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ON DERANGEMENT POLYNOMIALS OF TYPE B

CHAK-ON CHOW

ABSTRACT. Wachs [*Proc. Amer. Math. Soc.* **106** (1989), 273–278] studied the q-enumeration of derangements in the symmetric group $\mathfrak{S}_n$ by the major index and obtained a q-analogue of the classical derangement number. We consider in this work the q-enumeration of derangements in the hyperoctahedral group B_n by the flag major index and obtain a q-analogue of the type B derangement number.

Let $n \geqslant 1$. Let B_n be the nth hyperoctahedral group, which is a Coxeter group of rank n, consisting of signed permutations of $[n] := \{1, 2, \ldots, n\}$. We shall represent a signed permutation $\sigma \in B_n$ by the word $\sigma_1\sigma_2 \cdots \sigma_n$, where $\sigma_i = \sigma(i)$, $i = 1, 2, \ldots, n$. The (type A) descent set of σ is $D(\sigma) := \{i \in [n-1] \colon \sigma_i > \sigma_{i+1}\}$. A statistic derivable from $D(\sigma)$ is the usual (type A) major index: $\operatorname{maj}(\sigma) := \sum_{i\in D(\sigma)} i$. Let $N(\sigma) := \#\{i \in [n] \colon \sigma_i < 0\}$ be the number of negative letters of σ.

Two candidates for the type B major index, namely, the negative major index (nmaj) and the flag major index (fmaj), have recently been proposed and proven to be Mahonian, i.e.,

$$\sum_{\sigma\in B_n} q^{\operatorname{fmaj}(\sigma)} = \sum_{\sigma\in B_n} q^{\operatorname{nmaj}(\sigma)} = \sum_{\sigma\in B_n} q^{l_B(\sigma)} = [2]_q[4]_q \cdots [2n]_q,$$

where $[k]_q := 1 + q + q^2 + \cdots + q^{k-1}$ is a q-integer, $\operatorname{fmaj}(\sigma) := 2\operatorname{maj}(\sigma) + N(\sigma)$, and l_B the length function on B_n. By pairing with the negative descent number (ndes) and the flag descent number (fdes), respectively, these major indices are shown to satisfy a q-rational generating function generalizing the classical Carlitz identity. See [1, 2] for the definitions of undefined terms and the above mentioned results.

In a recent work, Chow and Gessel [5] studied the Euler-Mahonian pair $(\operatorname{des}_B, \operatorname{fmaj})$, where des_B is the type B descent number, on B_n and computed its q-rational generating function. This q-rational generating function reduces to the rational generating function for the type B Eulerian polynomial when $q \to 1$, thus providing a *natural* type B generalization of the classical Carlitz identity. It is from this point of view that the flag major index (fmaj) plays the same role on the hyperoctahedral group B_n as the usual major index (maj) does on the symmetric group $\mathfrak{S}_n$.

A number of statistical studies on B_n sequel to [1, 2, 5] have been pursued by various authors. See, e.g., [6, 7, 9] which considered the flag and negative major indices as well as other permutation statistics in the more general setting of signed words and wreath products.

For $n \geqslant 1$, let $\mathscr{D}_n := \{\sigma \in \mathfrak{S}_n \colon \sigma(i) \neq i \text{ for all } i \in [n]\}$ be the set of all derangements in $\mathfrak{S}_n$. Wachs [11] considered derangement polynomials defined by $d_n(q) := \sum_{\sigma\in\mathscr{D}_n} q^{\operatorname{maj}(\sigma)}$ and showed combinatorially that

$$d_n(q) = [n]_q! \sum_{k=0}^{n} \frac{(-1)^k q^{\binom{k}{2}}}{[k]_q!}, \tag{1}$$

where $[n]_q! := [1]_q[2]_q \cdots [n]_q$ is a q-factorial. Letting $q \to 1$, the above formula reduces to the formula of the nth classical derangement number: $d_n = n! \sum_{k=0}^{n} (-1)^k/k!$. It is from this point of view that Wachs' derangement polynomials are q-analogues of the derangement numbers.

We show in this work (Theorem 5) that $d_n^B(q) := \sum_{\sigma \in \mathscr{D}_n^B} q^{\mathrm{fmaj}(\sigma)}$ can be written as

$$d_n^B(q) = [2]_q[4]_q \cdots [2n]_q \sum_{k=0}^{n} \frac{(-1)^k q^{2\binom{k}{2}}}{[2]_q[4]_q \cdots [2k]_q},$$

where $\mathscr{D}_n^B := \{\sigma \in B_n \colon \sigma(i) \neq i \text{ for all } i \in [n]\}$ is the set of derangements in B_n. The first four $d_n^B(q)$ are given as follows:

$$\begin{aligned}
d_1^B(q) &= q,\\
d_2^B(q) &= q + 2q^2 + q^3 + q^4,\\
d_3^B(q) &= q + 3q^2 + 4q^3 + 5q^4 + 5q^5 + 4q^6 + 4q^7 + 2q^8 + q^9,\\
d_4^B(q) &= q + 4q^2 + 8q^3 + 13q^4 + 18q^5 + 22q^6 + 26q^7 + 28q^8 + 28q^9 + 25q^{10}\\
&\quad + 21q^{11} + 17q^{12} + 11q^{13} + 7q^{14} + 3q^{15} + q^{16}.
\end{aligned}$$

Following Wachs, for any signed permutation $\alpha \in B_A$, where $A = \{a_1 < a_2 < \cdots < a_k\}$, define the *reduction* of α to be the signed permutation in B_k obtained from α by replacing each letter α_j by $(\operatorname{sgn} \alpha_j)i$ if $|\alpha_j| = a_i$, $i = 1, 2, \ldots, k$. The *derangement part* of a signed permutation $\sigma \in B_n$, denoted $dp(\sigma)$, is the reduction of the subword of non-fixed points of σ. For example, $dp(5\bar{3}\bar{1}476\bar{2}) = 4\bar{3}\bar{1}5\bar{2}$. Note that the derangement part of a signed permutation is a signed derangement, and that conversely, any derangement in $\mathscr{D}_k^B$ and k-element subset of $[n]$ determine a signed permutation in B_n with $n-k$ fixed points. Hence, the number of signed permutations in B_n with a given derangement part in $\mathscr{D}_k^B$ is $\binom{n}{k}$.

Now let $\sigma = \sigma_1\sigma_2 \cdots \sigma_n \in B_n$. A letter σ_i of σ is an *excedant* (respectively *subcedant*) of σ if $\sigma_i > i$ (respectively $\sigma_i < i$). Let $s(\sigma)$ and $e(\sigma)$ be the number of subcedants and excedants of σ, respectively. It is clear that excedants of σ are necessarily positive. We now fix n and let $k \leqslant n$. For $\sigma \in B_k$, let $\tilde{\sigma}$ be the signed permutation of k letters obtained from σ by replacing its ith smallest (in absolute value) subcedant σ_j by $(\operatorname{sgn} \sigma_j)i$, $i = 1, 2, \ldots, s(\sigma)$, its ith smallest fixed point by $s(\sigma) + i$, $i = 1, 2, \ldots, k - s(\sigma) - e(\sigma)$, and its ith largest excedant by $n - i + 1$, $i = 1, 2, \ldots, e(\sigma)$. The map $\sigma \to \tilde{\sigma}$ restricted to the symmetric group $\mathfrak{S}_k$ is precisely the descent set preserving map used in [11]. If $k = n$ then $\tilde{\sigma} \in B_n$. If σ is a signed derangement, then $\tilde{\sigma} \in \mathscr{D}_A^B$, where $A = \{1, 2, \ldots, s(\sigma)\} \cup \{n - e(\sigma) + 1, n - e(\sigma) + 2, \ldots, n\}$.

Lemma 1. *Let $\sigma \in B_k$, $k \leqslant n$. Then $D(\sigma) = D(\tilde{\sigma})$ and $N(\sigma) = N(\tilde{\sigma})$.*

Proof. It is clear from the construction of $\tilde{\sigma}$ that $N(\sigma) = N(\tilde{\sigma})$. Let $\sigma = \sigma_1\sigma_2 \cdots \sigma_k$ and $\tilde{\sigma} = \tilde{\sigma}_1\tilde{\sigma}_2 \cdots \tilde{\sigma}_k$. For each $i \in [k-1]$, we shall show that $i \in D(\sigma)$ if and only if $i \in D(\tilde{\sigma})$, by considering the nine possible designations of subcedant (s), excedant (e), and fixed point (f)

to σ_i and σ_{i+1}. Since the letters $\tilde{\sigma}_i$ are determined according to whether σ_i is a subcedant (e), a fixed point (f), or an excedant (e), and in this order, $\tilde{\sigma}_i < \tilde{\sigma}_{i+1}$ if the designation of σ_i precedes that of σ_{i+1}.

Suppose that (σ_i, σ_{i+1}) is an (f, e) or (f, f) pair. Then $\sigma_i = i < i+1 \leqslant \sigma_{i+1}$ so that $\tilde{\sigma}_i < \tilde{\sigma}_{i+1}$. If (σ_i, σ_{i+1}) is an (f, s), then $\sigma_i = i > i-1 \geqslant \sigma_{i+1}$ so that $\tilde{\sigma}_i > s(\sigma) \geqslant |\tilde{\sigma}_{i+1}| \geqslant \tilde{\sigma}_{i+1}$.

Suppose now that (σ_i, σ_{i+1}) is an (s, f) or (s, e) pair. Then $\sigma_i < i < i+1 \leqslant \sigma_{i+1}$ so that $\tilde{\sigma}_i < \tilde{\sigma}_{i+1}$. If (σ_i, σ_{i+1}) is an (s, s), then since $\operatorname{sgn} \sigma_i = \operatorname{sgn} \tilde{\sigma}_i$, $\sigma_i \gtrless \sigma_{i+1} \gtrless 0$ if and only if $\tilde{\sigma}_i \gtrless \tilde{\sigma}_{i+1} \gtrless 0$, and $\sigma_i \gtrless 0 \gtrless \sigma_{i+1}$ if and only if $\tilde{\sigma}_i \gtrless 0 \gtrless \tilde{\sigma}_{i+1}$.

Suppose finally that (σ_i, σ_{i+1}) is an (e, f) pair. Then $\sigma_i \geqslant i+2 > i+1 = \sigma_{i+1}$ so that $\tilde{\sigma}_i > \tilde{\sigma}_{i+1}$; if (σ_i, σ_{i+1}) is an (e, s) pair, then $\sigma_i > i \geqslant \sigma_{i+1}$ so that $\tilde{\sigma}_i > \tilde{\sigma}_{i+1}$; consider now (σ_i, σ_{i+1}) being an (e, e) pair. If σ_i (respectively σ_{i+1}) is the jth (respectively kth) largest excedant of σ, then $\sigma_i \lesseqgtr \sigma_{i+1}$ if and only if $j \gtreqless k$ if and only if $\tilde{\sigma}_i = n-j+1 \lesseqgtr n-k+1 = \tilde{\sigma}_{i+1}$. □

Let $n_1, \ldots, n_r$ be non-negative integers such that $\sum_{i=1}^{r} n_i = n$. Recall the q-multinomial coefficient

$$\begin{bmatrix} n \\ n_1, \ldots, n_r \end{bmatrix}_q := \frac{[n]_q!}{[n_1]_q! \cdots [n_r]_q!}.$$

Now let $(N_1, \ldots, N_r)$ be a partition of the set $[n]$. For each $1 \leqslant i \leqslant r$ let π_i be a permutation of the elements of N_i. Recall that a permutation $\sigma \in \mathfrak{S}_n$ is a shuffle of $\pi_1, \pi_2, \ldots, \pi_r$ if, for every i, the letters of N_i appears in σ in the same order as the corresponding letters in π_i. The generating function of shuffles of permutations by their major index has been computed by Garsia and Gessel [8, Theorem 3.1].

Theorem 2 (Garsia-Gessel)**.** *Let* $\operatorname{sh}(\pi_1, \ldots, \pi_r)$ *be the collection of all shuffles of given permutations* $\pi_1, \pi_2, \ldots, \pi_r$*. Then*

$$\sum_{\sigma \in \operatorname{sh}(\pi_1, \ldots, \pi_r)} q^{\operatorname{maj}(\sigma)} = \begin{bmatrix} n \\ n_1, \ldots, n_r \end{bmatrix}_q q^{\operatorname{maj}(\pi_1) + \cdots + \operatorname{maj}(\pi_r)},$$

where n_i *is the length of* π_i $(1 \leqslant i \leqslant r)$*.*

Theorem 2 remains true if $\pi_1, \ldots, \pi_r$ are signed words of distinct letters from the totally ordered alphabet $\{-n < -n+1 < \cdots < -1 < 1 < 2 < \cdots < n\}$. It is clear in this case that $N(\sigma) = N(\pi_1) + \cdots + N(\pi_r)$. Replacing q by q^2, followed by multiplication by $q^{N(\sigma)}$ and noting that $\operatorname{fmaj}(\sigma) = 2 \operatorname{maj}(\sigma) + N(\sigma)$, we have the following identity:

$$\sum_{\sigma \in \operatorname{sh}(\pi_1, \ldots, \pi_r)} q^{\operatorname{fmaj}(\sigma)} = \begin{bmatrix} n \\ n_1, \ldots, n_r \end{bmatrix}_{q^2} q^{\operatorname{fmaj}(\pi_1) + \cdots + \operatorname{fmaj}(\pi_r)}, \tag{2}$$

where n_i is the length of π_i $(1 \leqslant i \leqslant r)$.

Theorem 3. *Let* $\alpha \in \mathscr{D}_k^B$, $k \leqslant n$, *and* $\gamma = (s(\alpha)+1)(s(\alpha)+2) \ldots (n - e(\alpha))$*. Then the map* $\varphi \colon \{\sigma \in B_n \colon dp(\sigma) = \alpha\} \to \operatorname{sh}(\tilde{\alpha}, \gamma)$ *defined by* $\varphi(\sigma) = \tilde{\sigma}$ *is a flag major index preserving bijection, i.e.,* $\operatorname{fmaj}(\sigma) = \operatorname{fmaj}(\varphi(\sigma))$*.*

Proof. The flag major index preserving property of φ is clear from Lemma 1. Since $\#\{\sigma \in B_n\colon\ dp(\sigma) = \alpha\} = \#\,\mathrm{sh}(\tilde{\alpha}, \gamma) = \binom{n}{k}$, to prove that φ is a bijection, it suffices to prove that $\varphi(\{\sigma \in B_n\colon\ dp(\sigma) = \alpha\}) = \mathrm{sh}(\tilde{\alpha}, \gamma)$.

First, we claim that if $\sigma \in B_n$ is such that $dp(\sigma) = \alpha$, then $\tilde{\sigma}$ is obtained from σ by replacing the subword of non-fixed points of σ by $\tilde{\alpha}$ and the subword of fixed points of σ by γ. Indeed, the subword of fixed points of σ is replaced by the word $(s(\sigma)+1)(s(\sigma)+2)\cdots(n-e(\sigma))$, which is precisely γ since $s(\sigma) = s(\alpha)$ and $e(\sigma) = e(\alpha)$. Also since α is the reduction of the subword of non-fixed points of σ, the position of the ith smallest subcedant of α is the same as the position of the i smallest subcedant of σ in its subword of non-fixed points. The same is true for the ith largest excedant. Hence each subcedant and excedant of σ is replaced by the same letter that replaces the corresponding subcedant and excedant of α, which means that the subword of subcedants and excedants of σ is replaced by $\tilde{\alpha}$. Thus $\varphi(\sigma) = \tilde{\sigma} \in \mathrm{sh}(\tilde{\alpha}, \gamma)$.

To finish the proof, we show that φ is surjective. Let $\tau \in \mathrm{sh}(\tilde{\alpha}, \gamma)$. Replace the $\tilde{\alpha}$ subword by the signed permutation, of the subword positions, whose reduction is α, and the letters of the γ subword by their positions, we obtain a unique signed permutation $\sigma \in B_n$ such that $dp(\sigma) = \alpha$ and $\varphi(\sigma) = \tau$. □

Proposition 4. *Let $\alpha \in \mathscr{D}_k^B$ and $0 \leqslant k \leqslant n$. Then*

$$\sum_{dp(\sigma)=\alpha,\, \sigma \in B_n} q^{\mathrm{fmaj}(\sigma)} = q^{\mathrm{fmaj}(\alpha)} \begin{bmatrix} n \\ k \end{bmatrix}_{q^2}.$$

Proof. This follows from

$$\sum_{dp(\sigma)=\alpha,\, \sigma \in B_n} q^{\mathrm{fmaj}(\sigma)} = \sum_{\sigma \in \mathrm{sh}(\tilde{\alpha},\gamma)} q^{\mathrm{fmaj}(\sigma)} = q^{\mathrm{fmaj}(\tilde{\alpha})} \begin{bmatrix} n \\ k \end{bmatrix}_{q^2} = q^{\mathrm{fmaj}(\alpha)} \begin{bmatrix} n \\ k \end{bmatrix}_{q^2},$$

where the first equality follows from Theorem 3, the second from (2), and the third from Lemma 1. □

Recall that the two q-exponential functions defined by

$$e(u;q) := \sum_{n \geqslant 0} \frac{u^n}{(q;q)_n}, \qquad E(u;q) := \sum_{n \geqslant 0} \frac{q^{\binom{n}{2}} u^n}{(q;q)_n},$$

satisfy $E(-u;q)e(u;q) = 1$ (which is what the so-called Gauß inversion [3, p. 96] amounts to), where

$$(u;q)_n := \begin{cases} 1 & \text{if } n = 0, \\ (1-u)(1-uq)\cdots(1-uq^{n-1}) & \text{if } n \geqslant 1. \end{cases}$$

Theorem 5. *For $n \geqslant 1$, we have*

(i) $d_n^B(q) = [2]_q[4]_q \cdots [2n]_q \displaystyle\sum_{k=0}^{n} \frac{(-1)^k q^{2\binom{k}{2}}}{[2]_q[4]_q \cdots [2k]_q}$,

(ii) $\displaystyle\sum_{n \geqslant 0} d_n^B(q) \frac{u^n}{[2]_q[4]_q \cdots [2n]_q} = \frac{E(-u(1-q);q^2)}{1-u}$,

(iii) $d_{n+1}^B(q) = [2n+2]_q d_n^B(q) + (-1)^{n+1} q^{2\binom{n+1}{2}}$.

Proof. By virtue of the preceding proposition, we have

$$
\begin{aligned}
[2]_q[4]_q\cdots[2n]_q &= \sum_{\sigma\in B_n} q^{\mathrm{fmaj}(\sigma)} \\
&= \sum_{k=0}^{n}\sum_{\alpha\in\mathscr{D}_k^B}\sum_{dp(\sigma)=\alpha} q^{\mathrm{fmaj}(\sigma)} \\
&= \sum_{k=0}^{n}\sum_{\alpha\in\mathscr{D}_k^B} q^{\mathrm{fmaj}(\alpha)} \begin{bmatrix} n \\ k \end{bmatrix}_{q^2} \\
&= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{q^2} d_k^B(q).
\end{aligned}
\tag{3}
$$

Since $[2]_q[4]_q\cdots[2n]_q = (1+q)^n[n]_{q^2}!$, (3) can be simplified as

$$(1+q)^n = \sum_{k=0}^{n} \frac{d_k^B(q)}{[k]_{q^2}![n-k]_{q^2}!}.$$

Multiplying through by $u^n/(1-q^2)^n$, followed by summing over $n \geqslant 0$, we get

$$e(u;q^2)\sum_{n\geqslant 0} d_n^B(q)\frac{u^n}{(q^2;q^2)_n} = \sum_{n\geqslant 0}\frac{u^n}{(1-q)^n},$$

since $(q^2;q^2)_n = (1-q^2)^n[n]_{q^2}!$. Multiplying the preceding equation by $E(-u;q^2)$, followed by extracting the coefficients of u^n, we have

$$d_n^B(q) = \sum_{k=0}^{n}\frac{(-1)^k q^{2\binom{k}{2}}(q^2;q^2)_n}{(q^2;q^2)_k(1-q)^{n-k}} = \sum_{k=0}^{n}\frac{(-1)^k q^{2\binom{k}{2}}[2]_q[4]_q\cdots[2n]_q}{[2]_q[4]_q\cdots[2k]_q},$$

which is (i). Since $(q^2;q^2)_n = (1-q)^n[2]_q[4]_q\cdots[2n]_q$, we have

$$\sum_{n\geqslant 0} d_n^B(q)\frac{u^n}{(1-q)^n[2]_q[4]_q\cdots[2n]_q} = \frac{E(-u;q^2)}{1-u/(1-q)}.$$

Replacing u by $u(1-q)$, we get (ii). (iii) is immediate from (i). □

Theorem 5(i) is the type B analogue of (1), in the sense that $[2]_q[4]_q\cdots[2n]_q$ (respectively $[n]_q!$) is the Poincaré series of B_n (respectively $\mathfrak{S}_n$). (See [4, Chapter 7].) By letting $q\to 1$, $E(-u(1-q);q^2)\to e^{-u/2}$ and Theorem 5 specializes to

$$
\begin{aligned}
\sum_{n\geqslant 0} d_n^B\frac{u^n}{2^n n!} &= \frac{e^{-u/2}}{1-u}, \\
d_{n+1}^B &= 2(n+1)d_n^B + (-1)^{n+1}, \\
d_n^B &= n!\sum_{k=0}^{n}\frac{(-1)^k 2^{n-k}}{k!},
\end{aligned}
$$

where $d_n^B = d_n^B(1)$ is the derangement number of B_n; the last formula can also be obtained by a routine application of the principle of inclusion-exclusion [10, Chapter 2].

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P.O. Box 91100, Tsimshatsui Post Office, Hong Kong
E-mail address: cchow@alum.mit.edu