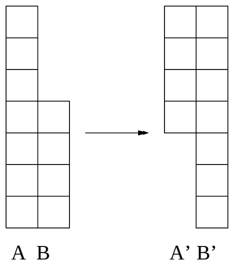
Now we compute:

$$M(T) \longrightarrow \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = M(U)$$

with $U = \begin{array}{|c|c|c|c|c|} \hline 5 & 5 \\ \cline{1-2} 2 & 2 & 5 & 5 \\ \cline{1-4} 1 & 1 & 2 & 2 & 5 \\ \hline \end{array}$, the left key of T.

The proof of the theorem starts with the following lemma.

Lemma 4. *When we apply the "jeu de taquin" to two-column tableaux to compute frank words:*



it is clear that if $b \in B \cap A$ then $b \in B' \cap A'$. If A and B do not commute, i.e., if $B - A \neq \emptyset$, let $\tilde{b} = max(B - A)$ and $\tilde{a} = max(\{a \in A - B,\ a \leq \tilde{b}\})$. We have

(1) *$\tilde{a} \in A'$;*
(2) *let $\tilde{B}$ be the column obtained by replacing $\tilde{b}$ by $\tilde{a}$ in B (maybe reordered), then if $A\tilde{B} \longrightarrow \tilde{A}'\tilde{B}'$, we have: $\tilde{A}' = A'$.*

Proof. For the first point, if we suppose that $\tilde{a}$ slides to B' during the "jeu de taquin," $\tilde{a}$ should have the hole on its right (and obviously below $\tilde{b}$). Let $\{x_1, x_2, \ldots, x_p\}$ be the entries in the left column above $\tilde{a}$ and not above $\tilde{b}$:

$$\begin{array}{|c|c|} \hline x_1 & \tilde{b} \\ \hline x_2 & \\ \hline \cdots & \\ \hline x_p & \\ \hline \tilde{a} & X \\ \hline \end{array}$$

where the X is the hole. Since for all $i \in \{1, \ldots, p\}$, $\tilde{a} < x_i \leq \tilde{b}$, and by definition of $\tilde{a}$, we have that all the $x_1, \ldots, x_p$ are in the right column. But these entries should be below the box that contains $\tilde{b}$ (since $x_i \leq \tilde{b}$) and above the hole (since $x_i \geq \tilde{a}$). This is impossible since we only have here $p - 1$ boxes.

For the second point, we observe that for any element $a \in A$, the fact that a stays in A' (or not) is the same whether B contains $\tilde{b}$ or $\tilde{a}$:

- for $a > \tilde{b}$, nothing changes;
- the case $a = \tilde{b}$ does not occur;
- if $\tilde{a} < a < \tilde{b}$, a is in A if and only if a is in B (or $\tilde{B}$), thus this is clear;
- for $a = \tilde{a}$, this is the first point of the lemma;
- for $a < \tilde{a}$, nothing changes.

$\square$

Proof of the Theorem 3. Since a sign matrix without any -1 is the matrix of a key, we only have to show that the elimination of a -1 does not change the left key.

Thus let T be a tableau and T' the tableau resulting from the elimination of a -1 (its removable -1). This elimination concerns an entry a which appeared in column j and disappeared in column i (we recall that this story reads from right to left).

The elimination of the -1 consists in replacing in T the entry a in the column k (for $i+1 \leq k \leq j$) by the greatest of all entries of C_i less or equal to a and that are not in C_k.

Now let us compare the computation of frank words for T and T'. For the left factors of size $\geq H_i$, nothing has changed from T to T'. Let $k \geq i+1$. By definition of the removable -1, there is no -1 in the rows of $M(T)$ above the row i. Thus the columns on the right of C_i commute, whether we consider T or T'. We have to deal with the action of the "jeu de taquin" on C_iC_k. In the case of T, we know from Lemma 4 that the entry a is replaced by the greatest of all entries of C_i less or equal to a and which are not in C_k. Moreover, for all entries less than a, the computation in the same for T and T'.

Thus T and T' have the same left key. ☐

Now we show how this process can be used to compute the right key of a Young tableau. As we shall see in Theorem 8, we can reduce it to the computation of a left key.

Let us introduce the notion of complement of a tableau, whose definition was already given in [11].

Definition 5. *Let T be a tableau with entries in $\{1, 2, \ldots, n\}$ and columns $T = C_1, \ldots, C_l$. The* complement *of T, denoted by $\mathcal{C}(T)$ is defined as $\mathcal{C}(T) = D_1, \ldots, D_l$ with for $1 \leq j \leq n$ and $1 \leq i \leq l$:*

$$j \in D_i \quad \textit{if and only if} \quad j \notin C_{l-i+1}.$$

For example, if $T = \begin{array}{|c|c|c|c|c|}\cline{1-2} 5&5&\multicolumn{3}{c}{}\\ \cline{1-4} 2&4&5&6&\multicolumn{1}{c}{}\\ \hline 1&2&4&4&6\\ \hline\end{array}$ then $\mathcal{C}(T) = \begin{array}{|c|c|c|c|c|}\cline{1-1} 5&\multicolumn{4}{c}{}\\ \cline{1-3} 4&5&6&\multicolumn{2}{c}{}\\ \hline 3&3&3&6&6\\ \hline 2&2&2&3&4\\ \hline 1&1&1&1&3\\ \hline\end{array}$.

Remark 6. The complement of a tableau T depends of course on the set of entries that we consider. For example, if we had considered that our previous example T has entries in $\{1, \ldots, 7\}$, the only difference would have been the presence of a box with a 7 in each column of $\mathcal{C}(T)$. Thus it should be clear that the next results (Proposition 2, Lemma 7 and Theorem 8) remain true if we add some entry to the set of entries.

Proposition 2. *If T is a Young tableau, then $\mathcal{C}(T)$ is also a Young tableau.*

Proof. This is a direct consequence of Lemma 7 below. ☐

Lemma 7. *Let T be a tableau with l columns and M its sign matrix. If M' is the sign matrix of its complement, then we have:*

$$\text{for all } i > 1 \text{ and for all } j,\ M_{i,j} = M'_{l-i+1,j}.$$

Proof. Let us introduce the columns: $T = C_1 \dots C_l$ and $\mathcal{C}(T) = D_1 \dots D_l$. For $i > 1$, we may write:

$$\begin{aligned} M_{i,j} = 1 &\iff j \in C_{l-i+1} - C_{l-i+2} \\ &\iff j \in D_i - D_{i-1} \\ &\iff M'_{l-i+1,j} = 1 \end{aligned}$$

and the same holds for -1's, which proves the lemma. □

Theorem 8. *Let T be a Young tableau. The complement of the right key of T is the left key of the complement of T.*

Proof. A first observation is that we can restrict to Young tableaux with exactly two columns, since frank words are computed by iteratively applying the jeu de taquin to pairs of columns.

A consequence of Lemma 4 is that the result of the action of the "jeu de taquin" $AB \longrightarrow A'B'$ depends only on $A \cap B = A' \cap B'$, $A - B$ and $B - A$, which we may write (where the symbol $\sqcup$ means "disjoint union"):

$$A' = (A \cap B) \sqcup T_1(A - B, B - A)$$
$$B' = (A \cap B) \sqcup T_2(A - B, B - A)$$

with $T_1(A - B, B - A) \cap T_2(A - B, B - A) = \emptyset$.

If we denote $CD = \mathcal{C}(AB)$, we have:

$$C - D = A - B \quad \text{and} \quad D - C = B - A.$$

We may write

$$C' = (C \cap D) \sqcup T_1(C - D, D - C)$$
$$D' = (C \cap D) \sqcup T_2(C - D, D - C)$$

and we want to check that $A' \sqcup D' = \{1, \dots, n\} = B' \sqcup C'$. By symmetry, we restrict to the first equality which is proved as follows:

$$\begin{cases} A' \cap D' &= \Big((A \cap B) \sqcup T_1(A - B, B - A\Big) \cap \Big((C \cap D) \sqcup T_2(C - D, D - C)\Big) \\ &= \Big((A \cap B) \sqcup T_1(A - B, B - A\Big) \\ & \qquad\qquad \cap \Big(\big(\{1, \dots, n\} - (A \cup B)\big) \sqcup T_2(A - B, B - A)\Big) \\ &= \emptyset \\ |A'| + |D'| &= n. \end{cases}$$

□

We deduce from Theorem 8 an easy way to compute the right key of a Young tableau by taking the complement, applying Theorem 3 to obtain the left key, then taking the complement again. For example, if we consider $T = \begin{array}{lllll} 5 & 5 & & & \\ 2 & 4 & 5 & 6 & \\ 1 & 2 & 4 & 4 & 6 \end{array}$, we obtain easily that the left key of its complement $\mathcal{C}(T) = \begin{array}{lllll} 5 & & & & \\ 4 & 5 & 6 & & \\ 3 & 3 & 3 & 6 & 6 \\ 2 & 2 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 & 3 \end{array}$ is $\begin{array}{lllll} 5 & & & & \\ 4 & 5 & 5 & & \\ 3 & 3 & 3 & 5 & 5 \\ 2 & 2 & 2 & 3 & 3 \\ 1 & 1 & 1 & 1 & 1 \end{array}$ thus its right key is $\begin{array}{lllll} 6 & 6 & & & \\ 4 & 4 & 6 & 6 & \\ 2 & 2 & 4 & 4 & 6 \end{array}$.

3. Keys of alternating sign matrices

Let us switch to the origin of this work, through A. Lascoux's paper [10], that is, to the context of alternating sign matrices. In this paper, using the notion of key, A. Lascoux obtains a description of Grothendieck polynomials (he already obtained a description of Schubert polynomials in [9]). In most questions where ASM's appear in algebraic combinatorics or theoretical physics, weights are assigned to ASM's. One of these weights can be specialized to 1, which leads to enumeration, and the end of this third section deals with enumeration. But this is not the case for the weight giving Grothendieck polynomials.

An *alternating sign matrix* (ASM for short) of size n is a square matrix with entries in $\{-1, 0, 1\}$ such that along each row and column:

- the sum of the entries is equal to 1;
- the non-zero entries alternate in sign.

It is clear that an ASM is in particular a sign matrix (in the sense defined in the previous section). Let us give an example of an ASM of size 5:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

The bijection defined in Section 2 between sign matrices and Young tableaux specializes here to the well-known ([10]) bijection between ASM's and monotone triangles. A *monotone triangle* is by definition a Young tableau of shape $(n, n-1, \ldots, 2, 1)$ with entries in $\{1, 2, \ldots, n\}$ with the property that the entries are non-decreasing along each diagonal (from South-East to North-West in French notation). As an example, we give the monotone triangle associated to the ASM given above:

5				
4	5			
3	4	4		
2	2	3	3	
1	1	1	2	2

In [10], A. Lascoux defined a process of elimination of the -1's of an ASM, to obtain an ASM without any -1, i.e., the matrix of a permutation, which Lascoux calls "the key of the ASM." We may temporarily use the notions of matrix-key (or M-key) to refer to the key defined in matrix form by Lascoux, and of tableau-key (or T-key) to refer to the key defined originally on tableaux by Lascoux and Schützenberger in [12]. Lascoux states without proof in [10] that the bijection between ASM's and monotone triangles exchanges these two notions. This assertion now becomes a corollary of Theorem 3.

Corollary 9. *Let M be an ASM and T its associated monotone triangle. If K and U are respectively the M-key of M and the T-key of T, then U is the monotone triangle in bijection with K.*

Proof. This is a direct consequence of Theorem 3, since the M-key is precisely computed by matrix rules defined in Section 2. □

Thus we may simply speak of "key" for ASM's or monotone triangles. We mention here that to index Schubert cells, Ehresmann [5] used combinatorial objects which can be identified with our keys, and showed that the order on keys corresponds to the natural order on cells (later called Bruhat order).

For combinatorial reasons, and to easily obtain the number of ASM's with exactly one or two -1's, we introduce the notion of pseudo-key of an ASM. The definition is almost the same as the definition of the key, but the process of elimination of the -1's is simpler.

Here we look at a -1 in position (i,j) in a given ASM M such that there is no other -1 in its North-West quadrant, i.e.,

$$\text{for all } k \leq i, l \leq j, (k,l) \neq (i,j), M_{k,l} \neq -1.$$

Now in the matrix M we simply replace the pattern

$$\begin{array}{ccc} 0 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & -1 \end{array}$$

by

$$\begin{matrix} 1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{matrix}.$$

The *pseudo-key* $pK(M)$ of an ASM M is obtained by iteratively removing all -1's according to this process. It is quite easy to check that the result does not depend on the order in which we perform these eliminations. It is clear that the pseudo-key can be defined for any sign matrix, and in the case of the tableau given on page 5, the matrix computation is the following, which shows that the pseudo-key is in general different from the key:

$$M(T) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = pK(M(T)).$$

Let us say a word about the lattice structure on ASM's. This notion comes from the bijection with monotone triangles. It is quite easy to define the supremum (respectively infimum) of a family of monotone triangles, by just taking the supremum (infimum) component-wise. The *lattice structure* on ASM's (which extends Bruhat's order) is then inherited from the lattice structure on monotone triangles. We mention the following fact, whose proof is straightforward.

Proposition 3. *Let M be an ASM, $K(M)$ its key and $pK(M)$ its pseudo-key. For the order in the lattice of ASM's, we have:*

$$pK(M) \leq K(M) \leq M.$$

We conclude our work with two enumerative results. Let us denote by $A_n^{(k)}$ the number of alternating sign matrices with exactly k entries equal to -1 (for example, $A_n^{(0)} = n!$).

Proposition 4. *We have for $n \geq 3$:*

$$A_n^{(1)} = \frac{(n!)^2}{(3!)^2(n-3)!}$$

and for $n \geq 6$:

$$A_n^{(2)} = (n!)^2\Big(\frac{1}{36(n-6)!} + \frac{1}{12(n-4)!} + \frac{11}{30(n-5)!}\Big).$$

Proof. We will only show how to prove the first result, the second result is proved in the same manner.

The proof will be in two steps: first we will prove that $A_n^{(1)}$ is equal to the total number of patterns 132 in all permutations of S_n, subsequently we will prove that this number is precisely $\frac{(n!)^2}{(3!)^2(n-3)!}$.

To begin with, we consider an ASM M with exactly one -1. We compute its pseudo-key $pK(M)$ which is a permutation matrix. This operation gives a pattern 132 in the permutation σ associated to the matrix $pK(M)$, i.e., a triple (i,j,k) such that:

$$i<j<k, \quad \text{and} \quad \sigma(i)<\sigma(k)<\sigma(j).$$

This fact is illustrated as:

$$\begin{array}{ccccccccccc} & & 1 & & & & \mathbf{1} & & 0 & & \\ & & \vdots & & & & & & \vdots & & \\ 1 & \cdots & -1 & \cdots & 1 & \longrightarrow & 0 & \cdots & 0 & \cdots & \mathbf{1} \\ & & \vdots & & & & & & \vdots & & \\ & & 1 & & & & & & \mathbf{1} & & \end{array}$$

where the three remaining 1's form the pattern 132. If we keep track of this triple (i,j,k), we obtain a bijection between ASM's with exactly one -1 and permutations with a marked pattern 132. This is the first step of our proof.

Next, to compute the total number of patterns 132 in all permutations of S_n, we observe that a pattern 132 is given by:

- the choice of a triple $1\le i<j<k\le n$, which gives $\binom{n}{3}$ possibilities;
- the choice of a triple $1\le \sigma(i)<\sigma(k)<\sigma(j)\le n$, which gives $\binom{n}{3}$ possibilities;
- the choice of any permutation in the remaining $n-3$ positions.

Thus the total number of patterns 132 is $\binom{n}{3}^2\times(n-3)! = \frac{(n!)^2}{(3!)^2(n-3)!}$, which completes the proof. $\square$

References

[1] O. Azenhas and R. Mamede, *Matrix realizations of pairs of Young tableaux, keys and shuffles*, Sém. Lotharingien Combin. **53** (2006), Article B53h, 22 pp.

[2] M. T. Batchelor, J. de Gier and B. Nienhuis, *The quantum symmetric XXZ chain at $\Delta=-1/2$, alternating sign matrices and plane partitions*, J. Phys. A: Math. Gen. **34** (2001), L265.

[3] D. Bressoud, *Proofs and Confirmations: The Story of the Alternating Sign Matrix Conjecture*, Cambridge, England: Cambridge University Press, 1999.

[4] P. Di Francesco and P. Zinn-Justin, *Around the Razumov–Stroganov conjecture: proof of a multi-parameter sum rule*, Electron. J. Comb. **12** (2005), Article #R6, 27 pp.

[5] C. Ehresmann, *Sur la topologie de certains espaces homogènes*, Annals of Math. (2) **35** (1934), 396–443.

[6] W. Fulton, *Young Tableaux*, London Mathematical Society Student Texts **35**, Cambridge University Press, 1997.

[7] G. Kuperberg, *Another proof of the alternating-sign matrix conjecture*, Internat. Math. Res. Notes, **3** (1996), 139–150.

[8] V. Lakshmibai, *Bases for quantum Demazure modules*, in *Representations of Groups*, Banff 1994, CMS Conference Proceedings, vol. 16, Canadian Math. Society, 1995.

[9] A. Lascoux, *The 6-vertex model*, SIGMA 3, volume dedicated to Vadim Kuznetsov, paper 029 (2007); arχiv:math.CO/0610719.

[10] A. Lascoux, *Chern and Yang through ice*, preprint.

[11] A. Lascoux and M.-P. Schützenberger, *Le monoïde plaxique*, in *Noncommutative structures in algebra and geometric combinatorics (Naples, 1978)*, Quad. della Ricerca Sci., vol. 109, CNR, Rome, 1981, pp. 129–156.

[12] A. Lascoux and M. P. Schützenberger, *Keys and standard bases*, in *Invariant Theory and Tableaux*, (Minneapolis MN, 1988) I.M.A. Vol. Math. Appl. **19**, Springer–Verlag, New York, 1990, pp. 125–144.

[13] A. Lascoux, M.-P. Schützenberger, *Arêtes et tableaux*, 20-ième Séminaire Lotharingien de Combinatoire, Alghero, Sept. 1988, 109–120; electronically available as Sém. Lotharingien Combin. **20** (1988), Article B20h, 10 pp.

[14] P. Littelmann, *A Littlewood–Richardson rule for symmetrizable Kac–Moody algebras*, Invent. Math. **116** (1994), 329–346.

[15] S. Mason, *An Explicit construction of type A Demazure atoms*, preprint.

[16] J. Propp, *The many faces of alternating-sign matrices*, in *Discrete Models: Combinatorics, Computation, and Geometry*, Discrete Math. Theoret. Computer Science (Proceedings) (2001), pp. 43–58; arχiv:math/0208125.

[17] A. V. Razumov and Y. G. Stroganov, *Combinatorial nature of ground state vector of $O(1)$ loop model*, Theor. Math. Phys. **138** (2004), 333–337.

[18] A. V. Razumov and Y. G. Stroganov, *Spin chains and combinatorics*, J. Phys. A: Math. Gen. **34** (2001), 3185–3190.

[19] M.-P. Schützenberger, *La correspondance de Robinson*, in *Combinatoire et représentation du groupe symétrique*, Lecture Notes Math., vol. 579, Springer–Verlag, 1977, pp. 59–113.

[20] D. Zeilberger, *Proof of the alternating sign matrix conjecture*, Electron. J. Comb. **3**, no. 2 (1996), Article #R13, 84 pp.

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