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COMPUTER-FREE EVALUATION OF AN INFINITE DOUBLE SUM VIA EULER SUMS

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Abstract. A short and computer-free proof (using Euler sums and multiple zeta functions) is provided for a double sum that was recently computed by Pemantle and Schneider using the computer software Sigma.

1. Introduction

The evaluation of the double sum

$$S := \sum_{j,k=1}^{\infty} \frac{H_j(H_{k+1}-1)}{jk(k+1)(j+k)} = -\zeta(2) - 2\zeta(3) + 4\zeta(2)\zeta(3) + 2\zeta(5), \tag{1}$$

(where $H_n := \sum_{k=1}^{n} \frac{1}{k}$ denote harmonic numbers) appears in [4] and was obtained using Carsten Schneider's software Sigma. Here, we will give a short proof that is completely computer-free. It uses Euler sums and multiple zeta functions.

The second order harmonic numbers which appear in the sequel are denoted by $H_n^{(2)} := \sum_{k=1}^{n} \frac{1}{k^2}$.

We split the sum S and apply partial fraction decomposition:

$$S = \sum_{k\ge1} \frac{H_{k+1}-1}{k(k+1)} \sum_{j\ge1} \frac{H_j}{j(j+k)} = \sum_{k\ge1} \frac{H_{k+1}-1}{k^2(k+1)} \sum_{j\ge1} H_j\Big(\frac{1}{j} - \frac{1}{j+k}\Big).$$

The inner sum is simplified as follows:

$$\sum_{j\ge1} H_j\Big(\frac{1}{j} - \frac{1}{j+k}\Big) = \sum_{j\ge1}\Big(\frac{1}{j} - \frac{1}{j+k}\Big)\sum_{l=1}^{j}\frac{1}{l} = \sum_{l\ge1}\frac{1}{l}\sum_{j\ge l}\Big(\frac{1}{j} - \frac{1}{j+k}\Big).$$

The second sum here telescopes, and only k summands remain:

$$\begin{aligned}
\sum_{l\ge1}\frac{1}{l}\sum_{j\ge l}\Big(\frac{1}{j} - \frac{1}{j+k}\Big) &= \sum_{l\ge1}\frac{1}{l}\sum_{j=0}^{k-1}\frac{1}{l+j} = \sum_{l\ge1}\frac{1}{l^2} + \sum_{j=1}^{k-1}\sum_{l\ge1}\frac{1}{l(l+j)} \\
&= \zeta(2) + \sum_{j=1}^{k-1}\frac{1}{j}\sum_{l\ge1}\Big(\frac{1}{l} - \frac{1}{l+j}\Big) \qquad \text{[partial fraction decomposition]} \\
&= \zeta(2) + \sum_{j=1}^{k-1}\frac{H_j}{j} \qquad \text{[again by telescoping]} \\
&= \zeta(2) + \frac{H_k^2 + H_k^{(2)}}{2} - \frac{H_k}{k}.
\end{aligned}$$

Thus the task reduces to evaluate the following single sum:

$$S = \sum_{k=1}^{\infty} \frac{H_{k+1}-1}{k^2(k+1)}\Big(\zeta(2) + \frac{H_k^2 + H_k^{(2)}}{2} - \frac{H_k}{k}\Big). \tag{2}$$

After partial fraction expansion (and shifting the index if necessary), sum S can be written as follows:

$$S = -2\zeta(2) + \frac{1}{2}\sum_{k\ge1}\frac{H_k H_k^{(2)}}{k^2} + \frac{1}{2}\sum_{k\ge1}\frac{H_k^3}{k^2} - \sum_{k\ge1}\frac{H_k^2}{k^3} + (\zeta(2)-1)\sum_{k\ge1}\frac{H_k}{k^2} - 2\sum_{k\ge1}\frac{1}{k^2}.$$

For the final computation of S we require the following evaluations of Euler sums via values of the Riemann zeta function:

$$\sum_{k\ge1}\frac{H_k H_k^{(2)}}{k^2} = \zeta(5) + \zeta(2)\zeta(3), \tag{3a}$$

$$\sum_{k\ge1}\frac{H_k^3}{k^2} = 10\zeta(5) + \zeta(2)\zeta(3), \tag{3b}$$

$$\sum_{k\ge1}\frac{H_k^2}{k^3} = \frac{7}{2}\zeta(5) - \zeta(2)\zeta(3), \tag{3c}$$

$$\sum_{k\ge1}\frac{H_k}{k^2} = 2\zeta(3), \tag{3d}$$

from which equation (1) then follows.

Equations (3c) and (3d) can be found explicitly in [3]. In [3] one also finds the identities

$$\sum_{k\ge1}\frac{H_k^3}{(k+1)^2} = \frac{15}{2}\zeta(5) + \zeta(2)\zeta(3), \tag{4a}$$

$$\sum_{k\ge1}\frac{H_k}{k^4} = 3\zeta(5) - \zeta(2)\zeta(3), \tag{4b}$$

and due to

$$\sum_{k\ge1}\frac{H_k^3}{k^2} = \sum_{k\ge1}\frac{H_k^3}{(k+1)^2} + 3\sum_{k\ge1}\frac{H_k^2}{k^3} - 3\sum_{k\ge1}\frac{H_k}{k^4} + \sum_{k\ge1}\frac{1}{k^5},$$

we obtain equation (3b) as well, using (4a), (4b) and (3c).

To show (3a) we will apply Theorem 2 of [2], which gives

$$\sum_{k\ge1}\frac{H_{k-1}H_{k-1}^{(2)}}{k^2} = \zeta(2,1,2) + \zeta(2,2,1) + \zeta(2,3),$$

where the *multiple zeta functions* are defined by

$$\zeta(a_1, a_2, \dots, a_m) = \sum_{k_1>k_2>\cdots>k_m\ge1} \frac{1}{k_1^{a_1}k_2^{a_2}\cdots k_m^{a_m}}.$$

Since

$$\begin{aligned}\sum_{k\ge1}\frac{H_kH_k^{(2)}}{k^2}&=\sum_{k\ge1}\frac{H_{k-1}H_{k-1}^{(2)}}{k^2}+\sum_{k\ge1}\frac{H_{k-1}^{(2)}}{k^3}+\sum_{k\ge1}\frac{H_{k-1}}{k^4}+\sum_{k\ge1}\frac{1}{k^5}\\&=\sum_{k\ge1}\frac{H_{k-1}H_{k-1}^{(2)}}{k^2}+\sum_{k>l\ge1}\frac{1}{k^3l^2}+\sum_{k>l\ge1}\frac{1}{k^4l}+\sum_{k\ge1}\frac{1}{k^5},\end{aligned}$$

we obtain

$$\sum_{k\ge1}\frac{H_kH_k^{(2)}}{k^2}=\zeta(2,1,2)+\zeta(2,2,1)+\zeta(2,3)+\zeta(3,2)+\zeta(4,1)+\zeta(5).$$

Using the following evaluations of the multiple zeta function given in [1] resp. [2]:

$$\begin{aligned}\zeta(2,1,2)=\zeta(2,3)&=\frac{9}{2}\zeta(5)-2\zeta(2)\zeta(3),\\\zeta(2,2,1)=\zeta(3,2)&=-\frac{11}{2}+3\zeta(2)\zeta(3),\\\zeta(4,1)&=2\zeta(5)-\zeta(2)\zeta(3),\end{aligned}$$

equation (3a) follows.

References

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