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AN EXTENSION OF THE FOATA MAP TO STANDARD YOUNG TABLEAUX

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Abstract. We define an inversion statistic on standard Young tableaux. We prove that this statistic has the same distribution over $SYT(\lambda)$ as the major index statistic by exhibiting a bijection on $SYT(\lambda)$ in the spirit of the Foata map on permutations.

1. Introduction

A permutation statistic is a combinatorial rule which associates an element of $\mathbb{N}$ to each element of the symmetric group S_n. Let $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ denote the element of S_n which sends i to σ_i for $1 \le i \le n$. An inversion of σ is a pair (i, j), $1 \le i < j \le n$ such that $\sigma_i > \sigma_j$. A descent of σ is an integer i, $1 \le i \le n-1$, for which $\sigma_i > \sigma_{i+1}$. The inversion statistic $\operatorname{inv}(\sigma)$ is defined to be the number of inversions of σ and the major index statistic $\operatorname{maj}(\sigma)$ is defined to be the sum of the descents of σ, i.e.

$$\operatorname{inv}(\sigma) = \sum_{i<j,\,\sigma_i>\sigma_j} 1, \qquad \operatorname{maj}(\sigma) = \sum_{i,\,\sigma_i>\sigma_{i+1}} i.$$

Major P. MacMahon [MacM] introduced the major index statistic and proved that, remarkably, its distribution over S_n is equal to the distribution of the inversion statistic over S_n. This raised the question of constructing a canonical bijection $\phi : S_n \to S_n$ such that $\operatorname{maj}(\sigma) = \operatorname{inv}(\phi(\sigma))$. Foata [Foa68] found such a map. (See also [FoSc78]). Note that the previous two statistics can be defined on each class $\tilde{C}$ of permutations of a sequence (with repetitions) $1^{m_1}2^{m_2}\cdots r^{m_r}$. The equidistribution of inv and maj also holds for such a class $\tilde{C}$, a result proved by MacMahon [MacM] and reproved by means of the Foata map [Foa68], when adequately defined on $\tilde{C}$. In this paper, we only need the case of permutations without repetitions.

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There is a version of maj for tableau which plays a prominent role in symmetric function theory (see for example [Sta, Chapter 7] or [Mac, Chapter 1]). One could also ask whether there is a natural version of inv which would play a similar role. In this article we advance a candidate tableau inversion statistic, which is defined in terms of a generalization of Foata's map. The statistic was discovered in the course of studying Macdonald polynomials (see the remark at the end of Section 4).

In section one, we review the algorithm describing the Foata map and the necessary background on tableaux. In section two we define the inversion statistic Inv and maps on tableaux that will feature in our Foata-type map. In section three we introduce our Foata-type map and prove that it is a bijection which sends a standard Young tableau with a given Inv to a standard Young tableau with the same major index. Section 5 contains some remarks about extending our map to skew shapes, and in what sense it generalizes Foata's map.

2. Definitions and Background

We begin by reviewing Foata's map $\phi : S_n \to S_n$. It can be described as follows. If $n \leq 2$, $\phi(\sigma) = \sigma$. If $n > 2$, we add numbers to ϕ one at a time: begin by setting $\phi^{(1)} = \sigma_1$, $\phi^{(2)} = \sigma_1\sigma_2$. To find $\phi^{(3)}$, start with $\sigma_1\sigma_2\sigma_3$. Then if $\sigma_3 > \sigma_2$, draw a bar after each element of $\sigma_1\sigma_2\sigma_3$ which is less than σ_3, while if $\sigma_3 < \sigma_2$, draw a bar after each element of $\sigma_1\sigma_2\sigma_3$ which is greater than σ_3. Also add a bar before σ_1. For example, if $\sigma = 4137562$, we now have $|41|3$. Now regard the numbers between two consecutive bars as "blocks", and in each block, move the last element to the beginning, and finally remove all of the bars. We end up with $\phi^{(3)} = 143$.

Proceeding inductively, we begin by adding σ_i to the end of $\phi^{(i-1)}$. Then if $\sigma_i > \sigma_{i-1}$, draw a bar after each element of $\phi^{(i-1)}$ which is less than σ_i, while if $\sigma_i < \sigma_{i-1}$, draw a bar after each element of $\phi^{(i-1)}$ which is greater than σ_i. Also draw a bar before $\phi_1^{(i-1)}$. Then in each block, move the last element to the beginning, and finally remove all of the bars. If $\sigma = 4137562$, the successive stages of the algorithm yield

$$\begin{aligned} 143 &= \phi^{(3)} \\ |1|4|3|7 \mapsto 1437 &= \phi^{(4)} \\ |1437|5 \mapsto 71435 &= \phi^{(5)} \\ |71|4|3|5|6 \ \mapsto 174356 &= \phi^{(6)} \\ |17|4|3|5|6|2 \mapsto 7143562 &= \phi^{(7)}, \end{aligned}$$

so $\phi(4137562) = 7143562$. Note that $\mathrm{maj}(4137562) = 11 = \mathrm{inv}(7143562)$.

Theorem 2.1. [Foa68] *The map $\phi : S_n \to S_n$ is a bijection and for all $\sigma \in S_n$,* $\mathrm{maj}(\sigma) = \mathrm{inv}(\phi(\sigma))$.

The inverse to Foata's map is as follows. Let $\psi = \phi^{-1}(\sigma)$. If $\sigma_n > \sigma_1$, draw a bar before each number in σ which is less than σ_n and also before σ_n. If $\sigma_n < \sigma_1$, draw a

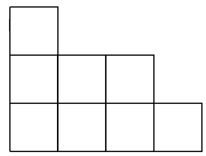

FIGURE 1. The Ferrers diagram of the partition $(4,3,1)$

bar before each number in σ which is greater than σ_n and also before σ_n. Next move each number at the beginning of a block to the end of the block. Remove the bars to obtain $\psi^{(1)}$. The last letter of ψ is now fixed. Now compare $\psi^{(1)}_{n-1}$ with $\psi^{(1)}_1$. Create blocks as above, drawing a bar before $\psi^{(1)}_{n-1}$. Proceed in this manner. For example, if $\sigma = 7143562$, the successive stages of the algorithm yield

$$\begin{aligned} |71|4|3|5|6|2 &\mapsto 1743562 = \psi^{(1)} \\ |17|4|3|5|62 &\mapsto 7143562 = \psi^{(2)} \\ |7143|562 &\mapsto 1437562 = \psi^{(3)} \\ |1|4|3|7562 &\mapsto 1437562 = \psi^{(4)} \\ |14|37562 &\mapsto 4137562 = \psi^{(5)} \\ & 4137562 = \psi^{(6)} = \psi^{(7)}, \end{aligned}$$

so $\phi^{-1}(7143562) = 4137562$. It turns out that the generalization of ϕ^{-1} to tableaux is simpler to describe than the generalization of ϕ.

Next we review the necessary background on tableaux. A partition of a positive integer n is a sequence $\lambda = (\lambda_1, \ldots, \lambda_k)$ of positive integers such that $\sum_{i=1}^{k} \lambda_i = n$ and $\lambda_1 \geq \cdots \geq \lambda_k$. If λ is a partition of n we write $\lambda \vdash n$ or $|\lambda| = n$. The λ_i are called the parts of λ.

We represent $\lambda = (\lambda_1, \ldots, \lambda_k)$ pictorially as follows. We assign (row,column)-coordinates to unit squares in the first quadrant, obtained by permuting the (x, y) coordinates of the upper right-hand corner of the square, so the lower left-hand square has coordinates $(1, 1)$, the square above it $(2, 1)$, etc., and a square (i, j) has ordinate i and abscissa j. Then the Ferrers diagram of λ, which we also denote by λ, is the set of squares, or "cells" $\{(i, j) \in \mathbb{N} \times \mathbb{N} : 1 \leq i \leq k,\ 1 \leq j \leq \lambda_i\}$. That is, the diagram of λ consists of k left-justified rows of squares in the first quadrant of the xy-plane with λ_i squares in the ith row from the bottom. See Figure 1.

The conjugate partition of $\lambda = (\lambda_1, \ldots, \lambda_k)$ is $\lambda' = (\lambda'_1, \ldots, \lambda'_j)$, where λ'_i is the number of parts of λ which are greater than or equal to i. Geometrically, the diagram of λ' is the reflection of the diagram of λ about the line $y = x$.

If $\lambda \vdash n$, a standard Young tableau of shape λ is a filling of the diagram of λ with the numbers $1, \ldots, n$ such that in each row, the numbers are increasing from left to right, and in each column, the numbers are increasing from bottom to top. The number filling a cell is called the content of the cell. We let $c(i, j)$ denote the content of the cell (i, j).

We let $SYT(\lambda)$ denote the set of standard Young tableaux of shape λ. For $T \in SYT(\lambda)$, the conjugate tableau $T' \in SYT(\lambda')$ is obtained by filling the cell (i, j) with the content of the cell (j, i) of T.

3. An Inv Statistic on Standard Young Tableaux

For a standard Young tableau T, the major index of T is given by

$$\mathrm{maj}(T) = \sum_{i \in \mathrm{Des}(T)} i,$$

where $\mathrm{Des}(T) = \{i \,|\, i+1 \text{ is in a row above the row containing } i \text{ in } T\}$. Here we define another statistic on standard Young tableaux which we call the Inv statistic. Let $(i, j), (k, l) \in T$ such that $c(i, j) > c(k, l)$. The cells (i, j) and (k, l) form an "inversion pair" if (k, l) is weakly SE of (i, j), and they do not form an inversion pair if (k, l) is weakly NW of (i, j). If (k, l) is strictly SW of (i, j), then whether or not they form an inversion pair is determined by a path in T called the "inversion path" of (i, j). By strictly SW, we mean Southwest but not due South or due West, i.e. $k < i$ and $l < j$. Below, we construct the set of of inversion paths for T. In the construction we define maps on standard Young tableaux which will be used in our extension of the Foata map.

Definition 3.1. Let $T \in SYT(\lambda)$, where $\lambda \vdash n$. For $k = 1, \ldots, n$, let $\pi(T, k)$ be the path in T constructed according to the following algorithm:

Start at the lower left-hand corner of the cell with content k. Suppose this cell has coordinates (i, j). If $i = 1$ or $j = 1$, proceed in a straight line until you reach the origin. If not, compare the contents of the cells $(i-1, j)$ and $(i, j-1)$. If $c(i-1, j) > c(i, j-1)$, take one unit step South. If $c(i-1, j) < c(i, j-1)$, take one unit step West. Arriving at the lower left-hand corner of a new cell (i', j'), iterate the algorithm. Proceed until you reach the origin.

For example, let T be the SYT in Figure 2 (the fact that the shape is a square is coincidental). To form the inversion path $\pi(T, 16)$, we start at the lower left-hand corner of the square containing 16, and draw a unit line segment left, since the square to the left of 16 contains 15, which is greater than the number in the square below (namely 14). Next we compare the numbers in the squares to the left and below the square containing 15. The largest is 13, so we now extend our inversion path by drawing another unit line segment, in this case downwards, ending at the lower left-hand corner of the square containing 13. We continue comparing squares left and below, moving one unit in the direction of the largest, until we reach the edge of the tableau (the lower left-hand corner of the square containing 3), at which time we go in the only direction we can, namely straight down, until we reach a square with no left or bottom neighbors (the square containing 1).

6	10	15	16
5	9	13	14
3	7	11	12
1	2	4	8

FIGURE 2. A standard Young tableau T and the path $\pi(T, 16)$

Note that the path $\pi(T, k)$ partitions the cells of T with content less than k into two sets: cells which are weakly NW of $\pi(T, k)$ and cells which are weakly SE of $\pi(T, k)$. We will say that cells weakly NW of $\pi(T, k)$ are above $\pi(T, k)$ and cells weakly SE of $\pi(T, k)$ are below $\pi(T, k)$. Two cells are on the same side of $\pi(T, k)$ if they are both above or both below $\pi(T, k)$. Note also that since $\pi(T, k)$ is constructed taking only South and West steps from the cell with content k, all cells weakly SE of the cell with content k are always below $\pi(T, k)$ and all cells weakly NW of the cell with content k are always above $\pi(T, k)$.

Definition 3.2. Let $T \in SYT(\lambda)$, where $\lambda \vdash n$. For $k = 1, \ldots, n$, let $\Psi_k(T)$ be the tableau constructed according to the following algorithm:

Construct the path $\pi(T, k)$. Partition the cells of T with content less than k into "blocks" $\{(i_1, j_1), \ldots, (i_m, j_m)\}$ of maximal length such that

(i) $c(i_l, j_l) = c(i_{l+1}, j_{l+1}) - 1$ for $1 \leq l \leq m - 1$.
(ii) (i_1, j_1) is on the same side of $\pi(T, k)$ as $(1, 1)$, and (i_l, j_l) is on the other side of $\pi(T, k)$ for $2 \leq l \leq m$,

Note that blocks may consist of only one cell. Next perform the following "cycling" procedure on the contents of the cells in each block:

(i) Replace $c(i_1, j_1)$ by $c(i_m, j_m)$.
(ii) For $2 \leq l \leq m$, replace $c(i_l, j_l)$ by $c(i_l, j_l) - 1$.

The resulting tableau is $\Psi_k(T)$.

In the next section, we prove that $\Psi_k(T) \in SYT(\lambda)$. Note that the maps Ψ_2, Ψ_1 are the identity.

Simply put, the blocks for the path $\pi(T, k)$ consist of cells containing consecutive numbers such that the smallest number is in a cell on the same side of the path as the cell containing 1 and all of the rest of the cells are on the other side of the path. In the example in Figure 2, if we denote cells by their contents, then the blocks for $\pi(T, 16)$ are $\{1\}$, $\{2, 3\}$, $\{4, 5, 6, 7\}$, $\{8, 9, 10\}$, $\{11\}$, $\{12\}$, $\{13\}$, $\{14, 15\}$.

If the contents of the cells in a block are $a, a+1, \ldots, a+j$, then the cycling procedure sends a to the cell which had content $a + 1$, etc. and finally $a + j$ to the cell which had content a. See Figure 3. A key observation about the cycling procedure is that it preserves the relative order of the contents of cells in different blocks.

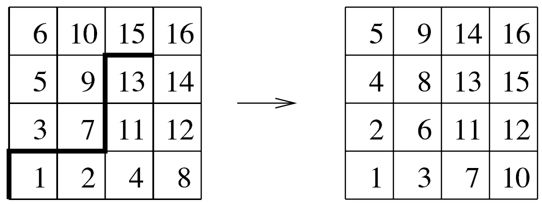

FIGURE 3. The tableau on the right is $\Psi_{16}(T)$

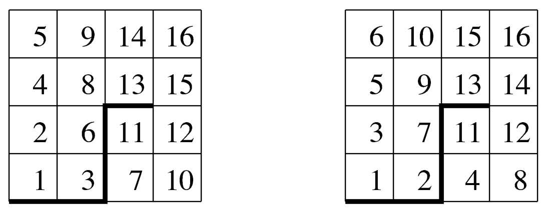

FIGURE 4. On the left: $\Psi_{16}(T)$ and the path $\pi(\Psi_{16}(T), 15)$. On the right: T and the inversion path of the cell $(3, 4)$.

Definition 3.3. The collection of paths

$$\{\pi(T, n), \pi(\Psi_n(T), n-1), \ldots, \pi(\Psi_3 \circ \cdots \circ \Psi_n(T), 2)\}$$

is the set of inversion paths for T. For $(i, j) \in T$, $(i, j) \neq (1, 1)$, the inversion path of (i, j) is the unique element of the set of inversion paths for T starting at the lower left-hand corner of the cell (i, j).

In our running example, the path $\pi(\Psi_{16}(T), 15)$ is the inversion path for the cell $(3, 4)$ (the cell with content 14) in T. See Figure 4. See also Figures 6 and 7 for a complete example.

Now we are ready to introduce the Inv statistic.

Definition 3.4. Let $T \in SYT(\lambda)$. An ordered pair of cells $((i, j), (k, l))$ in T is an inversion pair if $c(k, l) < c(i, j)$ and (k, l) is below the inversion path of (i, j). $\mathrm{Inv}(T)$ is the total number of inversion pairs in T.

In our running example, the cell with content 16 in T forms inversion pairs with the cells with contents $1, 2, 4, 8, 11, 12, 13, 14$ and the cell with content 14 in T forms inversion pairs with the cells with contents $4, 8, 11, 12$.

4. The bijection

Let $\lambda \vdash n$, and let Let $\Psi = \Psi_3 \circ \cdot \circ \Psi_n$. One of our main results is:

Theorem 4.1. *The map* $\Psi : SYT(\lambda) \to SYT(\lambda)$ *is a bijection and for* $T \in SYT(\lambda)$, $\mathrm{Inv}(T) = \mathrm{maj}(\Psi(T))$.

Most of this section is devoted to the proof of Theorem 4.1. First we prove that $\Psi_k : SYT(\lambda) \to SYT(\lambda)$. We will use the following lemma, which helps us understand the geometry of the blocks for a given path.

Lemma 4.2. *Let $T \in SYT(\lambda)$. Suppose that the cells with contents $a, a+1, \ldots, a+m$ form a block for $\pi(T,k)$. If the cell $(1,1)$ is below $\pi(T,k)$, then the cells containing $a+1, \ldots, a+m$ are strictly NW of the cell containing a. If the cell $(1,1)$ is above $\pi(T,k)$, then the cells containing $a+1, \ldots, a+m$ are strictly SE of the cell containing a.*

Proof. We assume k is not in the first row or first column of T because if it were, the lemma would be trivial as all cells would be on the same side of $\pi(T,k)$. Suppose the cell $(1,1)$ is below $\pi(T,k)$. Let (i,j) be the cell with content a and let (r,s) be the cell with content k. Since (i,j) is below $\pi(T,k)$, (i,j) must be either weakly SE or strictly SW of (r,s). In the first case, the remainder of the cells in the block must be in columns to the left of column j, since if they were not, they would be below $\pi(T,k)$. These cells must also be in rows above row i because all cells weakly SW of (i,j) have content $< a$. Next we consider the case in which (i,j) is strictly SW of (r,s). Since $\pi(T,k)$ consists of only South and West steps from (r,s), every cell weakly SE of (i,j) is also below $\pi(T,k)$. Thus, the remainder of the cells in the block must be in rows above row i. As $c(i,j+1) > a$ and the numbers $a+1, \ldots, a+m$ occur in rows above row i, $c(i,j+1) > a+m$. Hence, every cell weakly NE of $(i,j+1)$ has content $> a+m$. So the remainder of the cells in the block must be in column j or in columns to left of column j. Suppose $c(i+p,j) = a+l$, where $a+l$ is the smallest of the numbers $a+1, \ldots, a+m$ in column j. As $(i+p,j)$ is above $\pi(T,k)$ and (i,j) is below $\pi(T,k)$, there must be a West step of the path across column j. To take this West step, we must have had $c(i+q,j) > c(i+q-1,j+1)$ for some q between 1 and p. But this is a contradiction since $a < c(i+q,j) < a+m$ and $c(i+q-1,j+1) > a+m$. Thus the remainder of the cells in the block must be in columns to the left of column j.

The second statement follows since $(1,1)$ is below $\pi(T,k)$ if and only if $(1,1)$ is above $\pi(T',k)$, and the map $T \mapsto T'$ sends cells strictly NW of a given cell in T to cells strictly SE of the image of that cell in T'. □

Theorem 4.3. *For $k = 3, \ldots, n$, if $T \in SYT(\lambda)$, then $\Psi_k(T) \in SYT(\lambda)$.*

Proof. Since Ψ_k acts as the identity on all cells with content greater than $k-1$, it suffices to prove the theorem for $k = n$. As our argument involves the content of both cells in T and in $\Psi_n(T)$, it will be convenient to use the notation $c(i,j)_S$ to denote the content of the cell (i,j) in the tableau S. Let $(i,j) \in T$. Let $c(i,j)_T = p$ and $c(i,j)_{\Psi_n(T)} = p'$. We need to show that if $(i+1,j) \in T$, then $p' < c(i+1,j)_{\Psi_n(T)}$ and if $(i,j+1) \in T$, then $p' < c(i,j+1)_{\Psi_n(T)}$. Assume $(i+1,j) \in T$. Let $c(i+1,j)_T = r$ and $c(i+1,j)_{\Psi_n(T)} = r'$. If (i,j) and $(i+1,j)$ were in different blocks for $\pi(T,n)$, then $p' < r'$ since $p < r$ and the cycling procedure preserves the relative order of the contents of cells in different blocks. If (i,j) and $(i+1,j)$ were in the same block, then by Lemma 4.2, they had to be on the same side of $\pi(T,n)$. But then $p' = p-1$ and $r' = r-1$. So $p < r$ implies $p' < r'$. A similar argument works for the case $(i, j+1) \in T$. $\square$

Next we show that maps Ψ_k are bijections by explicitly constructing the inverse maps. Our starting point is the following lemma, which is key in reconstructing the path $\pi(\Psi_k^{-1}(S), k)$.

Lemma 4.4. *Let $T \in SYT(\lambda)$, where $\lambda \vdash n$. For $k = 2, \ldots, n$, $(1,1)$ is below $\pi(T,k)$ if and only if $k-1 \in \mathrm{Des}(\Psi_k(T))$.*

Proof. We may assume k is not in the first row or column of T, since in these cases the lemma is trivial. Suppose $(1,1)$ is below $\pi(T,k)$. Let a be the largest number below $\pi(T,k)$. So a is in a row below k. If $a = k-1$, then the cell containing a is in a block by itself, and the cycling procedure does not change the contents of this cell. If $a < k-1$, then the cells containing $a, \ldots, k-1$ form a block, and the cycling procedure sends $k-1$ to the cell which contained a. So in either case, $k-1$ is in a row below k in $\Psi_k(T)$. Using a similar argument, one shows that if $(1,1)$ is above $\pi(T,k)$, then $k-1$ is not in a row below k in $\Psi_k(T)$. $\square$

Now we are ready to describe the maps $\Phi_k = \Psi_k^{-1}$. Let $S \in SYT(\lambda)$, and suppose $c(r,s) = k$ and $c(x,y) = k-1$. We give an algorithm to obtain a new tableau from S and a path $\tilde{\pi}(S,k)$ starting at the lower left-hand corner of the cell (r,s) and ending at the origin:

Step 1: If $r = 1$ or $s = 1$, draw a straight line to the origin and stop. Otherwise:
If $k-1 \in \mathrm{Des}(S)$: Find all "simple blocks" $\{(i_1, j_1), \ldots, (i_m, j_m)\}$ of cells such that

(i) $c(r-1, s-1) < c(i_l, j_l) < k$ and $c(i_l, j_l) = c(i_{l+1}, j_{l+1}) + 1$ for all $1 \leq k \leq m$,
(ii) (i_1, j_1) is weakly SE of (r,s) and $(i_2, j_2), \ldots, (i_m, j_m)$ are weakly NW of (r,s),
(iii) the cell with content $c(i_m, j_m) - 1$ is weakly SE of (r,s).

If $k-1 \notin \mathrm{Des}(S)$: Find all simple blocks $\{(i_1, j_1), \ldots, (i_m, j_m)\}$ of cells such that

(i) $c(r-1, s-1) < c(i_l, j_l) < k$ and $c(i_l, j_l) = c(i_{l+1}, j_{l+1}) + 1$ for all $1 \leq k \leq m$,
(ii) (i_1, j_1) is weakly NW of (r,s) and $(i_2, j_2), \ldots, (i_m, j_m)$ are weakly SE of (r,s),
(iii) the cell with content $c(i_m, j_m) - 1$ is weakly NW of (r,s).

Step 2: Perform the following "reverse cycling" procedure on the contents of the cells of each block from Step 1: Replace $c(i_1, j_1)$ by $c(i_m, j_m)$, and for $2 \leq l \leq m$, replace $c(i_l, j_l)$ by $c(i_l, j_l) + 1$.
Comment: In Step 1, we are looking for the blocks of $\pi(\Psi_k^{-1}(S), k)$ which are easy to identify, in particular those which are independent of the path. In Step 2, we undo the cycling procedure of Ψ_k on these blocks. To identify the rest of the blocks, we must reconstruct the path.
Step 3: Construct a unit segment of $\tilde{\pi}(S,k)$ as follows:
If $k-1 \in \mathrm{Des}(S)$: Go South if $(r-1, s)$ is in one of the blocks from Step 2 and $c(r-1, s) > c(r, s-1)$. Otherwise go West.

If $k-1 \notin \mathrm{Des}(S)$: Go West if $(r, s-1)$ is in one of the blocks from Step 2 and $c(r, s-1) > c(r-1, s)$. Otherwise go South.
Step 4: A portion of $\tilde{\pi}(S, n)$ has been constructed and ends at the lower left-hand corner of the cell (u, v). If $u = 1$ or $v = 1$, draw a straight line to the origin and stop. Otherwise: Find all simple blocks $\{(i_1, j_1), \ldots, (i_m, j_m)\}$ of cells such that

(i) for all $1 \leq k \leq m$, (i_l, j_l) was not in a simple block at any previous stage in the algorithm,
(ii) $c(u-1, v-1) < c(i_l, j_l) < k$ and $c(i_l, j_l) = c(i_{l+1}, j_{l+1}) + 1$ for all $1 \leq k \leq m$,
(iii) (i_1, j_1) and (x, y) are on the same side of the portion of the path constructed so far and $(i_2, j_2), \ldots, (i_m, j_m)$ are on the other side,
(iv) the cell with content $c(i_m, j_m) - 1$ and (i_1, j_1) are on the same side of portion of the path constructed so far.

Step 5: Reverse cycle the contents of the cells of each block from Step 4.
Step 6: Construct a unit segment of $\tilde{\pi}(S, k)$ as follows:
If $n-1 \in \mathrm{Des}(S)$: Go South if $(u-1, v)$ has already appeared in a simple block and $c(u-1, v) > c(u, v-1)$. Otherwise go West.
If $n-1 \notin \mathrm{Des}(S)$: Go West if $(u, v-1)$ has already appeared in a simple block and $c(u, v-1) > c(u-1, v)$. Otherwise go South.
Arriving at the lower left-hand corner of a new cell, iterate the algorithm from Step 4.

Observe that if a cell with content a appears in a simple block, then all cells with content greater than a and less than n must have already appeared in simple blocks. Moreover, upon completion of the algorithm, every cell with content greater than 1 and less than n will appear in exactly one simple block.

In Figure 5, we give an example of the algorithm with $k = 16$ applied to the tableau $\Psi_{16}(T)$ from Figure 3.

Theorem 4.5. *For $k = 3, \ldots, n$, if T is the tableau obtained by applying the above algorithm to the tableau S, then $S = \Psi_k(T)$.*

Proof. It suffices to prove the result for $k = n$. First we show that $T \in SYT(\lambda)$. As in the proof that $\Psi_k : SYT(\lambda) \to SYT(\lambda)$, this follows from the geometry of the simple blocks. In particular, if the cells with contents $a, \ldots, a+m$ form a simple block at any stage of the algorithm, then the cells containing $a, \ldots, a+m-1$ are strictly NW, respectively strictly SE, of the cell containing $a+m$ if $n-1 \in \mathrm{Des}(S)$, respectively $n-1 \notin \mathrm{Des}(S)$. For example, suppose $n-1 \in \mathrm{Des}(S)$ and $c(i, j) = a+m$ and $c(i, j') = a+p$ for some p such that $1 \leq p \leq m-1$. Then $\tilde{\pi}(S, n)$ has a South segment across row i. Thus we must have had, at a previous run through the algorithm, (i, l) in a simple block for some l such that $j' < l \leq j$. This leads to a contradiction since $c(i, l) \leq a+m$, so (i, j) would have had to have been in a simple block before we constructed the South segment across row i. The rest of the details are left to the reader.

Next we prove that $S = \Psi_n(T)$. It is enough to show that $\tilde{\pi}(S, n) = \pi(T, n)$ because then by Lemma 4.4, the simple blocks in the above algorithm are (up to reordering)

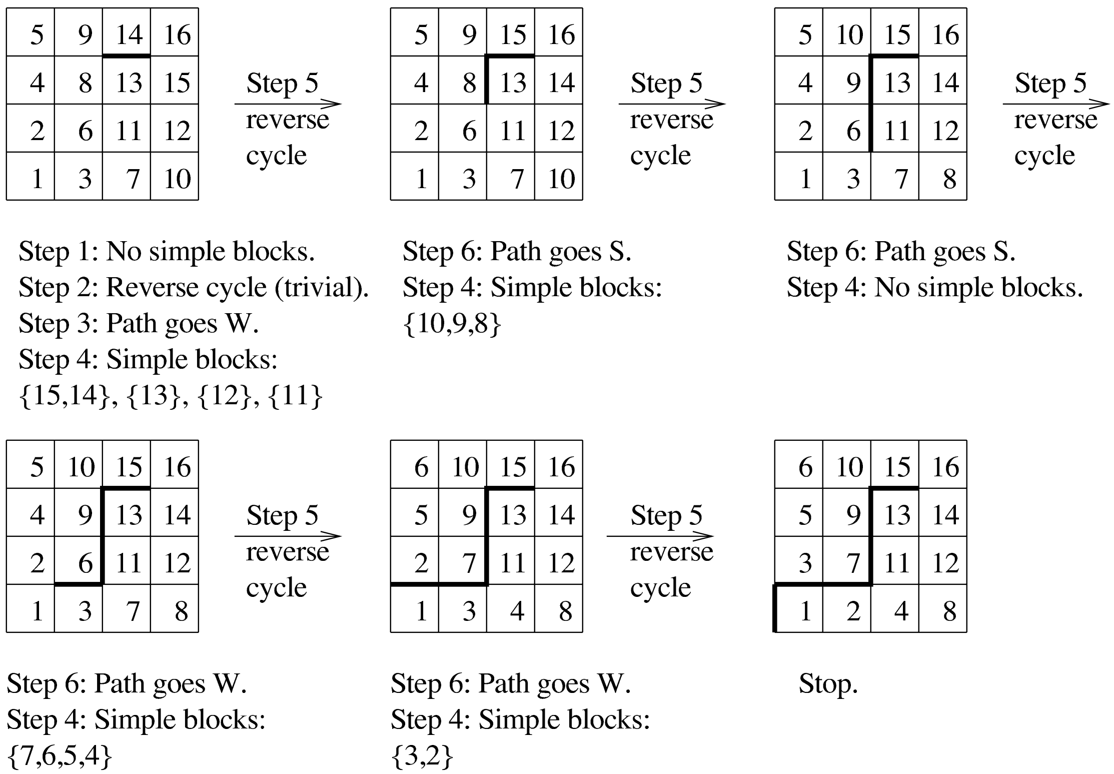

FIGURE 5. An example of the map Φ_n

the blocks for $\pi(T,n)$ with the exception of the block $\{(1,1)\}$, and the reverse cycling procedure undoes the cycling procedure of the map Ψ_n. We give the argument for the case $n-1 \in \mathrm{Des}(S)$; the other case follows by considering the conjugate tableaux. Suppose we have just completed either Step 2 or Step 4 of the algorithm and $\tilde{\pi}(S,n)$ and $\pi(T,n)$ are equal up to this point. We find ourselves at the lower left-hand corner of a cell (i,j) and we need to show that $\tilde{\pi}(S,n)$ and $\pi(T,n)$ go in the same direction. This is clear if $i=1$ or $j=1$, so we assume $i,j \neq 1$. Let p,q be the contents of $(i-1,j)$ and $(i,j-1)$, respectively, at this stage. Let $c(i-1,j)_T = p'$ and $c(i,j-1)_T = q'$. First we consider the case in which $(i-1,j)$ was in a simple block at a previous step in the algorithm. If $(i,j-1)$ was also in a simple block, then $p=p'$ and $q=q'$ so $\pi(T,n)$ and $\tilde{\pi}(S,n)$ go in the same direction. If $(i,j-1)$ was not yet in a simple block, then $q<p'=p$ and the cell with content q' will be in a simple block with $(i,j-1)$ at a later stage in the algorithm, so also $q'<p'$. Thus both paths go South. Now suppose $(i-1,j)$ has not yet appeared in a simple block. Then $c(i-1,j-1) = p-m$ for some m such that the cells with contents $p-1,\ldots,p-m+1$ are all above the portion $\tilde{\pi}(S,n)$ constructed so far. If $(i,j-1)$ was not yet in a simple block, then $q<p$, but also $q>p-m$, so $q=p-l$ for some l such that $1 \leq l \leq m-1$. By the geometry of the simple blocks, the cell $(i-1,j-1)$ cannot be in the block of $(i-1,j)$, so $p'=p-m+1<q'=p-l+1$. If $(i,j-1)$ was already in a simple block then $p<q=q'$, and the cell with content p' will be in a simple block with $(i-1,j)$ at a later stage, so also $p'<q'$. Thus, in both cases, $\pi(T,n)$ and $\tilde{\pi}(S,n)$ go West. □

Thus Ψ is a bijection with inverse $\Phi=\Phi_n\circ\cdots\circ\Phi_3$, and it remains only to prove the second statement of Theorem 4.1.

Proof of Theorem 4.1. Let $T\in SYT(\lambda)$. First we show that if $(1,1)$ is below $\pi(T,n)$, then $\mathrm{Inv}(T)=\mathrm{Inv}(\Psi_n(T)\backslash\{n\})+n-1$, where $\Psi_n(T)\backslash\{n\}$ is the standard Young tableau obtained from $\Psi_n(T)$ by deleting the cell with content n. We begin by observing that the set of inversion paths for $\Psi_n(T)\backslash\{n\}$ equals the set of inversion paths for T minus the inversion path of the cell containing n. Let $(i,j),(k,l)\in T$ such that $c(k,l)_T<c(i,j)_T<n$. First we consider the case in which (i,j) and (k,l) are in different blocks for $\pi(T,n)$. Because the cycling procedure preserves the relative ordering of the contents of cells within different blocks and the inversion paths of (i,j) for T and $\Psi_n(T)$ are the same, (i,j) and (k,l) form an inversion pair in T if and only if they form an inversion pair in $\Psi_n(T)$. Now suppose (i,j) and (k,l) are in the same block for $\pi(T,n)$. Say the contents of the cells in this block are $a,\ldots,a+m$. If (i,j) and (k,l) are both above $\pi(T,n)$, then $c(i,j)_{\Psi_n(T)}=c(i,j)_T-1$ and $c(k,l)_{\Psi_n(T)}=c(k,l)_T-1$. Thus, $c(k,l)_{\Psi_n(T)}<c(i,j)_{\Psi_n(T)}$. Again because the inversion paths of (k,l) for both tableaux are the same, (i,j) and (k,l) form an inversion pair in T if and only if they form an inversion pair in $\Psi_n(T)$. Finally, suppose (i,j) is above $\pi(T,n)$ and (k,l) is below $\pi(T,n)$. So $c(i,j)_T=a+p$ and $c(k,l)_T=a$ where $1\le p\le m$. By Lemma 4.2, (k,l) is strictly SE of (i,j), so (i,j) and (k,l) form an inversion pair in T. But $c(i,j)_{\Psi_n(T)}=a+p-1$ and $c(k,l)_{\Psi_n(T)}=a+m$, so (i,j) and (k,l) do not form an inversion pair in $\Psi_n(T)$. Thus the total number of inversion pairs in $\Psi_n(T)\backslash\{n\}$ equals the number of inversion pairs in T minus one inversion pair for every cell above $\pi(T,n)$ minus one inversion pair for every cell below $\pi(T,n)$, since all of these cells formed inversion pairs in T with the cell with content n. Hence, $\mathrm{Inv}(T)=\mathrm{Inv}(\Psi_n(T)\backslash\{n\})+n-1$.

Next we show that If $(1,1)$ is above $\pi(T,n)$, then $\mathrm{Inv}(T)=\mathrm{Inv}(\Psi_n(T)\backslash\{n\})$. Observe that two cells in T form an inversion pair if and only if they do not form an inversion pair in T', the conjugate tableau. Thus $\mathrm{Inv}(T)=\binom{n}{2}-\mathrm{Inv}(T')$. Similarly, $\mathrm{Inv}(\Psi_n(T)\backslash\{n\})=\binom{n-1}{2}-\mathrm{Inv}((\Psi_n(T)\backslash\{n\})')$. But $(1,1)$ is below $\pi(T',n)$, so

$$\mathrm{Inv}(\Psi_n(T)\backslash\{n\})=\binom{n-1}{2}-(\mathrm{Inv}(T')-(n-1))=\binom{n}{2}-\mathrm{Inv}(T')=\mathrm{Inv}(T).$$

Thus, by Lemma 4.4

$$\mathrm{Inv}(T)=\mathrm{Inv}(\Psi_n(T)\backslash\{n\})+(n-1)\chi\left(n-1\in\mathrm{Des}(\Psi_n(T))\right),$$

where for any logical statement L, $\chi(L)=1$ if L is true and $\chi(L)=0$ if L is false. But Ψ_k acts as the identity on all cells with content $\geq k$, so

$$\mathrm{Inv}(T)=\mathrm{Inv}(\Psi_n(T)\backslash\{n\})+(n-1)\chi\left(n-1\in\mathrm{Des}(\Psi(T))\right).$$

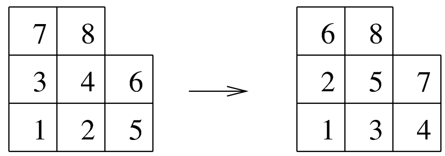

FIGURE 6. On the left: a tableau S. On the right: $\Psi(S)$.

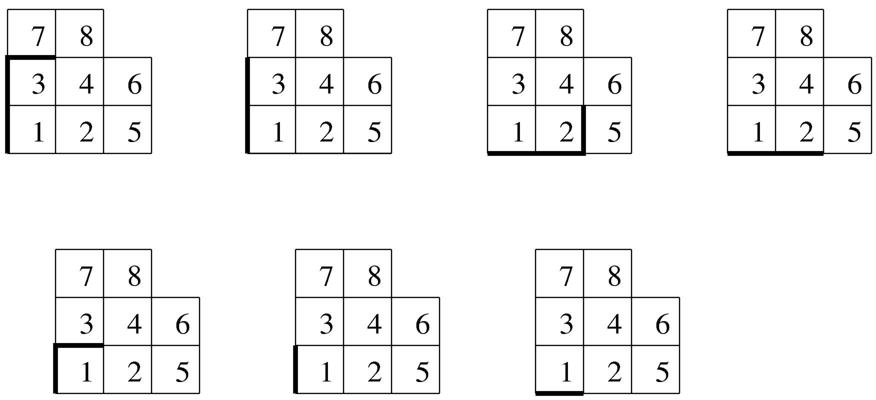

FIGURE 7. The set of inversion paths for S.

Iterating this argument, we see that

$$\mathrm{Inv}(T) = \sum_{j=1}^{n-1} j\chi\,(j \in \mathrm{Des}(\Psi(T))) = \mathrm{maj}(\Psi(T)).$$

$\square$

In Figure 6, we give an example of a tableau S and its image under the map Ψ. We have $\mathrm{maj}(\Psi(S)) = 1+4+5+7 = 17$. In Figure 7, we show the set of inversion paths for S. If we label cells by their contents, then inversion pairs for S are $(8,6),(8,5),(8,4),(8,3),(8,2),(8,1),(7,6),(7,5),(7,4),(7,3),(7,2),(7,1),(6,5),(4,2),$ $(4,1),(3,2),(3,1)$. So $\mathrm{Inv}(S) = 17$.

Corollary 4.6. *The statistics* maj *and* Inv *have equal distributions over $SYT(\lambda)$.*

Remark: Part of our motivation for defining an inversion statistic on SYT is based on recent work of Haglund, Haiman and Loehr [HHL], where it is shown that the coefficient of a monomial symmetric function in the Macdonald polynomial $\tilde{H}_\mu[X;q,t]$ can be expressed in terms of descents and inversion pairs for fillings of the Ferrers shape of μ by positive integers. One could hope that a similar phenomenon would apply to the coefficients $\tilde{K}_{\lambda,\mu}(q,t)$ in the expansion of $\tilde{H}_\mu$ into Schur functions (Macdonald showed that $\tilde{K}_{\lambda,\mu}(1,1)$ equals the number of $SYT(\lambda)$, and posed the question [Mac, p. 356] of whether $\tilde{K}_{\lambda,\mu}(q,t)$ could be written in the form

$$\tilde{K}_{\lambda,\mu}(q,t) = \sum_{T \in SYT(\lambda)} t^{\mathrm{tstat}(T,\mu)} q^{\mathrm{qstat}(T,\mu)} \tag{1}$$

for some tableau statistics tstat, qstat). In [Hag04, Conjecture 3] specific values for tstat, qstat are conjectured for all μ with at most three columns. The value of tstat is described in terms of descents of T, while the value of qstat is described in terms of certain "inversion pairs" of T. (The special case of this conjecture where μ has at most two columns is proved in [HHL, Proposition 9.2]). Although we have as yet been unable to extend this conjecture to general μ, Corollary 4.6 gives a way of expressing $\tilde{K}_{\lambda,(n)}(q,t)$ in this form, since it is known that

$$\tilde{K}_{\lambda,(n)}(q,t) = \sum_{T \in SYT(\lambda)} q^{\mathrm{maj}(T)}, \tag{2}$$

which by Corollary 4.6 also equals

$$\tilde{K}_{\lambda,(n)}(q,t) = \sum_{T \in SYT(\lambda)} q^{\mathrm{Inv}(T)}. \tag{3}$$

5. Extensions

Although we have assumed throughout this article that λ is a partition shape, all our results apply just as easily to skew shapes. For a given SYT T of skew shape $\lambda \vdash n$, note that our map Φ (and its inverse) both fix the largest element n in the tableau. Thus our inversion statistic is equidistributed with maj over the set of $SYT(\lambda)$ with n occurring in some fixed corner square. Let

$$\mathrm{comaj}(T) = \sum_{i \in \mathrm{Des}(T)} n - i.$$

It is known that comaj and maj have the same distribution over SYT of skew shape [Sta, Chapter 7]. We briefly describe a parallel version of our Foata map which is linked to comaj. If square (i,j) contains 1 in T, start at the NE corner of (i,j) and draw a unit line segment N if $c(i+1,j) > c(i,j+1)$, otherwise draw a unit line segment E, and now iterate, eventually ending up at the upper border of T. Next break up the numbers 2 through n into maximal blocks, as before, consisting of an integer k on the same side of the inversion path as n, and a consecutive sequence of integers $k-1, k-2, \ldots, k-a$ on the other side of the path, then cycling by moving $k-a$ to the square containing k, k to the square containing $k-1$, etc.. Thus each square ends up with an inversion path which starts at its NE corner and travels NE to the border, and the number of inversion pairs are equidistributed with comaj over the set of all $SYT(\lambda)$ with 1 occurring in some fixed corner square.

We now show that the special case of our map Φ when λ is a disjoint union of squares, i.e. $\lambda = (n, n-1, \ldots, 1)/(n-1, \ldots, 1)$, and T is the tableau obtained by filling these

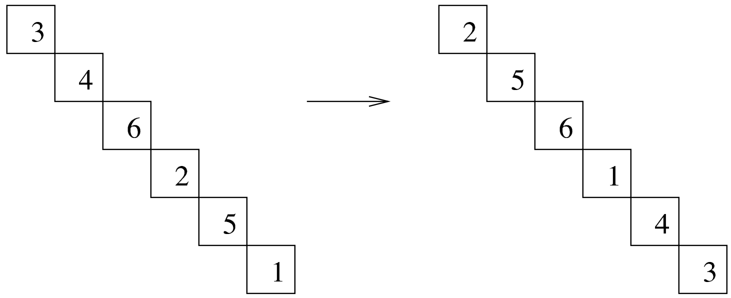

FIGURE 8. On the left: a tableau T of shape $(6,5,4,3,2,1)/(5,4,3,2,1)$ whose filling corresponds to the permutation $\sigma = 346251$. On the right: $\Phi(T)$.

squares with the numbers $\sigma_1, \ldots, \sigma_n$ (see Figure 8), is essentially the same as Foata's original map ϕ applied to σ^{-1}. We identify T with σ and use the notation $\Phi(\sigma)$. To find $\Phi_k(\sigma)$, partition the numbers $k-1, \ldots, 1$ into blocks $\{a, a-1, \ldots, a-j\}$ of maximal length such that a is on the same side of k as $k-1$ and $a-1, \ldots, a-j$ are on the other side of k. Then reverse cycle, e.g. $\Phi_4(346251) = 146352$. Let $\sigma^{-1} = \omega$, so ω_i is the position of i in σ. We now show that $\phi(\omega) = (\Phi(\sigma))^{-1}$. The main idea is to prove that for any k, $3 \leq k \leq n$, $\phi^{(k)}\omega_{k+1} \ldots \omega_n = (\Phi_k \circ \cdots \circ \Phi_3(\sigma))^{-1}$, where $\phi^{(k)} = \phi^{(k)}(\omega)$. This is straightforward when $k = 3$; $\Phi_3(\sigma) = \sigma$ unless 2 and 1 are on opposite sides of 3, in which case Φ_3 interchanges their positions and leaves all other numbers fixed, whereas $\phi^{(3)} = \omega_1\omega_2\omega_3$ only if ω_1 and ω_2 are both less than or greater than ω_3, otherwise $\phi^{(3)} = \omega_2\omega_1\omega_3$. Now assume that the claim is true for some k such that $3 \leq k \leq n-1$. Write $\Phi_k \circ \cdots \circ \Phi_3(\sigma) = \beta$ and $\phi^{(k)} = \alpha_1 \ldots \alpha_k$. Then $\{a, a-1, \ldots, a-j\}$ is a block in β if and only if $\alpha_{a-j} \ldots \alpha_{a-1}\alpha_a$ is a block in $\alpha_1 \ldots \alpha_k\omega_{k+1}$. The reverse cycling procedure sends $a-j$ to the position of a and $a, \ldots, a-j+1$ to the positions of $a-1, \ldots, a-j$, respectively, which corresponds to the Foata algorithm sending the block $\alpha_{a-j} \ldots \alpha_{a-1}\alpha_a$ to $\alpha_a\alpha_{a-j} \ldots \alpha_{a-1}$. Thus $\phi^{(k+1)}\omega_{k+1} \ldots \omega_n = (\Phi_{k+1}(\beta))^{-1}$. For the example in Figure 8, the sequence $\sigma, \Phi_3(\sigma), \Phi_4 \circ \Phi_3(\sigma), \Phi_5 \circ \Phi_4 \circ \Phi_3(\sigma), \Phi(\sigma)$ is

$$346251, 346251, 146352, 146253, 256143$$

whereas the sequence $\omega, \phi^{(3)}\omega_4\omega_5\omega_6, \phi^{(4)}\omega_5\omega_6, \phi^{(5)}\omega_6, \phi(\omega)$ is

$$641253, 641253, 164253, 146253, 416523.$$

6. CONCLUDING REMARKS

The authors would like to thank the referees for helpful suggestions on the exposition, as well as pointing out some interesting directions for future research. For example, MacMahon's result on the equidistribution of maj and inv holds for multiset permutations, and Foata showed how his map proves this more general fact bijectively. Perhaps our results on SYT also have versions for SSYT. We remark that the obvious thing to try, which is to start with a SSYT, standardize in the usual way, apply the Φ map, then "unstandardize", doesn't quite work, as this process will in general produce fillings which are not column strict, and hence not SSYT. One thing that does seem to have a natural analogue in our setting is that of the "inversion code" of a permutation, which is a sequence $x_1x_2\ldots x_n$ with $0 \le x_i \le i-1$ for $1 \le i \le n$, where x_i equals the number of inversion pairs of the permutation of the form (i,j) with $i > j$. Our tableaux inversion statistic is exactly the number of similar such pairs, as in the example just above Corollary 4.6. One could also hope to connect our ideas with other inversion statistics which occur in the theory of symmetric functions and Macdonald polynomials, such as the inversion statistic for tuples of SSYT which can be used to describe LLT polynomials (see [HHL]).

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