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COMBINATORICS, SUPERALGEBRAS, INVARIANT THEORY AND REPRESENTATION THEORY

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Abstract

We provide an elementary introduction to the (characteristic zero) theory of *Letterplace Superalgebras*, regarded as bimodules with respect to the *superderivation* actions of a pair of *general linear Lie superalgebras*, and discuss some applications.

Contents

1 Introduction

The purpose of the present work is to provide an elementary introduction to the (characteristic zero) theory of *Letterplace Superalgebras*, regarded as bimodules with respect to the *superderivation* actions of a pair of *general linear Lie superalgebras*, as well as to show that this theory yields (by specialization) a simple unified treatment of classical theories.

The idea of Lie superalgebras arises from Physics, since they implement "transitions of symmetry" and, more generally, *supersymmetry* (see, e.g., [52], [39], [88], [2], [29], [37], [60]).

General linear Lie superalgebra actions on letterplace superalgebras yield a natural setting that allows Capelli's method of *virtual (auxiliary) variables* to get its full effectiveness and suppleness. The superalgebraic version of Capelli's method was introduced by Palareti, Teolis and the present author in 1988 in order to prove the complete decomposition theorem for letterplace superalgebras [11], to introduce the notion of *Young–Capelli symmetrizers*, and to provide a "natural matrix form" of Schur superalgebras [12].

The theory of letterplace superalgebras, regarded as bimodules with respect to the action of a pair of general linear Lie superalgebras, is a fairly general one, and encompasses a variety of classical theories, by specialization and/or restriction.

We limit ourselves to mention the following:

- The ordinary representation theory of the symmetric group, up to the Young natural form of irreducible matrix representations.

- The classical representation theory of general linear and symmetric groups over tensor spaces, as well as its pioneering generalization to the $\mathbb{Z}_2$-graded case due to Berele and Regev [4], [5].

- Vector invariant theory (see, e.g., [92], [38], [34], [31]).

- After the work of Grosshans, Rota and Stein [46], letterplace superalgebras provide a unified language for the *symbolic representation* of invariants of symmetric tensors (Aronhold symbols) and skew-symmetric tensors (Weitzenböck's Komplex-Symbolik) (see, e.g., Weyl [92], Brini, Huang and Teolis [16], Grosshans [47]).

- The Deruyts theory of *covariants of weight zero* [33], 1892.

  Deruyts developed a theory of covariants of weight zero that anticipates by nearly a decade the main results of Schur's celebrated Dissertation on the irreducible polynomial representations of $GL_n(\mathbb{C})$.

  This work of Deruyts has been defined by Green *"a pearl of nineteenth century algebra"* ([45, page 249]).

- Letterplace superalgebras provide a natural setting to extend (in an effective and non-trivial way), the theory of *transvectants* from binary forms to n-ary forms, n an arbitrary positive integer (Brini, Regonati and Teolis [22]).

Two methodological remarks. Thanks to the systematic use of the superalgebraic version of Capelli's method of *virtual variables*, the theory described below turns out to be a quite compact one.

As a matter of fact, the whole theory relies upon two basic results: the (super) Straightening Formula (Grosshans, Rota and Stein [46]) and the Triangularity Theorem for the action of (superstandard) *Young–Capelli symmetrizers* on the basis of (superstandard) *symmetrized bitableaux* (Brini and Teolis [12]).

The (super) Straightening Formula admits a few lines proof in terms of virtual variables (see, e.g., [17], [19]). A quite direct proof of the Triangularity Theorem has been recently derived from a handful of combinatorial lemmas on Young tableaux (see, e.g., [21] and Regonati [69]).

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2 Synopsis

The work is organized as follows.

In Section 3, we recall some elementary definitions about associative and Lie superalgebras, and describe some basic examples.

In Section 4, Letterplace Superalgebras, regarded as bimodules with respect to the actions of a pair of general linear Lie superalgebras, are introduced by comparing two equivalent languages, namely, the *combinatorial* one ("signed alphabets") and the more traditional language of multilinear algebra ("$\mathbb{Z}_2$-graded vector spaces").

In Section 5, we summarize a handful of basic definitions and facts about Young tableaux on signed alphabets.

In Section 6, we provide an introduction to the superalgebraic version of Capelli's method of *virtual variables*.

Indeed, the basic operators one needs to manage are operators that induce "symmetries". The starting point of the method is that these operators, that we call *Capelli-type operators*, can be defined, in a quite natural and simple way, by appealing to "extra" variables (the *virtual variables*).

The true meaning of Theorem 6.1 is that the action of these operators is the *same* as the action of operators induced by the action of the enveloping algebra $\mathcal{U}(pl(V)$ of the general linear Lie superalgebra of a $\mathbb{Z}_2$-graded vector space V, and, therefore, they are indeed of representation-theoretical meaning.

The action of these "virtual" Capelli-type operators is much easier to study than the action of their "non-virtual" companions, and computations are consequently carried out in the virtual context.

In Sections 7, 8 and 9, by using the method of virtual variables, we introduce some crucial concepts of the theory, namely, the concepts of *biproducts*, *bitableaux*, and *left, right and doubly symmetrized bitableaux.*

Bitableaux provide the natural generalization of *bideterminants* of a pair of Young tableaux in the sense of [38] and [34].

Left, right and doubly symmetrized bitableaux generalize a variety of classical notions. We mention:

- Images of highest weight vectors, under the action of the "negative" *root spaces* of $sl_n(\mathbb{C})$ (see, e.g., [40]).
- Generators of the irreducible symmetry classes of tensors (see, e.g., [92], [4], [5]).
- Generators of minimal left ideals of the group algebra $\mathbb{K}[\mathbf{S}_n]$, $\mathbf{S}_n$ the symmetric group on n elements (see, e.g., [53]).

We provide an elementary proof of the (super)-Straightening Law of Grosshans, Rota and Stein [46] and exhibit four classes of "representation-theoretically" relevant bases of the letterplace superalgebra: the standard basis and three classes of Clebsch–Gordan–Capelli bases. The first basis is given by (super)standard bitableaux, while the others are given by (super)standard right, left and doubly symmetrized bitableaux, respectively. (We submit that there are deep relations among these bases. The Clebsch–Gordan–Capelli bases yield *complete decompositions* of the letterplace superalgebra as a module, while the standard basis yields an *invariant filtration* (this is a characteristic-free fact); in this filtration, the irreducible modules that appear in the decomposition associated to the Clebsch–Gordan–Capelli bases are complements of each invariant subspace to the preceding one).

In Section 10, we introduce the basic operators of the theory, the *Young–Capelli symmetrizers.* These operators are defined, via the method of virtual variables, as special Capelli-type operators and turn out to be a generalization of the classical Capelli operators as well as of the Young symmetrizers.

In spite of the fact that they could be represented as (extremely complicated) "polynomials" in the (proper) polarization operators, their virtual definition is quite simple and leads to the main result of the theory, the *Triangularity Theorem* (Theorem 10.1, Subsection 10.2).

In Section 11, Schur and Weyl modules are described as subspaces spanned by *right* and *left* symmetrized bitableaux.

Schur and Weyl modules provide the two basic classes of irreducible submodules of the letterplace superalgebra with respect to the action of a general linear Lie superalgebra. (In the special case of a trivial $\mathbb{Z}_2$-graduation, they yield the two basic constructions of the irreducible representations of the general linear group; see, e.g., [1], [85]).

In Section 12, complete decompositions of the letterplace superalgebra and of its Schur superalgebra are exhibited.

It is worth noticing that these decomposition results follow at once from the Triangularity Theorem about the action of (standard) Young–Capelli symmetrizers on (standard) symmetrized bitableaux. Furthermore, a fairly general version of the *double centralizer theorem* also follows from the same argument.

In Section 13, we derive from the Triangularity Theorem a *matrix form* for the irreducible representations of a general linear Lie superalgebra over a letterplace superalgebra.

The matrix entries are strictly related (Theorem 13.1) to the *symmetry transition coefficients* discussed in Subsection 8.2 that are, in turn, a generalization of the Désarménien straightening coefficients [35].

Furthermore, these matrix representations yield, as a very special case, the *Young natural form* of the irreducible matrix representations of the symmetric group $\mathbf{S}_n$.

The third part of the paper is devoted to the discussion of some applications of the general theory.

In Section 14, we derive an explicit decomposition result for spaces obtained by performing tensor products of spaces of *symmetric* tensors and *skew-symmetric* tensors, regarded as modules with respect to the classical diagonal action of the general linear group.

The present approach turns out to be a nice and transparent application of the Feynman–Rota idea of the entangling/disentangling operator in the language of letterplace superalgebras. Specifically, the components of the tensor product are *faithfully encoded* by *"places with multiplicity"*, and, therefore, the decomposition results are special cases of the general decomposition results for letterplace algebras. For example, Howe's version of the first fundamental theorem of the invariant theory of GL_n (Theorem 1.A, [49]) can be obtained from Corollary 14.1.

In Section 15, the classical representation theory of the special linear Lie algebra $sl_m(\mathbb{C})$

in terms of highest weight vectors and root spaces is derived in a short and concrete way.

We exhibit two characterizations of highest weight vectors in $Super_n[\mathcal{L}|\mathcal{P}]$ as linear combinations of standard bitableaux and of standard right symmetrized bitableaux whose left tableau is a fixed tableau of Deruyts type (i.e., column-constant in the first symbols of the letter alphabet $\mathcal{L}$) (Subsection 15.1).

These characterizations allow us to recognize that the action of the subalgebra n_m^- of strictly lower triangular matrices on the highest weight vectors just "replaces" the left Deruyts tableau by any symmetrized standard tableau (Subsection 15.2).

By combining these two facts with the Clebsch–Gordan–Capelli bases theorem, we infer that the b_m^--cyclic modules generated by the highest weight vectors are indeed *irreducible* $sl_m(\mathbb{C})$-modules, and provide a combinatorial description of two families of bases of these irreducible $sl_m(\mathbb{C})$-modules in terms of left symmetrized bitableaux and of doubly symmetrized bitableaux (Subsection 15.3).

In Section 16, we discuss Deruyts' theory of covariants, following along the lines of its admirable reconstruction provided by J.A. Green [45] (see also [30]).

In his remarkable but undervalued paper of 1892, Deruyts developed a theory of covariants of weight zero that yields, in modern language, an exhaustive description of the irreducible polynomial representations of $GL_n(\mathbb{C})$, as well as a proof of the complete reducibility of any polynomial representations of $GL_n(\mathbb{C})$, and, therefore, he anticipates by nearly a decade the main results of Schur's Dissertation.

His language is the language of invariant theory, and he makes little use of matrices. But we can now look back on Deruyts' work and find a wealth of methods which, to our eyes, are pure representation theory; some of these methods are still unfamiliar today (Green [45, p. 248]).

The starting point of Deruyts and Green's approach is the construction of an algebra isomorphism σ from a commutative letterplace algebra to the algebra of covariants of weight zero and the fact that, given any such covariant φ, its *left span* $L(\varphi)$ is a $\mathbb{K}[GL_n]$-module that turns out to be equal to the cyclic $\mathbb{K}[GL_n]$-module generated by the unique preimage γ of φ with respect to the isomorphism σ (γ is called the *source* of the covariant φ).

Thanks to this correspondence, one can develop the theory in the context of covariants of weight zero and, then, translate the main results in the language of their sources, where the GL_n-representation theoretic meaning of the main results is, in a sense, more transparent. This is precisely the strategy of Deruyts and Green.

In order to exploit the full power of the theory developed in Part II, we reverse this strategy and work directly in the algebra of sources. The main results (e.g., characterization of semiinvariants, construction of irreducible representations, complete reducibility) turn out to be simple applications of general results (again, the notion of symmetrized bitableau plays a central role). Then, by using the map σ, we translate these results into Deruyts' language of covariants.

We believe that letterplace superalgebras and virtual variables provide also a "quick and good way" to learn and understand classical theories. In order to support this assertion, in Sections 17 and 18 we describe the way to deduce, just by specialization, the pioneering theory of tensor representations of the symmetric group and general linear Lie superalgebras [4], [80], [5] (special case: general linear groups [92]), as well as the theory of regular representations of symmetric groups (see, e.g., [53, 72]).

In Section 17, we consider the special case of a positively $\mathbb{Z}_2$-trivially graded alphabet $\mathcal{L} = \{1, \ldots, n\}$, while $\mathcal{P}$ is any finite signed set.

Let $Super_n[\underline{\mathcal{L}}|\mathcal{P}]$ be the subspace of the letterplace algebra spanned by all the *multilinear* monomials of degree n in the symbols of $\mathcal{L}$; this subspace is isomorphic (via the so-called Feynman entangling/disentangling operators [38]) to the space $T^n[W_0 \oplus W_1]$ of n-tensors over the $\mathbb{Z}_2$-graded vector space $W_0 \oplus W_1$ whose basis is identified with the signed alphabet $\mathcal{P}$. There is a natural action of the symmetric group $\mathbf{S_n}$ on the subspace $Super_n[\underline{\mathcal{L}}|\mathcal{P}]$; via the linear isomorphism mentioned above, this action corresponds to the *Berele–Regev action* [4, 5] of $\mathbf{S_n}$ on the tensor space $T^n[W_0 \oplus W_1]$.

Here, the point of the specialization argument is the fact that the operator algebra induced by the action of the symmetric group $\mathbf{S_n}$ admits a simple description in terms of polarization operators (Proposition 17.1). Therefore, the Berele–Regev theory follows, as a special case, from the general theory of letterplace superalgebras (we recall that, as a further special case — *in the case of trivial positive $\mathbb{Z}_2$-graduation on $\mathcal{P}$* — the Berele–Regev theory reduces to the classical Schur–Weyl tensor representation theory of symmetric and general linear groups, see, e.g., [92]).

In Section 18, we further specialize the theory and derive the theory of regular representations of symmetric groups.

We consider the subspace $Super_n[\underline{\mathcal{L}}|\underline{\mathcal{P}}]$ of the letterplace superalgebra spanned by doubly multilinear monomials (on the negatively signed set $\mathcal{L} = \mathcal{P} = \{1, \ldots, n\}$). The space $Super_n[\underline{\mathcal{L}}|\underline{\mathcal{P}}]$ is a $\mathbb{K}[\mathbf{S_n}]$-module in an obvious way; here, the point of the specialization argument is that the operator algebra $\underline{\underline{\mathcal{B}}}_n$ induced by the action of $\mathbb{K}[\mathbf{S_n}]$ is an isomorphic copy of $\mathbb{K}[\mathbf{S_n}]$ and, again, it admits a simple description in terms of polarization operators (Proposition 18.1).

We recall that the $\mathbb{K}[\mathbf{S_n}]$-modules $Super_n[\underline{\mathcal{L}}|\underline{\mathcal{P}}]$ and $\mathbb{K}[\mathbf{S_n}]$ (regarded as left regular module) are naturally isomorphic.

Therefore, the decomposition theory of $\mathbb{K}[\mathbf{S_n}]$ as a left regular module is "the same" as the theory of the module $\mathbb{K}[\mathbf{S_n}] \cdot \mathbf{Super_n}[\underline{\mathcal{L}}|\underline{\mathcal{P}}] = \underline{\underline{\mathcal{B}}}_{\mathbf{n}} \cdot \mathbf{Super_n}[\underline{\mathcal{L}}|\underline{\mathcal{P}}]$ that, in turn, immediately follows by specializing the general theory.

In the isomorphic structures

$$\mathbb{K}[\mathbf{S_n}] \simeq Super_n[\underline{\mathcal{L}}|\underline{\mathcal{P}}] \simeq \underline{\underline{\mathcal{B}}}_n,$$

Young symmetrizers in $\mathbb{K}[\mathbf{S_n}]$ correspond, up to a sign, to "doubly multilinear" *symmetrized bitableaux* in $Super_n[\underline{\mathcal{L}}|\underline{\mathcal{P}}]$ and to "doubly multilinear" *Young–Capelli symmetrizers* in $\underline{\underline{\mathcal{B}}}_n$.v

As a consequence, the general natural form of irreducible matrix representations specializes to the classical Young natural form of irreducible matrix representations of symmetric groups (see, e.g., [72], [53], [41]).

Following a cogent suggestion of the referees, in the fourth part of the paper we briefly discuss how to connect the present theory with the general representation theory of Lie superalgebras (over the complex field $\mathbb{C}$) as developed by Kac, Brundan, Penkov, Serganova, Van der Jeugt and Zhang, to name but a few.

In Section 19 and 20, we provide a brief outline of the basic ideas of Kac's approach to the representation theory of finite dimensional Lie superalgebras (see, e.g., [54], [55], [56]), and describe in detail the case of general linear Lie superalgebras. The representation theory of these superalgebras is very close to the representation theory of the so-called *basic classical simple Lie superalgebras of type I* (see, e.g [54], [23], [24]).

The main constructions are, in this context, those of the *integral highest weight modules* and of the *Kac modules relative to dominant integral highest weights* (see Subsections 20.4 and 20.5). A deep result of Kac (see, e.g., [55], [56], [39]) states that highest weight modules and Kac modules coincide if and only if the highest weight Λ of the representation is a *typical* one (Subsection 20.5, Proposition 20.1).

It follows from Theorem 12.1 and from the results in Section 17 that the irreducible modules that appear in the theory of letterplace superalgebra representations are *covariant modules*, in the sense of [26], [66].

Covariant modules are finite dimensional highest weight representations but they are not, in general, Kac modules (since their highest weights can be *atypical* ones, see Remark 20.7).

In Subsection 20.6, we provide a detailed combinatorial analysis of covariant modules as highest weight representations, as well as a direct description of their highest weights and highest weight vectors; these results follow at once from the fact that covariant modules are Schur–Weyl modules (Section 11).

In Subsection 20.7, we show that the theory of letterplace representations yields — up to the action of the so-called umbral operator (see, e.g., [46], [16], [47] — the decomposition theory of the super-symmetric algebra $\mathbf{S}(\mathbf{S}^2(V))$ and of the super-antisymmetric algebra $\bigwedge(\mathbf{S}^2(V))$, recently rediscovered by Cheng and Wang ([26], [27]) and Sergeev ([81], [82]).

We provide a rather extensive bibliography. Some items are not mentioned in the text; they are books and papers of general or historical interest ([3], [9], [36], [43], [44], [48], [63], [64], [65], [68], [75], [86], [89], [90], [91], [93]) and a couple of papers that deal with some aspects of the theory not treated in this work ([14], [15]).

Part I

The General Setting

Throughout the paper, $\mathbb{K}$ will denote a field of characteristic zero, even if a substantial part of the theory below still holds, modulo suitable normalizations, over fields of arbitrary characteristic.

3 Superalgebras

A *superalgebra* A is simply a $\mathbb{Z}_2$-graded algebra, in symbols

$$A = A_0 \oplus A_1,$$

such that

$$A_i A_j \subseteq A_{i+j}, \quad i, j \in \mathbb{Z}_2.$$

Given a $\mathbb{Z}_2$-homogeneous element $a \in A$, its $\mathbb{Z}_2$-degree is denoted by $|a|$.

3.1 The supersymmetric superalgebra of a $\mathbb{Z}_2$-graded vector space

Given a $\mathbb{Z}_2$-graded vector space $U = U_0 \oplus U_1$, its *supersymmetric superalgebra* $Super[U]$ is the superalgebra

$$Super[U] = Sym[U_0] \otimes \Lambda[U_1],$$

where

$$\begin{aligned} Super[U] &= Super[U]_0 \oplus Super[U]_1, \\ Super[U]_0 &= Sym[U_0] \otimes \left(\bigoplus_{h \in \mathbb{N}} \Lambda^{2h}[U_1] \right), \\ Super[U]_1 &= Sym[U_0] \otimes \left(\bigoplus_{h \in \mathbb{N}} \Lambda^{2h+1}[U_1] \right). \end{aligned}$$

The supersymmetric algebra $Super[U]$ has a natural structure of a $\mathbb{Z}_2$-graded bialgebra (see Subsection 3.5).

3.2 Lie superalgebras

A *Lie superalgebra* is a superalgebra $L = L_0 \oplus L_1$ whose product (Lie superbracket) satisfies the following identities:

- $[x, y] = -(-1)^{|x||y|}[y, x]$
- $(-1)^{|x||z|}[x, [y, z]] + (-1)^{|z||y|}[z, [x, y]] + (-1)^{|y||x|}[y, [z, x]] = 0$ ($\mathbb{Z}_2$-graded Jacobi identity.)

3.3 Basic example: the general linear Lie superalgebra of a $\mathbb{Z}_2$-graded vector space

Given a $\mathbb{Z}_2$-graded vector space $U = U_0 \oplus U_1$, its *general linear Lie superalgebra* $pl(U)$ is the $\mathbb{Z}_2$-graded vector space

$$End_{\mathbb{K}}[U] = End_{\mathbb{K}}[U]_0 \oplus End_{\mathbb{K}}[U]_1,$$
$$End_{\mathbb{K}}[U]_i = \{\varphi \in End_{\mathbb{K}}[U]; \varphi[U_j] \subseteq U_{i+j},\ j \in \mathbb{Z}_2\}, \quad i \in \mathbb{Z}_2,$$

endowed with the *supercommutator*

$$[\varphi, \psi] = \varphi\psi - (-1)^{|\varphi||\psi|}\psi\varphi.$$

MATRIX VERSION: let $\mathcal{L} = \{x_1, \ldots, x_n\}$ be a $\mathbb{Z}_2$-homogeneous basis of $U = U_0 \oplus U_1$. A standard basis of $pl(U)$ is given by the set of linear endomorphisms (elementary matrices)

$$E_{x_i,x_j}, \quad E_{x_i,x_j}(x_k) = \delta_{j,k}x_i, \quad i, j, k = 1, \ldots, n.$$

The E_{x_i,x_j}'s are $\mathbb{Z}_2$-homogeneous elements of $End_{\mathbb{K}}[U]$ of degree $|x_i| + |x_j|$ and satisfy the identities

$$[E_{x_i,x_j}, E_{x_h,x_k}] = \delta_{j,h}E_{x_i,x_k} - (-1)^{(|x_i|+|x_j|)(|x_h|+|x_k|)}\delta_{i,k}E_{x_h,x_j}.$$

3.4 The supersymmetric superalgebra $Super[\mathcal{A}]$ of a signed set $\mathcal{A}$

A *signed set* (or, equivalently, a $\mathbb{Z}_2$-*graded set*) is a set $\mathcal{A}$ endowed with a *sign* map $|\ | : \mathcal{A} \to \mathbb{Z}_2$; the sets $\mathcal{A}_0 = \{a \in \mathcal{A}; |a| = 0\}$ and $\mathcal{A}_1 = \{a \in \mathcal{A}; |a| = 1\}$ are called the subsets of *positive* and *negative* symbols, respectively.

The *supersymmetric* $\mathbb{K}$-*superalgebra* $Super[\mathcal{A}]$ is the quotient algebra of the free associative $\mathbb{K}$-algebra with 1 generated by the signed set $\mathcal{A}$ modulo the bilateral ideal generated by the elements of the form:

$$xy - (-1)^{|x||y|}yx, \qquad x, y, \in \mathcal{A}.$$

Remark 3.1. 1. $Super[\mathcal{A}]$ is a $\mathbb{Z}_2$-graded algebra

$$Super[\mathcal{A}] = (Super[\mathcal{A}])_0 \oplus (Super[\mathcal{A}])_1,$$

where $(Super[\mathcal{A}])_i$ is the subspace of $Super[\mathcal{A}]$ spanned by the monomials m of $\mathbb{Z}_2$-degree $|m| = i$, where, for $m = a_{i_1} \cdots a_{i_n}$, $|m| = |a_{i_1}| + \cdots + |a_{i_n}|$.

With respect to this grading, $Super[\mathcal{A}]$ is supersymmetric, i.e.:

$$mm' = (-1)^{|m||m'|} m'm,$$

for every $\mathbb{Z}_2$-homogeneous elements $m, m' \in Super[\mathcal{A}]$.

2. $Super[\mathcal{A}]$ is an $\mathbb{N}$-graded algebra

$$Super[\mathcal{A}] = \bigoplus_{n \in \mathbb{N}} Super_n[\mathcal{A}]$$
$$Super_n[\mathcal{A}] = \langle a_{i_1} a_{i_2} \cdots a_{i_n};\ a_{i_h} \in \mathcal{A} \rangle_{\mathbb{K}};$$

3. the $\mathbb{Z}_2$-graduation and the $\mathbb{N}$-graduation of $Super[\mathcal{A}]$ are *coherent*, that is

$$(Super[\mathcal{A}])_i = \bigoplus_{n \in \mathbb{N}} (Super_n[\mathcal{A}])_i, \qquad i \in \mathbb{Z}_2.$$

Let $\mathcal{A} = \mathcal{A}_0 \cup \mathcal{A}_1$ be a *finite* signed set, and let $U = U_0 \oplus U_1$, where $U_0 = \langle \mathcal{A}_0 \rangle_{\mathbb{K}}$, $U_1 = \langle \mathcal{A}_1 \rangle_{\mathbb{K}}$. The superalgebra $Super[\mathcal{A}]$ is isomorphic to the supersymmetric algebra $Super[U]$ of the $\mathbb{Z}_2$-graded vector space $U = U_0 \oplus U_1$.

3.5 $\mathbb{Z}_2$-graded bialgebras. Basic definitions and the Sweedler notation

Let us consider an algebraic structure $(X, \pi, \Delta, \eta, \varepsilon)$ where:

- $X = X_0 \oplus X_1$ is a $\mathbb{Z}_2$- graded $\mathbb{K}$-vector space.
- π is an *associative product* on X, that is an even linear map

$$\pi : X \otimes X \to X$$

such that

$$\pi(I \otimes \pi) = \pi(\pi \otimes I).$$

- Δ is a *coassociative coproduct* on X, that is an even linear map
$$\Delta : X \to X \otimes X$$
such that
$$(\Delta \otimes I)\Delta = (I \otimes \Delta)\Delta.$$
The *Sweedler notation* [84] is a way to write the coproduct of an element $x \in X$, namely
$$\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}.$$
For example, in the Sweedler notation the fact that Δ is a coassociative coproduct reads as follows:
$$\sum_{(x)} x_{(1)} \otimes \left(\sum_{(x_{(2)})} x_{(2)_{(1)}} \otimes x_{(2)_{(2)}} \right) = \sum_{(x)} \left(\sum_{(x_{(1)})} x_{(1)_{(1)}} \otimes x_{(1)_{(2)}} \right) \otimes x_{(2)}.$$

- η is a *unit map*, that is an even linear map
$$\eta : \mathbb{K} \to X$$
such that
$$\eta(k)x = kx, \quad k \in \mathbb{K}, \quad x \in X.$$

- ε is a *counit map*, that is an even linear map
$$\varepsilon : X \to \mathbb{K}$$
such that
$$\sum_{(x)} \varepsilon(x_{(1)})x_{(2)} = \sum_{(x)} x_{(1)}\varepsilon(x_{(2)}) = x, \quad x \in X.$$

- $(\varepsilon \circ \eta)(1) = 1, \quad 1 \in \mathbb{K}.$

An algebraic structure $(X, \pi, \Delta, \eta, \varepsilon$ is said to be a $\mathbb{Z}_2$-*graded* $\mathbb{K}$-*bialgebra* whenever the following conditions hold:

- The coproduct Δ is an algebra morphism. Notice that $X \otimes X$ is meant as the *tensor product of* $\mathbb{Z}_2$-*graded algebras*, that is $(x \otimes x')(y \otimes y') = (-1)^{|x'||y|} xy \otimes x'y'$, x, y, x', y' $\mathbb{Z}_2$-homogeneous elements in X. In the Sweedler notation, the above condition reads as follows:
$$\Delta(xy) = \sum_{(xy)} (xy)_{(1)} \otimes (xy)_{(2)} = \sum_{(x),(y)} (-1)^{|x_{(2)}||y_{(1)}|} x_{(1)}y_{(1)} \otimes x_{(2)}y_{(2)} = \Delta(x)\Delta(y),$$
$x, y \in X$.

- The unit map η is a coalgebra morphism, that is

$$\Delta(\eta(1)) = \eta(1) \otimes \eta(1), \quad 1 \in \mathbb{K}.$$

- The counit map ε is an algebra morphism, that is

$$\varepsilon(xy) = \varepsilon(x)\varepsilon(y), \quad x, y \in X.$$

3.6 The superalgebra $Super[\mathcal{A}]$ as a $\mathbb{Z}_2$-graded bialgebra. Left and right derivations, coderivations and polarization operators

Let $\mathcal{A} = \mathcal{A}_0 \cup \mathcal{A}_1$ be a *finite* signed set, $\mathcal{A}_0 = \{a_1, a_2, \ldots, a_m\}$, $\mathcal{A}_1 = \{a_{m+1}, a_{m+2}, \ldots, a_{m+n}\}$, and let $U = U_0 \oplus U_1$, where $U_0 = \langle \mathcal{A}_0 \rangle_{\mathbb{K}}$, $U_1 = \langle \mathcal{A}_1 \rangle_{\mathbb{K}}$.

The superalgebra $Super[\mathcal{A}] \cong Super[U]$ is a $\mathbb{Z}_2$-graded bialgebra, where the structure maps are defined in the following way:

- $\Delta(1) = 1 \otimes 1, \quad \Delta(a_i) = a_i \otimes 1 + 1 \otimes a_i, \ a_i \in \mathcal{A}$;
- $\eta(1) = 1$;
- $\varepsilon(1) = 1, \quad \varepsilon(a_i) = 0, \ a_i \in \mathcal{A}$.

A linear map

$$D : Super[\mathcal{A}] \to Super[\mathcal{A}],$$

homogeneous of degree $d = |D| \in \mathbb{Z}_2$, i.e., such that $D(Super[\mathcal{A}])_i \subseteq (Super[\mathcal{A}])_{i+d}$, is a *left superderivation* if

$$D(m\ m') = D(m)\ m' + (-1)^{|D||m|} m\ D(m'),$$

for all monomials $m, m' \in Super[\mathcal{A}]$.

Let $a_i, a_j \in \mathcal{A}$. The *left superpolarization* D_{a_i,a_j} *of the letter* a_j *to the letter* a_i is the unique left superderivation of $\mathbb{Z}_2$-degree $|a_i| + |a_j|$, such that

$$D_{a_i,a_j}(a_h) = \delta_{j,h}\ a_i,$$

for every $a_h \in \mathcal{A}$.

Here and in the following the Greek letter δ will denote the Kronecker symbol.

Any linear map $D : Super[\mathcal{A}] \to Super[\mathcal{A}]$ may be extended to an operator

$$(D \oplus D) : Super[\mathcal{A}] \otimes Super[\mathcal{A}] \to Super[\mathcal{A}] \otimes Super[\mathcal{A}],$$

by the rule of "left superderivation", that is by setting

$$(D \oplus D)(m \otimes m') = D(m) \otimes m' + (-1)^{|D||m|} m \otimes D(m'),$$

for every $m, m' \in Super[\mathcal{A}]$. If no confusion arises, we will frequently write D in place of $D \oplus D$.

We notice that if (D is a left superderivation of $\mathbb{Z}_2$-degree $|D|$, then $(D \oplus D)$ is a left superderivation of $\mathbb{Z}_2$-degree $|D|$.

A linear map

$$D : Super[\mathcal{A}] \to Super[\mathcal{A}],$$

homogeneous of degree $|D| \in \mathbb{Z}_2$, is said to be a *left coderivation* if the following condition holds:

$$\Delta(D(m)) = (D \oplus D)(\Delta(m)),$$

for every $m \in Super[\mathcal{A}]$. In the Sweedler notation, the above condition reads as follows:

$$\Delta(D(m)) = \sum_{(m)} \left[D(m_{(1)}) \otimes m_{(2)} + (-1)^{|D||m_{(1)}|} m_{(1)} \otimes D(m_{(2)}) \right].$$

Proposition 3.1. *Any left superpolarization D_{a_i,a_j} is a left coderivation of the bialgebra $Super[\mathcal{A}]$.*

The next result is one of the basic tools of the *method of vitual variables* and exploits a deep connection between the language of superpolarizations and the language of bialgebras.

Corollary 3.1. *Let $m = a_{i_1} a_{i_2} \cdots a_{i_p} \in Super[\mathcal{A}]$ and let $a_j \in \mathcal{A}$, $|a_j| = 0$, such that $a_j \neq a_{i_h}$, $h = 1, 2, \ldots, p$. Then*

$$\Delta(m) = \frac{1}{p!} D_{a_{i_1},a_j} D_{a_{i_2},a_j} \cdots D_{a_{i_p},a_j} (\Delta((a_j)^p)).$$

Example 3.1. *Let $a_1, a_2, a_3 \in \mathcal{A}$, $|a_1| = 1, |a_2| = 1, |a_3| = 0$ and let $a \in \mathcal{A}$, $|a| = 0$, such that $a \neq a_i$, $i = 1, 2, 3$. Let $m = a_1 a_2 a_3$. We have the following identity:*

$$\begin{aligned}
\Delta(m) = \Delta(a_1)\Delta(a_2)\Delta(a_3) = \\
(a_1 \otimes 1 + 1 \otimes a_1)(a_2 \otimes 1 + 1 \otimes a_2)(a_3 \otimes 1 + 1 \otimes a_3) = \\
a_1 a_2 a_3 \otimes 1 + a_1 a_2 \otimes a_3 + a_1 a_3 \otimes a_2 - a_2 a_3 \otimes a_1 + \\
a_1 \otimes a_2 a_3 - a_2 \otimes a_1 a_3 + a_3 \otimes a_1 a_2 + 1 \otimes a_1 a_2 a_3 = \\
\frac{1}{3!} D_{a_1,a} D_{a_2,a} D_{a_3,a} (a^3 \otimes 1 + 3a^2 \otimes a + 3a \otimes a^2 + 1 \otimes a^3) = D_{a_1,a} D_{a_2,a} D_{a_3,a} (\frac{1}{3!} \Delta(a^3)).
\end{aligned}$$

In the following sections, we also need the notion of a right superderivation: a linear map

$$Super[\mathcal{A}] \leftarrow Super[\mathcal{A}] : \ \tilde{D},$$

homogeneous of degree $|\tilde{D}|$, is a *right superderivation* if

$$(-1)^{|\tilde{D}||m'|}(m)\tilde{D}\ m' + m\ (m')\tilde{D} = \ (m\ m')\tilde{D},$$

for all monomials $m, m' \in Super[\mathcal{A}]$.

Let $a_h, a_k \in \mathcal{A}$. The *right superpolarization* ${}_{a_h,a_k}D$ *of the letter* a_h *to the letter* a_k is the unique right superderivation of $\mathbb{Z}_2$-degree $|a_h| + |a_k|$, such that

$$(a_j)_{a_h,a_k}D = \delta_{j,h}\ a_k,$$

for every $a_j \in \mathcal{A}$.

Consider linear automorphism

$$R : Super[\mathcal{A}] \rightarrow Super[\mathcal{A}]$$

such that

$$R(a_{i_1}a_{i_2}\cdots a_{i_n}) = a_{i_n}a_{i_{n-1}}\cdots a_{i_1},$$

for every $a_{i_1}a_{i_2}\cdots a_{i_n} \in Super[\mathcal{A}]$.

Clearly, R is an involutorial map, that is $R^2 = id$, and $R(mm') = R(m')R(m)$, for every $m, m' \in Super[\mathcal{A}]$. It follows that the map

$$D \mapsto R \circ D \circ R = \ \tilde{D}$$

is an involutorial isomorphism from the vector space of all left superderivations to the vector space of all right superderivations.

Notice that the right superpolarization ${}_{a_h,a_k}D$ of the letter a_h to the letter a_k is the right superderivation

$${}_{a_h,a_k}D = R \circ D_{a_k,a_h} \circ R.$$

The next result follows from the definitions.

Proposition 3.2. *1.* ${}_{a_h,a_k}D$ *is a right coderivation of the bialgebra* $Super[\mathcal{A}]$*, that is*

$$\Delta((m)_{a_h,a_k}D) \\ = \sum_{(m)} \left((-1)^{(|a_h|+|a_k|)|m_{(2)}|}\ (m_{(1)})_{a_h,a_k}D\ \otimes m_{(2)} + m_{(1)} \otimes (m_{(2)})_{a_h,a_k}D\right),$$

for every $m \in Super[\mathcal{A}]$.

2. Let $m = a_{i_1}a_{i_2}\cdots a_{i_p} \in Super[\mathcal{A}]$ *and let* $a_j \in \mathcal{A}$*,* $|a_j| = 0$*, such that* $a_j \neq a_{i_h}$*,* $h = 1, 2, \ldots, p$*. Then*

$$\Delta(m) = \frac{1}{p!}(\Delta((a_j)^p)))_{a_j,a_{i_1}}D_{a_j,a_{i_2}}D\cdots_{a_j,a_{i_p}}D.$$

4 The Letterplace Superalgebra as a Bimodule

4.1 Letterplace superalgebras

In the following, we consider a pair of signed sets $\mathcal{X} = \mathcal{X}_0 \cup \mathcal{X}_1$ and $\mathcal{Y} = \mathcal{Y}_0 \cup \mathcal{Y}_1$, that we call the *letter set* and the *place set*, respectively. The *letterplace set*

$$[\mathcal{X} \,|\, \mathcal{Y}] = \{(x \,|\, y);\ x \in \mathcal{X},\ y \in \mathcal{Y}\}$$

inherits a sign by setting $|(x \,|\, y)| = |x| + |y| \in \mathbb{Z}_2$.

The *letterplace* $\mathbb{K}$*-superalgebra* $Super[\mathcal{X}|\mathcal{Y}]$ is the quotient algebra of the free associative $\mathbb{K}$-algebra with 1 generated by the letterplace alphabet $[\mathcal{X} \,|\, \mathcal{Y}]$ modulo the bilateral ideal generated by the elements of the form:

$$(x \,|\, y)(z \,|\, t) - (-1)^{(|x|+|y|)(|z|+|t|)}(z \,|\, t)(x \,|\, y), \qquad x, z \in \mathcal{X},\ y, t \in \mathcal{Y}.$$

In other words, the letterplace $\mathbb{K}$-superalgebra $Super[\mathcal{X}|\mathcal{Y}]$ is the supersymmetric superalgebra of the Z_2-graded set $\mathcal{A} = [\mathcal{X} \,|\, \mathcal{Y}]$ (see Subsection 3.4).

Remark 4.1.

1. $Super[\mathcal{X}|\mathcal{Y}]$ is a $\mathbb{Z}_2$-graded algebra

   $$Super[\mathcal{X}|\mathcal{Y}] = (Super[\mathcal{X}|\mathcal{Y}])_0 \oplus (Super[\mathcal{X}|\mathcal{Y}])_1,$$

   where $(Super[\mathcal{X}|\mathcal{Y}])_i$ is the subspace of $Super[\mathcal{X}|\mathcal{Y}]$ spanned by the letterplace monomials M of $\mathbb{Z}_2$-degree $|M| = i$, where, for $M = (x_{i_1} \,|\, y_{i_1}) \cdots (x_{i_n} \,|\, y_{i_n})$, $|M| = |(x_{i_1} \,|\, y_{i_1})| + \cdots + |(x_{i_n} \,|\, y_{i_n})|$.

   With respect to this grading, $Super[\mathcal{X}|\mathcal{Y}]$ is supersymmetric, i.e.:

   $$MN = (-1)^{|M||N|}NM,$$

   for every $\mathbb{Z}_2$-homogeneous elements $M, N \in Super[\mathcal{X}|\mathcal{Y}]$.

2. $Super[\mathcal{X}|\mathcal{Y}]$ is an $\mathbb{N}$-graded algebra

   $$Super[\mathcal{X}|\mathcal{Y}] = \bigoplus_{n \in \mathbb{N}} Super_n[\mathcal{X}|\mathcal{Y}]$$
   $$Super_n[\mathcal{X}|\mathcal{Y}] = \langle (x_{i_1} \,|\, y_{i_1})(x_{i_2} \,|\, y_{i_2}) \cdots (x_{i_n} \,|\, y_{i_n}),\ x_{i_h} \in \mathcal{X},\ y_{j_k} \in \mathcal{Y} \rangle_{\mathbb{K}};$$

3. the $\mathbb{Z}_2$-graduation and the $\mathbb{N}$-graduation of $Super[\mathcal{X}|\mathcal{Y}]$ are *coherent*, that is

   $$(Super[\mathcal{X}|\mathcal{Y}])_i = \bigoplus_{n \in \mathbb{N}} (Super_n[\mathcal{X}|\mathcal{Y}])_i, \qquad i \in \mathbb{Z}_2.$$

4.2 Superpolarization operators

Let $x', x \in \mathcal{X}$. The *superpolarization* $\mathcal{D}_{x',x}$ *of the letter* x *to the letter* x' is the unique left superderivation

$$\mathcal{D}_{x',x} : Super[\mathcal{X}|\mathcal{Y}] \to Super[\mathcal{X}|\mathcal{Y}]$$

of $\mathbb{Z}_2$-degree $|x'| + |x|$, such that

$$\mathcal{D}_{x',x}(z \,|\, t) = \delta_{x,z}(x' \,|\, t),$$

for every $(z \,|\, t) \in [\mathcal{X} \,|\, \mathcal{Y}]$.

Let $y, y' \in \mathcal{Y}$. The *superpolarization* ${}_{y,y'}\mathcal{D}$ *of the place* y *to the place* y' is the unique right superderivation

$$Super[\mathcal{X}|\mathcal{Y}] \leftarrow Super[\mathcal{X}|\mathcal{Y}] : \ {}_{y,y'}\mathcal{D}$$

of $\mathbb{Z}_2$-degree $|y| + |y'|$, such that

$$(z \,|\, t) \ {}_{y,y'}\mathcal{D} = \delta_{t,y}(z \,|\, y'),$$

for every $(z \,|\, t) \in [\mathcal{X} \,|\, \mathcal{Y}]$.

In passing, we point out that every letter-polarization operator *commutes* with every place-polarization operator.

4.3 Letterplace superalgebras and supersymmetric algebras: the classical description

Given a pair of *finite* alphabets $\mathcal{L} = \mathcal{L}_0 \cup \mathcal{L}_1$ and $\mathcal{P} = \mathcal{P}_0 \cup \mathcal{P}_1$, $\mathcal{L}_0 \subseteq \mathcal{X}_0$, $\mathcal{L}_1 \subseteq \mathcal{X}_1$, $\mathcal{P}_0 \subseteq \mathcal{Y}_0$, $\mathcal{P}_1 \subseteq \mathcal{Y}_1$, consider the $\mathbb{Z}_2$-graded vector spaces

$$V = V_0 \oplus V_1 = \langle \mathcal{L}_0 \rangle_{\mathbb{K}} \oplus \langle \mathcal{L}_1 \rangle_{\mathbb{K}}$$

and

$$W = W_0 \oplus W_1 = \langle \mathcal{P}_0 \rangle_{\mathbb{K}} \oplus \langle \mathcal{P}_1 \rangle_{\mathbb{K}}.$$

The tensor product $V \otimes W$ has a natural $\mathbb{Z}_2$-grading

$$V \otimes W = [(V_0 \otimes W_0) \oplus (V_1 \otimes W_1)] \oplus [(V_0 \otimes W_1) \oplus (V_1 \otimes W_0)]).$$

The *supersymmetric algebra* of the tensor product $V \otimes W$ is the superalgebra

$$Super[V \otimes W] = Sym[(V_0 \otimes W_0) \oplus (V_1 \otimes W_1)] \otimes \Lambda[(V_0 \otimes W_1) \oplus (V_1 \otimes W_0)].$$

Clearly, one has a natural isomorphism

$$Super[\mathcal{L}|\mathcal{P}] \cong Super[V \otimes W].$$

4.4 General linear Lie superalgebras, representations and polarization operators

Given a pair of *finite* alphabets $\mathcal{L} = \mathcal{L}_0 \cup \mathcal{L}_1$ and $\mathcal{P} = \mathcal{P}_0 \cup \mathcal{P}_1$, regard them as homogeneous bases of the pair of vector spaces $V = V_0 \oplus V_1$ and $W = W_0 \oplus W_1$.

The Lie superalgebras $pl(\mathcal{L})$ and $pl(\mathcal{P})$ are, by definition, the general linear Lie superalgebras $pl(V)$ and $pl(W)$ of V and W respectively. Therefore, we have the standard bases

$$\{E_{x,x'}; x, x' \in \mathcal{L}\}$$

and

$$\{E_{y,y'}; y, y' \in \mathcal{P}\}$$

of $pl(\mathcal{L}) = pl(V)$ and $pl(\mathcal{P}) = pl(W)$.

We recall the canonical isomorphism

$$Super[\mathcal{L}|\mathcal{P}] \cong Super[V \otimes W].$$

The (even) mappings

$$E_{x'x} \mapsto \mathcal{D}_{x',x}, \quad x, x' \in \mathcal{L}, \qquad E_{y'y} \mapsto \ {}_{y,y'}\mathcal{D}, \quad y, y' \in \mathcal{P}$$

induce Lie superalgebra actions of $pl(\mathcal{L})$ and $pl(\mathcal{P})$ over any $\mathbb{N}$-*homogeneous component* $Super_n[\mathcal{L}|\mathcal{P}]$ of the letterplace algebra.

In the following, we will denote by

$$\mathcal{B}_n, \ {}_n\mathcal{B}$$

the (finite dimensional) homomorphic images in $End_{\mathbb{K}}(Super_n[\mathcal{L}|\mathcal{P}])$ of the universal enveloping algebras $\mathcal{U}(pl(\mathcal{L}))$ and $\mathcal{U}(pl(\mathcal{P}))$, induced by the actions of $pl(\mathcal{L})$ and $pl(\mathcal{P})$, respectively.

The operator algebras $\mathcal{B}_n, \ {}_n\mathcal{B}$ are therefore the algebras generated by the proper letter and place polarization operators (restricted to $Super_n[\mathcal{L}|\mathcal{P}]$), respectively.

Furthermore, by the commutation property, $Super_n[\mathcal{L}|\mathcal{P}]$ is a bimodule over the universal enveloping algebras $\mathcal{U}(pl(\mathcal{L})))$ and $\mathcal{U}(pl(\mathcal{P}))$.

4.5 General linear groups and even polarization operators

Let $\mathcal{L} = \mathcal{L}_1 = \{x_1, \ldots, x_m\}$ be a finite alphabet of letters, let $\mathcal{P}$ be a finite alphabet of places; since $\mathcal{L}$ is trivially $\mathbb{Z}_2$-graded, the general linear Lie superalgebra $pl(\mathcal{L})$ reduces to the usual general linear Lie algebra $gl_m(\mathbb{K})$ of all square matrices of order m over

$\mathbb{K}$, and the letterplace superalgebra $Super[\mathcal{L}|\mathcal{P}]$ is a (left) $gl_m(\mathbb{K})$-module via the usual action, that is, a matrix $S = [s_{ij}]$ acts on a letterplace variable as

$$S(x_i \,|\, y) = \sum_{j=1}^{m} (x_j \,|\, y) \; s_{ji}$$

and is extended as a *derivation* (the action of any elementary matrix E_{hk} is implemented by the polarization operator $\mathcal{D}_{x_h,x_k}$, $|\mathcal{D}_{x_h,x_k}| = 0 \in \mathbb{Z}_2$). We denote by σ_n the corresponding representation of the universal enveloping algebra $\mathcal{U}[gl_m(\mathbb{K})]$ over the homogeneous component $Super_n[\mathcal{L}|\mathcal{P}]$.

Let $Gl_m(\mathbb{K})$ be the general linear group of nonsingular matrices of order m over $\mathbb{K}$; the letterplace superalgebra $Super[\mathcal{L}|\mathcal{P}]$ is a (left) $GL_m(\mathbb{K})$-module via the usual action, that is, a matrix $S = [s_{ij}]$ acts on a letterplace variable as

$$S(x_i \,|\, y) = \sum_{j=1}^{m} (x_j \,|\, y) \; s_{ji}$$

and is extended as an *algebra automorphism.* We denote by ρ_n the corresponding representation of the group algebra $\mathbb{K}[GL_m(\mathbb{K})]$ over the homogeneous component $Super_n[\mathcal{L}|\mathcal{P}]$.

The following standard result will be systematically used in Sections 15 and 16.

Proposition 4.1. *The algebra ρ_n $[\mathbb{K}[GL_m(\mathbb{K})]]$ generated by the action of the general linear group coincides with the algebra σ_n $[\mathcal{U}[gl_m(\mathbb{K})]]$ generated by the action of the general linear Lie algebra.*

Therefore, the algebra ρ_n $[\mathbb{K}[GL_m(\mathbb{K})]]$ is the algebra $\mathcal{B}_n$ generated by the letter polarization operators $\mathcal{D}_{x_i x_j}$, $x_i, x_j \in \mathcal{L}$.

Proof. Recall that the group $GL_m(\mathbb{K})$ is generated by the *transvections*, namely:

- $T_{ij}(\lambda) = I + \lambda \; E_{ij}$, with $i \neq j$;
- $T_{ii}(\lambda) = I + \lambda \; E_{ii}$, with $\lambda \neq -1$.

The statement now follows from a standard argument (the so-called Vandermonde matrix argument):

- The image under the representation ρ_n of any transvection $T_{ij}(\lambda) = I + \lambda \; E_{ij}$, with $i \neq j$, is a polynomial in the image under the representation σ_n of the elementary matrix E_{ij}. Specifically, we have:

$$\rho_n(T_{ij}(\lambda)) = \sum_{h=0}^{n} \lambda^h \frac{\sigma_n(E_{ij})^h}{h!};$$

by evaluating this relation in $n+1$ different values $\lambda = \lambda_1, \dots, \lambda_{n+1}$, one gets a system of $n+1$ linear relations that can be solved with respect to the divided powers of the representation of the elementary matrix E_{ij}.

- The same argument applies to the representation of any transvection $T_{ii}(\lambda)$. Specifically, we have:

$$\rho_n(T_{ii}(\lambda)) = \sum_{h=0}^{n} \lambda^h \begin{pmatrix} \sigma_n(E_{ii}) \\ h \end{pmatrix};$$

by evaluating this relation in $n+1$ different values $\lambda = \lambda_1, \dots, \lambda_{n+1}$, one gets a system of $n+1$ linear relations that can be solved with respect to the formal binomials of the representation of the elementary matrix E_{ii}.

$\square$

5 Tableaux

5.1 Young tableaux

We recall that *signed set* is a set $\mathcal{A}$ endowed with a *sign* map $|\ | : \mathcal{A} \to \mathbb{Z}_2$; the sets $\mathcal{A}_0 = \{a \in \mathcal{A}; |a| = 0\}$ and $\mathcal{A}_1 = \{a \in \mathcal{A}; |a| = 1\}$ are called the subsets of *positive* and *negative* symbols, respectively.

A *signed alphabet* is a linearly ordered signed set.

A *Young tableau* over a signed alphabet $\mathcal{A}$ is a sequence

$$S = (w_1, w_2, \dots, w_p)$$

of words $w_i = a_{i1}a_{i2}\dots a_{i\lambda_i}$, $a_{ij} \in \mathcal{A}$, whose lengths form a weakly decreasing sequence, i.e., a partition

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_p) = \mathrm{sh}(S),$$

called the *shape* of S. The concatenation of the words w_i

$$w = w_1 w_2 \dots w_p = w(S)$$

is called the *word* of S. If n is the length of w, then λ is a partition of n:

$$\lambda \vdash n.$$

The *content* $c(S)$ of a tableau S is the multiset of the symbols occurring in S.

We will frequently represent tableaux in the array notation:

$$S = (abb,\ bae,\ c) = \begin{array}{ccc} a & b & b \\ b & a & e \\ c & & \end{array}$$

The set of all the tableaux over $\mathcal{A}$ is denoted by $Tab(\mathcal{A})$.

5.2 Co-Deruyts and Deruyts tableaux

A tableau C is said to be of *co-Deruyts* type whenever any two symbols in the same row of C are equal, while any two symbols in the same column of C are distinct. For example:

$$C = \begin{matrix} a & a & a & a & a \\ b & b & b & & \\ c & & & & \end{matrix}$$

A tableau D is said to be of *Deruyts* type whenever any two symbols in the same column of D are equal, while any two symbols in the same row of D are distinct. For example:

$$D = \begin{matrix} a & b & c & d & e \\ a & b & c & & \\ a & & & & \end{matrix}$$

In the following, the symbol C will denote a co-Deruyts tableau filled with *positive* symbols, and the symbol D will denote a Deruyts tableau filled with *negative* symbols. In the formulas below, the shapes of the tableaux C and D, and the fact that the symbols were letter or place symbols should be easily inferred from the context.

5.3 Standard Young tableaux

Following Grosshans, Rota and Stein [46], a Young tableau S over a (linearly ordered) signed alphabet $\mathcal{A}$ is called *(super)standard* when each row of S is non-decreasing, with no negative repeated symbols and each column of S is non-decreasing, with no positive repeated symbols. For example, if $a < b < c < d$, the tableau

$$S = \begin{matrix} a & a & c & c & d \\ b & c & d & & \\ b & & & & \end{matrix} \qquad\qquad a, c \in \mathcal{A}_0,\ b, d \in \mathcal{A}_1$$

is a standard tableau. The set of all the standard tableaux over $\mathcal{A}$ is denoted by $Stab(\mathcal{A})$.

5.4 The Berele–Regev hook property

Assume now that the linearly ordered signed alphabet $\mathcal{A}$ is finite, with $|\mathcal{A}_0| = r$ and $|\mathcal{A}_1| = s$. The hook set of $\mathcal{A}$ is

$$H(\mathcal{A}) = \{(\lambda_1 \geq \lambda_2 \geq \ldots); \quad \lambda_{r+1} < s + 1\}.$$

Proposition 5.1. *There are some standard tableaux of shape λ over A if and only if $\lambda \in H(\mathcal{A})$. Furthermore, the number $p_\lambda(\mathcal{A})$ of standard tableaux of any given shape λ over $\mathcal{A}$ is independent of the linear order defined on $\mathcal{A}$.*

Proof. Let $\mathcal{A}_0 = \{x_1, x_2, \ldots, x_m\}$, $\mathcal{A}_1 = \{x_{m+1}, x_{m+2}, \ldots, x_{m+n}\}$. If $\mathcal{A}$ is endowed with a linear order such that $x_i < x_j$ for every $i = 1, 2, \ldots, m$ and $j = m+1, m+2, \ldots, m+n$, the proof follows from a straightforward argument (see, e.g., [4], [5]). If $\mathcal{A}$ is endowed with an arbitrary linear order, the proof follows from the previous assertion in combination with the standard basis theorem of Grosshans, Rota and Stein (see, e.g [46] and Proposition 11.1). □

5.5 Orders on tableaux

Let $\mathcal{L}$ be a finite signed alphabet. We define a *partial* order on the set of all standard tableaux over $\mathcal{L}$ which have a given content, and, therefore, have shapes which are partitions of a given integer n.

For every standard tableau S, we consider the sequence $S^{(p)}$, $p = 1, 2, \ldots$, of the subtableaux obtained from S by considering only the first p symbols of the alphabet, and consider the family $\mathrm{sh}(S^{(p)})$, $p = 1, 2, \ldots$, of the corresponding shapes. Since the alphabet is assumed to be finite, this sequence is finite and its last term is $\mathrm{sh}(S)$.

Then, for standard tableaux S, T, we set

$$S \leq T \;\Leftrightarrow\; \mathrm{sh}(S^{(p)}) \trianglelefteq \mathrm{sh}(T^{(p)}),\; p = 1, 2, \ldots,$$

where $\trianglelefteq$ stands for the *dominance order* on partitions. We recall that the dominance order on partitions is defined as follows: $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots) \trianglelefteq \mu = (\mu_1 \geq \mu_2 \geq \ldots)$ if and only if $\lambda_1 + \cdots + \lambda_i \leq \mu_1 + \cdots + \mu_i$, for every $i = 1, 2, \ldots$

We extend this partial order to the set of all standard tableaux on $\mathcal{L}$ which contain a total number (taking into account multiplicities) of n symbols simply by stating that two tableaux S and T such that $c(S) \neq c(T)$ are incomparable.

We define a *linear* order on the set of all tableaux over $\mathcal{L}$ which contain a total number (taking into account multiplicities) of n symbols, by setting $Q < Q'$ if and only if

$$\begin{aligned} & \mathrm{sh}(Q) <_l \mathrm{sh}(Q') \\ \textit{or} \quad & \mathrm{sh}(Q) = \mathrm{sh}(Q') \quad \textit{and } w(Q) >_l w(Q'), \end{aligned}$$

where the shapes and the words are compared in the lexicographic order.

We remark that this linear order, restricted to standard tableaux, is a linear extension of the partial order defined above.

Part II

The General Theory

6 The Method of Virtual Variables

6.1 The metatheoretic significance of Capelli's idea of *virtual* variables

Let $\mathcal{L} = \mathcal{L}_0 \cup \mathcal{L}_1 \subset \mathcal{X}$ and $\mathcal{P} = \mathcal{P}_0 \cup \mathcal{P}_1 \subset \mathcal{Y}$ be *finite* signed subsets of the "universal" signed letter and place alphabets $\mathcal{X}$ and $\mathcal{Y}$, respectively. The elements $x \in \mathcal{L}$ $(y \in \mathcal{P})$ are called *proper* letters (*proper* places), and the elements $x \in \mathcal{X} \setminus \mathcal{L}$ $(y \in \mathcal{Y} \setminus \mathcal{P})$ are called *virtual* letters (places). Usually we denote virtual symbols by Greek letters.

The signed subset $[\mathcal{L} \,|\, \mathcal{P}] = \{(x \,|\, y);\ x \in \mathcal{L},\ y \in \mathcal{P}\} \subset [\mathcal{X} \,|\, \mathcal{Y}]$ is called a *proper letterplace alphabet.*

Consider an operator of the form:

$$\mathcal{D}_{x_{i_1},\alpha_{i_1}} \cdots \mathcal{D}_{x_{i_n},\alpha_{i_n}} \cdot \mathcal{D}_{\alpha_{i_1},x_{j_1}} \cdots \mathcal{D}_{\alpha_{i_n},x_{j_n}}$$
$$(x_{i_1}, \ldots, x_{i_n}, x_{j_1}, \ldots, x_{j_n} \in \mathcal{L}, \text{ i.e., proper letters})$$

that is an operator that *creates* some virtual letters $\alpha_{i_1}, \ldots, \alpha_{i_n}$ (with prescribed multiplicities) *times* an operator that *annihilates* the *same* virtual letters (with the *same* prescribed multiplicities).

Such an operator will be called a (letter) *Capelli-type operator.*

Clearly, the proper letterplace superalgebra $Super[\mathcal{L}|\mathcal{P}]$ is *left invariant* under the action of a Capelli-type operator.

Theorem 6.1 ([12, 20])**.** *The action of a Capelli-type operator over the proper letterplace superalgebra Super*$[\mathcal{L}|\mathcal{P}]$ *is the same as the action of a "polynomial" operator in the proper polarizations* $\mathcal{D}_{x_{i_h},x_{i_k}}$*,* $x_{i_h}, x_{i_k} \in \mathcal{L}$*, that is an operator that does not involve virtual variables.*

Informally speaking, a Capelli-type operator is of $pl(\mathcal{L})$*-representation theoretic meaning*, and, in general is much more manageable than its "non-virtual companion".

In the following, we will write $\mathbf{T}_1 \cong \mathbf{T}_2$ to mean that two operators $\mathbf{T}_1, \mathbf{T}_2$ on $Super[\mathcal{X}|\mathcal{Y}]$ are the same when restricted to the proper letterplace algebra $Super[\mathcal{L}|\mathcal{P}]$ and say that the operators $\mathbf{T}_1, \mathbf{T}_2$ are $[\mathcal{L} \,|\, \mathcal{P}]$ – *equivalent.*

Example 6.1. *Let* $x, y \in \mathcal{L}_1$*, with* $x \neq y$*, and* $\alpha \in \mathcal{X} \setminus \mathcal{L}$*, with* $|\alpha| = 0$*. Then*

$$\begin{aligned}
\mathcal{D}_{y\alpha}\mathcal{D}_{x\alpha}\mathcal{D}_{\alpha x}\mathcal{D}_{\alpha y} &= -\mathcal{D}_{y\alpha}\mathcal{D}_{\alpha x}\mathcal{D}_{x\alpha}\mathcal{D}_{\alpha y} + \mathcal{D}_{y\alpha}\mathcal{D}_{xx}\mathcal{D}_{\alpha y} + \mathcal{D}_{y\alpha}\mathcal{D}_{\alpha\alpha}\mathcal{D}_{\alpha y} \\
&= +\mathcal{D}_{y\alpha}\mathcal{D}_{\alpha x}\mathcal{D}_{\alpha y}\mathcal{D}_{x\alpha} - \mathcal{D}_{y\alpha}\mathcal{D}_{\alpha x}\mathcal{D}_{xy} \\
&\quad - \mathcal{D}_{xx}\mathcal{D}_{\alpha y}\mathcal{D}_{y\alpha} + \mathcal{D}_{xx}\mathcal{D}_{yy} - \mathcal{D}_{xx}\mathcal{D}_{\alpha\alpha} \\
&\quad + \mathcal{D}_{y\alpha}\mathcal{D}_{\alpha y}\mathcal{D}_{\alpha\alpha} + \mathcal{D}_{y\alpha}\mathcal{D}_{\alpha y} \\
&\cong -\mathcal{D}_{y\alpha}\mathcal{D}_{\alpha x}\mathcal{D}_{xy} + \mathcal{D}_{xx}\mathcal{D}_{yy} + \mathcal{D}_{y\alpha}\mathcal{D}_{\alpha y} \\
&\cong \ldots \\
&\cong -\mathcal{D}_{yx}\mathcal{D}_{xy} + \mathcal{D}_{xx}\mathcal{D}_{yy} + \mathcal{D}_{yy}.
\end{aligned}$$

Here the identities $=$ are obtained by applying the commutator identity, and the $[\mathcal{L} \mid \mathcal{P}]$-equivalences $\cong$ are obtained by applying again the commutator identity and by deleting the summands which turn out to be the zero operator when restricted to the proper letterplace algebra $Super[\mathcal{L}|\mathcal{P}]$.

6.2 Tableau polarization monomials

Let u' and u be words of the same length m on the alphabet $\mathcal{X}$, say $u' = x_1'x_2' \cdots x_m'$, $u = x_1x_2 \cdots x_m$. The *letter polarization monomial* of the word u to the word u' is defined to be the $\mathbb{K}$-linear operator

$$\mathcal{D}_{u'u} = \mathcal{D}_{x_1'x_1}\mathcal{D}_{x_2'x_2} \ldots \mathcal{D}_{x_m'x_m} \in End_{\mathbb{K}}[Super_n[\mathcal{X}|\mathcal{Y}]].$$

Let $S', S \in Tab(\mathcal{X})$, with $\mathrm{sh}(S') = \mathrm{sh}(S) \vdash n$. The *letter polarization monomial* of the tableau S to the tableau S' is defined to be the $\mathbb{K}$-linear operator

$$\mathcal{D}_{S'S} = \mathcal{D}_{x_1'x_1}\mathcal{D}_{x_2'x_2} \ldots \mathcal{D}_{x_n'x_n} \in End_{\mathbb{K}}[Super_n[\mathcal{X}|\mathcal{Y}]],$$

where $x_1' \ldots x_n' = w(S')$ and $x_1 \ldots x_n = w(S)$.
In passing, we point out that, if $S' = (u_1', \ldots, u_p')$ and $S = (u_1, \ldots, u_p)$, then

$$\mathcal{D}_{S'S} = \mathcal{D}_{u_1'u_1}\mathcal{D}_{u_2'u_2} \ldots \mathcal{D}_{u_p'u_p}.$$

Example 6.2. *Let*

$$S' = \begin{matrix} x & y \\ x & z \end{matrix} \; , \; S = \begin{matrix} x & z \\ z & t \end{matrix} \; ;$$

then

$$\mathcal{D}_{S'S} \;=\; \mathcal{D}_{\begin{matrix} x & y \\ x & z \end{matrix}\, ,\, \begin{matrix} x & z \\ z & t \end{matrix}} \;=\; \mathcal{D}_{xx}\mathcal{D}_{yz}\mathcal{D}_{xz}\mathcal{D}_{zt}$$

Let v and v' be words of the same length m on the alphabet $\mathcal{Y}$, say $v = y_1 y_2 \cdots y_m$, $v' = y'_1 y'_2 \cdots y'_m$. The *place polarization monomial* of the word v to the word v' is defined to be the $\mathbb{K}$-linear operator

$$_{vv'}\mathcal{D} = _{y_1 y'_1}\mathcal{D}\ _{y_2 y'_2}\mathcal{D}\ \ldots\ _{y_m y'_m}\mathcal{D} \in End_{\mathbb{K}}[Super_n[\mathcal{X}|\mathcal{Y}]].$$

Let $V, V' \in Tab(\mathcal{Y})$, with $\mathrm{sh}(V) = \mathrm{sh}(V') \vdash n$. The *place polarization monomial* of the tableau V to the tableau V' is defined to be the $\mathbb{K}$-linear operator

$$_{VV'}\mathcal{D} = _{y_1 y'_1}\mathcal{D}\ _{y_2 y'_2}\mathcal{D}\ \ldots\ _{y_n y'_n}\mathcal{D} \in End_{\mathbb{K}}[Super_n[\mathcal{X}|\mathcal{Y}]],$$

where $y_1 \ldots y_n = w(V)$ and $y'_1 \ldots y'_n = w(V')$.

Clearly, if $V = (v_1, \ldots, v_p)$ and $V' = (v'_1, \ldots, v'_p)$, then

$$_{VV'}\mathcal{D} = _{v_1 v'_1}\mathcal{D}\ _{v_2 v'_2}\mathcal{D}\ \ldots\ _{v_p v'_p}\mathcal{D}.$$

Example 6.3. *Let*

$$V = \begin{matrix} a & c \\ b & c \end{matrix}\ , \quad V' = \begin{matrix} b & b \\ c & d \end{matrix}\ ;$$

then

$$_{VV'}\mathcal{D} \ = \ _{\begin{matrix} a & c \\ b & c \end{matrix}\ ,\ \begin{matrix} b & b \\ c & d \end{matrix}}\mathcal{D} \ = \ _{ab}\mathcal{D}_{cb}\mathcal{D}_{bc}\mathcal{D}_{cd}\mathcal{D}$$

6.3 Capelli bitableaux and Capelli rows

Among Capelli-type operators, a distinguished role is played by those involving virtual letters of the *same* sign. Specifically, let $U = (u_1, \ldots, u_p)$ and $V = (v_1, \ldots, v_p)$ be tableaux on the alphabet of proper letters $\mathcal{L}$, $\mathrm{sh}(U) = \mathrm{sh}(V) = \lambda = (\lambda_1, \ldots, \lambda_p)$, and let $\alpha_1, \ldots, \ldots, \alpha_p$ be p *distinct* virtual letters of the same sign.

The Capelli-type operator

$$[U\,|\,V] = \left[\begin{array}{c|c} u_1 & v_1 \\ u_2 & v_2 \\ \vdots & \vdots \\ u_p & v_p \end{array}\right] = \mathcal{D}_{u_1,\alpha_1^{\lambda_1}} \cdots \mathcal{D}_{u_p,\alpha_p^{\lambda_p}} \mathcal{D}_{\alpha_1^{\lambda_1},v_1} \cdots \mathcal{D}_{\alpha_p^{\lambda_p},v_p}$$

will be called a *Capelli bitableau.* A Capelli bitableau is said to be *positive* (*negative*) if the virtual letters $\alpha_1, \ldots, \ldots, \alpha_p$ have positive (negative) sign.

Example 6.4. *Let* $U = (u_1, u_2)$, $u_1 = xyz$, $u_2 = yz$ *and let* $V = (v_1, v_2)$, $v_1 = xzz$, $v_2 = xy$. *Then, given two positive virtual letters* α_1, α_2, *the operator*

$$[U\,|\,V] = \left[\begin{array}{c|c} u_1 & v_1 \\ u_2 & v_2 \end{array}\right] = \mathcal{D}_{u_1,\alpha_1^3}\mathcal{D}_{u_2,\alpha_2^2}\mathcal{D}_{\alpha_1^3,v_1}\mathcal{D}_{\alpha_2^2,v_p},$$

where

$$\begin{aligned}
\mathcal{D}_{u_1,\alpha_1^3} &= \mathcal{D}_{x\alpha_1}\mathcal{D}_{y\alpha_1}\mathcal{D}_{z\alpha_1} \quad \mathcal{D}_{u_2,\alpha_2^2} = \mathcal{D}_{y\alpha_2}\mathcal{D}_{z\alpha_2} \\
\mathcal{D}_{\alpha_1^3,v_1} &= \mathcal{D}_{\alpha_1 x}\mathcal{D}_{\alpha_1 z}\mathcal{D}_{\alpha_1 z} \quad \mathcal{D}_{\alpha_2^2,v_2} = \mathcal{D}_{\alpha_2 x}\mathcal{D}_{\alpha_2 y},
\end{aligned}$$

is a positive Capelli bitableau.

If U, V are tableaux with just one row, the Capelli bitableau $[U \,|\, V]$ is said to be a *Capelli row*. Note that a Capelli bitableau, is not, in general, a product of Capelli rows.

The following result is a more detailed reformulation of Proposition 4 of [13]; it provides the basic identity that relates Capelli bitableaux and products of Capelli rows.

Proposition 6.1. *Let*

$$u_1, u_2, \ldots u_p, u_{p+1}; \;\; v_1, v_2, \ldots v_p, v_{p+1}$$

be words over the proper letter alphabet $\mathcal{L}$, where v_i has the same length as u_i.

We have the following $[\mathcal{L} \,|\, \mathcal{P}]$-equivalence involving positive (negative) Capelli bitableaux:

$$\left[\begin{array}{c|c} u_1 & v_1 \\ u_2 & v_2 \\ \vdots & \vdots \\ u_p & v_p \end{array}\right] [u_{p+1} \,|\, v_{p+1}] \cong \sum c_* \left[\begin{array}{c|c} u_1 & v_1' \; v_{p+1,(1)} \\ u_2 & v_2' \; v_{p+1,(2)} \\ \vdots & \vdots \\ u_p & v_p' \; v_{p+1,(p)} \\ u_{p+1}' & v_{p+1,(p+1)} \end{array}\right]$$

where the c_*'s are rational coefficients and the sum is taken over all the $(p+1)$-tuples of subwords

$$u_{p+1}', v_1', v_2', \ldots v_p'$$

of the words $u_{p+1}, v_1, v_2, \ldots v_p$ such that the (multiset) difference of contents

$$c(u_{p+1}) - c(u_{p+1}')$$

equals the sum of (multiset) differences of contents

$$\sum_{i=1}^{p} \left(c(v_i) - c(v_i') \right),$$

and over all the splittings

$$v_{p+1,(1)}, v_{p+1,(2)}, \ldots, v_{p+1,(p+1)}$$

of the word v_{p+1}, such that

$$l(v_i' \; v_{p+1,(i)}) = l(u_i), \qquad i = 1, \ldots, p.$$

It is a simple fact that, in the above summation, the Capelli tableau

$$\left[\begin{array}{c|c} u_1 & v_1 \\ u_2 & v_2 \\ \vdots & \vdots \\ u_p & v_p \\ u_{p+1} & v_{p+1} \end{array}\right]$$

appears with coefficient ± 1.

Therefore, the identity in the previous proposition can be rewritten as

$$\pm \left[\begin{array}{c|c} u_1 & v_1 \\ u_2 & v_2 \\ \vdots & \vdots \\ u_p & v_p \\ u_{p+1} & v_{p+1} \end{array}\right] \cong \left[\begin{array}{c|c} u_1 & v_1 \\ u_2 & v_2 \\ \vdots & \vdots \\ u_p & v_p \end{array}\right] [u_{p+1} \,|\, v_{p+1}] - \sum c_* \left[\begin{array}{c|c} u_1 & v_1' \; v_{p+1,(1)} \\ u_2 & v_2' \; v_{p+1,(2)} \\ \vdots & \vdots \\ u_p & v_p' \; v_{p+1,(p)} \\ u_{p+1}' & v_{p+1,(p+1)} \end{array}\right],$$

where the sum is taken under the previous conditions, and u_{p+1}' ranges among the strict subwords of u_{p+1}.

By iterating this argument, we obtain the following theorem.

Theorem 6.2.

- *Every Capelli bitableau is $[\mathcal{L} \,|\, \mathcal{P}]$-equivalent to a linear combination of products of Capelli rows, with rational coefficients.*
- *Every product of Capelli rows is $[\mathcal{L} \,|\, \mathcal{P}]$-equivalent to a linear combination of Capelli bitableaux, with rational coefficients.*

6.4 Devirtualization of Capelli rows and Laplace expansion type identities

If we consider the restricted action of a Capelli operator on the proper superalgebra $Super[L|\mathcal{P}]$, we get the following Laplace expansion type identities.

Theorem 6.3 ([13, 20])**.** *Let $\alpha \in \mathcal{X} \setminus \mathcal{L}$ be a virtual letter, $|\alpha| = 0$; then*

$$\begin{aligned} &\mathcal{D}_{y_1\alpha}\mathcal{D}_{y_2\alpha}\ldots\mathcal{D}_{y_n\alpha}\cdot\mathcal{D}_{\alpha z_1}\mathcal{D}_{\alpha z_2}\ldots\mathcal{D}_{\alpha z_n} \\ &\qquad \cong \sum_{i=1}^{n} \pm(\mathcal{D}_{y_i z_1} - (-1)^{|y_i||z_1|}(n-1)\delta_{y_i z_1} I)\mathcal{D}_{y_1\alpha}\ldots\widehat{\mathcal{D}_{y_i\alpha}}\ldots\mathcal{D}_{y_n\alpha}\cdot\mathcal{D}_{\alpha z_2}\ldots\mathcal{D}_{\alpha z_n} \end{aligned}$$

where $\pm$ is the sign associated to the pair of words

$$y_1 \ldots y_i \ldots y_n z_1 \ldots z_n \quad \text{and} \quad y_i z_1 y_1 \ldots \widehat{y_i} \ldots y_n z_2 \ldots z_n,$$

that is

$$(-1)^{|z_1|(|y_1|+\ldots+|\widehat{y_i}|+\ldots+|y_n|)+|y_i|(|y_1|+\ldots+|y_{i-1}|)}.$$

A similar result holds in the case $|\alpha| = 1$.

By Theorem 6.2 and iterating the identity of Theorem 6.3, one can eliminate all the virtual variables α_i, in any Capelli bitableau, therefore obtaining a devirtualization of the associated operator. The crucial point of the virtual method is that, in the study of the actions, the virtual form is much more preferable than a devirtualized form.

6.5 Basic examples

- Let $x \neq y$ be proper letters, α a virtual letter, $|\alpha| = |x| = |y| = 0$; then

$$\mathcal{D}^n_{x\alpha}\mathcal{D}^n_{\alpha y} \cong n!\mathcal{D}^n_{xy}.$$

- Let x be a proper letter, α a virtual letter, $|\alpha| = |x| = 0$; then

$$\mathcal{D}^n_{x\alpha}\mathcal{D}^n_{\alpha x} \cong n!\mathcal{D}_{xx}(\mathcal{D}_{xx} - I)\ldots(\mathcal{D}_{xx} - (n-1)I) = (n!)^2\binom{\mathcal{D}_{xx}}{n}.$$

- Let $y_{i_1}, \ldots, y_{i_n}, x_{j_1}, \ldots, x_{j_n}$ be two n-tuples of proper letters of the same sign, say $|y_{i_h}| = |x_{j_k}| = 0$, for every $h, k = 1, \ldots, n$. Assume that the two n-tuples above have no letters in common. Then

$$\mathcal{D}_{y_{i_1}\alpha}\ldots\mathcal{D}_{y_{i_n}\alpha}\mathcal{D}_{\alpha x_{j_n}}\ldots\mathcal{D}_{\alpha x_{j_n}} = \begin{cases} \mathrm{per}[\mathcal{D}_{y_{i_h}x_{j_k}}]_{h,k} & \text{if } |\alpha| = 0, \\ \det[\mathcal{D}_{y_{i_h}x_{j_k}}]_{h,k} & \text{if } |\alpha| = 1. \end{cases}$$

- Let $\mathcal{L} = \mathcal{L}^- = \{x_1, \ldots x_m\}$ be a linearly ordered set of (distinct) negative proper letters; then, we get the classical Capelli operator:

$$\mathcal{D}_{x_m\alpha}\ldots\mathcal{D}_{x_1\alpha}\cdot\mathcal{D}_{\alpha x_1}\ldots\mathcal{D}_{\alpha x_m}$$
$$= \det\begin{bmatrix} \mathcal{D}_{x_1x_1} + (m-1)I & \mathcal{D}_{x_1x_2} & \ldots & \mathcal{D}_{x_1x_m} \\ \mathcal{D}_{x_2x_1} & \mathcal{D}_{x_2x_2} + (m-2)I & \ldots & \mathcal{D}_{x_2x_m} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{D}_{x_mx_1} & \mathcal{D}_{x_mx_2} & \ldots & \mathcal{D}_{x_mx_m} \end{bmatrix} = H_m,$$

  where the expansion of the "determinant" is by column from left to right (Weyl [92]).

7 Biproducts and Bitableaux in $Super[\mathcal{L}|P]$

7.1 Capelli rows and supersymmetries in $Super[\mathcal{L}|P]$

Let $x_1, \ldots, x_n \in \mathcal{L}$ and $y_1, \ldots, y_n \in \mathcal{P}$ be (not necessarily distinct) proper letters and places, respectively. Let $\alpha \in \mathcal{X}\backslash\mathcal{L}$ be a virtual letter. By applying the Capelli operator

$\mathcal{D}_{x_1\alpha}\mathcal{D}_{x_2\alpha}\ldots\mathcal{D}_{x_n\alpha}\cdot\mathcal{D}_{\alpha x_n}\ldots\mathcal{D}_{\alpha x_2}\mathcal{D}_{\alpha x_1}$ to the monomial $(x_1\,|\,y_1)(x_2\,|\,y_2)\ldots(x_n\,|\,y_n)\in Super_n[\mathcal{L}|\mathcal{P}]$, we get

$$\mathcal{D}_{x_1\alpha}\mathcal{D}_{x_2\alpha}\ldots\mathcal{D}_{x_n\alpha}\cdot\mathcal{D}_{\alpha x_n}\ldots\mathcal{D}_{\alpha x_2}\mathcal{D}_{\alpha x_1}(x_1\,|\,y_1)(x_2\,|\,y_2)\ldots(x_n\,|\,y_n)$$
$$= k\cdot\mathcal{D}_{x_1\alpha}\mathcal{D}_{x_2\alpha}\ldots\mathcal{D}_{x_n\alpha}(\alpha\,|\,y_1)(\alpha\,|\,y_2)\ldots(\alpha\,|\,y_n),\quad k\in\mathbb{Z}$$

which is

- supersymmetric in the x's and the y's if $|\alpha|=0$ (that is equals zero whenever the word $x_1\cdots x_n$ or the word $y_1\cdots y_n$ contain repeated negatively signed symbols);
- "dual" supersymmetric in the x's and the y's if $|\alpha|=1$ (that is equals zero whenever the word $x_1\cdots x_n$ or the word $y_1\cdots y_n$ contain repeated positively signed symbols).

Informally speaking, Capelli rows are supersymmetrization operators in disguise.

7.2 Biproducts as basic symmetrized elements in $Super[\mathcal{L}|\mathcal{P}]$

The argument of the previous subsection leads naturally to a virtual definition of the basic supersymmetric and of the basic dually supersymmetric objects in $Super_n[\mathcal{L}|\mathcal{P}]$, both associated to pairs of sequences of the same length in $\mathcal{L}$ and $\mathcal{P}$. These objects, here presented in their three different virtual forms [11], are called *biproducts* and **–biproducts*, respectively; in particular, the biproducts coincide, in characteristic 0, with the Grosshans–Rota–Stein biproducts [46].

Let $x_1,\ldots,x_n\in\mathcal{L}$ and $y_1,\ldots,y_n\in\mathcal{P}$ be proper letters and places, respectively. Let $\alpha\in\mathcal{X}\setminus\mathcal{L}$ be a virtual letter and $\beta\in\mathcal{Y}\setminus\mathcal{P}$ be a virtual place, with $|\alpha|=|\beta|$.

The element of $Super_n[\mathcal{L}|\mathcal{P}]$

$$\mathcal{D}_{x_1\alpha}\mathcal{D}_{x_2\alpha}\ldots\mathcal{D}_{x_n\alpha}(\alpha\,|\,y_1)(\alpha\,|\,y_2)\ldots(\alpha\,|\,y_n)$$
$$=\mathcal{D}_{x_1\alpha}\mathcal{D}_{x_2\alpha}\ldots\mathcal{D}_{x_n\alpha}\Big(\frac{(\alpha\,|\,\beta)^n}{n!}\Big)\ {}_{\beta y_1}\mathcal{D}_{\beta y_2}\mathcal{D}\ldots{}_{\beta y_n}\mathcal{D}$$
$$=(x_1\,|\,\beta)(x_2\,|\,\beta)\ldots(x_n\,|\,\beta)_{\beta y_1}\mathcal{D}_{\beta y_2}\mathcal{D}\ldots{}_{\beta y_n}\mathcal{D}$$

is called, for $|\alpha|=|\beta|=0$, the *biproduct* of $x_1\ldots x_n$ and $y_1\ldots y_n$, denoted by

$$(x_1x_2\ldots x_n\,|\,y_1y_2\ldots y_n),\tag{1}$$

and, for $|\alpha|=|\beta|=1$, the **-biproduct* of $x_1\ldots x_n$ and $y_1\ldots y_n$, denoted by

$$(x_1x_2\ldots x_n\,|\,y_1y_2\ldots y_n)^*.\tag{2}$$

Remark 7.1. The Laplace expansions of biproducts and *–biproducts correspond to the Leibniz rule for superderivations.

Example 7.1. *1. Let $x_1, \dots, x_n \in \mathcal{L}_0 = \mathcal{L}$ and $y_1, \dots, y_n \in \mathcal{P}_0 = \mathcal{P}$ be proper letters and places, respectively; then $Super_n[\mathcal{L}|\mathcal{P}] = Sym_n[\mathcal{L}|\mathcal{P}]$. We have*

$$(x_1 x_2 \dots x_n \,|\, y_1 y_2 \dots y_n) = \mathrm{per}\,((x_i \,|\, y_j))_{i,j=1,\dots,n}$$
$$(x_1 x_2 \dots x_n \,|\, y_1 y_2 \dots y_n)^* = (-1)^{\binom{n}{2}} \det\,((x_i \,|\, y_j))_{i,j=1,\dots,n}$$

2. Let $x_1, x_2 \in \mathcal{L}_0 = \mathcal{L}$, $y_1, y_2 \in \mathcal{P}_1 = \mathcal{P}$; then $Super[\mathcal{L}_0|\mathcal{P}_1] = \Lambda[\mathcal{L}|\mathcal{P}]$. We have

$$\begin{aligned}(x_1 x_2 \,|\, y_1 y_2) &= \mathcal{D}_{x_1\alpha}\mathcal{D}_{x_2\alpha}\,((\alpha \,|\, y_1)(\alpha \,|\, y_2)) \\ &= (x_2 \,|\, y_1)(x_1 \,|\, y_2) + (x_1 \,|\, y_1)(x_2 \,|\, y_2);\end{aligned}$$

on the other hand,

$$\begin{aligned}(x_1 x_2 \,|\, y_1 y_2) &= ((x_1 \,|\, \beta)(x_2 \,|\, \beta))\ {}_{\beta y_1}\mathcal{D}\ {}_{\beta y_2}\mathcal{D} \\ &= -(x_1 \,|\, y_2)(x_2 \,|\, y_1) + (x_1 \,|\, y_1)(x_2 \,|\, y_2).\end{aligned}$$

Note that $(x_1 x_2 \,|\, y_1 y_2)$ is symmetric in the x's and skew-symmetric in the y's.

The notion of biproduct is extended to a *bilinear map*

$$Super[\mathcal{L}] \times Super[\mathcal{P}] \to Super[\mathcal{L}|\mathcal{P}]$$

by setting

- If $\omega = x_{i_1} x_{i_2} \cdots x_{i_p}$ is a monomial of $\mathbb{Z}$-degree p in $Super[\mathcal{L}]$, and $\varpi = y_{j_1} y_{j_2} \cdots y_{_q}$ is a monomial of $\mathbb{Z}$-degree q in $Super[\mathcal{P}]$, with $p \neq q$, then $(\omega \,|\, \varpi) = 0$.
- $\left(\sum_h \ c_h \omega_h \,|\, \sum_k \ d_k \varpi_k\right) = \sum_{h,k} \ c_h d_k \ (\omega_h \,|\, \varpi_k)$.

The actions of superpolarizations on biproducts can be computed in a quite direct way.

Proposition 7.1. *Let $x_i, x_j \in \mathcal{L}$, $y_h, y_k \in \mathcal{P}$, $\omega \in Super[\mathcal{L}]$, $\varpi \in Super[\mathcal{P}]$. We have the following identities:*

$$\begin{aligned}\mathcal{D}_{x_i x_j}(\omega \,|\, \varpi)_{y_h y_k}\mathcal{D} &= (D_{x_i x_j}(\omega) \,|\, \varpi)_{y_h y_k}\mathcal{D} \\ &= \mathcal{D}_{x_i x_j}(\omega \,|\, (\varpi)_{y_h y_k} D) \\ &= (D_{x_i x_j}(\omega) \,|\, (\varpi)_{y_h y_k} D),\end{aligned}$$

where $D_{x_i x_j}$ and ${}_{y_h y_k}D$ are left and right superpolarizations on $Super[\mathcal{L}]$ and $Super[\mathcal{P}]$, respectively (see Subsection 3.6).

Example 7.2. *Let $x_1, x_2, x_3 \in \mathcal{L}$, $|x_1| = |x_3| = 1, |x_2| = 0$, $y_1, y_2, y_3 \in \mathcal{L}$, $|y_1| = |y_2| = 1, |y_3| = 0$. We have the following identities:*

$$\begin{aligned}\mathcal{D}_{x_3 x_2}(x_1 x_2 \,|\, y_1 y_2)_{y_1 y_3}\mathcal{D} &= (D_{x_3 x_2}(x_1 x_2) \,|\, y_1 y_2)_{y_1 y_3}\mathcal{D} \\ &= -(x_1 x_3 \,|\, y_1 y_2)_{y_1 y_3}\mathcal{D} \\ &= -(x_1 x_3 \,|\, (y_1 y_2)_{y_1 y_3} D) \\ &= (x_1 x_3 \,|\, y_3 y_2).\end{aligned}$$

Proposition 7.1 may be rephrased as a "representation-theoretical" result.

Recall that $Super[\mathcal{L}]$ is $pl(\mathcal{L})$-module and $Super[\mathcal{P}]$ is $pl(\mathcal{P})$-module, where the actions of $pl(\mathcal{L})$ and $pl(\mathcal{P})$ are implemented by left and right superpolarizations, respectively. Thus, the tensor product $Super[\mathcal{L}] \otimes Super[\mathcal{P}]$ is a $(pl(\mathcal{L}), pl(\mathcal{P}))$-bimodule

$$pl(\mathcal{L}) \cdot Super[\mathcal{L}] \otimes Super[\mathcal{P}] \cdot pl(\mathcal{P}),$$

since the actions of $pl(\mathcal{L})$ and $pl(\mathcal{P})$ clearly commute.

Corollary 7.1. *The biproduct induces a $(pl((L), pl((P))$-equivariant linear map*

$$\begin{gathered} Super[\mathcal{L}] \otimes Super[\mathcal{P}] \rightarrow Super[\mathcal{L}|\mathcal{P}], \\ \omega \otimes \varpi \mapsto (\omega \,|\, \varpi), \end{gathered}$$

$\omega \in Super[\mathcal{L}]$, $\varpi \in Super[\mathcal{P}]$.

7.3 Bitableau monomials

For any words $x' = x_1'x_2' \cdots x_n'$ and $x = x_1x_2 \cdots x_n$ on the letter alphabet $\mathcal{X}$, any words $y = y_1y_2 \cdots y_n$ and $y' = y_1'y_2' \cdots y_n'$ on the place alphabet $\mathcal{Y}$, all of the same length, we set

$$\begin{aligned} \mathcal{D}_{x'x} &= \mathcal{D}_{x_1'x_1}\ \mathcal{D}_{x_2'x_2} \cdots \mathcal{D}_{x_n'x_n}, \\ \langle x \,|\, y\rangle &= (x_1 \,|\, y_1)\ (x_2 \,|\, y_2) \cdots (x_n \,|\, y_n), \\ {}_{yy'}\mathcal{D} &= {}_{y_1y_1'}\mathcal{D}\ {}_{y_2y_2'}\mathcal{D}\ \cdots {}_{y_ny_n'}\mathcal{D}. \end{aligned}$$

For any pair of tableaux S' and S on the letter alphabet $\mathcal{X}$, and any pair of tableaux T and T' on the place alphabet $\mathcal{Y}$, all of the same shape, we set

$$\begin{aligned} \mathcal{D}_{S'S} &= \mathcal{D}_{x'x}, & x' = w(S'),\ x = w(S), \\ \langle S \,|\, T\rangle &= \langle x \,|\, y\rangle, & x = w(S),\ y = w(T), \\ {}_{TT'}\mathcal{D} &= {}_{yy'}\mathcal{D}, & y = w(T),\ y' = w(T'), \end{aligned}$$

where x', x, y, y' are the row words of the tableaux S', S, T, T', respectively. We recall that the row word $w(U)$ of a tableau U is the word obtained by reading the entries of U row by row, from left to right and from top to bottom.

We will often use the short forms

$$\begin{aligned} S'S &\quad for \quad \mathcal{D}_{S'S}, \\ \underline{ST} &\quad for \quad \langle S \,|\, T\rangle, \\ TT' &\quad for \quad {}_{TT'}\mathcal{D}. \end{aligned}$$

The definition of biproduct can also be written in the following form:

$$\mathcal{D}_{x\alpha^n}\ \langle \alpha^n \,|\, y\rangle = \mathcal{D}_{x\alpha^n}\ \frac{\langle \alpha^n\,|\,\beta^n\rangle}{n!}\ {}_{\beta^n y}\mathcal{D} = \langle x\,|\,\beta^n\rangle\ {}_{\beta^n y}\mathcal{D},$$

or in an even shorter form:

$$x\alpha^n\ \underline{\alpha^n y} = \frac{1}{n!}\ x\alpha^n\ \underline{\alpha^n\beta^n}\ \beta^n y = \underline{x\beta^n}\ \beta^n y;$$

in these formulas x and y denote words over the proper alphabets $\mathcal{L}$ and $\mathcal{P}$, while α and β denote constant words of positive virtual symbols.

7.4 Bitableaux in $Super[\mathcal{L}|\mathcal{P}]$

For every partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p) \vdash n$, and every pair of tableaux $T \in Tab(\mathcal{L})$, $U \in Tab(\mathcal{P})$, with $\mathrm{sh}(T) = \lambda = \mathrm{sh}(U)$, we define the *bitableau* [46, 11]

$$(T\,|\,U) \in Super_n[\mathcal{L}|\mathcal{P}]$$

as the common value of the expressions

$$TC_1\ \underline{C_1 U} = \frac{1}{\lambda!}\ TC_1\ \underline{C_1 C_2}\ C_2 U, = \underline{TC_2}\ C_2 U,$$

where C_1 is any virtual letter tableau of co-Deruyts type, C_2 is any virtual place tableau of co-Deruyts type, all of shape λ, and $\lambda! = \prod_i \lambda_i!$.

If $\mathrm{sh}(T) \neq \mathrm{sh}(U)$, the bitableau $(T\,|\,U)$ is set to be zero.

Example 7.3. *In the following, let $|\alpha_i| = 0$ and $|\beta_i| = 0$ be any virtual positive symbols.*

$$\left(\begin{array}{cc|cc} x & y & a & b \\ x & z & a & c \\ y & & c & \end{array}\right) = \mathcal{D}_{\begin{smallmatrix} x & y & \alpha_1 & \alpha_1 \\ x & z & \alpha_2 & \alpha_2 \\ y & & \alpha_3 & \end{smallmatrix}} \left\langle \begin{array}{cc|cc} \alpha_1 & \alpha_1 & a & b \\ \alpha_2 & \alpha_2 & a & c \\ \alpha_3 & & c & \end{array}\right\rangle$$

$$= \frac{1}{2!2!1!}\ \mathcal{D}_{\begin{smallmatrix} x & y & \alpha_1 & \alpha_1 \\ x & z & \alpha_2 & \alpha_2 \\ y & & \alpha_3 & \end{smallmatrix}} \left\langle \begin{array}{cc|cc} \alpha_1 & \alpha_1 & \beta_1 & \beta_1 \\ \alpha_2 & \alpha_2 & \beta_2 & \beta_2 \\ \alpha_3 & & \beta_3 & \end{array}\right\rangle {}_{\begin{smallmatrix} \beta_1 & \beta_1 & a & b \\ \beta_2 & \beta_2 & a & c \\ \beta_3 & & c & \end{smallmatrix}}\mathcal{D}$$

$$= \left\langle \begin{array}{cc|cc} x & y & \beta_1 & \beta_1 \\ x & z & \beta_2 & \beta_2 \\ y & & \beta_3 & \end{array}\right\rangle {}_{\begin{smallmatrix} \beta_1 & \beta_1 & a & b \\ \beta_2 & \beta_2 & a & c \\ \beta_3 & & c & \end{smallmatrix}}\mathcal{D}.$$

Proposition 7.2. *Let $T = (\omega_1, \ldots, \omega_p)$, $U = (\omega'_1, \ldots, \omega'_p)$ be tableaux of the same shape. Then the bitableau $(T \,|\, U)$ equals, up to a sign, the product of the biproducts $(\omega_1 \,|\, \omega'_1), \ldots, (\omega_p \,|\, \omega'_p)$. In symbols*

$$(T \,|\, U) = \left(\begin{array}{c|c} \omega_1 & \omega'_1 \\ \omega_2 & \omega'_2 \\ \vdots & \vdots \\ \omega_p & \omega'_p \end{array} \right) = (-1)^{|\omega_2||\omega'_1| + \cdots + |\omega_p|(|\omega'_1| + \cdots + |\omega'_{p-1}|)} (\omega_1|\omega'_1) \cdots (\omega_p|\omega'_p).$$

Example 7.4.

$$\left(\begin{array}{cc|cc} x & y & a & b \\ x & z & a & c \\ y & & c & \end{array} \right) = \pm (xy \,|\, ab) \; (xz \,|\, ac) \; (y \,|\, c)$$

where the sign is given by the parity of $(|xz|)(|ab|) + |y|(|ab| + |ac|)$.

Remark 7.2. The bitableaux of shape $(1, 1, \ldots, 1)$, i.e., *column-bitableaux*, are monomials and, conversely, monomials can be written, up to a sign, as column bitableaux. Bitableaux of shape (n), i.e., *row-bitableaux*, are biproducts.

The actions of superpolarizations on bitableaux can be computed in a quite direct way.

Proposition 7.3. *Let $x_i, x_j \in \mathcal{L}$, $y_h, y_k \in \mathcal{P}$, and let $S = (\omega_1, \ldots, \omega_p) \in Tab(\mathcal{L})$, $T = (\omega'_1, \ldots, \omega'_p) \in Tab(\mathcal{P})$. We have the following identity:*

$$\begin{aligned} \mathcal{D}_{x_i x_j} (T \,|\, U)_{y_h y_k} \mathcal{D} \; &= \; \mathcal{D}_{x_i x_j} \left(\begin{array}{c|c} \omega_1 & \omega'_1 \\ \omega_2 & \omega'_2 \\ \vdots & \vdots \\ \omega_p & \omega'_p \end{array} \right) {}_{y_h y_k} \mathcal{D} \\ &= \sum_{s,t=1}^{p} (-1)^{(|x_i| + |x_j|)\epsilon_s} (-1)^{(|y_h| + |y_k|)\epsilon'_t} \left(\begin{array}{c|c} \omega_1 & \omega'_1 \\ \omega_2 & \omega'_2 \\ \vdots & \vdots \\ \vdots & (\omega'_t)_{y_h y_k} D \\ D_{x_i x_j}(\omega_s) & \vdots \\ \vdots & \vdots \\ \omega_p & \omega'_p \end{array} \right), \end{aligned}$$

where

$$\begin{aligned} \epsilon_s &= |\omega_1| + \cdots + |\omega_{s-1}|, \quad s = 2, \ldots, p, \\ \epsilon'_t &= |\omega'_{t+1}| + \cdots + |\omega'_p|, \quad t = 1, 2, \ldots, p-1, \end{aligned}$$

and $D_{x_i x_j}$, ${}_{y_h y_k}D$ are left and right superpolarizations on $Super[\mathcal{L}]$ and $Super[\mathcal{P}]$, respectively (see Subsection 3.6).

Proposition 7.3 may be rephrased as a "representation-theoretical" result.

Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_p) \vdash n$ be a partition. Set

$$Super^{\lambda}[\mathcal{L}] = Super_{\lambda_1}[\mathcal{L}] \otimes Super_{\lambda_2}[\mathcal{L}] \otimes \cdots \otimes Super_{\lambda_p}[\mathcal{L}],$$

and

$$Super^{\lambda}[\mathcal{P}] = Super_{\lambda_1}[\mathcal{P}] \otimes Super_{\lambda_2}[\mathcal{P}] \otimes \cdots \otimes Super_{\lambda_p}[\mathcal{P}].$$

The tensor product $Super^{\lambda}[\mathcal{L}]$ is $pl(\mathcal{L})$-module and the tensor product $Super^{\lambda}[\mathcal{P}]$ is $pl(\mathcal{P})$-module, where the actions of $pl(\mathcal{L})$ and $pl(\mathcal{P})$ are implemented by left and right superpolarizations and extended as left and right superderivations , respectively. Thus, the tensor product $Super^{\lambda}[\mathcal{L}] \otimes Super^{\lambda}[\mathcal{P}]$ is a $(pl(\mathcal{L}), pl(\mathcal{P}))$-bimodule

$$pl(\mathcal{L}) \cdot Super^{\lambda}[\mathcal{L}] \otimes Super^{\lambda}[\mathcal{P}] \cdot pl(\mathcal{P}),$$

since the actions of $pl(\mathcal{L})$ and $pl(\mathcal{P})$ clearly commute.

Corollary 7.2. *The map*

$$\omega_1 \otimes \omega_2 \otimes \cdots \otimes \omega_p \otimes \omega'_1 \otimes \omega'_2 \otimes \cdots \otimes \omega'_p \mapsto \left(\begin{array}{c|c} \omega_1 & \omega'_1 \\ \omega_2 & \omega'_2 \\ \vdots & \vdots \\ \omega_p & \omega'_p \end{array} \right),$$

induces a $(pl((L), pl((P))$-equivariant linear map

$$Super^{\lambda}[\mathcal{L}] \otimes Super^{\lambda}[\mathcal{P}] \to Super_n[\mathcal{L}|\mathcal{P}],$$

$(\omega_1, \omega_2, \cdots, \omega_p) \in Tab_{\lambda}((L),\ (\omega'_1, \omega'_2, \cdots, \omega'_p) \in Tab_{\lambda}((P).$

8 The Standard Basis

8.1 The Straightening Law of Grosshans, Rota and Stein.

Theorem 8.1 (Straightening Law, general form [46]). *For all monomials $u, v, w \in Super(\mathcal{L}),\ x, y \in Super(\mathcal{P})$,*

$$\sum_{(v)} \left(\begin{array}{c|c} uv_{(1)} & x \\ v_{(2)}w & y \end{array} \right) = (-1)^{|u||v|} \sum_{(u)\ (y)} (-1)^{l(u_{(2)})} \left(\begin{array}{c|c} vu_{(1)} & xy_{(1)} \\ u_{(2)}w & y_{(2)} \end{array} \right).$$

In the statement above, the summations are meant with respect to the *coproducts* of the supersymmetric algebras $Super(\mathcal{L})$ and $Super(\mathcal{P})$, regarded as $\mathbb{Z}_2$-bialgebras (Subsection 3.6).

In the following, for any positive symbol ξ and any natural number n, we set $\xi^{(n)} = \frac{\xi^n}{n!}$; notice that

$$\xi^{(n)}\xi^{(m)} = \binom{n+m}{n}\xi^{(n+m)}, \qquad \Delta\left(\xi^{(n)}\right) = \sum_{k=0}^{n} \xi^{(k)} \otimes \xi^{(n-k)}.$$

Proof. [17] First we prove the identity in the case in which the words u, v, w, x, y are powers of positive symbols: let $\alpha, \beta, \gamma, \delta, \epsilon$ be positive virtual symbols, and a, b, d, e natural numbers such that $b \geq d$; then

$$\left(\begin{array}{c|c} \alpha^{(a)}\beta^{(b-d)} & \gamma^{(a+b-d)} \\ \beta^{(d)}\epsilon^{(e)} & \delta^{(d+e)} \end{array}\right) = \sum_{k=0}^{\min(a,d)} (-1)^k \left(\begin{array}{c|c} \beta^{(b)}\alpha^{(a-k)} & \gamma^{(a+b-d)}\delta^{(d-k)} \\ \alpha^{(k)}\epsilon^{(e)} & \delta^{(k+e)} \end{array}\right). \qquad (*)$$

Indeed, starting from the right-hand side

$$\sum_{k=0}^{\min(a,d)} (-1)^k \left(\beta^{(b)}\alpha^{(a-k)} \,|\, \gamma^{(a+b-d)}\delta^{(d-k)}\right)\left(\alpha^{(k)}\epsilon^{(e)} \,|\, \delta^{(k+e)}\right)$$

we get

$$\begin{aligned}
&\sum_{k=0}^{\min(a,d)} (-1)^k \sum_{s=0}^{\min(a-k,d-k)} (\beta\,|\,\gamma)^{(b-d+k+s)}(\beta\,|\,\delta)^{(d-k-s)}(\alpha\,|\,\gamma)^{(a-k-s)}(\alpha\,|\,\delta)^{(s)}(\alpha\,|\,\delta)^{(k)}(\epsilon\,|\,\delta)^{(e)} \\
&= \sum_{k=0}^{\min(a,d)} (-1)^k \sum_{s=0}^{\min(a-k,d-k)} \binom{s+k}{k}(\beta\,|\,\gamma)^{(b-d+k+s)}(\beta\,|\,\delta)^{(d-k-s)} \\
&\qquad\qquad \cdot (\alpha\,|\,\gamma)^{(a-k-s)}(\alpha\,|\,\delta)^{(s+k)}(\epsilon\,|\,\delta)^{(e)} \\
&= \sum_{k=0}^{\min(a,d)} (-1)^k \sum_{t=k}^{\min(a,d)} \binom{t}{k}(\beta\,|\,\gamma)^{(b-d+t)}(\beta\,|\,\delta)^{(d-t)}(\alpha\,|\,\gamma)^{(a-t)}(\alpha\,|\,\delta)^{(t)}(\epsilon\,|\,\delta)^{(e)} \\
&= \sum_{t=0}^{\min(a,d)} \sum_{k=0}^{t}(-1)^k \binom{t}{k}(\beta\,|\,\gamma)^{(b-d+t)}(\beta\,|\,\delta)^{(d-t)}(\alpha\,|\,\gamma)^{(a-t)}(\alpha\,|\,\delta)^{(t)}(\epsilon\,|\,\delta)^{(e)} \\
&= (\beta\,|\,\gamma)^{(b-d)}(\beta\,|\,\delta)^{(d)}(\alpha\,|\,\gamma)^{(a)}(\epsilon\,|\,\delta)^{(e)},
\end{aligned}$$

which equals the left-hand side

$$\left(\begin{array}{c|c} \alpha^{(a)}\beta^{(b-d)} & \gamma^{(a+b-d)} \\ \beta^{(d)}\epsilon^{(e)} & \delta^{(d+e)} \end{array}\right) = \left(\alpha^{(a)}\beta^{(b-d)} \,|\, \gamma^{(a+b-d)}\right)\left(\beta^{(d)}\epsilon^{(e)} \,|\, \delta^{(d+e)}\right).$$

Consider the letter polarization monomial

$$\mathcal{D}_{u\alpha^{(a)}}\mathcal{D}_{v\beta^{(b)}}\mathcal{D}_{w\epsilon^{(e)}} = (-1)^{|u||v|}\mathcal{D}_{v\beta^{(b)}}\mathcal{D}_{u\alpha^{(a)}}\mathcal{D}_{w\epsilon^{(e)}}$$

and the place polarization monomial

$$ {}_{\gamma^{(a+b-d)}x}\mathcal{D}_{\delta^{(d+e)}y}\mathcal{D}. $$

and apply these operators on both sides of the positive identity (∗).

Since letter and place polarizations operators are *coderivations* (see Subsection 3.6, and, for further details, [18] and [19]), we get:

$$
\begin{aligned}
\left(\begin{array}{c|c} uv_{(1)} & x \\ v_{(2)}w & y \end{array}\right) &= \mathcal{D}_{u\alpha^{(a)}}\mathcal{D}_{v\beta^{(b)}}\mathcal{D}_{w\epsilon^{(e)}} \left(\begin{array}{c|c} \alpha^{(a)}\beta^{(b-d)} & \gamma^{(a+b-d)} \\ \beta^{(d)}\epsilon^{(e)} & \delta^{(d+e)} \end{array}\right) {}_{\gamma^{(a+b-d)}x}\mathcal{D}\ {}_{\delta^{(d+e)}y}\mathcal{D} \\
&= (-1)^{|u||v|}\mathcal{D}_{v\beta^{(b)}}\mathcal{D}_{u\alpha^{(a)}}\mathcal{D}_{w\epsilon^{(e)}} \\
&\quad \times \sum_{k=0}^{\min(a,d)} (-1)^k \left(\begin{array}{c|c} \beta^{(b)}\alpha^{(a-k)} & \gamma^{(a+b-d)}\delta^{(d-k)} \\ \alpha^{(k)}\epsilon^{(e)} & \delta^{(k+e)} \end{array}\right) {}_{\gamma^{(a+b-d)}x}\mathcal{D}\ {}_{\delta^{(d+e)}y}\mathcal{D} \\
&= (-1)^{|u||v|} \sum_{(u)\ (y)} (-1)^{l(u_{(2)})} \left(\begin{array}{c|c} vu_{(1)} & xy_{(1)} \\ u_{(2)}w & y_{(2)} \end{array}\right).
\end{aligned}
$$

□

8.2 Triangularity and nondegeneracy results

For the sake of readability, we recall the partial order already defined in Subsection 5.5. This order is defined on the set of all standard tableaux over $\mathcal{L}$ of a given content; note that the shapes of these tableaux are partitions of the same integer.

For every standard tableau S, we consider the sequence $S^{(p)}$ $p = 1, 2, \ldots,$ of the subtableaux obtained from S by considering only the first p symbols of the alphabet, and consider the family $\mathrm{sh}(S^{(p)}),\ p = 1, 2, \ldots$ of the corresponding shapes. Since the alphabet is assumed to be finite, this sequence is finite and its last term is $\mathrm{sh}(S)$.

Then, for standard tableaux S, T, we set

$$ S \leq T \ \Leftrightarrow\ \mathrm{sh}(S^{(p)}) \trianglelefteq \mathrm{sh}(T^{(p)}),\ p = 1, 2, \ldots, $$

where $\trianglelefteq$ stands for the *dominance order* on partitions. We recall that the dominance order on partitions is defined as follows: $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots) \leq \mu = (\mu_1 \geq \mu_2 \geq \ldots)$ if and only if

$$ \lambda_1 + \cdots + \lambda_i \leq \mu_1 + \cdots + \mu_i, $$

for every $i = 1, 2, \ldots$.

Lemma 8.1 ([21]). *Let S, T be standard letter tableaux of the same content, D a virtual Deruyts letter tableau, C a co-Deruyts virtual place tableau, with* $\mathrm{sh}(D) = \mathrm{sh}(S)$ *and* $\mathrm{sh}(C) = \mathrm{sh}(T)$.

- *$DS\ \underline{TC} = 0$, for $S \not\geq T$;*
- *$DS\ \underline{TC} = \theta^{-+}_{ST}\ \underline{DC}$, for $\mathrm{sh}(S) = \mathrm{sh}(T)$.*

Notice that θ^{-+}_{ST} are uniquely determined integer coefficient, since $\underline{DC} \neq 0$.

For the proof see [69].

The coefficients θ^{-+}_{ST} are called *symmetry transition coefficients* and turn out to be a generalization of the " Désarménien coefficients" [35].

Lemma 8.2 ([21]). *For every pair of standard letter tableaux S, T of the same content, with $\mathrm{sh}(S) = \mathrm{sh}(T)$, we have*

$$\theta^{-+}_{ST} \begin{cases} = 0 & \text{if } S \not\geq T, \\ \neq 0 & \text{if } S = T. \end{cases}$$

Moreover, each diagonal coefficient θ^{-+}_{SS} is, up to a sign, the product of the factorials of the multiplicities of positive symbols in each row and of negative symbols in each column of the tableau S.

For the proof see [69].

We have also the following nondegeneracy result.

Lemma 8.3 ([21]). *Let C, C_1 be co-Deruyts tableaux and D_1, D be Deruyts tableaux, all of the same shape $\lambda \vdash n$. Then*

$$CD_1\ \underline{D_1C_1}\ C_1D = (-1)^{\binom{n}{2}} h_\lambda\ \underline{CD},$$

where h_λ is a positive coefficient; indeed, it equals the product of the hook lengths of the shape λ.

For the proof, see Subsection 18.3.

All these results have many variations.

For example, interchanging the role of C and D, we have the following analogue of Lemma 8.1.

Lemma 8.4. *Let S, T be standard letter tableaux of the same content, C a virtual co-Deruyts letter tableau, D a virtual Deruyts place tableau, with $\mathrm{sh}(C) = \mathrm{sh}(S)$ and $\mathrm{sh}(D) = \mathrm{sh}(T)$.*

- *$CS\ \underline{TD} = 0$, for $S \not\leq T$;*
- *$CS\ \underline{TD} = \theta^{+-}_{ST}\ \underline{CD}$, for $\mathrm{sh}(S) = \mathrm{sh}(T)$.*

Notice that the two types of symmetry transition coefficients are related by

$$\theta^{+-}_{ST} = \theta^{-+}_{TS}.$$

As another example, we have the following operator analogue of Lemma 8.1.

Lemma 8.5 ([21])**.** *Let S, T be standard letter tableaux of the same content, D a virtual Deruyts letter tableau, C a virtual co-Deruyts letter tableau, with* $\mathrm{sh}(D) = \mathrm{sh}(S)$ *and* $\mathrm{sh}(C) = \mathrm{sh}(T)$.

- $DS\ TC = 0$, *for* $S \not\geq T$;
- $DS\ TC = \theta^{-+}_{ST}\ DC$, *for* $\mathrm{sh}(S) = \mathrm{sh}(T)$.

The coefficients θ^{-+}_{ST} are indeed the symmetry transition coefficients, as before.

8.3 The standard basis

On the set of all the letter tableaux of a given content, we consider the linear order in which

$$P \leq P' \quad \Leftrightarrow \quad \begin{array}{c} \mathrm{sh}(P) \leq_l \mathrm{sh}(P') \\ w(P) \geq_l w(P') \end{array},$$

where the shapes and the words are compared in the lexicographic order. We recall that this order, restricted to standard tableaux, is a linear extension of the partial order defined above (see Subsection 5.5). We consider an analogous order on the set of all the place tableaux of a given content.

On the set of all the pairs (P, Q), where P ranges over the letter tableaux of a given content, with sum of multiplicities n, where Q ranges over the place tableaux of a given content, with sum of multiplicities n, and $\mathrm{sh}(P) = \mathrm{sh}(Q)$, we consider the partial order

$$(P, Q) \leq (P', Q') \quad \Leftrightarrow \quad \begin{array}{c} \mathrm{sh}(P) = \mathrm{sh}(Q) \leq_l \mathrm{sh}(P') = \mathrm{sh}(Q') \\ w(P) \geq_l w(P') \\ w(Q) \geq_l w(Q') \end{array}.$$

Theorem 8.2 ([46])**.** *Every bitableau $(P \,|\, Q) \in Super_n[\mathcal{L}|\mathcal{P}]$ can be written as a linear combination, with rational coefficients,*

$$(P \,|\, Q) = \sum_{S,T \ standard} c_{S,T}\ (S \,|\, T)$$

of standard bitableaux $(S \,|\, T) \in Super_n[\mathcal{L}|\mathcal{P}]$, where the standard tableaux S have the same content as the content of the tableau P, the standard tableaux T have the same content of the tableau Q, and $(P, Q) \leq (S, T)$.

Proof. (Sketch) We consider the following rewriting rules on a bitableau $(A \mid B)$:

1. if the bitableau $(A \mid B)$ is not row-ordered, rewrite it as a row-ordered bitableau $\pm(A' \mid B')$;

2. if the row-ordered bitableau $(A \mid B)$ has a column violation in the tableau A, perform the following process. Notice that in this case $(A \mid B)$ has a factor of the form
$$\left(\begin{array}{cccccccc|ccc} u_1 & \dots & u_p & v_{p+1} & v_{p+2} & \dots & v_m & & x_1 & \dots & x_m \\ v_1 & \dots & v_p & v^*_{p+1} & w_{p+2} & \dots & w_n & & y_1 & \dots & y_n \end{array} \right), \qquad (*)$$
where
$$v_1 \leq \ldots \leq v_p \leq v^*_{p+1} \leq v_{p+1} \leq v_{p+2} \leq \ldots v_m,$$
and $v^*_{p+1} < v_{p+1}$ or $v^*_{p+1} = v_{p+1}$, v^*_{p+1} a positive letter. Consider the straightening law
$$\sum_{(v)} \left(\begin{array}{c|c} uv_{(1)} & x \\ v_{(2)}w & y \end{array} \right) = (-1)^{|u||v|} \sum_{(u)\ (y)} (-1)^{l(u_{(2)})} \left(\begin{array}{c|c} vu_{(1)} & xy_{(1)} \\ u_{(2)}w & y_{(2)} \end{array} \right),$$
where
$$\begin{aligned} u &= u_1 \cdots u_p, \\ v &= v_1 \cdots v_p v^*_{p+1} v_{p+1} v_{p+2} \cdots v_m, \\ w &= w_{p+2} \cdots w_n, \\ x &= x_1 \cdots x_m, \\ y &= y_1 \cdots y_n; \end{aligned}$$
Notice that the bitableau $(*)$ appears in the left-hand side with a nonzero integral coefficient, thus it can be rewritten as a linear combination, with rational coefficients, of the other bitableaux in the identity; notice that these are strictly bigger than $(*)$ in the order of pairs of tableaux. Then rewrite the bitableau $(A \mid B)$ by replacing its factor $(*)$ by this linear combination.

3. if the row-ordered bitableau $(A \mid B)$ has a column violation in the tableau B, perform a process analogous to the previous one. Use the variation of the straightening law obtained by interchanging the roles of letters and places.

Notice that each rule replaces a nonstandard bitableau $(A \mid B)$ with a linear combination, with rational coefficients, of bitableaux $(A' \mid B')$ where A' has the same content as the content of A, B' has the same content as the content of B, and $(A, B) < (A', B')$. Starting with a bitableau $(P \mid Q)$, the iteration of these rewriting rules gives, in a finite number of steps, a rewriting of $(P \mid Q)$ as a linear combination, with rational coefficients, of standard bitableaux $(S \mid T)$, where S has the same content as the content of P, T has the same content as the content of Q, and $(P, Q) \leq (S, T')$. $\square$

Theorem 8.3 (The Grosshans, Rota, Stein standard basis theorem for $Super_n[\mathcal{L}|\mathcal{P}]$, [46]). *The following set is a basis of* $Super_n[\mathcal{L}|\mathcal{P}]$*:*

$$\{(S\,|\,T);\ S \in Stab_\lambda(\mathcal{L}),\ T \in Stab_\lambda(\mathcal{P}),\ \lambda \vdash n,\ \lambda \in H(\mathcal{L}) \cap H(\mathcal{P})\},$$

where $H(\mathcal{L})$ *and* $H(\mathcal{P})$ *are the hook sets defined in Subsection 5.4.*

Proof. The standard bitableaux whose shape are partitions of n span $Super_n[\mathcal{L}|\mathcal{P}]$ since they span a subspace containing all the bitableaux whose shapes are partition of n,; among these tableaux there are all the bitableaux of shape $1, 1, \ldots, 1$ which are the monomials of order n.

The standard bitableaux whose shape is a partition of n are linearly independent. Assume, for the purpose of contradiction, that there is a nontrivial linear relation

$$\sum c_{ST}\ (S\,|\,T) = 0$$

among standard bitableaux. Let S_0 be minimal among the letter tableaux S such that $c_{ST} \neq 0$ for some place tableaux T, and let T_0 be minimal among the place tableaux T such that $c_{S_0T} \neq 0$. Denote by λ_0 the shape of S_0 and T_0, let D_1 be a letter Deruyts tableau of shape λ_0, and let D_2 be a place Deruyts tableau of shape λ_0.

By applying the polarization monomials D_1S_0 and T_0D_2 on both sides of the nontrivial relation, we get the contradiction

$$\begin{aligned}
0 &= D_1S_0\left(\sum_{S,T} c_{ST}\ (S\,|\,T)\right)\ T_0D_2\\
&= \sum_{S,T} c_{ST}\ D_1S_0\ \underline{SC}\ CT\ T_0D_2\\
&= \sum_{S\le S_0,\ T\le T_0} c_{ST}\ D_1S_0\ \underline{SC}\ CT\ T_0D_2\\
&= \theta^{-+}_{S_0S_0}\ c_{S_0T_0}\ \theta^{+-}_{T_0T_0}\ D_1C\ \underline{CD_2} = \theta^{-+}_{S_0S_0}\ c_{S_0T_0}\ \theta^{+-}_{T_0T_0}\ (D_1\,|\,D_2)\\
&\neq 0.
\end{aligned}$$

□

An alternative proof of the linear independence of standard bitableaux follows from the superalgebraic version of the Robinson–Schensted correspondence [10], [61].

8.4 An invariant filtration associated to the standard basis

Consider the linear order — defined in Subsection 5.5 — on the place standard tableaux whose shapes are partitions of n.

We recall that $Q < Q'$ if and only if

$$\begin{array}{ll} & \mathrm{sh}(Q) <_l \mathrm{sh}(Q') \\ or & \mathrm{sh}(Q) = \mathrm{sh}(Q') \quad and\ w(Q) >_l w(Q'). \end{array}$$

Let

$$T_1, T_2, \ldots, T_f$$

be the list of the standard place tableaux, whose shapes are partitions of n, in ascending order, and let

$$Super_n[\mathcal{L}|\mathcal{P}] = V_0 \supset V_1 \ldots \supset V_f = (0)$$

be the chain of the subspaces

$$V_i = \langle (S \,|\, T);\ S \text{ standard},\ T > T_i \rangle.$$

Each subspace V_i is a $pl(\mathcal{L})$-submodule; indeed for any letter polarization $\mathcal{D}$, and any basis element $(S \,|\, T)$, with S, T standard and $T > T_i$, we have

$$\mathcal{D}\ (S \,|\, T) = \sum_P (P \,|\, T) = \sum_P \sum_{U,V \text{ standard}} c^P_{UV} (U \,|\, V),$$

where, by the standard expansion theorem, $V \geq T > T_i$.

Thus, the chain $Super_n[\mathcal{L}|\mathcal{P}] = V_0 \supset V_1 \ldots \supset V_f = (0)$ is indeed a $pl(\mathcal{L})$-invariant filtration of $Super_n[\mathcal{L}|\mathcal{P}]$. In the associated graded module

$$V_0/V_1 \oplus V_1/V_2 \oplus \cdots \oplus V_{f-1}/V_f$$

each summand V_{i-1}/V_i has a basis $\{(S \,|\, T_i) + V_i; S \text{ standard}, \mathrm{sh}(S) = \mathrm{sh}(T_i)\}$.

An analogous argument gives a $pl(\mathcal{P})$-invariant filtration of $Super_n[\mathcal{L}|\mathcal{P}]$.

9 Clebsch–Gordan–Capelli Expansions

9.1 Right symmetrized bitableaux

For every $\lambda \vdash n$ and every $T \in Tab(\mathcal{L})$, $U \in Tab(\mathcal{P})$, with $\mathrm{sh}(T) = \lambda = \mathrm{sh}(U)$, we define the *right symmetrized bitableau* $\left(T \,\middle|\, \boxed{U}\right) \in Super_n[\mathcal{L}|\mathcal{P}]$ by setting

$$\begin{aligned} \left(T \,\middle|\, \boxed{U}\right) &= \mathcal{D}_{TC}\ \langle C \,|\, D \rangle\ _{DU}\mathcal{D} \\ &= TC\ \underline{CD}\ DU \end{aligned}$$

where C is any virtual tableau of co-Deruyts type and D is any virtual tableau of Deruyts type, all of shape λ.

A quite useful, but not trivial, fact is that right symmetrized bitableaux admit different equivalent definitions.

Proposition 9.1 ([11, 20, 21]). *For every $\lambda \vdash n$ and every $T \in Tab(\mathcal{L})$, $U \in Tab(\mathcal{P})$, with $\mathrm{sh}(T) = \lambda = \mathrm{sh}(U)$, we have*

$$
\begin{aligned}
\left(T \,\middle|\, \boxed{U}\right) &= \underline{TC_1}\ C_1D\ DU \\
&= TC\ \underline{CD}\ DU \\
&= TC\ CD_1\ \underline{D_1U},
\end{aligned}
$$

where C, C_1 are any virtual tableaux of co-Deruyts type and D, D_1 are any virtual tableaux of Deruyts type, all of shape λ.

Example 9.1. *In the following, let $|\alpha_i| = |\alpha_i'| = 0$ and $|\beta_i| = |\beta_i'| = 1$ be virtual symbols.*

$$
\left(\begin{array}{ll} x & y \\ x & z \\ y & \end{array} \,\middle\|\, \boxed{\begin{array}{ll} a & b \\ a & c \\ c & \end{array}} \right)
$$

$$
= \left\langle \begin{array}{ll} x & y \\ x & z \\ y & \end{array} \,\middle|\, \begin{array}{ll} \alpha_1' & \alpha_1' \\ \alpha_2' & \alpha_2' \\ \alpha_3' & \end{array} \right\rangle
\begin{array}{llll} \alpha_1' & \alpha_1' & \beta_1 & \beta_2 \\ \alpha_2' & \alpha_2' & \beta_1 & \beta_2 \\ \alpha_3' & & \beta_1 & \end{array}^{\mathcal{D}}
\begin{array}{llll} \beta_1 & \beta_2 & a & b \\ \beta_1 & \beta_2 & a & c \\ \beta_1 & & c & \end{array}^{\mathcal{D}}
$$

$$
= {}^{\mathcal{D}}\begin{array}{llll} x & y & \alpha_1 & \alpha_1 \\ x & z & \alpha_2 & \alpha_2 \\ y & & \alpha_3 & \end{array}
\left\langle \begin{array}{ll} \alpha_1 & \alpha_1 \\ \alpha_2 & \alpha_2 \\ \alpha_3 & \end{array} \,\middle|\, \begin{array}{ll} \beta_1 & \beta_2 \\ \beta_1 & \beta_2 \\ \beta_1 & \end{array} \right\rangle
\begin{array}{llll} \beta_1 & \beta_2 & a & b \\ \beta_1 & \beta_2 & a & c \\ \beta_1 & & c & \end{array}^{\mathcal{D}}
$$

$$
= {}^{\mathcal{D}}\begin{array}{llll} x & y & \alpha_1 & \alpha_1 \\ x & z & \alpha_2 & \alpha_2 \\ y & & \alpha_3 & \end{array}
\ {}^{\mathcal{D}}\begin{array}{llll} \alpha_1 & \alpha_1 & \beta_1' & \beta_2' \\ \alpha_2 & \alpha_2 & \beta_1' & \beta_2' \\ \alpha_3 & & \beta_1' & \end{array}
\left\langle \begin{array}{ll} \beta_1' & \beta_2' \\ \beta_1' & \beta_2' \\ \beta_1' & \end{array} \,\middle|\, \begin{array}{ll} a & b \\ a & c \\ c & \end{array} \right\rangle .
$$

Informally speaking, the right symmetrized bitableau $\left(T \,\middle|\, \boxed{U}\right)$ is supersymmetric in the rows of T and dual supersymmetric in the columns of U: as a matter of fact, $\left(T \,\middle|\, \boxed{U}\right)$ is zero if T has a row repetition of negative letters or U has a column repetition of positive places.

Notice that the right symmetrized bitableau $\left(T \,\middle|\, \boxed{U}\right)$ can be also regarded as the result of applying a place polarization operator to a bitableau: more specifically,

$$
\left(T \,\middle|\, \boxed{U}\right) = (T \,|\, D)_{DU}\mathcal{D},
$$

D any virtual place tableau of Deruyts type, $\mathrm{sh}(U) = \mathrm{sh}(D)$.

Therefore, a right symmetrized bitableau is a linear combination of bitableaux, all of the same shape.

Example 9.2. *In the following all the proper symbols are negative and* $|\beta_i| = 1$.

$$\left(\begin{array}{ll} x & y \\ x & z \\ y & \end{array}\,\middle\|\,\boxed{\begin{array}{ll} a & b \\ a & c \\ c & \end{array}}\right) = \left(\begin{array}{ll} x & y \\ x & z \\ y & \end{array}\middle| \begin{array}{ll} \beta_1 & \beta_2 \\ \beta_1 & \beta_2 \\ \beta_1 & \end{array}\right)_{\begin{array}{ll} \beta_1 & \beta_2 \\ \beta_1 & \beta_2 \\ \beta_1 & \end{array}, \begin{array}{ll} a & b \\ a & c \\ c & \end{array}}^{\mathcal{D}}$$

$$= 2\left(\begin{array}{ll} x & y \\ x & z \\ y & \end{array}\middle| \begin{array}{ll} a & b \\ a & c \\ c & \end{array}\right) + 2\left(\begin{array}{ll} x & y \\ x & z \\ y & \end{array}\middle| \begin{array}{ll} a & b \\ c & c \\ a & \end{array}\right) + 2\left(\begin{array}{ll} x & y \\ x & z \\ y & \end{array}\middle| \begin{array}{ll} c & b \\ a & c \\ a & \end{array}\right)$$

$$+ 2\left(\begin{array}{ll} x & y \\ x & z \\ y & \end{array}\middle| \begin{array}{ll} a & c \\ a & b \\ c & \end{array}\right) + 2\left(\begin{array}{ll} x & y \\ x & z \\ y & \end{array}\middle| \begin{array}{ll} a & c \\ c & b \\ a & \end{array}\right) + 2\left(\begin{array}{ll} x & y \\ x & z \\ y & \end{array}\middle| \begin{array}{ll} c & c \\ a & b \\ a & \end{array}\right).$$

Indeed, in the case of negatively signed alphabets, our definition coincides with the classical one [28].

Remark 9.1. Let D be a *Deruyts tableau* of shape λ on the proper place alphabet $\mathcal{P}$. The *right* symmetrized bitableau $\left(T \,\middle|\, \boxed{D}\right)$ satisfies the identity:

$$\left(T \,\middle|\, \boxed{D}\right) = \tilde{\lambda}_1! \cdots \tilde{\lambda}_q! \; (T \,|\, D),$$

for every $T \in Tab(\mathcal{L})$.

9.2 Left symmetrized bitableaux

For every $\lambda \vdash n$ and every $T \in Tab(\mathcal{L})$, $U \in Tab(\mathcal{P})$, with $\mathrm{sh}(T) = \lambda = \mathrm{sh}(U)$, we define the *left symmetrized bitableau* $\left(\boxed{T} \,\middle|\, U\right) \in Super_n[\mathcal{L}|\mathcal{P}]$ by setting

$$\begin{aligned} \left(\boxed{T} \,\middle|\, U\right) &= \mathcal{D}_{TD} \; \langle D \,|\, C \rangle \; {}_{CU}\mathcal{D} \\ &= TD \; \underline{DC} \; CU, \end{aligned}$$

where D is any virtual tableau of Deruyts type and C is any virtual tableau of co-Deruyts type, all of shape λ.

A quite useful, but not trivial, fact is that left symmetrized bitableaux admit different equivalent definitions.

Proposition 9.2 ([11, 20, 21])**.** *For every* $\lambda \vdash n$ *and every* $T \in Tab(\mathcal{L})$, $U \in Tab(\mathcal{P})$, *with* $\mathrm{sh}(T) = \lambda = \mathrm{sh}(U)$, *we have*

$$\begin{aligned} \left(\boxed{T} \,\middle|\, U\right) &= \underline{TD_1} \; D_1C \; CU \\ &= TD \; \underline{DC} \; CU \\ &= TD \; DC_1 \; \underline{C_1U} \end{aligned}$$

where D, D_1 are virtual tableaux of Deruyts type and C, C_1 are virtual tableaux of co-Deruyts type, all of shape λ.

Informally speaking, the left symmetrized bitableau $\left(\boxed{T}\,\middle|\, U\right)$ is dual supersymmetric in the columns of T and supersymmetric in the rows of U: as a matter of fact, $\left(\boxed{T}\,\middle|\, U\right)$ is zero if T has a column repetition of positive letters or U has a row repetition of negative places.

Notice that the left symmetrized bitableau $\left(\boxed{T}\,\middle|\, U\right))$ can be also regarded as the result of applying a letter polarization operator to a bitableau: more specifically,

$$\left(\boxed{T}\,\middle|\, U\right) = \mathcal{D}_{TD}(D\,|\,U),$$

D any virtual letter tableau of Deruyts type, $\mathrm{sh}(T) = \mathrm{sh}(D)$.

Therefore, a left symmetrized bitableau is a linear combination of bitableaux, all of the same shape.

Remark 9.2. Let D be a *Deruyts tableau* of shape λ on the proper letter alphabet $\mathcal{L}$. The *left* symmetrized bitableau $\left(\boxed{D}\,\middle|\, T\right)$ satisfies the identity:

$$\left(\boxed{D}\,\middle|\, T\right) = \tilde{\lambda}_1! \cdots \tilde{\lambda}_q!\ (D\,|\,T),$$

for every $T \in Tab(\mathcal{P})$.

9.3 Doubly symmetrized bitableaux

For every $\lambda \vdash n$ and every $T \in Tab(\mathcal{L})$, $U \in Tab(\mathcal{P})$, with $\mathrm{sh}(T) = \lambda = \mathrm{sh}(U)$, we define the *doubly symmetrized bitableau* $\left(\boxed{T}\,\middle|\,\boxed{U}\right) \in Super_n[\mathcal{L}|\mathcal{P}]$ by setting

$$\left(\boxed{T}\,\middle|\,\boxed{U}\right) = \mathcal{D}_{TD_1}\ (D_1\,|\,D_2)\ {}_{D_2U}\mathcal{D}$$

where D_1 is any virtual (letter) tableau of Deruyts type, D_2 is any virtual (place) tableau of Deruyts type, both of shape λ.

9.4 Clebsch–Gordan–Capelli bases of $Super_n[\mathcal{L}|\mathcal{P}]$

Theorem 9.1 ([11, 20, 21]). *The following sets are bases for $Super_n[\mathcal{L}|\mathcal{P}]$:*

1. $\left\{\left(S\,\middle|\,\boxed{T}\right);\ S \in Stab_\lambda(\mathcal{L}),\ T \in Stab_\lambda(\mathcal{P}),\ \lambda \vdash n,\ \lambda \in H(\mathcal{L}) \cap H(\mathcal{P})\right\}$;
2. $\left\{\left(\boxed{S}\,\middle|\, T\right);\ S \in Stab_\lambda(\mathcal{L}),\ T \in Stab_\lambda(\mathcal{P}),\ \lambda \vdash n,\ \lambda \in H(\mathcal{L}) \cap H(\mathcal{P})\right\}$;

3. $\left\{\left(\boxed{S}\middle|\boxed{T}\right);\ S \in Stab_\lambda(\mathcal{L}),\ T \in Stab_\lambda(\mathcal{P}),\ \lambda \vdash n,\ \lambda \in H(\mathcal{L}) \cap H(\mathcal{P})\right\}$;

Proof. We limit ourselves to the set of standard right symmetrized bitableaux.

This set is linearly independent. Assume, for the purpose of contradiction, that there is a nontrivial linear relation

$$\sum c_{ST} \left(S \middle| \boxed{T}\right) = 0$$

among standard right symmetrized bitableaux. Let S_0 be minimal among the letter tableaux S such that $c_{ST} \neq 0$ for some place tableau T, and let T_0 be maximal among the place tableaux T such that $c_{S_0T} \neq 0$. Denote by λ_0 the shape of S_0 and T_0, let D_0 be a letter Deruyts tableau of shape λ_0, and let C_0 be a place co-Deruyts tableau of shape λ_0.

By applying the polarization monomials D_0S_0 and T_0C_0 on both sides of the nontrivial relation, we get the contradiction

$$\begin{aligned}
0 &= D_0S_0 \left(\sum_{S,T} c_{ST} \left(S \middle| \boxed{T}\right) \right) T_0C_0 \\
&= \sum_{S,T} c_{ST}\ D_0S_0\ SC\ \underline{CD}\ DT\ T_0C_0 \\
&= \sum_{S \le S_0,\ T \ge T_0} c_{ST}\ D_0S_0\ SC\ \underline{CD}\ DT\ T_0C_0 \\
&= \theta^{-+}_{S_0S_0}\ c_{S_0T_0}\ \theta^{-+}_{T_0T_0}\ D_0C\ \underline{CD}\ DC_0 \\
&= (-1)^{\binom{n}{2}} h_{\lambda_0}\ \theta^{-+}_{T_0T_0}\ c_{S_0T_0}\ \theta^{-+}_{S_0S_0}\ \underline{D_0C_0} \\
&\neq 0.
\end{aligned}$$

The linearly independent set of standard right symmetrized bitableaux spans $Super_n[\mathcal{L}|\mathcal{P}]$, since it has the same cardinality as the basis of standard bitableaux. □

10 Young–Capelli Symmetrizers and Orthonormal Generators

10.1 Young–Capelli symmetrizers

Let $\lambda \vdash n$ and let $S', S \in Tab(\mathcal{L})$, with $\mathrm{sh}(S') = \lambda = \mathrm{sh}(S)$. The product of letter bitableau polarization monomials $\mathcal{D}_{S'C} \cdot\ \mathcal{D}_{CD} \cdot\ \mathcal{D}_{DS}$ defines, by restriction, a linear operator

$$\gamma_n\left(S', \boxed{S}\right) = \mathcal{D}_{S'C} \cdot\ \mathcal{D}_{CD} \cdot\ \mathcal{D}_{DS}\ \in End_{\mathbb{K}}[Super_n[\mathcal{L}|\mathcal{P}]],$$

which is independent of the choice of the virtual tableau C of co-Deruyts type and of the virtual tableau D of Deruyts type, both of shape λ. The operator $\gamma_n\big(S', \boxed{S}\big)$ is called a *right Young–Capelli symmetrizer* [12, 20, 21].

Example 10.1. *Let* $|\alpha_i| = 0$ *and* $|\beta_i| = 1$.

$$\gamma_n\left(\begin{matrix} x & y \\ x & z \\ y & \end{matrix}\,,\ \boxed{\begin{matrix} x & y \\ y & z \\ z & \end{matrix}}\right)$$

$$= \mathcal{D}_{\begin{matrix} x & y \\ x & z \\ y & \end{matrix}\,,\ \begin{matrix} \alpha_1 & \alpha_1 \\ \alpha_2 & \alpha_2 \\ \alpha_3 & \end{matrix}} \cdot \mathcal{D}_{\begin{matrix} \alpha_1 & \alpha_1 \\ \alpha_2 & \alpha_2 \\ \alpha_3 & \end{matrix}\,,\ \begin{matrix} \beta_1 & \beta_2 \\ \beta_1 & \beta_2 \\ \beta_1 & \end{matrix}} \cdot \mathcal{D}_{\begin{matrix} \beta_1 & \beta_2 \\ \beta_1 & \beta_2 \\ \beta_1 & \end{matrix}\,,\ \begin{matrix} x & y \\ y & z \\ z & \end{matrix}}$$

By the metatheoretic significance of the method of virtual variables (see, e.g., Subsection 6.1), the crucial fact is that $\gamma_n\big(S', \boxed{S}\big)$ belongs to the subalgebra $\mathcal{B}_n$, the algebra generated by proper letter polarizations. In plain words, even though the operator $\gamma_n\big(S', \boxed{S}\big)$ is defined by using virtual variables, it admits presentations involving only superpolarizations between proper letters.

A right Young–Capelli symmetrizer is called *standard* when both its tableaux are standard.

In an analogous way, we define the *left Young–Capelli symmetrizers*. Let $\lambda \vdash n$ and let $S', S \in Tab(\mathcal{L})$, with $\mathrm{sh}(S') = \lambda = \mathrm{sh}(S)$. The product of letter bitableau polarization monomials $\mathcal{D}_{S'D} \cdot \ \mathcal{D}_{DC} \cdot \ \mathcal{D}_{CS}$ defines, by restriction, a linear operator

$$\gamma_n\Big(\boxed{S'}, S\Big) = \mathcal{D}_{S'D} \cdot \ \mathcal{D}_{DC} \cdot \ \mathcal{D}_{CS} \ \in End_{\mathbb{K}}[Super_n[\mathcal{L}|\mathcal{P}]],$$

which is independent of the choice of the virtual tableau D of Deruyts type and of the virtual tableau C of co-Deruyts type, both of shape λ.

10.2 The Triangularity Theorem

Theorem 10.1 ([12, 20, 21])**.** *The action of standard right Young–Capelli symmetrizers on standard right symmetrized bitableaux is given by*

$$\gamma_n\Big(S', \boxed{S}\Big)\Big(T \,\Big|\, \boxed{U}\Big) = \begin{cases} (-1)^{\binom{n}{2}} h_\lambda \theta_{ST}^{-+} \Big(S' \,\Big|\, \boxed{U}\Big), & \mathrm{sh}(S) = \mathrm{sh}(T) = \lambda, \\ 0 & \textit{otherwise}, \end{cases}$$

where θ_{ST}^{-+} *are the symmetry transition coefficients from Lemma 8.1, thus integers satisfying the triangularity conditions* $\theta_{ST}^{-+} = 0$ *unless* $S \geq T$, $\theta_{ST}^{-+} \neq 0$ *for* $S = T$, *and* h_λ *is a positive integer which depends on the shape* λ. *Furthermore, the integer* h_λ *equals the product of the hook lengths of the shape* λ.

Proof. (Sketch) [21] First of all, by definition, we have

$$\gamma_n\Big(S', \boxed{S}\Big)\Big(T \,\Big|\, \boxed{U}\Big) = S'C_1 \ \ (C_1D_1 \ \ (D_1S \quad \underline{TC_2}\ ) \ C_2D_2) \ \ D_2U.$$

We note that:

$$\text{if } D_1S\ \underline{TC_2} \neq 0 \text{ then } \operatorname{sh}(D_1) \geq \operatorname{sh}(C_2);$$
$$\text{if } C_1D_1\ D_1S\ \underline{TC_2}\ C_2D_2 \neq 0 \text{ then } \operatorname{sh}(C_1) \leq \operatorname{sh}(D_2).$$

Thus, the whole expression is nonzero only if S and T have the same shape, say $\operatorname{sh}(S) = \operatorname{sh}(T) = \lambda$. Under this condition, we have, using Lemmas 8.1, 8.2 and 8.3 from Subsection 8.2,

$$\begin{aligned}
\gamma_n\Big(S', \boxed{S}\Big)\Big(T \,\Big|\, \boxed{U}\Big) &= S'C_1\ C_1D_1 \ \ \big(D_1S\ \underline{TC_2}\big) \ \ C_2D_2\ D_2U \\
&= \theta^{-+}_{ST} S'C_1\ (C_1D_1\ \underline{D_1C_2}\ C_2D_2)\ D_2U \\
&= (-1)^{\binom{n}{2}} h_\lambda \theta^{-+}_{ST} S'C_1\ \underline{C_1D_2}\ D_2U \\
&= (-1)^{\binom{n}{2}} h_\lambda \theta^{-+}_{ST} \Big(S' \,\Big|\, \boxed{U}\Big).
\end{aligned}$$

The coefficients θ^{-+}_{ST} and h_λ satisfy the triangularity and nondegeneracy conditions by Lemmas 8.2 and 8.3.

The last assertion will be proved in Subsection 18.3. □

Given any linear extension of the partial order defined above, the matrix $[\theta^{-+}_{S,T}]$ is lower triangular with nonzero integral diagonal entries; the matrix

$$[\varrho^{-+}_{S,T}] = [\theta^{-+}_{S,T}]^{-1}$$

is called the *Rutherford matrix.*

Remark 10.1. The action of standard left Young–Capelli symmetrizers on standard left symmetrized bitableaux is given by

$$\gamma_n\Big(\boxed{S'}, S\Big)\Big(\boxed{T} \,\Big|\, U\Big) = \begin{cases} (-1)^{\binom{n}{2}} h_\lambda \theta^{+-}_{ST} \Big(\boxed{S'} \,\Big|\, U\Big), & \operatorname{sh}(S) = \operatorname{sh}(T) = \lambda, \\ 0 & \textit{otherwise}, \end{cases}$$

where θ^{+-}_{ST} are the symmetry transition coefficients defined in Subsection 8.2, thus integers satisfying the triangularity conditions $\theta^{+-}_{ST} = 0$ unless $S \leq T$, $\theta^{+-}_{ST} \neq 0$ for $S = T$, and h_λ is the product of the hook lengths of the shape λ.

10.3 Orthonormal generators

For every $S', S \in Stab(\mathcal{L})$, with $\mathrm{sh}(S') = \mathrm{sh}(S) = \lambda \vdash n$, we define the *right orthonormal generator* $Y_n\big(S', \boxed{S}\big) \in \mathcal{B}_n$ by setting [12]

$$Y_n\Big(S', \boxed{S}\Big) = \frac{(-1)^{\binom{n}{2}}}{h_\lambda} \sum_{T \in Stab(\mathcal{L})} \varrho^{-+}_{ST}\ \gamma_n\Big(S', \boxed{T}\Big).$$

From Theorem 10.1 and the definitions above, the next result follows immediately.

Theorem 10.2. *The action of the right orthonormal generators on the standard right symmetrized bitableaux is given by*

$$Y_n\Big(S', \boxed{S}\Big)\Big(T \,\Big|\, \boxed{U}\Big) = \delta_{S,T}\Big(S' \,\Big|\, \boxed{U}\Big).$$

Therefore, the orthonormal generators

$$Y_n\Big(S', \boxed{S}\Big) \quad S,\ S' \in Stab_\lambda(\mathcal{L}), \quad \lambda \in H(\mathcal{L}) \cap H(\mathcal{P})$$

form a $\mathbb{K}$-linear basis of the algebra $\mathcal{B}_n$.

Remark 10.2. In an analogous way, we can define the *left orthonormal generators* $Y_n\big(\boxed{S'}, S\big)$, which act orthogonally on left symmetrized bitableaux.

10.4 Factorization properties

From the operator triangularity in Lemma 8.5, we have the following result.

Proposition 10.1 (Factorization theorem, [12, 20, 69])**.** *For any standard tableau T, $\mathrm{sh}(T) = \lambda$, with no letter in common with the tableau C and D, the Young–Capelli symmetrizer $\gamma_n\big(S', \boxed{S}\big)$ can be factorized as follows:*

$$\gamma_n\Big(S', \boxed{S}\Big) = \theta^{-+}_{TT}\ S'C\ CT\ TD\ DS.$$

Remark 10.3. Note that, if the tableau T has no letter in common with the tableaux S' and S, then the operator $S'C\ CT$ can be expressed as the product of the Capelli operators relative to pairs of corresponding rows of S' and T; analogously the operator $TD\ DS$ can be expressed as the product of the Capelli operators relative to pairs of corresponding columns of T and S.

10.5 Place operators

Interchanging the roles of letters and places, we can define the place Young–Capelli symmetrizers

$$\left(T, \boxed{T'}\right)_n \gamma, \qquad \left(\boxed{T}, T'\right)_n \gamma,$$

and the corresponding place orthonormal generators

$$\left(T, \boxed{T'}\right)_n Y, \qquad \left(\boxed{T}, T'\right)_n Y,$$

which act orthogonally on the corresponding symmetrized bitableaux

$$\left(U, \boxed{V}\right), \qquad \left(\boxed{U}, V\right).$$

11 Schur and Weyl Modules

From the results of Sections 9 and 10, one easily infers the following results.

11.1 Schur modules

Let $\lambda \vdash n$ be a partition of n, $\lambda \in H(\mathcal{L}) \cap H(\mathcal{P})$. Given a standard tableau $T \in Stab_\lambda(\mathcal{P})$, the subspace $\mathcal{S}_{\lambda T}$ generated by the set

$$\left\{ \left(U \,\middle|\, \boxed{T}\right);\ U \in Tab_\lambda(\mathcal{L}) \right\}$$

is called the (letter-)*Schur module* parametrized by the place-tableau T.

Proposition 11.1. *The Schur modules have the following properties.*

1. $\mathcal{S}_{\lambda T}$ *is a* $pl(\mathcal{L})$*-invariant subspace of* $Super_n[\mathcal{L}|\mathcal{P}]$;
2. *the set* $\left\{ \left(S \,\middle|\, \boxed{T}\right);\ S \in Stab_\lambda(\mathcal{L}) \right\}$ *is a* $\mathbb{K}$*-linear basis of* $\mathcal{S}_{\lambda T}$;
3. *the set* $\left\{ \left(\boxed{S} \,\middle|\, \boxed{T}\right);\ S \in Stab_\lambda(\mathcal{L}) \right\}$ *is a* $\mathbb{K}$*-linear basis of* $\mathcal{S}_{\lambda T}$;
4. $\mathcal{S}_{\lambda T}$ *is an irreducible submodule;*
5. $\mathcal{S}_{\lambda T}$ *and* $\mathcal{S}_{\lambda' T'}$ *are isomorphic* $pl(\mathcal{L})$*-modules if and only if* $\lambda = \lambda'$.

11.2 Weyl modules

Let $\lambda \vdash n$ be a partition of n, $\lambda \in H(\mathcal{L}) \cap H(\mathcal{P})$. Given a standard tableau $T \in Stab_\lambda(\mathcal{P})$, the subspace $\mathcal{W}_{\lambda T}$ generated by the set

$$\left\{ \left(\boxed{U} \middle| T \right);\ U \in Tab_\lambda(\mathcal{L}) \right\}$$

is called the (letter-) *Weyl module* parametrized by the place-tableau T.

Proposition 11.2. *The Weyl modules have the following properties.*

1. $\mathcal{W}_{\lambda T}$ *is a* $pl(\mathcal{L})$*-invariant subspace of* $Super_n[\mathcal{L}|\mathcal{P}]$;
2. *the set* $\left\{ \left(\boxed{S} \middle| T \right);\ S \in Stab_\lambda(\mathcal{L}) \right\}$ *is a* $\mathbb{K}$*-linear basis of* $\mathcal{W}_{\lambda T}$;
3. $\mathcal{W}_{\lambda T}$ *is an irreducible submodule;*
4. $\mathcal{W}_{\lambda T}$ *and* $\mathcal{W}_{\lambda' T'}$ *are isomorphic* $pl(\mathcal{L})$*-modules if and only if* $\lambda = \lambda'$.

11.3 The Schur–Weyl correspondence

Proposition 11.3. *Let* $T_1, T_2 \in Stab_\lambda(\mathcal{P})$ *and consider the (negative) place Capelli operator*

$$\mathcal{S}_{\lambda T_2} \leftarrow \mathcal{W}_{\lambda T_1} \ : T_1 D\ DT_2.$$

- $\left(\boxed{U} \middle| T_1 \right) T_1 D\ DT_2 = K_{T_1} \left(\boxed{U} \middle| \boxed{T_2} \right), \quad K_{T_1} \in \mathbb{K}^*,$ *for all* $U \in Tab_\lambda(\mathcal{L})$;
- $T_1 D\ DT_2$ *is a* $pl(\mathcal{L})$*-equivariant isomorphism.*

Proposition 11.4. *Let* $T_1, T_2 \in Stab_\lambda(\mathcal{P})$ *and consider the (positive) place Capelli operator*

$$\mathcal{W}_{\lambda T_1} \leftarrow \mathcal{S}_{\lambda T_2} \ : T_1 C\ CT_2.$$

- $\left(\boxed{U} \middle| \boxed{T_2} \right) T_2 C\ CT_1 = K_{T_2} \left(\boxed{U} \middle| T_1 \right), \quad K_{T_2} \in \mathbb{K}^*,$ *for all* $U \in Tab_\lambda(\mathcal{L})$;
- $T_2 C\ CT_1$ *is a* $pl(\mathcal{L})$*-equivariant isomorphism.*

12 Decomposition Theorems

Theorem 12.1 ([11, 20])**.** *We have the following complete decompositions of the semisimple $pl(\mathcal{L})$-module $Super_n[\mathcal{L}|P]$:*

$$\begin{aligned}
Super_n[\mathcal{L}|P] &= \bigoplus_{\substack{\lambda\in H(\mathcal{L})\cap H(\mathcal{P})\\ \lambda\vdash n}} \quad \bigoplus_{\substack{T\in Stab(\mathcal{P})\\ \mathrm{sh}(T)=\lambda}} S_{\lambda T}\\
&= \bigoplus_{\substack{\lambda\in H(\mathcal{L})\cap H(\mathcal{P})\\ \lambda\vdash n}} \quad \bigoplus_{\substack{T\in Stab(\mathcal{P})\\ \mathrm{sh}(T)=\lambda}} \left\langle \left(S \,\middle|\, \boxed{T}\right);\quad S \in Stab(\mathcal{L})\right\rangle\\
&= \bigoplus_{\substack{\lambda\in H(\mathcal{L})\cap H(\mathcal{P})\\ \lambda\vdash n}} \quad \bigoplus_{\substack{T\in Stab(\mathcal{P})\\ \mathrm{sh}(T)=\lambda}} \left\langle \left(\boxed{S} \,\middle|\, \boxed{T}\right);\quad S \in Stab(\mathcal{L})\right\rangle;
\end{aligned}$$

$$\begin{aligned}
Super_n[\mathcal{L}|\mathcal{P}] &= \bigoplus_{\substack{\lambda\in H(\mathcal{L})\cap H(\mathcal{P})\\ \lambda\vdash n}} \quad \bigoplus_{\substack{T\in Stab(\mathcal{P})\\ \mathrm{sh}(T)=\lambda}} W_{\lambda T}\\
&= \bigoplus_{\substack{\lambda\in H(\mathcal{L})\cap H(\mathcal{P})\\ \lambda\vdash n}} \quad \bigoplus_{\substack{T\in Stab(\mathcal{P})\\ \mathrm{sh}(T)=\lambda}} \left\langle \left(\boxed{S} \,\middle|\, T\right);\quad S \in Stab(\mathcal{L})\right\rangle;
\end{aligned}$$

where the outer sum indicates the isotypic decomposition of the semisimple module, and the inner sum describes a complete decomposition of each isotypic component into irreducible submodules.

Remark 12.1. We recall (see Subsection 8.4) that the basis of standard bitableaux gives rise to a $pl(\mathcal{L})$-invariant filtration

$$Super_n[\mathcal{L}|\mathcal{P}] = V_0 \supset V_1 \ldots \supset V_f = (0),$$

where $V_i = \langle (S\,|\,T);\ S \text{ standard},\ T > T_i\rangle$. Notice that each term V_i in the invariant filtration admits the Weyl module $\mathcal{W}_{\lambda T_i}$, $\mathrm{sh}(T_i) = \lambda$, as a complementary invariant subspace in the preceding term V_{i-1}:

$$V_{i-1} = V_i \oplus \mathcal{W}_{\lambda T_i}.$$

Therefore, the standard basis theorem must be regarded as a "weak form" of the Clebsch–Gordan–Capelli basis theorem.

Theorem 12.2 ([12, 20])**.** *We have the following complete decomposition for the operator algebra $\mathcal{B}_n$ generated by the letter polarization operators acting over $Super_n[\mathcal{L}|\mathcal{P}]$:*

$$\mathcal{B}_n = \bigoplus_{\substack{\lambda\in H(\mathcal{L})\cap H(\mathcal{P})\\ \lambda\vdash n}} \quad \bigoplus_{\substack{S\in Stab(\mathcal{L})\\ \mathrm{sh}(S)=\lambda}} \left\langle Y_n\left(S', \boxed{S}\right);\quad S' \in Stab(\mathcal{L})\right\rangle,$$

where the outer sum indicates the isotypic decomposition of the semisimple algebra, and the inner sum describes a complete decomposition of each simple subalgebra into minimal left ideals.

Corollary 12.1 (Structure Theorem)**.**

$$\mathcal{B}_n \cong \bigoplus_{\substack{\lambda \in H(\mathcal{L}) \cap H(\mathcal{P}) \\ \lambda \vdash n}} M_{f_\lambda(\mathcal{L})}(\mathbb{K}),$$

where $M_{f_\lambda(\mathcal{L})}(\mathbb{K})$ is the full matrix algebra of square matrices of order $f_\lambda(\mathcal{L})$.

A completely parallel theory holds for the operator algebra ${}_n\mathcal{B}$ generated by the proper place polarization operators acting over $Super_n[\mathcal{L}|\mathcal{P}]$.

It should be clear from the preceding discussion how to define *letter left* orthonormal generators $Y\big(\boxed{S'}, S\big)$ and *place left* and *right* orthonormal generators $\big(\boxed{T'}, T\big)Y$ and $\big(T', \boxed{T}\big)Y$.

We have the following "symmetric" version of Theorem 10.2.

Proposition 12.1. *Let $S', S, U \in Stab(\mathcal{L})$ and $V, T, T' \in Stab(\mathcal{P})$. Then:*

$$Y\Big(S', \boxed{S}\Big)\Big(U, \boxed{V}\Big)\Big(T, \boxed{T'}\Big)Y = \delta_{SU}\delta_{VT}\Big(S', \boxed{T'}\Big),$$
$$Y\Big(\boxed{S'}, S\Big)\Big(\boxed{U}, V\Big)\Big(\boxed{T}, T'\Big)Y = \delta_{SU}\delta_{VT}\Big(\boxed{S'}, T'\Big).$$

As a consequence, we get the double centralizer theorem.

Theorem 12.3 (The Double Centralizer Theorem [12])**.** *The algebras*

$$\mathcal{B}_n,\ {}_n\mathcal{B} \subset End_{\mathbb{K}}(Super_n[\mathcal{L}|\mathcal{P}])$$

are the centralizers of each other.

13 The Natural Form of Irreducible Matrix Representations of Schur Superalgebras

Let $\mathcal{L}, \mathcal{P}$ be arbitrary finite signed alphabets, and $\lambda \vdash n$ a partition of n. In the following we will write $S_1, S_2, \ldots, S_{p_\lambda}$ to mean the list of all standard tableaux of shape λ over $\mathcal{L}$, sorted with respect to the linear order defined in Subsection 5.5. For sake of simplicity, we will write

$$\theta^\lambda_{ij} \quad \text{in place of} \quad \theta^{-+}_{S_i S_j}, \qquad \text{and} \quad \varrho^\lambda_{ij} \quad \text{in place of} \quad \varrho^{-+}_{S_i S_j}.$$

For every operator $G \in \mathcal{B}_n = \oplus_{\lambda \vdash n} \mathcal{B}_\lambda$, we will denote by G_λ its component in the simple subalgebra B_λ (Theorem 12.2).

On the one hand, we have

$$Y_n\Big(S_h, \boxed{S_h}\Big)\ G\ Y_n\Big(S_k, \boxed{S_k}\Big) = d^\lambda_{hk}(G)\ Y_n\Big(S_h, \boxed{S_k}\Big), \qquad d^\lambda_{hk}(G) \in \mathbb{K};$$

notice that the coefficients $d^\lambda_{hk}(G)$ are precisely the coefficients that appear in the expansion of G_λ with respect to the basis $\big\{Y_n\big(S_h, \boxed{S_k}\big);\ h, k = 1, 2, \ldots p_\lambda\big\}$ of $\mathcal{B}_\lambda$.

On the other hand, given any standard tableau T of shape λ over $\mathcal{P}$, we have

$$G\Big(S_k \Big| \boxed{T}\Big) = G_\lambda\Big(S_k \Big| \boxed{T}\Big) = \sum_h c^T_{hk}(G)\ \Big(S_h \Big| \boxed{T}\Big), \qquad c^T_{hk}(G) \in \mathbb{K}.$$

Clearly, we have:

Proposition 13.1. *For every standard tableau T over $\mathcal{P}$ with $\mathrm{sh}(T) = \lambda$, we have*

$$d^\lambda_{hk}(G_\lambda) = d^\lambda_{hk}(G) = c^T_{hk}(G) = c^T_{hk}(G_\lambda),$$

for every $G \in \mathcal{B}_n$, and for every $h, k = 1, 2, \ldots p_\lambda$.

In the following, we write c^λ_{hk} in place of c^T_{hk}.

We remark that, for every operator $G \in \mathcal{B}_n$, and for every tableau $S_j \in Stab(\mathcal{L})$ of shape λ, the following identity holds:

$$DS_h\ G\ \underline{S_j C} = \ \theta^\lambda_{hj}(G)\ \underline{DC},$$

where $\theta^\lambda_{hj}(G)$ is a uniquely determined scalar coefficient (here, as usual, D denotes any virtual letter Deruyts tableau of shape λ and C denotes any virtual place co-Deruyts tableau of shape λ).

Notice that

$$[\theta^\lambda_{ij}] = [\theta^\lambda_{ij}(I)], \qquad [\varrho^\lambda_{ij}] = [\theta^\lambda_{ij}(I)]^{-1},$$

where I denotes the identity in $\mathcal{B}_n$.

Theorem 13.1 ([21])**.** *We have:*

$$c^\lambda_{ij}(G) = \sum_h \varrho^\lambda_{ih}\ \theta^\lambda_{hj}(G),$$

or, in matrix form:

$$C^\lambda(G) = \Theta^\lambda(I)^{-1} \times \Theta^\lambda(G).$$

Proof. (Sketch) We start from the definition of the coefficients $c^{\lambda}_{hk}(G)$:

$$G\Big(S_k\,\Big|\,\boxed{T}\Big) = \sum_{h=1}^{p_\lambda} c^{\lambda}_{hk}(G)\ \Big(S_h\,\Big|\,\boxed{T}\Big);$$

we apply the Young–Capelli symmetrizer $\gamma_n\big(S_l\,\big|\,\boxed{S_l}\big)$ to the left-hand side:

$$\begin{aligned}\gamma_n\Big(S_l\,\Big|\,\boxed{S_l}\Big)\ G\ \Big(S_k\,\Big|\,\boxed{T}\Big) &= S_lC\ CD\ DS_l\ G\ \underline{S_kC}\ CD\ DT\\ &= \theta^{\lambda}_{lk}(G)\ S_lC\ CD\ \underline{DC}\ CD\ DT\\ &= (-1)^{\binom{n}{2}}h_\lambda\ \theta^{\lambda}_{lk}(G)\ S_lC\ \underline{CD}\ DT\\ &= (-1)^{\binom{n}{2}}h_\lambda\ \theta^{\lambda}_{lk}(G)\ \Big(S_l\,\Big|\,\boxed{T}\Big);\end{aligned}$$

we apply the Young–Capelli symmetrizer $\gamma_n\big(S_l\,\big|\,\boxed{S_l}\big)$ to the right-hand side:

$$\begin{aligned}\gamma_n\Big(S_l\,\Big|\,\boxed{S_l}\Big)\ \sum_{h=1}^{p_\lambda} c^{\lambda}_{hk}(G)\ \Big(S_h\,\Big|\,\boxed{T}\Big) &= \sum_{h=1}^{p_\lambda} c^{\lambda}_{hk}(G)\ \gamma_n\Big(S_l\,\Big|\,\boxed{S_l}\Big)\ \Big(S_h\,\Big|\,\boxed{T}\Big)\\ &= (-1)^{\binom{n}{2}}h_\lambda\ \sum_{h=1}^{p_\lambda} \theta^{\lambda}_{lh}\ c^{\lambda}_{hk}(G)\ \Big(S_l\,\Big|\,\boxed{T}\Big).\end{aligned}$$

In the end, we have

$$\theta^{\lambda}_{lk}(G) = \sum_{h=1}^{p_\lambda} \theta^{\lambda}_{lh}\ c^{\lambda}_{hk}(G).$$

$\square$

For every $\lambda \vdash n$ such that $\lambda \in H(\mathcal{L}) \cap H(\mathcal{P})$ and every standard place tableau T of shape λ, the module structure $\mathcal{U}(pl(\mathcal{L}))\cdot\mathcal{S}_T$ on the irreducible Schur module $\mathcal{S}_T$ induces a surjective algebra morphism

$$\nu_T :\ \mathcal{U}(pl(\mathcal{L})) \to End_{\mathbb{K}}[\mathcal{S}_T];$$

by choosing the basis of the standard symmetrized bitableaux $\big(S_i\,\big|\,\boxed{T}\big)$ in $\mathcal{S}_T$, the morphism ν_T induces an irreducible matrix representation

$$\overline{\nu}_T :\ \mathcal{U}(pl(\mathcal{L})) \to M_{p_\lambda}$$

where, for every $\mathcal{G} \in \mathcal{U}(pl(\mathcal{L}))$,

$$\overline{\nu}_T(\mathcal{G}) = [c^{\lambda}_{ij}(\nu_T(\mathcal{G}))] = [c^{\lambda}_{ij}(G)],$$

where $G = \nu_T(\mathcal{G})$. Therefore, this irreducible representation has the matrix form described in the preceding theorem.

Part III

Applications

14 Decomposition of Tensor Products of Spaces of Symmetric and Skew-Symmetric Tensors

Let V be a vector space of dimension m, and let $\{x_1, \ldots, x_m\}$ be a basis of V. Consider a pair of multiindices

$$I = (i_1, \ldots, i_s) \in \mathbb{Z}^{+s}, \quad J = (j_1, \ldots, j_t) \in \mathbb{Z}^{+t},$$

and set $|I| = i_1 + \cdots + i_s$, $|J| = j_1 + \cdots + j_t$. The space

$$W^{I,J} = \Lambda^{i_1}(V) \otimes \cdots \otimes \Lambda^{i_s}(V) \otimes Sym^{j_1}(V) \otimes \cdots \otimes Sym^{j_t}(V)$$

is a $gl(m)$-module as well as a $GL(m)$-module, and, by a standard argument, the operator algebras induced by the action of $gl(m)$ and $GL(m)$ are the same.

By way of application, we will derive an explicit complete decomposition result for the $GL(m)$-module $W^{I,J}$.

Consider the negatively signed letter alphabet $\mathcal{L} = \mathcal{L}_1 = \{x_1, \ldots, x_m\}$ and the signed place alphabet $\mathcal{P} = \mathcal{P}_0 \cup \mathcal{P}_1$, where

$$\mathcal{P}_0 = \{\varphi_1, \ldots, \varphi_s\}, \quad \mathcal{P}_1 = \{\psi_1, \ldots, \psi_t\},$$

and the order is given by

$$\varphi_1 < \ldots < \varphi_s < \psi_1 < \ldots < \psi_t.$$

Let $Super^{I,J}[\mathcal{L}|\mathcal{P}]$ be the "homogeneous" subspace of $Super[\mathcal{L}|\mathcal{P}]$ spanned by all the monomials of content (I, J), that is, of content i_h in each positive place φ_h, for every $h = 1, \ldots, s$, and of content $j_{k'}$ in each negative place ψ_k, for every $k = 1, \ldots, t$.

Clearly, $Super^{I,J}[\mathcal{L}|\mathcal{P}]$ is a $gl(m)$-module as well as a $GL(m)$-module, and again, by a standard argument (see, e.g., Subsection 4.5), the operator algebras induced by the action of $gl(m)$ and $GL(m)$ are the same.

Let

$$F : W^{I,J} \to Super^{I,J}[\mathcal{L}|\mathcal{P}]$$

be the map such that

$$\begin{aligned} F(x_{p_{11}} \wedge \cdots \wedge x_{p_{1i_1}} \otimes \cdots \otimes x_{p_{s1}} \wedge \cdots \wedge x_{p_{si_s}} \otimes x_{q_{11}} \cdots x_{q_{1j_1}} \otimes \cdots \otimes x_{q_{t1}} \cdots x_{q_{tj_t}}) \\ = (x_{p_{11}} \cdots x_{p_{1i_1}} \,|\, \varphi_1^{i_1}) \cdots (x_{p_{s1}} \cdots x_{p_{si_s}} \,|\, \varphi_s^{i_s})(x_{q_{11}} \,|\, \psi_1) \cdots (x_{q_{1j_1}} \,|\, \psi_1) \\ \cdots (x_{q_{t1}} \,|\, \psi_t) \cdots (x_{q_{tj_t}} \,|\, \psi_t). \end{aligned}$$

Proposition 14.1. *The map $F : W^{I,J} \to Super^{I,J}[\mathcal{L}|\mathcal{P}]$ is a $GL(m)$-equivariant isomorphism.*

By specializing Theorem 12.1, we get:

Corollary 14.1. *We have the following complete $GL(m)$-module decomposition*

$$W^{I,J} = \bigoplus_{\substack{\lambda \vdash |I|+|J| \\ \lambda \in H(\mathcal{L}) \cap H(\mathcal{P})}} \bigoplus_{T \in Stab_\lambda^{I,J}(\mathcal{P})} F^{-1}[S_{\lambda T}],$$

where the outer sum indicates the isotypic decomposition of the semisimple module, and the inner sum, which describe a complete decomposition of each isotypic component into $GL(m)$-irreducible Schur modules, is over the set $Stab_\lambda^{I,J}(\mathcal{P})$ of all standard place tableaux of content (I, J).

We remark that a weaker version of the preceding result — just up to isomorphism — could be derived by iterated applications of Pieri's rule.

15 Letterplace Algebras, Highest Weight Vectors and $sl_m(\mathbb{C})$-Irreducible Modules

15.1 Representations of $sl_m(\mathbb{C})$: basic definitions and results

Let $sl_m(\mathbb{C})$ be the special linear Lie algebra of $m \times m$ *traceless* matrices with complex entries. Let $\mathbf{h} \subseteq sl_m(\mathbb{C})$ be the *Cartan subalgebra* of all diagonal matrices.

For every $i = 1, \ldots, m$, let ε_i be the linear function

$$\varepsilon_i : gl_m(\mathbb{C}) \to \mathbb{C}$$

such that

$$\varepsilon_i(M) = m_{ii},$$

for every matrix $M = (m_{ij})_{i,j=1,\ldots,m} \ \in gl_m(\mathbb{C})$.

Thus, $\mathbf{h}^* = \mathbb{C}\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m\}/\langle \varepsilon_1 + \varepsilon_2 + \ldots + \varepsilon_m \rangle$. We often write ε_i for the image of ε_i in $\mathbf{h}^*$.

Consider the Cartan decomposition

$$sl_m(\mathbb{C}) = \mathbf{h} \oplus \langle E_{i,j}; \quad i \neq j \rangle;$$

the one-dimensional spaces $\langle E_{i,j} \rangle$ are invariant subspaces of $sl_m(\mathbb{C})$ with respect to the adjoint action of the subalgebra $\mathbf{h}$, since $[H, E_{i,j}] = (\varepsilon_i - \varepsilon_j)(H) E_{i,j}$ for every $H \in \mathbf{h}$.

The spaces $\langle E_{i,j} \rangle$ are called the *root spaces* and the elements $\varepsilon_i - \varepsilon_j \in \mathbf{h}^*$ are called the *roots* of the Lie algebra $sl_m(\mathbb{C})$.

As usual, let

$$\{\varepsilon_i - \epsilon_j ; i < j\}$$

be the set of *positive* roots of $sl_m(\mathbb{C})$; thus, $\{\langle E_{i,j} \rangle ; i < j\}$ is the set of positive root spaces, and $\{\langle E_{i,j} \rangle ; i > j\}$ is the set of negative root spaces.

Let $\mathcal{V}$ be a finite-dimensional representation of $sl_m(\mathbb{C})$ and φ an element of $\mathbf{h}^*$; a non-trivial subspace $\mathcal{V}(\varphi) = \{v \in \mathcal{V}; H(v) = \varphi(H)v, \ for\ every\ H \in \mathbf{h}\}$ is called a *weight subspace* of the representation $\mathcal{V}$, and φ is called the *weight* of $\mathcal{V}(\varphi)$.

It is easy to see that $\mathcal{V}$ decomposes into the (finite) direct sum of its weight spaces, in symbols $\mathcal{V} = \oplus_\varphi \mathcal{V}(\varphi)$.

A *weight vector* is a vector $v \in \mathcal{V}$ which belongs to a weight subspace. A *highest weight vector* is a weight vector which is annihilated by the action of any positive root space.

Proposition 15.1 (see, e.g., [40]). *1. Every finite-dimensional representation $\mathcal{V}$ of $sl_m(\mathbb{C})$ possesses a highest weight vector.*

2. *The subspace $\mathcal{W}$ generated by the images of a highest weight vector v under successive applications of negative root spaces is an irreducible subrepresentation.*

3. *An irreducible representation possesses a unique highest weight vector, up to scalar factors.*

4. *Two irreducible representations $\mathcal{W}$ and $\mathcal{W}'$ of $sl_m(\mathbb{C})$ are isomorphic if and only if they have the same highest weight (as an element of $\mathbf{h}^*$).*

It follows from assertion 3) of the above proposition that the *highest weight* of an irreducible $sl_m(\mathbb{C})$-representation may be unambiguously defined as the weight of its highest weight vector.

15.2 $sl_m(\mathbb{C})$-irreducible modules and $gl_m(\mathbb{C})$-irreducible modules

We recall that $gl_m(\mathbb{C}) = sl_m(\mathbb{C}) \oplus \mathbb{C}I_m$, I_m the identity $m \times m$ matrix (this is an instance of the so-called *Levi decomposition theorem*). The subalgebra $\mathbb{C}I_m$ is the *radical ideal* of $gl_m(\mathbb{C})$ and $sl_m(\mathbb{C})$ is its semisimple part. As a matter of fact, $sl_m(\mathbb{C})$ is a *simple* Lie algebra and it is also an ideal of $gl_m(\mathbb{C})$; furthermore, the non-trivial ideals of $gl_m(\mathbb{C})$ are precisely $\mathbb{C}I_m$ and $sl_m(\mathbb{C})$.

The following assertions are special instances of general results (see, e.g., [40], [42], [88]).

Proposition 15.2 (see, e.g., [40]). *1. Every irreducible representation $\mathcal{V}$ of $gl_m(\mathbb{C})$ is of the form $\mathcal{V} = \mathcal{V}_0 \otimes L$, where $\mathcal{V}_0$ is an irreducible representation of $sl_m(\mathbb{C})$ (i.e., a representation of $gl_m(\mathbb{C})$ that is trivial on the radical $\mathbb{C}I_m$), and L is a one-dimensional representation of $gl_m(\mathbb{C})$.*

2. *Since the subalgebra $sl_m(\mathbb{C})$ is a simple one, it acts on L in the trivial way, and, then, each element of $gl_m(\mathbb{C})$ acts on L just by multiplying by a complex coefficient.*

3. *It follows from the preceding item that any irreducible representation of $gl_m(\mathbb{C})$ restricts to an irreducible representation of $sl_m(\mathbb{C})$, and any irreducible representation of $sl_m(\mathbb{C})$ extends to an irreducible representation of $gl_m(\mathbb{C})$.*

Claim 15.1. *From the preceding proposition, it follows that, if $\mathcal{W}$ and $\mathcal{W}'$ are isomorphic $sl_m(\mathbb{C})$-modules, it is not in general true that $\mathcal{W}$ and $\mathcal{W}'$ are isomorphic $gl_m(\mathbb{C})$-modules.*

15.3 Letterplace algebras and representations of $sl_m(\mathbb{C})$

Let $\mathcal{L} = \mathcal{L}_1 = \{x_1, \ldots, x_m\}$ be a negatively signed alphabet of letters, and let $\mathcal{P}$ be any finite $\mathbb{Z}_2$-graded alphabet of places.

Set $\mathbb{K} = \mathbb{C}$.

In the notation of Subsection 4.3, we have

$$Super[\mathcal{L}|\mathcal{P}] \simeq Sym[V \otimes W_1] \otimes \Lambda[V \otimes W_0],$$

and

$$pl(\mathcal{L}) = pl(V) = gl_m(\mathbb{C}),$$

the usual Lie algebra of $m \times m$ matrices.

We will describe the connection between our combinatorial approach and the classical theory (highest weight vectors) of the Lie module

$$sl_m(\mathbb{C}) \cdot Super[\mathcal{L}|\mathcal{P}] \simeq sl_m(\mathbb{C}) \cdot Sym[V \otimes W_1] \otimes \Lambda[V \otimes W_0].$$

Recall that any homogeneous component $Super_n[\mathcal{L}|\mathcal{P}]$ is a $gl_m(\mathbb{C})$-submodule, and that the action of $gl_m(\mathbb{C})$ is implemented by letter polarization operators; since $\mathcal{L}$ is a trivially $\mathbb{Z}_2$-graded alphabet, these polarization operators are derivations in the usual sense (Subsection 4.4).

Claim 15.2. *The identity matrix I_m acts on the homogeneous component $Super_n[\mathcal{L}|\mathcal{P}]$ as the polarization operator*

$$\mathcal{D}_{x_1,x_1} + \mathcal{D}_{x_2,x_2} + \cdots + \mathcal{D}_{x_m,x_m},$$

and, then, by multiplying each element by the integer n.

Thus, every $sl_m(\mathbb{C})$-submodule of $Super_n[\mathcal{L}|\mathcal{P}]$ is a $gl_m(\mathbb{C})$-submodule.

Corollary 15.1. *Every irreducible $gl_m(\mathbb{C})$-submodule of $Super_n[\mathcal{L}|\mathcal{P}]$ is an irreducible $sl_m(\mathbb{C})$-submodule.*

In the case of irreducible $gl_m(\mathbb{C})$-submodules of $Super_n[\mathcal{L}|\mathcal{P}]$, assertion 1) of Proposition 15.2 reduces to a quite simple construction.

Corollary 15.2. *Every irreducible $gl_m(\mathbb{C})$-submodule $\mathcal{V}$ of $Super_n[\mathcal{L}|\mathcal{P}]$ is of the form $\mathcal{V} = \mathcal{V}_0 \otimes \mathbb{C}$, where:*

- *$\mathcal{V}_0$ is the irreducible $sl_m(\mathbb{C})$-submodule obtained from $\mathcal{V}$ by restriction of the action of $gl_m(\mathbb{C})$ to $sl_m(\mathbb{C})$.*
- *$\mathbb{C}$ is the one-dimensional $gl_m(\mathbb{C})$-module such that the action of $sl_m(\mathbb{C})$ is trivial and $I_m \cdot \boldsymbol{1} = n\boldsymbol{1}$.*

Claim 15.3. *Let $\underline{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in \mathbb{N}^m$ be an m-multiindex such that $\alpha_1 + \alpha_2 + \ldots + \alpha_m = n$. Given a monomial*

$$M = (x_{i_1} \,|\, y_{j_1})(x_{i_2} \,|\, y_{j_2}) \cdots (x_{i_n} \,|\, y_{j_n}) \in Super_n[\mathcal{L}|\mathcal{P}],$$

we say that M has letter content $\underline{\alpha}$, and write $c(M) = \underline{\alpha}$, if

$$\#\{x_{i_h} = x_k; h = 1, 2, \ldots, n\} = \alpha_k,$$

for every $k = 1, 2, \ldots, m$.
Let

$$Super_{\underline{\alpha},n}[\mathcal{L}|\mathcal{P}]$$

be the subspace of $Super_n[\mathcal{L}|\mathcal{P}]$ spanned by the set

$$\{M = (x_{i_1} \,|\, y_{j_1})(x_{i_2} \,|\, y_{j_2}) \cdots (x_{i_n} \,|\, y_{j_n});\ c(M) = \underline{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m)\}.$$

Note that $Super_{\underline{\alpha},n}[\mathcal{L}|\mathcal{P}]$ may be trivial.
A weight space the $sl_m(\mathbb{C})$-module $Super_n[\mathcal{L}|\mathcal{P}]$ is a non-trivial subspace of the form $Super_{\underline{\alpha},n}[\mathcal{L}|\mathcal{P}]$, and its weight is the element

$$\alpha_1\varepsilon_1 + \alpha_2\varepsilon_2 + \cdots + \alpha_m\varepsilon_m \in \boldsymbol{h}^*.$$

Thus, the weight space decomposition of $Super_n[\mathcal{L}|\mathcal{P}]$ is given by the following formula:

$$Super_n[\mathcal{L}|\mathcal{P}] = \bigoplus_{\underline{\alpha}} Super_{\underline{\alpha},n}[\mathcal{L}|\mathcal{P}],$$

where the direct sum is over the set of all multiindices $\underline{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in \mathbb{N}^m$ such that the subspaces $Super_{\underline{\alpha},n}[\mathcal{L}|\mathcal{P}]$ are non-trivial.
Furthermore, every weight space $Super_{\underline{\alpha},n}[\mathcal{L}|\mathcal{P}]$ possesses four classes of bases associated to pairs of standard Young tableaux. By specializing Theorem 9.1, we infer that the sets:

- $\{(S\,|\,T);\ S \in Stab_\lambda(\mathcal{L}), T \in Stab_\lambda(\mathcal{P}), \lambda \vdash n, c(S) = \underline{\alpha}\}$,
- $\big\{\big(S\,|\,\boxed{T}\big);\ S \in Stab_\lambda(\mathcal{L}), T \in Stab_\lambda(\mathcal{P}), \lambda \vdash n, c(S) = \underline{\alpha}\big\}$,
- $\big\{\big(\boxed{S}\,|\,T\big);\ S \in Stab_\lambda(\mathcal{L}), T \in Stab_\lambda(\mathcal{P}), \lambda \vdash n, c(S) = \underline{\alpha}\big\}$,
- $\big\{\big(\boxed{S}\,|\,\boxed{T}\big);\ S \in Stab_\lambda(\mathcal{L}), T \in Stab_\lambda(\mathcal{P}), \lambda \vdash n, c(S) = \underline{\alpha}\big\}$,

are $\mathbb{K}$-bases of $Super_{\underline{\alpha},n}[\mathcal{L}|\mathcal{P}]$, where $c(S)$ denotes the content of $S \in Stab_\lambda(\mathcal{L})$ (see Subsection 5.1).

15.4 The action of upper polarizations

We recall that the finite set of all the standard tableaux over $\mathcal{L}$ with total content the fixed integer n is meant to be the partially ordered with respect to the order defined in Subsection 5.5.

The polarizations $\mathcal{D}_{x_i x_j}$ with $1 \leq i \leq j \leq m$ will be called *upper polarizations*; if $i < j$, the operator $\mathcal{D}_{x_i x_j}$ will be said to be a *strictly upper polarization.*

In the following, we will often use the following identity. For any Deruyts tableau of shape λ, we have

$$\big(\boxed{D}\,|\,T\big) = \left(\prod_i \tilde{\lambda}_i!\right)\ (D\,|\,T).$$

First of all, we establish a triangularity result.

Proposition 15.3. *The action of the standard Capelli bitableaux of the type*

$$[\tilde{D}_\lambda\,|\,\tilde{S}]_- = \pm D_\lambda D\ DS$$

with $\mathrm{sh}(D) = \mathrm{sh}(S) \vdash n$, *on the standard bitableaux* $(T\,|\,U)$, *with* $\mathrm{sh}(T) = \mathrm{sh}(U) \vdash n$, *satisfies the nondegenerate triangularity conditions*

$$[\tilde{D}\,|\,\tilde{S}]_-\ (T\,|\,U) = \begin{cases} c_S\ (D\,|\,U) & \textit{for } S = T, \\ 0 & \textit{for } S \not\geq T, \end{cases}$$

where c_S are nonzero integers.

Proof. By definition, we have

$$[\tilde{D}\,|\,\tilde{S}]_-\ (T\,|\,U) = \pm DD_1\ D_1S\ \underline{TC}\ CU,$$

for some virtual tableau D_1 (Deruyts) and C (co-Deruyts). By the triangularity and nondegeneracy results of Subsection 8.2, this expression vanishes for $S \not\geq T$, while for $S = T$ becomes

$$DD_1\ D_1S\ \underline{SC}\ CU = \theta^{-+}_{SS}\ DD_1\ \underline{D_1C}\ CU = \pm(\tilde{\lambda})!\ \theta^{-+}_{SS}\ \underline{DC}\ CU = c_S\ (D\,|\,U),$$

where $c_S \neq 0$. ∎

In the following, given a shape $\lambda = (\lambda_1, \ldots, \lambda_p) \vdash n$, $\lambda_1 \leq m$, we denote by D_λ the Deruyts tableau of shape λ filled, in order, with the first λ_1 symbols of the alphabet $\mathcal{L}$: for example

$$D_{322} = \begin{matrix} x_1 & x_2 & x_3 \\ x_1 & x_2 & \\ x_1 & x_2 & \end{matrix} .$$

Proposition 15.4. *Let $\lambda \vdash n$ and S be any standard tableau of shape λ. Then the action of the Capelli bitableau*

$$[\tilde{D}_\lambda \,|\, \tilde{S}]_- = \pm D_\lambda D \ DS$$

on $Super_n[\mathcal{L}|\mathcal{P}]$ is the same as the action of a linear combination of products of upper polarizations. Furthermore, if $S \neq D_\lambda$, each term of this linear combination can be written as a product of upper polarizations, with a strictly upper polarization as right-most factor.

Proof. (Sketch) The proof is based on three steps.

Step 1. By Proposition 6.1, we infer:

$$[\tilde{D}_\lambda \,|\, \tilde{S}]_- = \left[\begin{array}{l|l} x_1^{\tilde{\lambda}_1} & \tilde{w}_1 \\ x_2^{\tilde{\lambda}_2} & \tilde{w}_2 \\ \vdots & \vdots \\ x_{p-1}^{\tilde{\lambda}_{p-1}} & \tilde{w}_{p-1} \\ x_p^{\tilde{\lambda}_p} & \tilde{w}_p \end{array}\right]_- \cong \left[\begin{array}{l|l} x_1^{\tilde{\lambda}_1} & \tilde{w}_1 \\ x_2^{\tilde{\lambda}_2} & \tilde{w}_2 \\ \vdots & \vdots \\ x_{p-1}^{\tilde{\lambda}_{p-1}} & \tilde{w}_{p-1} \end{array}\right]_- \left[x_p^{\tilde{\lambda}_p} \,|\, \tilde{w}_p\right]$$

$$- \sum c_* \left[\begin{array}{l|l} x_1^{\tilde{\lambda}_1} & \tilde{w}'_1 \ \tilde{w}_{p,(1)} \\ x_2^{\tilde{\lambda}_2} & \tilde{w}'_2 \ \tilde{w}_{p,(2)} \\ \vdots & \vdots \\ x_{p-1}^{\tilde{\lambda}_{p-1}} & \tilde{w}'_{p-1} \ \tilde{w}_{p,(p-1)} \\ x_p^{m} & \tilde{w}_{p,(p)} \end{array}\right]_- ,$$

where the sum is taken under the conditions of the same proposition, and $m < \tilde{\lambda}_p$.

Since S is a standard tableau, each column word $\tilde{w}_i$ of $\tilde{S}$ contains only letters x_j, with $j \geq i$. Furthermore, if $S \neq D_\lambda$, there is at least one word $\tilde{w}_i$ of $\tilde{S}$ that contains a letter x_j with $j > i$. Notice that every summand in the preceding identity has the same properties.

By iterating this identity, one recognizes that $[\tilde{D}_\lambda \,|\, \tilde{S}]$ is $[\mathcal{L} \,|\, \mathcal{P}]$-equivalent to a linear combination of products of Capelli rows of the form

$$[x_i \ldots x_i \,|\, w];$$

if $S \neq D_\lambda$, in each of these products there is at least one factor whose right-hand word w contains a letter x_j with $j > i$.

Step 2. By Theorem 6.3, every Capelli row $[x_i \ldots x_i \,|\, w]$ produced by Step 1 is $[\mathcal{L} \,|\, \mathcal{P}]$-equivalent to a linear combination of products of upper polarizations. Thus $[\tilde{D}_\lambda \,|\, \tilde{S}]$ is $[\mathcal{L} \,|\, \mathcal{P}]$-equivalent to a linear combination of products of upper polarizations. Furthermore, if $S \neq D_\lambda$, each summand in this linear combination is a product of upper polarizations, and contains at least a strictly upper polarization as a factor.

Step 3. If $S \neq D_\lambda$, by iterating the commutator identity, each of the above products can be turned into a linear combination of products of upper polarizations, with a *strictly* upper polarization as right-most factor. □

The elements of $Super_n[\mathcal{L}|\mathcal{P}]$ which are annihilated by every strictly upper polarization operators are characterized by the following theorem.

Theorem 15.1. *For any element F of $Super_n[\mathcal{L}|\mathcal{P}]$, the following statements are equivalent.*

1. *F is annihilated by every strictly upper polarization.*
2. *The expansion of F as a linear combination of standard bitableaux is of the form*
$$F = \sum_{\lambda \vdash n} \sum_{U \in Stab_\lambda(\mathcal{P})} c_U \; (D_\lambda \,|\, U).$$
3. *The expansion of F as a linear combination of standard right symmetrized bitableaux is of the form*
$$F = \sum_{\lambda \vdash n} \sum_{U \in Stab_\lambda(\mathcal{P})} d_U \; \left(D_\lambda \,\middle|\, \boxed{U}\right).$$

Proof. It is clear that 2) implies 1) and that 3) implies 1). We limit ourselves to proving that 1) implies 2), since the implication 1) $\Rightarrow$ 3) follows from a similar argument. By way of contradiction, we assume that there is an element F of $Super_n[\mathcal{L}|\mathcal{P}]$ which is annihilated by all the upper polarizations and has a standard expansion of the form

$$F = \sum_{\lambda \vdash n} \sum_{U} c_U \; (D_\lambda \,|\, U) + \sum_{T,U} c_{TU} \; (T \,|\, U),$$

where T ranges in a set $\mathcal{T}$ of standard tableaux different from any tableau D_λ and all the coefficients c_{TU} are nonzero. Let S be any minimal tableau in $\mathcal{T}$, of shape μ, say, and consider the Capelli bitableau

$$[\tilde{D}_\mu \,|\, \tilde{S}]_- = \pm D_\mu D \; DS.$$

By the preceding proposition, $[\tilde{D}_\mu \,|\, \tilde{S}]_-$ can be expressed as a linear combination of products of upper polarizations, where each product has a strictly upper polarization as right-most factor. We have

$$
\begin{aligned}
0 &= [\tilde{D}_\mu \,|\, \tilde{S}]_- \left(\sum_{\lambda \vdash m} \sum_{U} c_U \, (D_\lambda \,|\, U) + \sum_{T,U} c_{TU} \, (T \,|\, U) \right) \\
&= [\tilde{D}_\mu \,|\, \tilde{S}]_- \left(\sum_{T,U} c_{TU} \, (T \,|\, U) \right) \\
&= [\tilde{D}_\mu \,|\, \tilde{S}]_- \left(\sum_{T,U:\ S \geq T} c_{TU} \, (T \,|\, U) \right) \\
&= [\tilde{D}_\mu \,|\, \tilde{S}]_- \left(\sum_{U} c_{SU} \, (S \,|\, U) \right) \\
&= c_S \, \left(\sum_{U} c_{SU} \, (D_\mu \,|\, U) \right),
\end{aligned}
$$

where c_S is a nonzero integer, in contradiction with the linear independence of standard bitableaux. $\square$

In the language of Subsection 15.1, Theorem 15.1 reads as follows:

Corollary 15.3. *For any element F of $Super_n[\mathcal{L}|\mathcal{P}]$, the following statements are equivalent.*

1. *F is a highest weight vector.*
2. *The expansion of F as a linear combination of standard bitableaux is of the form*

$$F = \sum_{U \in Stab_\lambda(\mathcal{P})} c_U \, (D_\lambda \,|\, U),$$

   where λ is some partition of n.

3. *The expansion of F as a linear combination of standard right symmetrized bitableaux is of the form*

$$F = \sum_{U \in Stab_\lambda(\mathcal{P})} d_U \, \left(D_\lambda \,\middle|\, \boxed{U} \right),$$

   where λ is some partition of n.

15.5 The action of lower polarizations

The polarizations $\mathcal{D}_{x_i x_j}$ with $m \geq i \geq j \geq 1$ will be called *lower polarizations*; if $i > j$, $\mathcal{D}_{ij}$ will be called a *strictly lower polarization.*

Let $\lambda = (\lambda_1, \ldots, \lambda_p) \vdash n$ be a partition of the integer n, and let $\tilde{\lambda} = (\tilde{\lambda}_1, \ldots, \tilde{\lambda}_q)$ be its *conjugate* partition. Let $S = (\omega_1, \ldots, \omega_p)$ be a standard tableau of shape λ on the letter alphabet $\mathcal{L} = \mathcal{L}_1 = \{x_1, \ldots, x_m\}$, and let $\tilde{S} = (\tilde{\omega}_1, \ldots, \tilde{\omega}_q)$ be its *conjugate* tableau.

For every pair (i, j), with $i = 1, \ldots, q$, and $j = 1, \ldots, m$, denote by $c_{j,i}$ the number of occurrences of x_j in the word $\tilde{\omega}_i$.

Let

$$\mathfrak{P}_S = \mathcal{D}^{c_{2,1}}_{x_2,x_1} \cdots \mathcal{D}^{c_{m,1}}_{x_m,x_1} \mathcal{D}^{c_{3,2}}_{x_3,x_2} \cdots \mathcal{D}^{c_{m,2}}_{x_m,x_2} \cdots \mathcal{D}^{c_{q+1,q}}_{x_{q+1,q}x_q} \cdots \mathcal{D}^{c_{m,q}}_{x_m,x_q}.$$

Proposition 15.5. *For every standard letter tableau S and every standard place tableau T, both of shape λ, we have*

$$\mathfrak{P}_S\left(\boxed{D_\lambda}\,\middle|\, T)\right) = \left(\prod_{i=1}^{q} \frac{\tilde{\lambda}_i!}{c_{i,i}!}\right) \left(\boxed{S}\,\middle|\, T\right).$$

Corollary 15.4. *Denote by n_m^- be the maximal nilpotent subalgebra of $sl_m(\mathbb{C})$ spanned by all strictly lower triangular matrices.*

1. *Let F be a highest weight vector of $Super_n[\mathcal{L}|\mathcal{P}]$ whose expansion as a linear combination of standard bitableaux is of the form*

   $$F = \sum_{U \in Stab_\lambda(\mathcal{P})} c_U \ (D_\lambda \mid U), \quad \lambda \vdash n.$$

   Then, the cyclic submodule $n_m^- \cdot F$ is an irreducible $sl_m(\mathbb{C})$-module, with basis

   $$\left\{ \sum_{U \in Stab_\lambda(\mathcal{P})} c_U \ \left(\boxed{S}\,\middle|\, U\right);\ S \in Stab_\lambda(\mathcal{L}) \right\}.$$

   Furthermore, F is (up to a scalar factor) the unique highest weight vector of this irreducible $sl_m(\mathbb{C})$-module.

2. *Let F' be a highest weight vector of $Super_n[\mathcal{L}|\mathcal{P}]$ whose expansion as a linear combination of standard right symmetrized bitableaux is of the form*

   $$F' = \sum_{U \in Stab_\lambda(\mathcal{P})} d_U \ \left(D_\lambda\,\middle|\, \boxed{U}\right), \quad \lambda \vdash n.$$

   Then, the cyclic submodule $n_m^- \cdot F'$ is an irreducible $sl_m(\mathbb{C})$-module, with basis

   $$\left\{ \sum_{U \in Stab_\lambda(\mathcal{P})} d_U \ \left(\boxed{S}\,\middle|\, \boxed{U}\right);\ S \in Stab_\lambda(\mathcal{L}) \right\}.$$

   Furthermore, F' is (up to a scalar factor) the unique highest weight vector of this irreducible $sl_m(\mathbb{C})$-module.

Proof. We prove assertion 1. By the preceding proposition, we infer:

$$n_m^- \cdot F \supseteq \left\langle \sum_{U \in Stab_\lambda(\mathcal{P})} c_U \left(\boxed{S} \middle| U \right);\ S \in Stab_\lambda(\mathcal{L}) \right\rangle_{\mathbb{K}}.$$

On the other hand, since F equals, up to a scalar factor, we have

$$\sum_{U \in Stab_\lambda(\mathcal{P})} c_U \left(\boxed{D_\lambda} \middle| U \right) \in \left\langle \sum_{U \in Stab_\lambda(\mathcal{P})} c_U \left(\boxed{S} \middle| U \right);\ S \in Stab_\lambda(\mathcal{L}) \right\rangle_{\mathbb{K}}$$

and

$$b_m^- \cdot F \subseteq \left\langle \sum_{U \in Stab_\lambda(\mathcal{P})} c_U \left(\boxed{S} \middle| U \right);\ S \in Stab_\lambda(\mathcal{L}) \right\rangle_{\mathbb{K}}.$$

Finally, the module

$$\left\langle \sum_{U \in Stab_\lambda(\mathcal{P})} c_U \left(\boxed{S} \middle| U \right);\ S \in Stab_\lambda(\mathcal{L}) \right\rangle_{\mathbb{K}}$$

is a $gl_m(\mathbb{C})$-irreducible module and, hence, an $sl_m(\mathbb{C})$-irreducible module.

Suppose now that $\overline{F}$ is another highest weight vector, $\overline{F} \in n_m^- \cdot F$. On the one hand, by Corollary 15.3, $\overline{F}$ is of the form

$$\overline{F} = \sum_{U \in Stab_\mu(\mathcal{P})} \overline{c}_U \ (D_\mu \,|\, U),$$

for some partition $\mu \vdash n$, and, on the other hand,

$$\overline{F} = \sum_{S \in Stab_\lambda(\mathcal{L})} a_S \sum_{U \in Stab_\lambda(\mathcal{P})} c_U \left(\boxed{S} \middle| U \right).$$

This implies $\lambda = \mu$ and $a_S = 0$ whenever $S \neq D_\lambda$. Therefore, $\overline{F}$ is a scalar multiple of F. ∎

As a matter of fact, Corollary 15.4 describes *all* the irreducible $sl_m(\mathbb{C})$-submodules of $Super_n[\mathcal{L}|\mathcal{P}]$. In the language of Subsection 15.1, we have:

Proposition 15.6. *Let W be an irreducible $sl_m(\mathbb{C})$-submodule of $Super_n[\mathcal{L}|\mathcal{P}]$. Then, there exists a highest weight vector F such that*

$$W = n_m^- \cdot F = sl_m(\mathbb{C}) \cdot F.$$

Proof. Thanks to the claim at the beginning of this section, we can treat W as an irreducible gl_m-submodule. Let

$$f = \sum_{\lambda \vdash n} \sum_{S,T} c_{S,T} \left(S \middle| \boxed{T} \right), \qquad S \in Stab_\lambda(\mathcal{L}),\ T \in Stab_\lambda(\mathcal{P})$$

be a non-zero element of W. Let S be a standard tableau such that there exists at least one coefficient $c_{S,T}$ different from zero. Then, the element

$$Y\left(D_\lambda, \boxed{S}\right) f = \sum_T c_{S,T} \left(D_\lambda \middle| \boxed{T} \right) \neq 0$$

is a highest weight vector that belongs to W. The cyclic gl_m-submodule (equivalently, the cyclic $sl_m(\mathbb{C})$-submodule) of $Super_n[\mathcal{L}|\mathcal{P}]$ generated by this highest weight vector is a submodule of W. Since W is supposed to be irreducible, these modules coincide. □

Claim 15.4. *The set of all highest weight vectors of a given weight in $Super_n[\mathcal{L}|\mathcal{P}]$ is a Schur irreducible $pl(\mathcal{P})$-module, namely, a vector subspace of the form*

$$\langle (D_\lambda \,|\, U), \quad U \in Stab(\mathcal{P}) \rangle_{\mathbb{K}} = \left\langle \left(D_\lambda \middle| \boxed{U} \right), \quad U \in Stab(\mathcal{P}) \right\rangle_{\mathbb{K}},$$

with $\lambda \vdash n,\ \lambda \in H(\mathcal{L}) \cap H(\mathcal{P})$. In particular, this means that each $sl_m(\mathbb{C})$-irreducible submodule has a unique (up to a scalar factor) highest weight vector, but it can be expressed in two ways, namely, by using the basis $\{(D_\lambda \,|\, U), \quad U \in Stab(\mathcal{P})\}$ and by using the basis $\left\{ \left(D_\lambda \middle| \boxed{U}\right), \quad U \in Stab(\mathcal{P}) \right\}$.

15.6 Highest weight vectors and complete decompositions

By combining 1) and 3) of Theorem 9.1 (Clebsch–Gordan–Capelli bases) and Corollary 15.3, one immediately infers the following result.

Proposition 15.7. *The following sets are maximal sets of linearly independent highest weight vectors in $Super_n[\mathcal{L}|\mathcal{P}]$:*

1. $\{(D_\lambda \,|\, T);\ \lambda \vdash n,\ T \in Stab_\lambda(\mathcal{P}),\ \lambda \in H(\mathcal{L}) \cap H(\mathcal{P})\}$;
2. $\left\{ \left(D_\lambda \middle| \boxed{T}\right);\ \lambda \vdash n,\ T \in Stab_\lambda(\mathcal{P}),\ \lambda \in H(\mathcal{L}) \cap H(\mathcal{P}) \right\}$.

The preceding facts are to be read as follows:

1. Given a highest weight vector in $Super_n[\mathcal{L}|\mathcal{P}]$ of the form

   $$(D_\lambda \,|\, T),\ T \in Stab_\lambda(\mathcal{P}),\ \lambda \in H(\mathcal{L}) \cap H(\mathcal{P}),$$

   successive applications of the polarization operators $D_{x_i x_j},\ i > j$, generate all the standard left symmetrized bitableaux

   $$\left(\boxed{S} \middle| T \right),\ S \in Stab_\lambda(\mathcal{L}),\ T \in Stab_\lambda(\mathcal{P}),\ \lambda \in H(\mathcal{L}) \cap H(\mathcal{P});$$

2. Given a highest weight vector in $Super_n[\mathcal{L}|\mathcal{P}]$ of the form

$$\left(D_\lambda \,\middle|\, \boxed{T}\right),\ T \in Stab_\lambda(\mathcal{P}),\ \lambda \in H(\mathcal{L}) \cap H(\mathcal{P}),$$

successive applications of the polarization operators $D_{x_i x_j}$, $i > j$, generate all the standard doubly symmetrized bitableaux

$$\left(\boxed{S} \,\middle|\, \boxed{T}\right),\ S \in Stab_\lambda(\mathcal{L}),\ T \in Stab_\lambda(\mathcal{P}),\ \lambda \in H(\mathcal{L}) \cap H(\mathcal{P});$$

Therefore:

1. For every $T \in Stab_\lambda(\mathcal{P}),\ \lambda \in H(\mathcal{L}) \cap H(\mathcal{P})$, the irreducible $sl_m(\mathbb{C})$-submodule generated by the highest weight vector $(D_\lambda \,|\, T)$ is the Weyl module $W_{\lambda T}$ parametrized by the place-tableau T.

2. For every $T \in Stab_\lambda(\mathcal{P}),\ \lambda \in H(\mathcal{L}) \cap H(\mathcal{P})$, the irreducible $sl_m(\mathbb{C})$-submodule generated by the highest weight vector $\left(D_\lambda \,\middle|\, \boxed{T}\right)$ is the Schur module $S_{\lambda T}$ parametrized by the place-tableau T (recall that the set $\left\{\left(\boxed{S} \,\middle|\, \boxed{T}\right),\ S \in Stab_\lambda(\mathcal{L}),\ T \in Stab_\lambda(\mathcal{P}),\ \lambda \in H(\mathcal{L}) \cap H(\mathcal{P})\right\}$ is a basis of $S_{\lambda T}$).

Finally, we have the following results (compare with Theorem 12.1 and Proposition 15.7).

Proposition 15.8. *We have:*

1. *The complete decomposition of the semisimple $sl_m(\mathbb{C})$-module $Super_n[\mathcal{L}|\mathcal{P}]$:*

$$Super_n[\mathcal{L}|\mathcal{P}] = \bigoplus_\lambda \left(\bigoplus_{T \in Stab_\lambda(\mathcal{P})} W_{\lambda T} \right)$$

is the complete decomposition corresponding to the maximal set of linearly independent highest weight vectors in $Super_n[\mathcal{L}|\mathcal{P}]$:

$$\{(D_\lambda \,|\, T);\quad T \in Stab_\lambda(\mathcal{P}),\ \lambda \in H(\mathcal{L}) \cap H(\mathcal{P})\};$$

2. *The complete decomposition of the semisimple $sl_m(\mathbb{C})$-module $Super_n[\mathcal{L}|\mathcal{P}]$:*

$$Super_n[\mathcal{L}|\mathcal{P}] = \bigoplus_\lambda \left(\bigoplus_{T \in Stab_\lambda(\mathcal{P})} S_{\lambda T} \right)$$

is the complete decomposition corresponding to the maximal set of linearly independent highest weight vectors in $Super_n[\mathcal{L}|\mathcal{P}]$:

$$\left\{\left(D_\lambda \,\middle|\, \boxed{T}\right);\ T \in Stab_\lambda(\mathcal{P}),\ \lambda \in H(\mathcal{L}) \cap H(\mathcal{P})\right\}.$$

15.7 Letterplace algebras and complete sets of pairwise non-isomorphic irreducible $sl_m(\mathbb{C})$-representations

We know from Subsection 15.1 that every irreducible representation of $sl_m(\mathbb{C})$ is uniquely determined by its highest weight vector, or, equivalently, by its highest weight.

From the "abstract" representations theory of $sl_m(\mathbb{C})$, we get the following result (see, e.g., [40], [42], [88]):

Proposition 15.9. *Given an irreducible $sl_m(\mathbb{C})$-module, its highest weight is of the form*

$$a_1\varepsilon_1 + a_2\varepsilon_2 + \cdots + a_m\varepsilon_m,$$

with $a_i \in \mathbb{Z}$ and $a_1 \geq a_2 \geq \cdots \geq a_m$.

Since a weight in $\mathbf{h}^*$ is defined up to a sum with a scalar multiple of the linear functional $\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_m$, the preceding proposition implies the following assertion.

Corollary 15.5. *Given an irreducible $sl_m(\mathbb{C})$-module, its highest weight is of the form*

$$a_1\varepsilon_1 + a_2\varepsilon_2 + \cdots + a_{m-1}\varepsilon_{m-1},$$

with $a_i \in \mathbb{N}$ and $a_1 \geq a_2 \geq \cdots \geq a_{m-1}$.

Hence, any irreducible $sl_m(\mathbb{C})$-representation can be realized as Weyl and Schur submodules of a suitable letterplace algebra.

More precisely, we have the following result.

Proposition 15.10. *Let $\mathcal{L} = \mathcal{L}_1 = \{x_1, \ldots, x_m\}$ be a negatively signed alphabet of proper letters, and let $\mathcal{P} = \mathcal{P}_1 = \{y_1, \ldots, y_m\}$ be a negatively signed alphabet of proper places. Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p)$ be a partition such that $\lambda_1 \leq m$, and let $\tilde{\lambda} = (\tilde{\lambda}_1 \geq \tilde{\lambda}_2 \geq \cdots \geq \tilde{\lambda}_q)$ be the conjugate partition of λ.*

Consider the (irreducible) Weyl submodule $W_{\lambda T}$ and the (irreducible) Schur submodule $S_{\lambda T}$ of $Super[\mathcal{L}|\mathcal{P}]$, $T \in Stab_\lambda(\mathcal{P})$.

The highest weight of both $W_{\lambda T}$ and $S_{\lambda T}$ is the element

$$(\tilde{\lambda}_1 - \tilde{\lambda}_m)\varepsilon_1 + (\tilde{\lambda}_2 - \tilde{\lambda}_m)\varepsilon_2 + \cdots + (\tilde{\lambda}_m - \tilde{\lambda}_m)\varepsilon_m \in \boldsymbol{h}^*,$$

where $\tilde{\lambda}_s$ is set to be zero whenever $s > q$.

16 Deruyts' Theory of Covariants (after J. A. Green)

Let m, n be positive integers. In general, given a positive integer r, we write $\underline{r}$ to mean the linearly ordered set $\{1, 2, \ldots, r\}$.

Let $\mathcal{A} = \{a_1, \ldots, a_m\}$, $\mathcal{X} = \{x_1, \ldots, x_n\}$, and $\mathcal{P} = \underline{n} = \{1, 2, \ldots, n\}$ be negatively signed alphabets.

Consider the (positively signed) letterplace alphabets

$$[\mathcal{A} \mid \underline{n}] = \{(a_i \mid \mu);\ i = 1, \ldots, m,\ \mu = 1, \ldots, n\},$$
$$[\underline{n} \mid \mathcal{X}] = \{(\mu \mid x_j);\ \mu,\ j = 1, \ldots, n\},$$

and the tensor product of *commutative* letterplace superalgebras

$$Super[\mathcal{A} | \underline{n}] \otimes Super[\underline{n} | \mathcal{X}].$$

In the following we write $\mathbb{K}[\mathcal{A}]$ for $Super[\mathcal{A}| \underline{n}]$, and $\mathbb{K}[\mathcal{X}]$ for $Super[\underline{n}| \mathcal{X}]$, respectively, and $\mathbb{K}[\mathcal{A}] \otimes \mathbb{K}[\mathcal{X}]$ for the tensor product $Super[\mathcal{A}| \underline{n}] \otimes Super[\underline{n}| \mathcal{X}]$.

16.1 Left and right actions of $GL_n(\mathbb{K})$ on $\mathbb{K}[\mathcal{A}]$ and $\mathbb{K}[\mathcal{X}]$

Let $S = (s_{\mu\nu})$ be a matrix in $GL_n(\mathbb{K})$.

We define a left action $GL_n(\mathbb{K}) \circ \mathbb{K}[\mathcal{A}]$ by setting, for every matrix $S = (s_{\mu\nu}) \in GL_n(\mathbb{K})$:

$$S \circ (a_i \mid \nu) = \sum_{\mu=1}^{n} (a_i \mid \mu)\ s_{\mu\nu}, \quad i = 1, \ldots, m, \quad \nu = 1, \ldots, n,$$

and extending as an algebra endomorphism.

We define a right action $\mathbb{K}[\mathcal{X}] \circ GL_n(\mathbb{K})$ by setting:

$$(\mu \mid x_j) \circ S = \sum_{\nu=1}^{n} s_{\mu\nu}\ (\nu \mid x_j), \quad j, \mu = 1, \ldots, n,$$

and extending as an algebra endomorphism.

Clearly, the actions of $GL_n(\mathbb{K})$ on $\mathbb{K}[\mathcal{A}]$ and $\mathbb{K}[\mathcal{X}]$, regarded as actions on the tensor product $\mathbb{K}[\mathcal{A}] \otimes \mathbb{K}[\mathcal{X}]$, *commute* and, therefore, they may be combined to make $\mathbb{K}[\mathcal{A}] \otimes \mathbb{K}[\mathcal{X}]$ into a $GL_n(\mathbb{K}) - GL_n(\mathbb{K})$ - bimodule by the rule:

$$S \circ (\alpha \otimes \xi) \circ S' = (S \circ \alpha) \otimes (\xi \circ S'),$$

for all $S, S' \in GL_n(\mathbb{K})$, $\alpha \in \mathbb{K}[\mathcal{A}]$, $\xi \in \mathbb{K}[\mathcal{X}]$.

16.2 Invariants and covariants

Given an integer ω, an element $\gamma \in \mathbb{K}[\mathcal{A}]$ is called an *invariant of weight* ω if

$$S \circ \gamma = |S|^{\omega}\ \gamma, \quad \text{for all } S \in GL_n(\mathbb{K}),$$

$|S|$ the determinant of S.

An element $\varphi \in \mathbb{K}[\mathcal{A}] \otimes \mathbb{K}[\mathcal{X}]$ is called a *covariant of weight* ω if

$$S \circ \varphi \circ S^{-1} = |S|^{\omega}\ \varphi, \quad \text{for all } S \in GL_n(\mathbb{K}).$$

16.3 Isobaric elements and semiinvariants. Weights

An element $\gamma \in \mathbb{K}[\mathcal{A}]$ is said to be *isobaric* of *weight* $\pi = (\pi_1, \ldots, \pi_n) \in \mathbb{N}^n$ if the identity

$$S \circ \gamma = d_1^{\pi_1} \cdots d_n^{\pi_n} \cdot \gamma$$

holds for all diagonal matrices $S = Diag(d_1, \ldots, d_n)$, $d_1, \ldots, d_n \in \mathbb{K}^*$.

The *Borel* subgroup B_n of $GL_n(\mathbb{K})$ is the subgroup which consists of all "upper triangular" (non-singular) $n \times n$ matrices.

An element $\gamma \in \mathbb{K}[\mathcal{A}]$ is called a *semiinvariant* of weight π if

$$S \circ \gamma = d_1^{\pi_1} \cdots d_n^{\pi_n} \cdot \gamma,$$

for all $S \in B_n$, $(d_1, \ldots, d_n) \in (\mathbb{K}^*)^n$ the n-tuple of diagonal entries of S.

Remark 16.1. Clearly, semiinvariants are isobaric elements. In particular, a semiinvariant of weight π is a homogeneous polynomial of degree $d = \pi_1 + \pi_2 + \cdots + \pi_n$ of $\mathbb{K}[\mathcal{A}]$.

Note that the remarks in Subsection 15.4 apply to the study of semiinvariants, since they are homogenous elements of $\mathbb{K}[\mathcal{A}]$; the next result is just a reformulation of Theorem 15.1 and Corollary 15.3.

We recall that, given a shape $\lambda = (\lambda_1, \ldots, \lambda_p) \vdash d$, $\lambda_1 \leq n$, we denote by D_λ the Deruyts tableau of shape λ filled, in order, with the first λ_1 symbols of the alphabet.

Theorem 16.1. *Let γ be a homogeneous element of degree d of $\mathbb{K}[\mathcal{A}]$.*

The following statements are equivalent.

1. *γ is a semiinvariant of weight $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$.*

2. *The expansion of γ as a linear combination of standard bitableaux is of the form*

$$\gamma = \sum_{\mathrm{sh}(V)=\lambda} c_V \ (V \mid D_\lambda), \qquad \lambda \vdash d, \quad \lambda_1 \leq m,$$

   with $\pi = \tilde{\lambda}$.

3. *The expansion of γ as a linear combination of standard left symmetrized bitableaux is of the form*

$$\gamma = \sum_{\mathrm{sh}(V)=\lambda} d_V \left(\boxed{V} \,\middle|\, D_\lambda \right), \qquad \lambda \vdash d, \quad \lambda_1 \leq m,$$

   with $\pi = \tilde{\lambda}$.

Claim 16.1. *The preceding result implies that the weight π of a semiinvariant is an n-tuple of natural integers $\pi = (\pi_1, \ldots, \pi_n)$, with $\pi_1 \geq \pi_2 \geq \cdots \geq \pi_n$.*

16.4 The map $\sigma : \mathbb{K}[\mathcal{A}] \to \mathbb{K}[\mathcal{A}] \otimes \mathbb{K}[\mathcal{X}]$

Given $a_i \in \mathcal{A}$ and $x_j \in \mathcal{X}$, we set:

$$\langle a_i \,|\, x_j \rangle = \sum_{\mu=1}^{n} (a_i \,|\, \mu) \otimes (\mu \,|\, x_j), \quad i = 1, \ldots, m \quad j = 1, \ldots, n.$$

We explicitly note that the elements $\langle a_i \,|\, x_j \rangle$ are covariants of weight zero in $\mathbb{K}[\mathcal{A}] \otimes \mathbb{K}[\mathcal{X}]$.

Define a map

$$\sigma : \mathbb{K}[\mathcal{A}] \to \mathbb{K}[\mathcal{A}] \otimes \mathbb{K}[\mathcal{X}]$$

by setting

$$\sigma : (a_i \,|\, j) \to \langle a_i \,|\, x_j \rangle, \quad i = 1, \ldots, m, \quad j = 1, \ldots, n$$

and extending as an algebra morphism.

Remark 16.2. Since $\mathbb{K}$ is an infinite field, we regard an element $\varphi \in \mathbb{K}[\mathcal{A}] \otimes \mathbb{K}[\mathcal{X}]$ as a polynomial function in the n^2 variables $(\mu \,|\, x_j)$ from the space $M_n(\mathbb{K})$ to $\mathbb{K}[\mathcal{A}]$ (evaluate any "variable" $(\mu \,|\, x_j)$ on the entry $h_{\mu j}$ of a matrix $H \in M_n(\mathbb{K}),\ \mu,\ j = 1, \ldots, n$).

The following facts follow from the definitions.

1. Let $\gamma \in \mathbb{K}[\mathcal{A}]$. Then $(\sigma\gamma)(S) = S \circ \gamma$, for all $S \in GL_n(\mathbb{K})$.
2. Let $\varphi \in \mathbb{K}[\mathcal{A}] \otimes \mathbb{K}[\mathcal{X}]$. Then

$$(\varphi \circ S)(H) = \varphi(SH) \in \mathbb{K}[\mathcal{A}],$$

   for all $S \in GL_n(\mathbb{K}),\ H \in M_n(\mathbb{K})$.

Theorem 16.2. *The map σ is a $\mathbb{K}$-algebra monomorphism.*

Furthermore, $Im[\sigma]$ is the set of all $\varphi \in \mathbb{K}[\mathcal{A}] \otimes \mathbb{K}[\mathcal{X}]$ which satisfy

$$S \circ \varphi = \varphi \circ S, \quad for\ all\ S \in GL_n(\mathbb{K}).$$

In other words, $Im[\sigma]$ is the set of all covariants of weight zero in $\mathbb{K}[\mathcal{A}] \otimes \mathbb{K}[\mathcal{X}]$.

Proof. Notice that, since σ is an algebra morphism, it maps any element $\gamma \in \mathbb{K}[\mathcal{A}]$ to a covariant of weight zero.

Since

$$\sigma(a_i \,|\, j)(S) = \langle a_i \,|\, x_j \rangle(S) = S \circ (a_i \,|\, j), \qquad \text{for all } S \in GL_n(\mathbb{K}),$$

it follows that

$$(\sigma\gamma)(S) = S \circ \gamma, \qquad \text{for all } \gamma \in \mathbb{K}[\mathcal{A}],\ \text{for all } S \in GL_n(\mathbb{K}).$$

In particular, the "evaluation of the covariant $(\sigma\gamma)(I)$ at the identity matrix $I \in GL_n(\mathbb{K})$" equals γ, in symbols:

$$(\sigma\gamma)(I) = \gamma, \qquad \text{for all } \gamma \in \mathbb{K}[\mathcal{A}],$$

Hence, the map σ is injective.

Let φ be a covariant of weight zero. By the previous remark, we know that

$$(S \circ \varphi)(H) = (\varphi \circ S)(H) = \varphi(SH),$$

for all $S \in GL_n(\mathbb{K}),\ H \in M_n(\mathbb{K})$.

By setting $H = I$, we have

$$\varphi(S) = (S \circ \varphi)(I) = S \circ \varphi(I), \qquad \text{for all } S \in GL_n(\mathbb{K}),$$

with $\varphi(I) \in \mathbb{K}[\mathcal{A}]$. Since

$$\varphi(S) = S \circ \varphi(I) = [\sigma(\varphi(I))](S), \qquad \text{for all } S \in GL_n(\mathbb{K}),$$

and $GL_n(\mathbb{K})$ is a Zariski open set in $M_n(\mathbb{K})$, this implies the identity

$$\varphi = \sigma(\varphi(I)) \quad \text{in} \quad \mathbb{K}[\mathcal{A}] \otimes \mathbb{K}[\mathcal{X}],$$

and, therefore, the map σ is surjective on the subalgebra of covariants of weight zero.

□

If φ is a covariant of weight zero, then, by Theorem 16.2, there exists a unique element $\gamma \in \mathbb{K}[\mathcal{A}]$ such that $\sigma(\gamma) = \varphi$. The element γ is called the *source* of the covariant φ, and $\gamma = \varphi(I)$.

16.5 Left and right spans of covariants of weight zero

Consider an element $\varphi \in \mathbb{K}[\mathcal{A}] \otimes \mathbb{K}[\mathcal{X}]$, and write

$$\varphi = \alpha_1 \otimes \beta_1 + \cdots + \alpha_r \otimes \beta_r, \quad \alpha_s \in \mathbb{K}[\mathcal{A}], \quad \beta_s \in \mathbb{K}[\mathcal{X}];$$

let $r(\varphi)$ be the minimal length of all such expansions for φ. If $r = r(\varphi)$, we say that the above expansion is *minimal.* As a matter of fact an expansion is minimal if and only if $\{\alpha_1, \ldots, \alpha_r\}$ and $\{\beta_1, \ldots, \beta_r\}$ are linearly independent sets. The vector spaces

$$L(\varphi) = \langle \alpha_1, \ldots, \alpha_r \rangle_{\mathbb{K}}$$

and

$$R(\varphi) = \langle \beta_1, \ldots, \beta_r \rangle_{\mathbb{K}}$$

are called the *left span* and the *right span* of φ, respectively.

The following result follows directly from the definitions.

Proposition 16.1. *Let $\varphi \in \mathbb{K}[\mathcal{A}] \otimes \mathbb{K}[\mathcal{X}]$ be a covariant of weight zero. Then $L(\varphi)$ and $R(\varphi)$ are left and right $\mathbb{K}[GL_n(\mathbb{K})]$- submodules of $\mathbb{K}[\mathcal{A}]$ and $\mathbb{K}[\mathcal{X}]$, respectively.*

The following result easily follows from a standard argument (see, e.g., Green [45, p. 261]).

Proposition 16.2. *If $\varphi = \sigma(\gamma)$, $\gamma \in \mathbb{K}[\mathcal{A}]$, then*

$$L(\varphi) = \mathbb{K}[GL_n(\mathbb{K})] \circ \gamma,$$

the cyclic $\mathbb{K}[GL_n(\mathbb{K})]$- submodule of $\mathbb{K}[\mathcal{A}]$ generated by γ.

16.6 Primary covariants and irreducible $GL_n(\mathbb{K})$-representations

A non-zero function $\varphi \in \mathbb{K}[\mathcal{A}] \otimes \mathbb{K}[\mathcal{X}]$ is called a *primary covariant* if φ is a covariant of weight zero and $\varphi = \sigma(\gamma)$, where the source γ is a semiinvariant.

Theorem 16.3. *If $\varphi \in \mathbb{K}[\mathcal{A}] \otimes \mathbb{K}[\mathcal{X}]$ is a primary covariant, then $L(\varphi)$ is an irreducible left $\mathbb{K}[GL_n(\mathbb{K})]$- submodule of $\mathbb{K}[\mathcal{A}]$.*

Proof. By the preceding proposition,

$$L(\varphi) = \mathbb{K}[GL_n(\mathbb{K})] \circ \gamma,$$

and the source γ is a semiinvariant of a given weight π. Therefore, by Theorem 16.1, γ is of the form

$$\gamma = \sum_{\mathrm{sh}(V)=\lambda} c_V \; (V \mid D_\lambda),$$

where λ is the conjugate shape of π.

By Corollary 15.4, the cyclic module $\mathbb{K}[GL_n(\mathbb{K})] \circ \gamma$ is the irreducible module spanned, as a vector space, by the basis

$$\left\{ \sum_{\mathrm{sh}(V)=\lambda} c_V \left(V \,\middle|\, \boxed{T} \right); T \text{ standard}, \ \mathrm{sh}(T) = \lambda \right\}.$$

□

Given a weight $\pi = (\pi_1 \geq \pi_2 \geq \cdots \geq \pi_n)$, $\pi_h \in \mathbb{N}$, consider the semiinvariant

$$a(\pi) = (D_\lambda \mid D_\lambda) \in \mathbb{K}[\mathcal{A}],$$

where the left-most D_λ denotes the Deruyts tableau of shape λ filled, in order, with the first letters of $\mathcal{A}$, the right-most D_λ denotes the Deruyts tableau of shape λ filled, in order, with the first places of $\underline{n}$, and λ is the conjugate shape of π.

Example 16.1. *Let* $\pi = (3,3,1)$. *Then*

$$a(\pi) = (D_{(3,2,2)} \,|\, D_{(3,2,2)}) = \left(\begin{array}{c|l} a_1 a_2 a_3 & 123 \\ a_1 a_2 & 12 \\ a_1 a_2 & 12 \end{array} \right).$$

Proposition 16.3. *We have:*

- *Given any semiinvariant* $\gamma = \sum_{\mathrm{sh}(V)=\lambda} c_V \ (V \,|\, D_\lambda) \in \mathbb{K}[\mathcal{A}]$ *of weight* $\pi = \tilde{\lambda}$, *we have the following identity:*

$$\gamma = \left(\sum_{\mathrm{sh}(V)=\lambda} c_V \ Y\Big(V \,\Big|\, \boxed{D_\lambda}\Big) \right) (a(\pi)),$$

  where the $\mathcal{A}$*-letter polarization operator* $\sum_{\mathrm{sh}(V)=\lambda} c_V \ Y\big(V \,\big|\, \boxed{D_\lambda}\big)$ *is a* $\mathbb{K}[GL_n(\mathbb{K})]$*-equivariant operator.*

- *Every non-zero semiinvariant* $\gamma \in \mathbb{K}[\mathcal{A}]$ *of weight* π *generates an irreducible left* $\mathbb{K}[GL_n(\mathbb{K})]$*- submodule* $\mathbb{K}[GL_n(\mathbb{K})] \circ \gamma$ *which is isomorphic to* $\mathbb{K}[GL_n(\mathbb{K})] \circ a(\pi)$.

The proof follows immediately from Theorem 16.1 and Theorem 16.3.

16.7 The Deruyts–Capelli expansion

Theorem 16.4. *Let* γ *be an element of* $\mathbb{K}[\mathcal{A}]$, *homogeneous of degree* $d \in \mathbb{Z}^+$. *Consider the canonical expansion*

$$\gamma = \sum_{\substack{\lambda \vdash d \\ \lambda_1 \le min(m,n)}} \ \sum_{\mathrm{sh}(T)=\lambda} \ \sum_{\mathrm{sh}(S)=\lambda} c^{\gamma}_{ST} \ \Big(S \,\Big|\, \boxed{T}\Big)$$

with respect to the Clebsch–Gordan–Capelli basis of standard right symmetrized bitableaux, and set

$$\gamma_{\lambda,T} = \sum_{\mathrm{sh}(S)=\lambda} c^{\gamma}_{ST} \ \Big(S \,\Big|\, \boxed{T}\Big),$$

$$\gamma^{D_\lambda}_{\lambda,T} = \sum_{\mathrm{sh}(S)=\lambda} c^{\gamma}_{ST} \ (S \,|\, D_\lambda),$$

for every $\lambda \vdash d$ *and* $T \in Stab_\lambda(\mathcal{P})$.

Then:

1. $\gamma_{\lambda,T}$ *belongs to the cyclic* $\mathbb{K}[GL_n(\mathbb{K})]$*-module generated by* $\gamma \in \mathbb{K}[\mathcal{A}]$;

2. *$\gamma_{\lambda,T}$ belongs to the irreducible cyclic $\mathbb{K}[GL_n(\mathbb{K})]$-module generated by the semiinvariant $\gamma_{\lambda,T}^{D_\lambda}$;*

3. *the semiinvariant $\gamma_{\lambda,T}^{D_\lambda}$ belongs to the cyclic $\mathbb{K}[GL_n(\mathbb{K})]$-module generated by $\gamma_{\lambda,T}$;*

4. *the cyclic modules $\mathbb{K}[GL_n(\mathbb{K})] \circ \gamma_{\lambda,T}$ and $\mathbb{K}[GL_n(\mathbb{K})] \circ \gamma_{\lambda,T}^{D_\lambda}$ are the same.*

Proof. By Proposition 12.1, we have:

1. $\gamma_{\lambda,T}$ is the image of γ under the operator $\big(T, \boxed{T}\big)Y$ induced by the (place) action of $\mathbb{K}[GL_n(\mathbb{K})]$.

2. $\gamma_{\lambda,T}$ is the image of $\gamma_{\lambda,T}^{D_\lambda}$ under the operator $\big(D_\lambda, \boxed{T}\big)Y$ induced by the (place) action of $\mathbb{K}[GL_n(\mathbb{K})]$, up to a non-zero scalar factor.

3. $\gamma_{\lambda,T}^{D_\lambda}$ is the image of $\gamma_{\lambda,T}$ under the operator $\big(T, \boxed{D_\lambda}\big)Y$ induced by the (place) action of $\mathbb{K}[GL_n(\mathbb{K})]$, up to a non-zero scalar factor.

□

The next reformulation of Theorem 16.4 describes the *Deruyts–Capelli expansion* of a polynomial $\gamma \in \mathbb{K}[\mathcal{A}]$. As a matter of fact, the first statement is properly the Deruyts Expansion (cfr. Green, [45, Theorem 12.1]), and the second statement is the core of the crucial Capelli's refinement of the result of Deruyts (Capelli's polar expansion formula [25]).

Corollary 16.1 (The Deruyts–Capelli Expansion). *Let γ be an element of $\mathbb{K}[\mathcal{A}]$, γ homogeneous of degree d. Then:*

1. *for every $\lambda \vdash d$ and $T \in Stab_\lambda(\mathcal{P})$, there exist semiinvariants*
$$\gamma_{\lambda,T}^{D_\lambda},$$
and elements
$$\Lambda_{\lambda,T} \in \mathbb{K}[GL_n(\mathbb{K})],$$
such that:
$$\gamma = \sum_{\lambda \vdash d} \sum_{\mathrm{sh}(T)=\lambda} \Lambda_{\lambda,T} \circ \gamma_{\lambda,T}^{D_\lambda};$$

2. *the semiinvariants*
$$\gamma_{\lambda,T}^{D_\lambda}$$
belong to the cyclic $\mathbb{K}[GL_n(\mathbb{K})]$-module generated by $\gamma \in \mathbb{K}[\mathcal{A}]$.

A more explicit formulation and a more direct proof of the preceding result can be found in [21].

16.8 On the complete reducibility of the span of covariants of weight zero. A "pre-Schur" description of polynomial $GL_n(\mathbb{K})$- irreducible representations

In this subsection, we will show that any left span $L(\varphi)$ can be decomposed into a direct sum of irreducible representations.

Since $L(\varphi) = \mathbb{K}[GL_n(\mathbb{K})] \circ \gamma$, $\sigma(\gamma) = \varphi$, the problem can be solved in the algebra $\mathbb{K}[\mathcal{A}]$.

Let $\varphi \in \mathbb{K}[\mathcal{A}] \otimes \mathbb{K}[\mathcal{X}]$ be a covariant of weight zero, and let $\gamma \in \mathbb{K}[\mathcal{A}]$ be its source. Without loss of generality, we assume that γ is a "homogeneous polynomial" of degree d.

Let

$$\gamma = \sum_{\lambda \vdash d} \sum_{\mathrm{sh}(T)=\lambda} \gamma_{\lambda,T}, \qquad \gamma_{\lambda,T} \neq 0,$$

$$\gamma_{\lambda,T} = \sum_{\mathrm{sh}(S)=\lambda} c^{\gamma}_{ST} \cdot \Big(S \Big| \boxed{T} \Big),$$

be the canonical expression of γ as a linear combination of standard right symmetrized bitableaux.

Corollary 16.2 (Decomposition theorem)**.** *Let $\varphi \in \mathbb{K}[\mathcal{A}] \otimes \mathbb{K}[\mathcal{X}]$ be a covariant of weight zero, and let $\gamma \in \mathbb{K}[\mathcal{A}]$ be its source, that is $\varphi = \sigma(\gamma)$. Then:*

$$\begin{aligned} L(\varphi) &= \mathbb{K}[GL_n(\mathbb{K})] \circ \gamma \\ &= \bigoplus_{\lambda \vdash d} \Big(\sum_{\mathrm{sh}(T)=\lambda} \mathbb{K}[GL_n(\mathbb{K})] \circ \gamma^{D_\lambda}_{\lambda T} \Big) \\ &= \bigoplus_{\lambda \vdash d} \Big(\sum_{\mathrm{sh}(T)=\lambda} L(\sigma(\gamma_{\lambda T})) \Big), \end{aligned}$$

where $L(\sigma(\gamma_{\lambda T})) = \mathbb{K}[GL_n(\mathbb{K})] \circ \gamma^{D_\lambda}_{\lambda T}$.

16.9 Weyl's First Fundamental Theorem

Let V be a vector space of finite dimension m. Since $char(\mathbb{K}) = 0$, we may identify the algebra $\mathbb{K}[\mathcal{A}]$ with the algebra $\mathbb{K}[V^{\oplus n}]$ of polynomial functions on the vector space $V^{\oplus n}$, by reading the variable $(a_i \,|\, \mu)$ as the *i-th coordinate function* on the μ-th copy of V in the direct sum $V^{\oplus n}$, for every $i = 1, 2, \ldots, m$, $\mu = 1, 2, \ldots, n$.

In plain words, we identify the algebra $\mathbb{K}[\mathcal{A}]$ with the algebra $\mathbb{K}[V^{\oplus n}] = Sym[(V^*)^{\oplus n}]$.

We recall the *contravariant* action of the general linear group $GL_m(\mathbb{K})$ on

$$\mathbb{K}[\mathcal{A}] = \mathbb{K}[V^{\oplus n}] = Sym[(V^*)^{\oplus n}]$$

is defined in the following way:

$$(\mathbf{g} \circ \gamma)(v_1, \ldots, v_n) = \gamma(\mathbf{g}^{-1}(v_1), \ldots, \mathbf{g}^{-1}(v_n)),$$

for all $\mathbf{g} \in GL_m(\mathbb{K})$, $\gamma \in \mathbb{K}[\mathcal{A}]$, $(v_1, \ldots, v_n) \in V^{\oplus n}$.

In matrix notation, the *diagonal* action of a matrix $T = (t_{hk}) \in GL_m(\mathbb{K})$ on $\mathbb{K}[\mathcal{A}]$ is given by

$$T \circ (a_i \,|\, \mu) = \sum_{k=1}^{m} \bar{t}_{ik}\ (a_k \,|\, \mu), \quad i = 1, \ldots, m, \quad \mu = 1, \ldots, n,$$

where

$$(\bar{t}_{hk}) = T^{-1}.$$

The algebra $\mathbb{K}[\mathcal{A}]$ is both a left $GL_n(\mathbb{K})$-module (see Subsection 16.1) and a left $GL_m(\mathbb{K})$-module, and the two actions clearly commute.

Furthermore, the action of the general linear group $GL_m(\mathbb{K})$ on every homogeneous component of $\mathbb{K}[\mathcal{A}]$ is implemented by the algebra of $\mathcal{A}$-letter polarizations, and the action of the general linear group $GL_n(\mathbb{K})$ is implemented by the algebra of $\mathcal{P}$-place polarizations, $\mathcal{P} = \underline{n} = \{1, 2, \ldots, n\}$.

Let $m \leq n$.

Informally speaking, the Deruyts–Capelli expansion formula says that any homogeneous polynomial function γ in n vector variables in dimension m may be expressed as a linear combination of polarized $GL_n(\mathbb{K})$-semiinvariants and, by Theorem 16.1, these $GL_n(\mathbb{K})$-semiinvariants involve only the first m^2 letterplace variables

$$(a_1 \,|\, \mu), \ldots, (a_m \,|\, \mu), \quad \mu = 1, \ldots, m;$$

furthermore, these $GL_n(\mathbb{K})$-semiinvariants may be obtained, in turn, by applying place polarization operators to the original polynomial function γ.

Since the place polarization process is a $GL_m(\mathbb{K})$-invariantive process (i.e., the actions of $GL_m(\mathbb{K})$ and $GL_n(\mathbb{K})$ commute), the study of $GL_m(\mathbb{K})$-invariant polynomial functions in n vector variables in dimension m is reduced to the study of invariant homogeneous polynomial functions in $m - 1$ vector variables.

To be precise, we recall the following definitions:

- Let G be a subgroup of the general linear group $GL_m(\mathbb{K})$.

  A polynomial function $\gamma \in \mathbb{K}[\mathcal{A}] = \mathbb{K}[V^{\oplus n}]$ is said to be a (formal) *relative G-invariant* if and only if the following condition holds:

  $$(\mathbf{g} \circ \gamma) = \lambda(\mathbf{g})\gamma,$$

  for all $\mathbf{g} \in G$, $\lambda(\mathbf{g}) \in \mathbb{K}$.

- The *bracket*
$$[j_1,\ldots,j_m], \quad j_h \in \mathcal{P} = \underline{n} = \{1,2,\ldots,n\}$$
is defined as follows:
$$[j_1,\ldots,j_m] = \det[(a_i\,|\,j_h)]_{i=1,\ldots,m,\ h=1,\ldots,m,} = (-1)^{\binom{m}{2}}(a_1a_2\cdots a_m\,|\,j_1j_2\cdots j_m).$$

Clearly, any bracket $[j_1,\ldots,j_m]$ is a relative G-invariant, G a subgroup of the general linear group $GL_m(\mathbb{K})$.

Let $\gamma \in \mathbb{K}[\mathcal{A}] = \mathbb{K}[V^{\oplus n}]$ be a polynomial function, γ homogeneous of degree d. Thanks to the Deruyts–Capelli expansion formula (Corollary 16.1), γ is expanded in the following form:
$$\gamma = \sum_{\substack{\lambda \vdash d \\ \lambda_1 \leq m}} \ \sum_{\mathrm{sh}(T)=\lambda} \Lambda_{\lambda,T} \circ \gamma^{D_\lambda}_{\lambda,T}, \qquad \Lambda_{\lambda,T} \in \mathbb{K}[GL_n(\mathbb{K})], \quad T \in Stab(\mathcal{P}),$$

where any
$$\gamma^{D_\lambda}_{\lambda,T} = \sum_{\mathrm{sh}(S)=\lambda} c^{\gamma}_{ST}\ (S\,|\,D_\lambda)$$
belongs to the cyclic $\mathbb{K}[GL_n(\mathbb{K})]$-module generated by $\gamma \in \mathbb{K}[\mathcal{A}]$.

Since the actions of $GL_m(\mathbb{K})$ and $GL_n(\mathbb{K})$ commute, if γ is a relative G-invariant (G a subgroup of $GL_m(\mathbb{K})$), the $GL_n(\mathbb{K})$-semiinvariants $\gamma^{D_\lambda}_{\lambda,T}$ are relative G-invariants.

Note that any $\gamma^{D_\lambda}_{\lambda,T}$ can be written in the form
$$\gamma^{D_\lambda}_{\lambda,T} = [1,2,\ldots,m]^{q_\lambda}\varphi_T,$$
where
$$q_\lambda = \#\{i;\ \lambda_i = m\},$$
and
$$\varphi_T = \pm \sum_S c^{\gamma}_{ST}\ (S^*\,|\,D_{\lambda^*}),$$
where $\lambda^* = (\lambda_{q_\lambda+1}, \lambda_{q_\lambda+2}, \ldots)$ and the tableaux S^* of shape λ^* are obtained from the tableaux S of shape λ by deleting the first q_λ top rows.

Since $\lambda^*_1 < m$, any φ_T is a polynomial in the $m \times m-1$ letterplace variables
$$\begin{array}{cccc}
(a_1\,|\,1), & (a_2\,|\,1), & \ldots, & (a_m\,|\,1) \\
(a_1\,|\,2), & (a_2\,|\,2), & \ldots, & (a_m\,|\,2) \\
\vdots & \vdots & & \vdots \\
(a_1\,|\,m-1), & (a_2\,|\,m-1), & \ldots, & (a_m\,|\,m-1)
\end{array}$$
or, equivalently, φ_T is a polynomial function
$$\varphi_T(v_1, v_2, \ldots, v_{m-1}) \in \mathbb{K}[V^{\oplus(m-1)}]$$

in the vector variables $v_1, v_2, \ldots, v_{m-1}$.

Hence, we have the following result.

Corollary 16.3 (Weyl's First Fundamental Theorem). *Let G be a subgroup of $GL_m(\mathbb{K})$, and let γ be a d-homogeneous relative G-invariant polynomial function in n vector variables, $n \geq m$. Then,*

$$\gamma = \sum_{\substack{\lambda \vdash d \\ \lambda_1 \leq m}} \sum_{T \in Stab_\lambda(P)} \Lambda_{\lambda,T} \circ ([1, 2, \ldots m]^{q_\lambda} \varphi_T),$$

where

$$\Lambda_{\lambda,T} \in \mathbb{K}[GL_n(\mathbb{K})],$$

and the φ_T's are both relative G-invariant polynomial functions in the vector variables $v_1, v_2, \ldots, v_{m-1}$ and $GL_n(\mathbb{K})$-semiinvariants.

For example, since the $GL_m(\mathbb{K})$-invariant polynomial functions which involve at most $m-1$ vector variables are just the constant functions (see, e.g., Theorem 10.2 and Remark 10.1) and polarizations of brackets yield linear combinations of brackets, the following well-known result immediately follows.

Corollary 16.4 (The first fundamental theorem for vector $GL_m(\mathbb{K})$-invariants). *Let γ be a relative $GL_m(\mathbb{K})$-invariant polynomial function on n vector variables, $n \geq m$. Then γ can be written as a homogeneous polynomial in the brackets*

$$[j_1, \ldots, j_m], \quad j_h \in \mathcal{P} = \underline{n} = \{1, 2, \ldots, n\}.$$

16.10 The Capelli identities

For several decades, the Capelli identities were regarded as the main technical tool to prove the Deruyts–Capelli Expansion formula and Weyl's theorem; however, as we recognized in Subsection 16.7 and 16.9, these results can be easily derived from general arguments.

The standard proofs of the Capelli identities are rather complicated (see., e.g., [40], [92]; in this subsection, by way of application of the method of virtual variables, we provide an elementary and mechanical proof of them.

Let $\mathcal{A} = \{a_1, a_2, \ldots, a_m\}$ and $\mathcal{P} = \underline{n} = \{1, 2, \ldots, n\}$ be negatively signed alphabets and let $\mathbb{K}[\mathcal{A}] = Super[\mathcal{A}|\ \underline{n}]$ be the *commutative* letterplace algebra generated by the positively signed letterplace alphabet $[\mathcal{A}\mid\ \underline{n}]$.

Let H_m be the *Capelli operator* in m variables:

$$H_m = \det \begin{bmatrix} \mathcal{D}_{a_1a_1} + (m-1)I & \mathcal{D}_{a_1a_2} & \ldots & \mathcal{D}_{a_1a_m} \\ \mathcal{D}_{a_2a_1} & \mathcal{D}_{a_2a_2} + (m-2)I & \ldots & \mathcal{D}_{a_2a_m} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{D}_{a_ma_1} & \mathcal{D}_{a_ma_2} & \ldots & \mathcal{D}_{a_ma_m} \end{bmatrix},$$

where the expansion of the determinant is by column from left to right.

Let Ω_n be the *Cayley operator* in dimension n:

$$\Omega_n = \det \begin{bmatrix} \frac{\partial}{\partial(a_1\,|\,1)} & \frac{\partial}{\partial(a_1\,|\,2)} & \cdots & \frac{\partial}{\partial(a_1\,|\,n)} \\ \frac{\partial}{\partial(a_2\,|\,1)} & \frac{\partial}{\partial(a_2\,|\,2)} & \cdots & \frac{\partial}{\partial(a_2\,|\,n)} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial}{\partial(a_n\,|\,1)} & \frac{\partial}{\partial(a_n\,|\,2)} & \cdots & \frac{\partial}{\partial(a_n\,|\,n)} \end{bmatrix},$$

where the symbol $\frac{\partial}{\partial(a_i\,|\,\mu)}$ denotes the (formal) partial derivative with respect to the variable $(a_i\,|\,\mu)$.

The following result is due to Capelli (see, e.g., [25], [92]).

Theorem 16.5. *Let $\gamma \in \mathbb{K}[\mathcal{A}]$. We have the following identities:*

$$H_m(\gamma) = \begin{cases} 0 & \textit{if } m > n, \\ [a_1, a_2, \ldots, a_n]\ \Omega_n(\gamma) & \textit{if } m = n, \end{cases}$$

where the bracket $[a_1, a_2, \ldots, a_n]$ denotes the determinant

$$\det[(a_i\,|\,\mu)]_{i,\ \mu=1,2,\ldots,m=n}.$$

Proof. We regard the algebra $\mathbb{K}[\mathcal{A}] = Super[\mathcal{A}|\ \underline{n}]$ as a subalgebra of the letterplace superalgebra $Super[\mathcal{A} \cup \{\alpha\}|\ \underline{n}]$, α a (virtual) positive letter.

From Subsection 6.6, we recall that the action on $\mathbb{K}[\mathcal{A}]$ of the Capelli operator H_m is the same as the action of the operator

$$\mathcal{D}_{a_1,\alpha}\mathcal{D}_{a_2,\alpha}\ldots\mathcal{D}_{a_m,\alpha}\mathcal{D}_{\alpha,a_m}\mathcal{D}_{\alpha,a_{m-1}}\ldots\mathcal{D}_{\alpha,a_1}.$$

Consider a momomial $\mathbf{m} = \prod_{i,\mu}(a_i\,|\,\mu)^{d_{i\mu}}$, with $d_{i\mu} \in \mathbb{N}$, in $\mathbb{K}[\mathcal{A}]$.

Clearly, all the monomials in the polynomial

$$\mathcal{D}_{\alpha,a_m}\mathcal{D}_{\alpha,a_{m-1}}\ldots\mathcal{D}_{\alpha,a_1}(\mathbf{m}) \in Super[\mathcal{A} \cup \{\alpha\}|\ \underline{n}] \qquad (*)$$

contain exactly m occurrences of the positive letter α.

Note that the letterplace variables $(\alpha\,|\,\mu)$ are of $\mathbb{Z}_2$-degree 1.

In the case $m > n$, all the monomials in the polynomial $(*)$ must contain at least a square of such a letterplace variable $(\alpha\,|\,\mu)$; thus,

$$\mathcal{D}_{\alpha,a_m}\mathcal{D}_{\alpha,a_{m-1}}\ldots\mathcal{D}_{\alpha,a_1}(\mathbf{m}) = 0,$$

and the first assertion is proved, by linearity.

Let us now consider the case $m = n$. Set $(a_i \,|\, \mu)^{d_{i\mu}-1} = 0$ whenever $d_{i\mu} = 0$, and recall that $(\alpha \,|\, \mu)^2 = 0$ for $\mu = 1, 2, \ldots, n$. We have the identity

$$\begin{aligned}
&\mathcal{D}_{\alpha,a_n}\mathcal{D}_{\alpha,a_{n-1}} \ldots \mathcal{D}_{\alpha,a_1}(\mathbf{m}) \\
&\qquad = \sum_{\sigma}\Bigg(\prod_{i,\mu} d_{i,\sigma(i)}(a_i \,|\, \mu)^{d_{i,\mu}-\delta_{i,\sigma(i)}}(\alpha \,|\, \sigma(n))(\alpha \,|\, \sigma(n-1)) \ldots (\alpha \,|\, \sigma(1))\Bigg),
\end{aligned}$$

where the summation is over all the permutations σ of the set $\{1, 2, \ldots, n\}$ and δ is the Kronecker symbol.

The last expression can be rewritten in the form

$$\sum_{\sigma}\left[\frac{\partial}{\partial(a_1 \,|\, \sigma(1))}a\Bigg(\prod_{\mu}(a_1 \,|\, \mu)^{d_{1\mu}}\Bigg) \cdots \frac{\partial}{\partial(a_n \,|\, \sigma(n))}\Bigg(\prod_{\mu}(a_n \,|\, \mu)^{d_{n\mu}}\Bigg)\right]$$
$$(\alpha \,|\, \sigma(n)) \ldots (\alpha \,|\, \sigma(1)).$$

Since the variables $(\alpha \,|\, \sigma(\mu))$ anticommute, we get the identities

$$\begin{aligned}
&\mathcal{D}_{\alpha,a_n}\mathcal{D}_{\alpha,a_{n-1}} \ldots \mathcal{D}_{\alpha,a_1}(\mathbf{m}) \\
&\qquad = (\alpha \,|\, n)(\alpha \,|\, n-1) \ldots (\alpha \,|\, 1)\left(\sum_{\sigma}(-1)^{|\sigma|}\frac{\partial}{\partial(a_1 \,|\, \sigma(1))} \cdots \frac{\partial}{\partial(a_n \,|\, \sigma(n))}\right)(\mathbf{m}) \\
&\qquad = (\alpha \,|\, n)(\alpha \,|\, n-1) \ldots (\alpha \,|\, 1)\ \Omega_n(\mathbf{m}).
\end{aligned}$$

Since

$$\mathcal{D}_{a_1,\alpha}\mathcal{D}_{a_2,\alpha} \cdots \mathcal{D}_{a_n,\alpha}(\alpha \,|\, n)(\alpha \,|\, n-1) \ldots (\alpha \,|\, 1) = [a_1, a_2, \ldots, a_n],$$

the second assertion is proved, by linearity. □

17 $\mathbb{Z}_2$-Graded Tensor Representations: the Berele–Regev Theory

From now on let $\mathcal{L} = \{1, 2, \ldots, n\} = \mathcal{L}_0$ be an alphabet of n positive letters, and $\mathcal{P} = \mathcal{P}_0 \cup \mathcal{P}_1$ a finite alphabet of places, with $|\mathcal{P}_0| = r$, $|\mathcal{P}_1| = s$.

A monomial of $Super_n[\mathcal{L}|\mathcal{P}]$ is said to be letter-*multilinear* whenever it contains each letter of $\mathcal{L}$ exactly once; a Young tableau T over $\mathcal{L}$ is said to be *multilinear* whenever each letter of $\mathcal{L}$ appears exactly once in T; obviously, if T is multilinear over $\mathcal{L}$, then $\mathrm{sh}(T) \vdash n$.

We consider the following structures:

- $Super_n[\underline{\mathcal{L}}|\mathcal{P}]$, the subspace of $Super_n[\mathcal{L}|\mathcal{P}]$ freely generated by the letter-multilinear monomials or, equivalently, by the standard symmetrized bitableaux $\left(S \,\middle|\, \boxed{T}\right)$, where S is a multilinear tableau on $\mathcal{L}$;

- $\underline{\mathcal{B}}_n$, the operator algebra linearly generated by the orthonormal letter generators $Y_n\big(S', \boxed{S}\big)$, with S and S' multilinear tableaux over $\mathcal{L}$, restricted to the subspace $Super_n[\underline{\mathcal{L}}|\mathcal{P}]$.
- ${}_n\mathcal{B}'$, the operator algebra generated by the proper place polarizations, or, equivalently, by the place orthonormal generators $\big(T, \boxed{T'}\big)_n Y$, $T, T' \in Stab_\lambda(\mathcal{P})$, restricted to the subspace $Super_n[\underline{\mathcal{L}}|\mathcal{P}]$.

Note that $Super_n[\underline{\mathcal{L}}|\mathcal{P}]$ is invariant under the action of $\underline{\mathcal{B}}_n$, ${}_n\mathcal{B}'$, so we have the bimodule

$$\underline{\mathcal{B}}_n \cdot Super_n[\underline{\mathcal{L}}|\mathcal{P}] \cdot_n \mathcal{B}'.$$

By specialization of the general theory, we have that: the space $Super_n[\underline{\mathcal{L}}|\mathcal{P}]$ is a semisimple $\underline{\mathcal{B}}_n$-module, and it is a semisimple ${}_n\mathcal{B}'$-module; the operator algebras $\underline{\mathcal{B}}_n$ and ${}_n\mathcal{B}'$ are semisimple, and one is the centralizer of the other in the endomorphism algebra of $Super_n[\underline{\mathcal{L}}|\mathcal{P}]$. The basic elements and operators, that is, the orthonormal letter generators $Y_n\big(S', \boxed{S}\big)$, the symmetrized bitableaux $\big(S \,\big|\, \boxed{T}\big)$, and the place orthonormal generators $\big(T, \boxed{T'}\big)_n Y$, are parametrized by tableaux such that

$$S', S \in Stab_\lambda(L), \quad S', S \text{ multilinear}, \qquad T, T' \in Stab_\lambda(\mathcal{P}),$$

where the partitions $\lambda \vdash n$ satisfy the hook condition

$$\lambda_{r+1} < s + 1.$$

17.1 The letter multilinear subspace as a $\mathbb{K}[\mathbf{S}_n]$-module

Note that a group action of the symmetric group $\mathbf{S}_n$ on $Super_n[\underline{\mathcal{L}}|\mathcal{P}]$ is consistently defined by setting

$$\sigma \cdot (i_1 \,|\, y_{j_1})(i_2 \,|\, y_{j_2}) \ldots (i_n \,|\, y_{j_n}) = (\sigma(i_1) \,|\, y_{j_1})(\sigma(i_2) \,|\, y_{j_2}) \ldots (\sigma(i_n) \,|\, y_{j_n}).$$

This action defines a representation

$$\rho : \mathbb{K}[\mathbf{S}_n] \to End_{\mathbb{K}}[Super_n[\underline{\mathcal{L}}|\mathcal{P}]].$$

Notice that the representation of any permutation σ can be regarded as a multilinear letter Capelli column:

$$\left[\begin{array}{c|c} \sigma(1) & 1 \\ \vdots & \vdots \\ \sigma(n) & n \end{array}\right] = \mathcal{D}_{\sigma(1)\alpha_1} \cdots \mathcal{D}_{\sigma(n)\alpha_n} \mathcal{D}_{\alpha_n n} \cdots \mathcal{D}_{\alpha_1 1} = \rho(\sigma).$$

Indeed, we have the following result.

Proposition 17.1. *The operator algebra induced by the action of the symmetric group coincides with the operator algebra generated by the multilinear orthonormal letter generators:*

$$\underline{\mathcal{B}}_n = \rho(\mathbb{K}[S_n]).$$

Proof. [20] To begin with, we show that each operator associated to a permutation belongs to the algebra $\underline{\mathcal{B}}_n$. Indeed, for any permutation $\sigma \in S_n$, and for any multilinear standard symmetrized bitableau $\left(T \,\middle|\, \boxed{U}\right)$, we have

$$\begin{aligned}
\sigma \cdot \left(T \,\middle|\, \boxed{U}\right) &= \left(\sigma \cdot T \,\middle|\, \boxed{U}\right) \\
&= \sum_{S'} c^{\sigma}_{TS'} \left(S' \,\middle|\, \boxed{U}\right) \\
&= \left(\sum_{S',S} c^{\sigma}_{SS'}\; Y_n(S', S)\right)\left(T \,\middle|\, \boxed{U}\right),
\end{aligned}$$

where S' and S range over all the multilinear standard letter tableaux.

On the other hand, any multilinear Young–Capelli symmetrizer belongs to the subalgebra $\rho(\mathbb{K}[S_n])$. Indeed, by Proposition 10.1 (Factorization Theorem), the action of $\gamma_n(S', S)$ on $Super_n[\underline{\mathcal{L}}|\mathcal{P}]$ is the same, up to a scalar factor, as the action of an operator of the form

$$\mathcal{D}_{S'C}\mathcal{D}_{CT}\mathcal{D}_{TD}\mathcal{D}_{DS},$$

where T denotes any multilinear standard tableau of the same shape as S' and S, filled with positive virtual letters $\alpha_1, \ldots, \alpha_n$ not appearing in the tableaux C and D. Furthermore, by the remark at the end of Subsection 10.4 and the third example in Subsection 6.6, the action of this operator is the same, up to a scalar factor, as the action of an operator of the form

$$\begin{aligned}
&\sum_{\sigma} \mathcal{D}_{\sigma(1)\alpha_1}\mathcal{D}_{\sigma(2)\alpha_2}\ldots\mathcal{D}_{\sigma(n)\alpha_n} \sum_{\tau} (-1)^{\tau} \mathcal{D}_{\alpha_1\tau(1)}\mathcal{D}_{\alpha_2\tau(2)}\ldots\mathcal{D}_{\alpha_n\tau(n)} \\
&\qquad = \sum_{\sigma,\tau} \pm \mathcal{D}_{\sigma(1)\alpha_1}\mathcal{D}_{\sigma(2)\alpha_2}\ldots\mathcal{D}_{\sigma(n)\alpha_n}\mathcal{D}_{\alpha_1\tau(1)}\mathcal{D}_{\alpha_2\tau(2)}\ldots\mathcal{D}_{\alpha_n\tau(n)}.
\end{aligned}$$

where σ, τ range over suitable sets of permutations. Finally, notice that each summand acts on $Super_n[\underline{\mathcal{L}}|\mathcal{P}]$ in the same way as a permutation. □

Remark 17.1. 1. The action of $\gamma_n\left(S', \boxed{S}\right)$ is, up to sign, the same as the action of

$$\pi_{S'S} \sum_{\substack{\tau \in R(S) \\ \xi \in C(S)}} (-1)^{|\xi|} \tau\xi,$$

where $R(S)$ and $C(S)$ are the row-stabilizer and the column-stabilizer of the tableau S, respectively, and $\pi_{S'S} \in \mathbf{S}_n$ is the permutation such that $\pi_{S'S}(S) = S'$.

The element

$$e^*_{S'S} = \pi_{S'S} \sum_{\substack{\tau \in R(S) \\ \xi \in C(S)}} (-1)^{|\xi|} \tau\xi \in \mathbb{K}[\mathbf{S}_n]$$

is called a generalized (dual) Young symmetrizer. Clearly, $e^*_{S'S} = \pi_{S'S} e^*_{SS}$, for any pair of multilinear tableaux (S', S).

2. Let $S' \in Tab(\mathcal{L})$ be any multilinear tableau. Then,we have the following result:

$$\left(S' \,\middle|\, \boxed{T}\right) = \pm\gamma_n \left(S', \boxed{S}\right) \langle S \,|\, T \rangle,$$

where $\langle S \,|\, T \rangle$ is the tableau monomial in $Super_n[\underline{\mathcal{L}}|\mathcal{P}]$ associated to the pair (S, T) and $S \in Tab(\mathcal{L})$ is *any* multilinear tableau such that $sh(S) = sh(T)$. As a matter of fact, (in short notation) we have

$$\left(S' \,\middle|\, \boxed{T}\right) = S'C \; CD \; \underline{DT} = \pm S'C \; CD \; DS \; \underline{ST} = \pm\gamma_n \left(S', \boxed{S}\right) \langle S \,|\, T \rangle.$$

17.2 The $\mathbb{Z}_2$-graded tensor representation theory

Let

$$W = \langle \mathcal{P} \rangle_{\mathbb{K}} = \langle \mathcal{P}_0 \rangle_{\mathbb{K}} \oplus \langle \mathcal{P}_1 \rangle_{\mathbb{K}} = W_0 \oplus W_1,$$

then

$$Super_n[\underline{\mathcal{L}}|\mathcal{P}] \cong T^n[W_0 \oplus W_1]$$

via the linear isomorphism

$$\mathcal{F} : (1 \,|\, y_{i_1}) \ldots (n \,|\, y_{i_n}) \mapsto y_{i_1} \otimes \ldots \otimes y_{i_n}.$$

Therefore, any construction and any result pertaining to the multilinear letter subspace $Super_n[\underline{\mathcal{L}}|\mathcal{P}]$ may be carried over to the space $T^n[W_0 \oplus W_1]$ of the n-tensors over the $\mathbb{Z}_2$-graded vector space $W = W_0 \oplus W_1$.

The natural action of the symmetric group $\mathbf{S}_n$ on $Super_n[\underline{\mathcal{L}}|\mathcal{P}]$, gives rise to a representation

$$\mathbb{K}[S_n] \rightarrow End_{\mathbb{K}}[T^n[W_0 \oplus W_1]],$$

which is called a *Berele–Regev* $\mathbb{Z}_2$*-graded representation* of $\mathbf{S}_n$.

Example 17.1. *Let* $y_1, y_2, y_3 \in \mathcal{P}$, $|y_1| = 0, |y_2| = |y_3| = 1$. *Let* $n = 3$. *We have:*

$$\begin{aligned}(12)\cdot y_1\otimes y_2\otimes y_3 &= \mathcal{F}\left((12)\cdot(1\,|\,y_1)(2\,|\,y_2)(3\,|\,y_3)\right)\\ &= \mathcal{F}\left((2\,|\,y_1)(1\,|\,y_2)(3\,|\,y_3)\right)\\ &= \mathcal{F}\left((1\,|\,y_2)(2\,|\,y_1)(3\,|\,y_3)\right) = y_2\otimes y_1\otimes y_3.\end{aligned}$$

$$\begin{aligned}(23)\cdot y_1\otimes y_2\otimes y_3 &= \mathcal{F}\left((23)\cdot(1\,|\,y_1)(2\,|\,y_2)(3\,|\,y_3)\right)\\ &= \mathcal{F}\left((1\,|\,y_1)(3\,|\,y_2)(2\,|\,y_3)\right)\\ &= \mathcal{F}\left(-(1\,|\,y_1)(2\,|\,y_3)(3\,|\,y_2)\right) = -y_1\otimes y_3\otimes y_2.\end{aligned}$$

On the other hand, the action of $pl(\mathcal{P})$ on $Super_n[\underline{\mathcal{L}}|\mathcal{P}]$ gives rise to a representation

$$End_{\mathbb{K}}[T^n[W_0\oplus W_1]] \leftarrow pl(W)$$

of the general linear Lie superalgebra $pl(W)$. Therefore, by specializing the results of Section 12 to the bimodule

$$\mathbb{K}[S_n]\cdot T^n[W_0\oplus W_1]\cdot pl(W_0\oplus W_1),$$

we get the following results.

Theorem 17.1.

1. *The subalgebras of $End_{\mathbb{K}}[T^n[W_0\oplus W_1]$ induced by the actions of*

$$\mathbb{K}[\mathbf{S}_n] \quad \textit{and} \quad pl(W_0\oplus W_1)$$

   are the centralizers of each other.

2. *A complete decomposition of $T^n[W_0\oplus W_1]$ with respect to the action of the symmetric group S_n is given by:*

$$\begin{aligned}T^n[W_0\oplus W_1] &= \mathcal{F}\left[Super_n[\underline{\mathcal{L}}|\mathcal{P}]\right]\\ &= \bigoplus_{\substack{\lambda\in H(\mathcal{P})\\ \lambda\vdash n}} \bigoplus_{\substack{T\in Stab(\mathcal{P})\\ \mathrm{sh}(T)=\lambda}} \mathcal{F}\left[\left\langle\left(S\,\middle|\,\boxed{T}\right),\ \ S\in Stab(\mathcal{L}),\ S\ \textit{multilinear}\right\rangle\right];\end{aligned}$$

3. *A complete decomposition of $T^n[W_0\oplus W_1]$ with respect to the action of the general linear Lie superalgebra $pl(W_0\oplus W_1)$ is given by:*

$$\begin{aligned}T^n[W_0\oplus W_1] &= \mathcal{F}\left[Super_n[\underline{\mathcal{L}}|\mathcal{P}]\right]\\ &= \bigoplus_{\substack{\lambda\in H(\mathcal{P})\\ \lambda\vdash n}} \bigoplus_{\substack{S\in Stab(\mathcal{L}),\ S\ \textit{multilinear}\\ \mathrm{sh}(S)=\lambda}} \mathcal{F}\left[\left\langle\left(S\,\middle|\,\boxed{T}\right),\ \ T\in Stab(\mathcal{P})\right\rangle\right].\end{aligned}$$

The irreducible $\mathbb{K}[\mathbf{S}_n]$-submodules and $pl(W_0 \oplus W_1)$-submodules which appear in the preceding complete decomposition theorem admit a direct description as subspaces of the tensor space $T^n[W_0 \oplus W_1]$.

First of all, we notice that any decomposable tensor $y_{i_1} \otimes \ldots \otimes y_{i_n} \in T^n[W_0 \oplus W_1]$ is (up to a sign) the image under $\mathcal{F}$ of a bitableau monomial $\langle S \,|\, T \rangle$, where $S \in Tab(\mathcal{L})$ is a multilinear tableau, $T \in Tab(\mathcal{P})$, $sh(S) = sh(T) \vdash n$. Clearly, this representation is not unique.

For example, let $y_1, y_2, y_3 \in \mathcal{P}$, $|y_1| = 0, |y_2| = |y_3| = 1$; then

$$\begin{aligned} y_1 \otimes y_2 \otimes y_3 &= \mathcal{F}\left(\left\langle\ 123 \,\middle|\, y_1 y_2 y_3 \ \right\rangle\right) = \mathcal{F}\left((1 \,|\, y_1)(2 \,|\, y_2)(3 \,|\, y_3)\right) \\ &= \mathcal{F}\left(-\left\langle \begin{array}{c} 13 \\ 2 \end{array} \middle| \begin{array}{l} y_1 y_3 \\ y_2 \end{array} \right\rangle\right) = \mathcal{F}\left(-(1 \,|\, y_1)(3 \,|\, y_3)(2 \,|\, y_2)\right). \end{aligned}$$

Thanks to Remark 17.1, we get the following results.

Remark 17.2. The irreducible $\mathbb{K}[\mathbf{S}_n]$-submodule

$$\mathcal{F}\left[\left\langle \left(S \,\middle|\, \boxed{T} \right), \quad S \in Stab_\lambda(\mathcal{L}),\ S \text{ multilinear} \right\rangle\right]$$

equals

$$\{\alpha \cdot \mathcal{F}\left(\langle U \,|\, T \rangle\right);\ \alpha \in \mathbb{K}[\mathbf{S}_n] e^*_{UU}\} \subseteq T^n[W_0 \oplus W_1],$$

for every multilinear tableau $U \in Stab_\lambda(\mathcal{L})$ and, thus, it is $\mathbb{K}[\mathbf{S}_n]$-isomorphic to the (minimal) left ideal $\mathbb{K}[\mathbf{S}_n] e^*_{UU}$ of $\mathbb{K}[\mathbf{S}_n]$.

Example 17.2. *Let $y_1, y_2, y_3 \in \mathcal{P}$, and let $|y_1| = 0, |y_2| = |y_3| = 1$. Consider the tableaux $U, U' \in Tab(\mathcal{L})$ and $T \in Stab(\mathcal{P})$,*

$$U = \begin{array}{c} 12 \\ 34 \end{array} \qquad U' = \begin{array}{c} 13 \\ 24 \end{array} \qquad T = \begin{array}{l} y_1 y_3 \\ y_2 y_3 \end{array}.$$

Note that

$$\mathcal{F}\left(\left\langle \begin{array}{c} 12 \\ 34 \end{array} \middle| \begin{array}{l} y_1 y_3 \\ y_2 y_3 \end{array} \right\rangle\right) = y_1 \otimes y_3 \otimes y_2 \otimes y_3, \quad \mathcal{F}\left(\left\langle \begin{array}{c} 13 \\ 24 \end{array} \middle| \begin{array}{l} y_1 y_3 \\ y_2 y_3 \end{array} \right\rangle\right) = -y_1 \otimes y_2 \otimes y_3 \otimes y_3.$$

The irreducible $\mathbb{K}[\mathbf{S}_n]$-submodule

$$\mathcal{F}\left[\left\langle \left(S \,\middle|\, \boxed{T} \right), \quad S \in Stab_{(2,2)}(\mathcal{L}),\ S \textit{ multilinear} \right\rangle\right] \subseteq T^4[W_0 \oplus W_1]$$

equals

$$\{\alpha \cdot (y_1 \otimes y_3 \otimes y_2 \otimes y_3);\ \alpha \in \mathbb{K}[\mathbf{S}_n] e^*_{UU}\} = \{\alpha \cdot (-y_1 \otimes y_2 \otimes y_3 \otimes y_3);\ \alpha \in \mathbb{K}[\mathbf{S}_n] e^*_{U'U'}\}.$$

Remark 17.3. The irreducible $pl(W_0 \oplus W_1)$-submodule

$$\mathcal{F}\left[\left\langle \left(S \,\middle|\, \boxed{T} \right), \quad T \in Stab_\lambda(\mathcal{P}) \right\rangle\right]$$

equals

$$e^*_{SS}\left[T^n[W_0 \oplus W_1]\right],$$

the image of the tensor space $T^n[W_0 \oplus W_1]$ under the action of the (dual) Young symmetrizer e^*_{SS}.

Remark 17.4. If $\mathcal{P}$ is trivially $\mathbb{Z}_2$-graded, it follows from Subsection 4.5 that the preceding results yield the classical Schur–Weyl tensor representation theory of the general linear group $GL(W)$ and of the symmetric group $\mathbf{S}_n$ ([76], [77], [92]).

18 The Symmetric Group

From now on let $\mathcal{L} = \mathcal{L}_1 = \mathcal{P} = \mathcal{P}_1 = \{1, 2, \ldots, n\}$, an alphabet of n negative symbols, and consider the letterplace algebra (in *commutative* letterplace variables) $Super_n[\mathcal{L}|\mathcal{P}]$.

In the sequel, given a partition $\lambda \vdash n$, we will write $S_{\lambda 1}, S_{\lambda 2}, \ldots, S_{\lambda \underline{f}_\lambda}$ to mean the list of all multilinear standard tableaux of shape λ over $\mathcal{L} = \mathcal{P} = \{1, 2, \ldots, n\}$ sorted with respect to the linear order defined in Subsection 5.5. When the shape is clear from the context, we write simply S_i instead of $S_{\lambda i}$.

We recall that the "doubly indexed" *Young symmetrizers* (of shape $\lambda \vdash n$) are the elements of the group algebra $\mathbb{K}[\mathbf{S_n}]$ defined as follows:

$$e_{ij} = \pi_{ij}\Bigg(\sum_{\substack{\tau \in R(S_j) \\ \xi \in C(S_j)}} (-1)^{|\tau|} \tau \xi \Bigg),$$

where $R(S_j)$ and $C(S_j)$ are the row-stabilizer and the column-stabilizer of S_j, respectively, and $\pi_{ij} \in \mathbf{S}_n$ is the (unique) permutation such that

$$\pi_{ij}(S_j) = S_i.$$

In the classical notation, $e_{ij} = \pi_{ij} e_j$, where e_j denotes the *Young symmetrizer* associated to the tableau S_j (see, e.g., [53]).

We consider the module

$$\underline{\underline{\mathcal{B}}}_n \cdot Super_n[\underline{\mathcal{L}}|\underline{\mathcal{P}}],$$

where

- $Super_n[\underline{\mathcal{L}}|\underline{\mathcal{P}}]$ is the subspace of $Super_n[\mathcal{L}|\mathcal{P}]$ spanned by all the doubly multilinear monomials, that is, monomials of the form

$$(\tau(1)\,|\,1)(\tau(2)\,|\,2)\cdots(\tau(n)\,|\,n), \quad \tau \in \mathbf{S}_n,$$

or, equivalently, by the doubly multilinear symmetrized bitableaux $\left(S \,\middle|\, \boxed{T}\right)$, S and T multilinear tableaux on $\mathcal{L} = \mathcal{P}$. Notice that the set of right symmetrized bitableux

$$\left(S_{\lambda i}, \boxed{S_{\lambda j}}\right), \qquad i, j = 1, 2, \ldots, \underline{f}_\lambda, \quad \lambda \vdash n,$$

is a $\mathbb{K}$-linear basis of $Super_n[\underline{\mathcal{L}}|\underline{\mathcal{P}}]$.

- $\underline{\underline{\mathcal{B}}}_n$ is the algebra of operators on $Super_n[\underline{\mathcal{L}}|\underline{\mathcal{P}}]$ linearly generated by the restrictions of the orthonormal letter generators $Y_n\big(S', \boxed{S}\big)$, S and S' multilinear tableaux on $\mathcal{L}$. $Y_n\big(S', \boxed{S}\big)$, We hardly need to recall that the set of restricted operators

$$Y_n\left(S_{\lambda i}, \boxed{S_{\lambda j}}\right), \qquad i, j = 1, \ldots, \underline{f}_\lambda \quad \lambda \vdash n$$

is a $\mathbb{K}$-linear basis of $\underline{\underline{\mathcal{B}}}_n$. Therefore, this algebra is a semisimple subalgebra of $End_{\mathbb{K}}(Super_n[\underline{\mathcal{L}}|\underline{\mathcal{P}}])$, and admits a complete decomposition

$$\underline{\underline{\mathcal{B}}}_n = \bigoplus_{\lambda \vdash n} \bigoplus_{j=1,2,\ldots,\underline{f}_\lambda} \left\langle Y_n\left(S_{\lambda i}, \boxed{S_{\lambda j}}\right),\ i = 1, 2, \ldots, \underline{f}_\lambda \right\rangle_{\mathbb{K}} = \bigoplus_{\lambda \vdash n} \underline{\underline{\mathcal{B}}}_\lambda,$$

where each simple component $\underline{\underline{\mathcal{B}}}_\lambda$ is isomorphic to $M_{\underline{f}_\lambda}$, the full $\mathbb{K}$-algebra of square matrices of order $\underline{f}_\lambda$.

18.1 The doubly multilinear subspace as a $\mathbb{K}[\mathbf{S_n}]$-module

The action of the symmetric group $\mathbf{S}_n$ on $Super_n[\underline{\mathcal{L}}|\underline{\mathcal{P}}]$ defined by setting

$$\sigma \cdot ((\tau(1)\,|\,1)((\tau(2)\,|\,2)\cdots(\tau(n)\,|\,n)) = (\sigma\tau(1)\,|\,1)(\sigma\tau(2)\,|\,2)\cdots(\sigma\tau(n)\,|\,n)$$

induces a faithful representation

$$\underline{\rho}: \ \mathbb{K}[\mathbf{S}_n] \to End_{\mathbb{K}}[Super_n[\underline{\mathcal{L}}|\underline{\mathcal{P}}]].$$

By specializing the argument of Proposition 17.1, we get:

Proposition 18.1. *The operator algebra induced by the action of the symmetric group coincides with the operator algebra generated by the multilinear orthonormal letter generators:*

$$\underline{\rho}[\mathbb{K}[\mathbf{S}_n]] = \underline{\underline{\mathcal{B}}}_n,$$

Furthermore, $\underline{\rho}$ *induces a* $\mathbb{K}$*-algebra isomorphism* $\mathbb{K}[\mathbf{S}_n] \cong \underline{\underline{\mathcal{B}}}_n$.

We remark that the Young–Capelli symmetrizers are, up to a sign, the representations of the classical two-parameter Young symmetrizers:

$$\underline{\rho}(e_{ij}) = (-1)^{\binom{n}{2}} \gamma_n\left(S_i, \boxed{S_j}\right).$$

18.2 Complete decompositions of the group algebra $\mathbb{K}[S_n]$ as a left regular module.

The map

$$F : Super_n[\underline{\mathcal{L}}|\underline{\mathcal{P}}] \to \mathbb{K}[\mathbf{S}_n], \qquad (\tau(1)\,|\,1)(\tau(2)\,|\,2)\cdots(\tau(n)\,|\,n) \mapsto \tau$$

is an isomorphism from the module

$$\mathbb{K}[\mathbf{S}_n] \cdot Super_n[\underline{\mathcal{L}}|\underline{\mathcal{P}}]$$

to the module

$$\mathbb{K}[\mathbf{S}_n] \cdot \mathbb{K}[\mathbf{S}_n].$$

The symmetrized bitableaux are sent, by this isomorphism to the classical two-parameter Young symmetrizers:

$$F\left(\left(S_i \,\middle|\, \boxed{S_j}\right)\right) = (-1)^{\binom{n}{2}} e_{ij}.$$

By combining Theorem 12.1 with Proposition 18.1, we get immediately the following theorem.

Theorem 18.1 (Complete decomposition of $\mathbb{K}[\mathbf{S}_n]$ as a left regular module).

$$\begin{aligned}
\mathbb{K}[\mathbf{S}_n] &= F[Super_n[\underline{\mathcal{L}}|\underline{\mathcal{P}}]] \\
&= F\left[\bigoplus_{\lambda\vdash n} \bigoplus_{j=1,2,\ldots,\underline{f}_\lambda} \left\langle \left(S_{\lambda i} \,\middle|\, \boxed{S_{\lambda j}}\right),\ i=1,2,\ldots,\underline{f}_\lambda \right\rangle_{\mathbb{K}}\right] \\
&= \bigoplus_{\lambda\vdash n} \bigoplus_{j=1,2,\ldots,\underline{f}_\lambda} F\left[\left\langle \left(S_{\lambda i} \,\middle|\, \boxed{S_{\lambda j}}\right),\ i=1,2,\ldots,\underline{f}_\lambda \right\rangle_{\mathbb{K}}\right] \\
&= \bigoplus_{\lambda\vdash n} \bigoplus_{j=1,2,\ldots,\underline{f}_\lambda} \langle \pi_{ij} e_j;\ i=1,2,\ldots,\underline{f}_\lambda \rangle_{\mathbb{K}} \\
&= \bigoplus_{\lambda\vdash n} \bigoplus_{j=1,2,\ldots,\underline{f}_\lambda} \mathbb{K}[\mathbf{S}_n] e_j,
\end{aligned}$$

where the outer sum is the isotypic decomposition, and

$$\langle \pi_{ij} e_j;\ i=1,2,\ldots,\underline{f}_\lambda \rangle_{\mathbb{K}} = \mathbb{K}[\mathbf{S}_n] e_j, \qquad j=1,2,\ldots,f_\lambda^M,$$

are minimal left ideals of $\mathbb{K}[\mathbf{S}_n]$.

The irreducible $\mathbb{K}[\mathbf{S}_n]$-module

$$\mathcal{S}_{\lambda j} = \left\langle \left(S_{\lambda i} \,\middle|\, \boxed{S_{\lambda j}}\right),\ i=1,2,\ldots,\underline{f}_\lambda \right\rangle_{\mathbb{K}} = F^{-1}[\mathbb{K}[\mathbf{S}_n] \cdot e_j]$$

is the *Specht module* (of the first kind) associated to the multilinear standard tableau $S_{\lambda j}$ of shape λ (see, e.g., [28], [73]).

18.3 On the coefficients h_λ (Lemma 8.3 and Theorem 10.1).

Proposition 18.2. *The coefficient h_λ which appears in Lemma 8.3 and Theorem 10.1 equals the product of hook lengths of the shape λ.*

Proof. We already know that the coefficient h_λ depends only on the shape λ.

Let S_i be the i-th multilinear standard tableau of shape λ.

The Triangularity Theorem (Theorem 10.1) specializes to the following identity:

$$\gamma_n\left(S_i, \boxed{S_i}\right)\left(S_i \,\middle|\, \boxed{S_i}\right) = (-1)^{\binom{n}{2}} \theta^{-+}_{S_i S_i} h_\lambda \left(S_i \,\middle|\, \boxed{S_i}\right),$$

with $\theta^{-+}_{S_i S_i} = 1$.

Then, the following identities hold:

$$\begin{aligned} F\left(\underline{\rho}(e_{ii})\left(S_i \,\middle|\, \boxed{S_i}\right)\right) &= F\left((-1)^{\binom{n}{2}} \gamma_n\left(S_i, \boxed{S_i}\right)\left(S_i \,\middle|\, \boxed{S_i}\right)\right) \\ &= F\left(h_\lambda\left(S_i \,\middle|\, \boxed{S_i}\right)\right) \\ &= (-1)^{\binom{n}{2}} h_\lambda e_{ii}. \end{aligned}$$

On the other hand, we have:

$$F\left(\underline{\rho}(e_{ii})\left(S_i \,\middle|\, \boxed{S_i}\right)\right) = e_{ii} \cdot F\left(\left(S_i \,\middle|\, \boxed{S_i}\right)\right) = (-1)^{\binom{n}{2}} e_{ii}^2.$$

Hence $e_{ii}^2 = h_\lambda e_{ii}$.

Since it is well-known that $e_{ii}^2 = \prod (\text{hook lenghts of } \lambda)\, e_{ii}$, (see, e.g., [40], [73], [53], [72]), the assertion follows.

$\square$

18.4 The Young natural form of irreducible representations

In the following, we specialize the constructions and the results of Section 13.

Given any partition $\lambda \vdash n$ and any multilinear standard place tableau $S_{\lambda j}$ of shape λ, the module structure $\mathbb{K}[\mathbf{S}_n] \cdot \mathcal{S}_{\lambda j}$ induces a surjective algebra morphism

$$\underline{\nu}_{\lambda j} : \ \mathbb{K}[\mathbf{S}_n] \to End_{\mathbb{K}}[\mathcal{S}_{\lambda j}].$$

By choosing the basis of the $\left(S_{\lambda i} \,\middle|\, \boxed{S_{\lambda j}}\right)$'s in $\mathcal{S}_{\lambda j}$, the morphism $\underline{\nu}_{\lambda,j}$ induces an irreducible matrix representation

$$\overline{\underline{\nu}}_{\lambda j} : \ \mathbb{K}[\mathbf{S}_n] \to M_{\underline{f}_\lambda},$$

where, for every $\sigma \in \mathbf{S}_n$,

$$\overline{\underline{\nu}}_{\lambda j}(\sigma) = [c^{\lambda}_{hk}(\underline{\nu}_{\lambda j}(\sigma))].$$

For every $\lambda \vdash n$, the module structure $\mathbb{K}[\mathbf{S}_n] \cdot Super^M_n[\underline{\mathcal{L}}|\underline{\mathcal{P}}]$ induces a surjective algebra morphism

$$\underline{\rho}_{\lambda} : \mathbb{K}[\mathbf{S}_n] \to \underline{\underline{\mathcal{B}}}_{\lambda};$$

by choosing the basis of the $Y\left(S_{\lambda i} \,\middle|\, \boxed{S_{\lambda j}}\right)$ in $\underline{\underline{\mathcal{B}}}_n$, the morphism $\underline{\rho}_{\lambda}$ induces an irreducible matrix representation

$$\overline{\underline{\rho}}_{\lambda} : \mathbb{K}[\mathbf{S}_n] \to M_{\underline{f}_{\lambda}},$$

where, for every $\sigma \in \mathbf{S}_n$,

$$\overline{\underline{\rho}}_{\lambda}(\sigma) = [d^{\lambda}_{hk}(\underline{\rho}_{\lambda}(\sigma))].$$

From Proposition 13.1, it follows that the irreducible representations $\overline{\underline{\rho}}_{\lambda}$ and $\overline{\underline{\nu}}_{\lambda,j}$ are equal.

We now specialize Theorem 13.1 to the multilinear case, thereby obtaining a simple combinatorial interpretation of the coefficients

$$c^{\lambda}_{hk}(\underline{\nu}_{\lambda j}(\sigma)) = d^{\lambda}_{hk}(\underline{\rho}_{\lambda}(\sigma)).$$

We have:

$$C^{\lambda}(\sigma) = \Theta^{\lambda}(I)^{-1} \times \Theta^{\lambda}(\sigma),$$

where the entries $\theta^{\lambda}_{ij}(\sigma)$ of the matrix $\Theta^{\lambda}(\sigma)$ are the symmetry transition coefficients $\theta^{-+}_{S_i S_j}(\sigma)$, defined by the relations

$$DS\ \sigma \cdot \underline{TC} = \theta^{-+}_{ST}(\sigma)\ \underline{DC}.$$

Notice that

$$\theta^{-+}_{PQ}(\sigma) = \theta^{-+}_{P\ \sigma Q}.$$

These coefficients admit a simple combinatorial description; specifically

$$\theta^{-+}_{PQ} = \begin{cases} (-1)^{|\beta|} & \text{if there exists a (unique) } \alpha \in C(P),\ \beta \in R(Q):\ \alpha P = \beta Q, \\ 0 & \text{otherwise,} \end{cases}$$

where $R(Q)$ and $C(P)$ are the subgroups of $\mathbf{S}_n$ which are the row-stabilizer of Q and the column-stabilizer of P, respectively.

The matrix $\Theta^{\lambda}(I)^{-1}$ is the same as the transition matrix from the (normalized) generalized Young symmetrizers $\frac{1}{h_{\lambda}}\, e_{ij}$ to the *Young natural units* γ_{ij} in $\mathbb{K}[\mathbf{S}_n]$, (see e.g., [53, 72]). As a matter of fact, we have

$$\underline{\rho}(\gamma_{ij}) = \underline{\rho}\left(\sum_h \varrho^{\lambda}_{ih} \frac{1}{h_{\lambda}} e_{hj} \right) = \sum_h \varrho^{\lambda}_{ih} \frac{(-1)^{\binom{n}{2}}}{h_{\lambda}}\ \gamma\left(S_h, \boxed{S_j}\right) = Y\left(S_i, \boxed{S_j}\right).$$

Therefore, $\overline{\underline{\rho}}_{\lambda} = \overline{\underline{\nu}}_{\lambda,j}$ is indeed the *Young natural form* of the irreducible matrix representations of $\mathbf{S}_n$, for every $\lambda \vdash n$ (see, e.g., [53, 41]).

Part IV

A Glimpse of the General Representation Theory of Finite Dimensional Lie Superalgebras

19 A Brief Historical Outline of the Theory of Representations of Finite Dimensional Lie Superalgebras over the Complex Field (after V. G. Kac)

The classification of complex finite dimensional simple Lie superalgebras was published by Kac in 1977 [54]. Shortly afterwards, Kac founded the representation theory of these Lie superalgebras (cf. [55], [56]).

The list of complex finite dimensional simple Lie superalgebras $L = L_0 \oplus L_1$ consists of two essentially different parts, namely, *classical* and *Cartan* Lie superalgebras; classical Lie superalgebras are those for which the Lie subalgebra L_0 is *reductive.*

The list of classical Lie superalgebras is in turn divided into two parts, *basic* and *strange* Lie superalgebras.

The basic classical Lie superalgebras are the classical Lie superalgebras which admit a non degenerate invariant bilinear form $(\ ,\ )$; they are from many points of view the closest to the ordinary simple Lie superalgebras (see, e.g., [56]).

A basic classical Lie superalgebra $L = L_0 \oplus L_1$ is said to be of *type I* whenever its $\mathbb{Z}_2$ homogeneous component L_1, regarded as an L_0-module with respect to the adjoint representation, turns out to be the direct sum of two irreducible representations.

Our work essentially deals with general linear Lie superalgebras which are not simple ones; however, their representation theory is still “essentially the same” as the representation theory of the basic classical Lie superalgebras $\mathbf{A(m,n)}$, the super-analogues of the special linear Lie algebras (for details, see, e.g., [54], [55], [56], [39]).

In the following section, we will recall some of the basic results of the general representation theory of finite dimensional Lie superalgebras in the special case of general linear Lie superalgebras. We will follow along the lines of the recent approaches of Brundan ([23], [24]) and Soergel [83] which are, in turn, inspired by the work of Bernstein, Gelfand and Gelfand (see [6], [7]).

20 General Linear Lie Superalgebras. Highest Weight Modules, Kac Modules, Typical Modules and Covariant Modules

Over the past years, a substantial part of the work on the representation theory of general linear Lie superalgebras was concerned with the problem of computing the characters of the finite dimensional irreducible representations. This problem was raised originally by Kac in 1977 (cf. [55], [56]), who also introduced the two main classes of modules, namely, the class of *highest weight representations* and the class of modules known nowadays as *Kac modules*.

The highest weight representations $\mathcal{V}(\Lambda)$ (*with integral weights*) are irreducible modules, but they are not, in general, of finite dimension. Given a general linear Lie superalgebra $\mathbf{gl}(m|n)$, let $\mathcal{O}_{m|n}$ be the category whose objects are all $\mathbb{Z}_2$-graded $\mathbf{gl}(m|n)$-modules with integral weights which are finitely generated $\mathbf{gl}(m|n)$-modules and are locally finite dimensional over a *distinguished* Borel subalgebra; any irreducible module $M \in \mathcal{O}_{m|n}$ is isomorphic to an integral highest weight module $\mathcal{V}(\Lambda)$ (see, e.g., [55], [56], [39], [23], [24], [83]).

The Kac modules $\overline{\mathcal{V}}(\Lambda)$, Λ a *dominant integral* weight, are finite dimensional, but they are not, in general, irreducible.

The main connection between highest weight modules and Kac modules is that any finite dimensional integral highest weight module $\mathcal{V}(\Lambda)$ is a quotient module of the Kac module $\overline{\mathcal{V}}(\Lambda)$, Λ a *dominant integral* weight, and the module $\overline{\mathcal{V}}(\Lambda)$ is isomorphic to the highest weight module $\mathcal{V}(\Lambda)$ (that is, $\overline{\mathcal{V}}(\Lambda)$ is an irreducible module) if and only if its highest weight Λ is a *typical* weight.

Furthermore, any finite dimensional irreducible $\mathbf{gl}(m|n)$)-module is either typical or it can be obtained from $\mathcal{V}(\Lambda)$ for some integral dominant weight Λ by tensoring with a one-dimensional representation.

The characters of the modules $\overline{\mathcal{V}}(\Lambda)$ were computed by Kac (cf. [55], [56]) in the case of typical highest weights. After that, there were several conjectures and partial results dealing with atypical highest weights (see, e.g., [8], [5], [80], [87], [51], [57], [67]), before the complete solution of the problem was given by Serganova (see [78], [79]), using a mixture of algebraic and geometric techniques (see, e.g., [23]).

Inspired by some ideas of borrowed from the work of Lascoux, Leclerc and Thibon [62], Brundan [23] obtained a purely algebraic version of Serganova's results; in addition, the work of Brundan exploits a deep connection between the representation theory of general linear Lie superalgebras and the Kazhdan–Lusztig representation theory of Coxeter groups and Hecke algebras [59].

20.1 $\mathbb{Z}_2$-homogeneous bases of $V = V_0 \oplus V_1$ and consistent $\mathbb{Z}$-graduations of a general linear Lie superalgebra $\mathbf{gl}(m|n) = pl(V)$

In the remainder of this section, we will denote by the symbols $\overline{0}, \overline{1}$ the elements of the field $\mathbb{Z}_2$.

Let $V = V_{\overline{0}} \oplus V_{\overline{1}}$ be a $\mathbb{Z}_2$-graded vector space, $\dim(V_{\overline{0}}) = m$, $\dim(V_{\overline{1}}) = n$, and let $\mathcal{L} = \{x_1, \ldots, x_m, x_{m+1}, \ldots, x_{m+n}\}$ be a $\mathbb{Z}_2$-homogeneous basis of V, $|x_i| = \overline{0}$ for every $i = 1, \ldots, m$, $|x_i| = \overline{1}$ for every $i = m+1, \ldots, m+n$.

In the following, we will write

$$\mathbf{gl}(m|n) = \mathbf{gl}(m|n)_{\overline{0}} \oplus \mathbf{gl}(m|n)_{\overline{1}}$$

for the general linear Lie superalgebra $pl(V)$ of $V = V_{\overline{0}} \oplus V_{\overline{1}}$.

In matrix notation (see Subsection 3.3) — with respect to the choice of the $\mathbb{Z}_2$-homogeneous basis $\mathcal{L} = \{x_1, \ldots, x_m, x_{m+1}, \ldots, x_{m+n}\}$ — the Lie superalgebra

$$\mathbf{gl}(m|n) = \mathbf{gl}(m|n)_{\overline{0}} \oplus \mathbf{gl}(m|n)_{\overline{1}}$$

is the algebra of all $(m+n) \times (m+n)$-square matrices, where the set

$$\{E_{x_i,x_j}; i,j = 1, \ldots, m\} \cup \{E_{x_i,x_j}; i,j = m+1, \ldots, m+n\}$$

is a basis of $\mathbf{gl}(m|n)_{\overline{0}}$, and

$$\{E_{x_i,x_j}; i = 1, \ldots, m,\ j = m+1, \ldots, m+n\} \cup \{E_{x_i,x_j}; i = m+1, \ldots, m+n,\ j = 1, \ldots, m\}$$

is a basis of $\mathbf{gl}(m|n)_{\overline{1}}$.

The choice of the basis $\mathcal{L} = \{x_1, \ldots, x_m, x_{m+1}, \ldots, x_{m+n}\}$ induces a *consistent* $\mathbb{Z}$-graduation on $\mathbf{gl}(m|n)$ (see, e.g., [74]). Specifically, we have the following direct sum decomposition:

$$\mathbf{gl}(m|n) = \bigoplus_{k \in \mathbb{Z}} \mathbf{gl}(m|n)_k, \quad k \in \mathbb{Z},$$

where

- $\mathbf{gl}(m|n)_{-1}$ is the span of the set $\{E_{x_i,x_j}; i = 1, \ldots, m,\ j = m+1, \ldots, m+n\}$,
- $\mathbf{gl}(m|n)_0$ is the span of the set $\{E_{x_i,x_j}; i,j = 1, \ldots, m\} \cup \{E_{x_i,x_j}; i,j = m+1, \ldots, m+n\}$,
- $\mathbf{gl}(m|n)_1$ is the span of the set $\{E_{x_i,x_j}; i = m+1, \ldots, m+n,\ j = 1, \ldots, m\}$,
- and $\mathbf{gl}(m|n)_k = (0)$, for every $k \neq -1, 0, 1$.

Remark 20.1 (Scheunert, [74]). One may define another consistent $\mathbb{Z}$-graduation on $\mathbf{gl}(m|n)$ by inverting the roles of -1 and 1. As a matter of fact, the two definitions are equivalent in the following sense: consider the $\mathbb{Z}$-graded algebra

$$\mathbf{g}' = \mathbf{g}'_{-1} \oplus \mathbf{g}'_0 \oplus \mathbf{g}'_1,$$

where

$$\mathbf{g}'_{-1} = \mathbf{gl}(m|n)_1, \quad \mathbf{g}'_0 = \mathbf{gl}(m|n)_0, \quad \mathbf{g}'_1 = \mathbf{gl}(m|n)_{-1}$$

and the *supertransposition* map

$$M \to M^T$$

from $\mathbf{gl}(m|n)$ to $\mathbf{g}'$ such that

$$E^T_{x_i,x_j} = (-1)^{(|x_i|+|x_j|)|x_i|}\ E_{x_j,x_i}, \quad i,j = 1,2,\ldots,m+n.$$

The map

$$M \to -M^T$$

is an *isomorphism* of $\mathbb{Z}$-graded Lie superalgebras from $\mathbf{gl}(m|n)$ to $\mathbf{g}'$.

20.2 The supertrace. A consistent, supersymmetric, invariant bilinear form on $\mathbf{gl}(m|n)$

Let $M = (m_{ij})_{i,j=1,\ldots,m+m}$ be a matrix in $\mathbf{gl}(m|n) = \mathbf{gl}(m|n)_{\overline{0}} \oplus \mathbf{gl}(m|n)_{\overline{1}}$. The *supertrace* of M is the number

$$str(M) = \sum_{i=1}^{m} m_{ii} - \sum_{i=m+1}^{m+n} m_{ii}.$$

Following Kac [54], we define a bilinear form on $\mathbf{gl}(m|n)$ by setting

$$(M,N) = str(MN), \quad M,N \in \mathbf{gl}(m|n).$$

This bilinear form $(\ ,\ )$ is

- *consistent*, that is

$$(M,N) = 0, \quad for\ all\ M \in \mathbf{gl}(m|n)_{\overline{0}}, \quad N \in \mathbf{gl}(m|n)_{\overline{1}};$$

- *supersymmetric*, that is

$$(M,N) = (-1)^{|M||N|}\ (N,M), \quad for\ all\ \mathbb{Z}_2 - homogeneous\ M,N \in \mathbf{gl}(m|n);$$

- *invariant with respect to the adjoint action*, that is

$$([M,P],N) = (M,[P,N]), \quad for\ all\ M,P,N \in \mathbf{gl}(m|n).$$

20.3 Distinguished triangular decompositions and roots of the general linear Lie superalgebra $\mathbf{gl}(m|n)$

Let $\mathbf{h} \subseteq \mathbf{gl}(m|n)_{\overline{0}} \subset \mathbf{gl}(m|n)$ be the *Cartan subalgebra* of all diagonal matrices, and let

$$\{\varepsilon_i; i = 1, \ldots, m+n\}$$

be the canonical basis of the dual space $\mathbf{h}^*$ (in plain words, the evaluation of ε_i on a diagonal matrix in $\mathbf{h}$ equals its i-th diagonal entry).

Since the adjoint representation of $\mathbf{h}$ in $\mathbf{gl}(m|n)$ is diagonalizable, we may consider the so-called *distinguished choice* for a triangular decomposition of $\mathbf{gl}(m|n)$ (see, e.g., [54], [66]). This decomposition is characterized by the following conditions:

- $\mathbf{gl}(m|n) = \mathbf{n}^- \oplus \mathbf{h} \oplus \mathbf{n}^+$, where $\mathbf{n}^-$ and $\mathbf{n}^+$ are subalgebras such that $[\mathbf{h}, \mathbf{n}^+] \subset \mathbf{n}^+$, $[\mathbf{h}, \mathbf{n}^-] \subset \mathbf{n}^-$,
- $\mathbf{gl}(m|n)_{-1} \subset \mathbf{n}^+$ and $\mathbf{gl}(m|n)_1 \subset \mathbf{n}^-$.

In plain words, the subalgebras $\mathbf{n}^+$ and $\mathbf{n}^-$ are the algebras of all strictly upper and lower matrices, respectively.

The algebra $\mathbf{b} = \mathbf{h} \oplus \mathbf{n}^+$ is the distinguished Borel subalgebra and the set of *positive roots* - with respect to the choice of the Borel subalgebra $\mathbf{b}$ - is the set of linear functionals

$$\Delta_+ = \Delta_{\overline{0},+} \cup \Delta_{\overline{1},+} \subset \mathbf{h}^*,$$

where

$$\Delta_{\overline{0},+} = \{\varepsilon_i - \varepsilon_j; 1 \leq i < j \leq m\} \cup \{\varepsilon_i - \varepsilon_j; m+1 \leq i < j \leq m+n\}$$

is the set of positive roots of $\mathbb{Z}_2$-degree $\overline{0}$, and

$$\Delta_{\overline{1},+} = \{\varepsilon_i - \varepsilon_j; i = 1, \ldots, m, \ j = m+1, \ldots, m+n\},$$

is the set of positive roots of $\mathbb{Z}_2$-degree $\overline{1}$.

The corresponding set of simple roots (the *distinguished set*, see, e.g., [66]) is given by

$$\Pi = \{\alpha_1 = \varepsilon_1 - \varepsilon_2, \ldots, \alpha_m = \varepsilon_m - \varepsilon_{m+1}, \ldots, \alpha_{m+n-1} = \varepsilon_{m+n-1} - \varepsilon_{m+n}\}.$$

Thus, in the distinguished set there is only one simple root of $\mathbb{Z}_2$-degree $\overline{1}$, the root $\varepsilon_m - \varepsilon_{m+1}$.

As usual, we put

$$\rho_{\overline{0}} = \frac{1}{2} \sum_{\alpha \in \Delta_{\overline{0},+}} \alpha, \quad \rho_{\overline{1}} = \frac{1}{2} \sum_{\alpha \in \Delta_{\overline{1},+}} \alpha, \quad \rho = \rho_{\overline{0}} - \rho_{\overline{1}}.$$

Consider the symmetric bilinear form $(\ ,\ )$ on $\mathbf{h}^*$ induced by the invariant supersymmetric bilinear form on $\mathbf{gl}(m|n)$; in the natural basis $\{\varepsilon_i; i = 1, \ldots, m+n\}$ it takes the form

- $(\varepsilon_i, \varepsilon_j) = \delta_{ij}, \quad \textit{for all } i, j = 1, \ldots, m,$
- $(\varepsilon_i, \varepsilon_j) = 0, \quad \textit{for all } i = 1, \ldots, m,\ j = m+1, \ldots, m+n,$
- $(\varepsilon_i, \varepsilon_j) = -\delta_{ij}, \quad \textit{for all } i, j = m+1, \ldots, m+n.$

Note that the positive roots of $\mathbb{Z}_2$-degree $\overline{1}$ are *isotropic*, that is $(\alpha, \alpha) = 0$, for all $\alpha \in \Delta_{\overline{1},+}$.

Remark 20.2.

1. $$(\rho, \alpha_i) = \frac{1}{2}(\alpha_i, \alpha_i) = \begin{cases} 1 & \text{for } 1 \le i < m, \\ 0 & \text{for } i = m, \\ -1 & \text{for } m < i < m+n. \end{cases}$$

2. $$\begin{aligned} (\rho, \epsilon_i - \epsilon_j) &= \sum_{h=i}^{j-1} (\rho, \alpha_h) = \sum_{h=i}^{j-1} \frac{1}{2}(\alpha_h, \alpha_h) \\ &= \begin{cases} j - i & \text{for } 1 \le i < j \le m, \\ 2m + 1 - i - j & \text{for } 1 \le i \le m < j \le m+n, \\ i - j & \text{for } m < i < j \le m+n. \end{cases} \end{aligned}$$

Remark 20.3.

1. The supersymmetric bilinear form $(\ ,\ )$ is non-degenerate on $\mathbf{gl}(m|n)$.
2. Regarded as a $\mathbf{gl}(m|n)_{\overline{0}}$-module (with respect to the adjoint action), $\mathbf{gl}(m|n)_{\overline{1}}$ splits into the direct sum of the irreducible submodules:
$$\mathbf{gl}(m|n)_{\overline{1}} = \mathbf{gl}(m|n)_{-1} \oplus \mathbf{gl}(m|n)_1, \quad \overline{1} \in \mathbb{Z}_2, \quad -1, 1 \in \mathbb{Z}.$$
3. The irreducible $\mathbf{gl}(m|n)_{\overline{0}}$-modules $\mathbf{gl}(m|n)_{-1}$ and $\mathbf{gl}(m|n)_1$ are the irreducible $\mathbf{gl}(m|n)_{\overline{0}}$-modules with highest weight vectors $E_{x_1, x_{m+n}}$ and E_{x_{m+1}, x_m}, respectively.

The representation theory of the general linear Lie superalgebras $\mathbf{gl}(m|n)$ looks like the representation theory of basic classical simple Lie superalgebras of type I.

20.4 Highest weight modules with integral weights and Verma modules

Let $\mathbf{h} \subset \mathbf{gl}(m|n)$ be the Cartan subalgebra of all diagonal matrices, and let $\mathbf{b} = \mathbf{h} \oplus \mathbf{n}^+$ be the distinguished Borel subalgebra of all upper triangular matrices.

Let

$$X(m|n) = \{\mu \in \mathbf{h}^*;\ \mu = \sum_{i=1}^{m+n} \mu_i\, \varepsilon_i,\ \mu_i \in \mathbb{Z}\}$$

be the set of all linear functionals on $\mathbf{h}$ with integral coefficients with respect to the standard basis $\{\varepsilon_i;\ i = 1, \ldots, m+n\}$.

Given a $\mathbf{gl}(m|n)$-supermodule M and an *integral weight* $\mu \in X(m|n)$, define the μ-weight subspace $M(\mu)$ of M with respect to $\mathbf{h}$ as usual:

$$M(\mu) = \{m \in M; h \cdot m = \mu(h)m \ for\ all\ h \in \mathbf{h}\}.$$

A $\mathbf{gl}(m|n)$-supermodule M is said to be a *module with integral weights* whenever it satisfies the following condition:

$$M = \bigoplus_{\mu \in X(m|n)} M(\mu).$$

Following Brundan ([23], [24]) and Soergel [83], the category $\mathcal{O}_{m|n}$ is the category of all $\mathbb{Z}_2$-graded $\mathbf{gl}(m|n)$-modules M with integral weights which are finitely generated $\mathbf{gl}(m|n)$-modules and are locally finite dimensional over $\mathbf{b}$ (that is, all finitely generated $\mathbf{b}$-submodules of M are finite dimensional $\mathbb{C}$-vector spaces). The category $\mathcal{O}_{m|n}$ is the (integral weight) analogue of the category $\mathcal{O}$ for semisimple Lie algebras, introduced by Bernstein, Gelfand and Gelfand in [7].

For every $\Lambda = \sum_{i=1}^{m+n} \Lambda_i \varepsilon_i \in X(m|n)$, define a $\mathbb{Z}_2$-graded one-dimensional $\mathbf{b}$-module $< v_\Lambda >$ by setting

$$\begin{aligned} h(v_\Lambda) &= \Lambda(h) v_\Lambda, \quad h \in \mathbf{h} \\ \mathbf{n}^+(v_\Lambda) &= 0, \end{aligned}$$

where v_Λ is a $\mathbb{Z}_2$-homogeneous vector of degree $\sum_{i=m+1}^{m+n} \Lambda_i$ (mod 2).

Define the *Verma module* $\tilde{\mathcal{V}}(\Lambda)$ as the induced $\mathbf{gl}(m|n)$-module

$$\tilde{\mathcal{V}}(\Lambda) = \mathcal{U}(\mathbf{gl}(m|n)) \otimes_{\mathcal{U}(\mathbf{b})}\ < v_\Lambda > \in \mathcal{O}_{m|n},$$

where $\mathcal{U}(\mathbf{gl}(m|n))$ and $\mathcal{U}(\mathbf{b})$ are the universal enveloping algebras of $\mathbf{gl}(m|n)$ and $\mathbf{b}$, respectively.

Remark 20.4. The fact that the Verma modules are objects in the category $\mathcal{O}_{m|n}$ follows from the Poincaré–Birkhoff–Witt theorem (see, e.g., [54], [74]).

The $\mathbf{gl}(m|n)$-module $\tilde{\mathcal{V}}(\Lambda)$ contains a unique maximal submodule $I(\Lambda)$ and, therefore, the quotient module

$$\mathcal{V}(\Lambda) = \tilde{\mathcal{V}}(\Lambda)/I(\Lambda)$$

is an irreducible $\mathbf{gl}(m|n)$-module.

The modules $\mathcal{V}(\Lambda)$ are usually called *highest weight modules (with integral weights)* (see, e.g., [39]); the highest weight of the module $\mathcal{V}(\Lambda)$ is clearly the linear functional $\Lambda \in \mathbf{h}$ and its unique — up to a scalar factor — highest weight vector is the class vector of the vector $\mathbf{1} \otimes_{\mathcal{U}(\mathbf{b})} < v_\Lambda > \in \tilde{\mathcal{V}}(\Lambda)$.

The following result is essentially due to Kac (cf. [55], [56]).

Theorem 20.1 (see, e.g., [23], [24])**.** *The set*

$$\{\mathcal{V}(\Lambda);\ \Lambda \in X(m|n)\}$$

is a complete set of pairwise non-isomorphic irreducibles in $\mathcal{O}_{m|n}$.

20.5 Kac modules and typical modules

The category $\mathcal{F}_{m|n}$ is the category of all finite dimensional $\mathbf{gl}(m|n)$-modules with integral weights.

We recall that an integral weight $\Lambda \in \mathbf{h}^*$, $\Lambda = \sum_{i=1}^{m+n} \Lambda_i\, \varepsilon_i \in X(m|n)$ is said to be an *integral dominant weight* whenever it satisfies the condition

$$a_i = \frac{2(\Lambda, \alpha_i)}{(\alpha_i, \alpha_i)} \geq 0,$$

for every simple root $\alpha_i \in \Pi$, $i \neq m$, that is,

$$\Lambda_1 \geq \Lambda_2 \geq \cdots \geq \Lambda_m, \quad \textit{and} \quad \Lambda_{m+1} \geq \Lambda_{m+2} \geq \cdots \geq \Lambda_{m+n}.$$

The set of all integral dominant weights in $X(m|n)$ is denoted by the symbol $X^+(m|n)$.

Let $\mathcal{V}_{\overline{0}}(\Lambda)$ be the irreducible $\mathbf{gl}(m|n)_{\overline{0}}$-module with highest integral dominant weight $\Lambda \in X^+(m|n)$; $\mathcal{V}_{\overline{0}}(\Lambda)$ is turned into a $\mathbb{Z}_2$-graded module by specifying that its highest weight is a $\mathbb{Z}_2$-homogeneous vector of degree $\sum_{i=m+1}^{m+n} \Lambda_i$ (mod 2).

Consider the subalgebra $\mathbf{p} = \mathbf{gl}(m|n)_0 \oplus \mathbf{gl}(m|n)_{-1} \subset \mathbf{gl}(m|n)$.

The $\mathbf{gl}(m|n)_{\overline{0}}$-module $\mathcal{V}_{\overline{0}}(\Lambda)$ is extended to a $\mathbf{p}$-module by setting

$$\mathbf{gl}(m|n)_{-1} \cdot \mathcal{V}_{\overline{0}}(\Lambda) = 0.$$

The induced $\mathbf{gl}(m|n)$-module

$$\overline{\mathcal{V}}(\Lambda) = \mathcal{U}(\mathbf{gl}(m|n)) \ \otimes_{\mathcal{U}(\mathbf{p})} \ \mathcal{V}_{\overline{0}}(\Lambda)$$

is said to be a *Kac module.*

Remark 20.5. The Kac module $\overline{\mathcal{V}}(\Lambda)$ is a finite dimensional vector space isomorphic to the tensor product $\bigwedge(\mathbf{gl}(m|n)_1)\bigotimes\mathcal{V}_{\overline{0}}(\Lambda)$.

The Kac modules are not, in general, irreducible $\mathbf{gl}(m|n)$-modules; however, if $\overline{\mathcal{V}}(\Lambda)$ is not an irreducible module, then it contains a unique maximal submodule $\overline{\mathcal{I}}(\Lambda)$ and the quotient module $\overline{\mathcal{V}}(\Lambda)\big/\overline{\mathcal{I}}(\Lambda)$ is an irreducible module isomorphic to the module $\mathcal{V}(\Lambda)$ defined in the preceding subsection.

In general, we have the following result.

Proposition 20.1 ([55], [56], [39], [23]). *Let $\Lambda\in X^+(m|n)$ be an integral dominant weight. Then, the following statements are equivalent.*

1. $\mathcal{V}(\Lambda)=\overline{\mathcal{V}}(\Lambda)$.
2. *The Kac module $\overline{\mathcal{V}}(\Lambda)$ is an irreducible $\boldsymbol{gl}(m|n)$-module.*
3. $(\Lambda+\rho,\alpha)\neq 0$*, for all positive roots $\alpha\in\Delta_+$.*

A finite dimensional irreducible $\mathbf{gl}(m|n)$-module $\mathcal{V}(\Lambda)$ with highest weight Λ is called *typical* whenever one of the equivalent conditions 1–3 of Proposition 20.1 is satisfied.

The following result is essentially due to Kac (cf. [55], [56]).

Theorem 20.2 (see, e.g., [23], [24]).

- *The set*
$$\{\mathcal{V}(\Lambda);\ \Lambda\in X^+(m|n)\}$$
is a complete set of pairwise non-isomorphic irreducibles in $\mathcal{F}_{m|n}$.
- *Any finite dimensional irreducible $\boldsymbol{gl}(m|n)$-module is either typical or it can be obtained from $\mathcal{V}(\Lambda)$ for some integral dominant weight $\Lambda\in X^+(m|n)$ by tensoring with a one dimensional representation.*

20.6 Covariant modules, Schur modules and letterplace superalgebras

Let $V=V_{\overline{0}}\oplus V_{\overline{1}}$ be a $\mathbb{Z}_2$-graded vector space, $\dim(V_{\overline{0}})=m$, $\dim(V_{\overline{1}})=n$, and let $\mathcal{L}=\{x_1,\ldots,x_m,x_{m+1},\ldots,x_{m+n}\}$ be a *distinguished* $\mathbb{Z}_2$-homogeneous basis of V, $|x_i|=\overline{0}$ for every $i=1,\ldots,m$, $|x_i|=\overline{1}$ for every $i=m+1,\ldots,m+n$.

Let $\lambda\vdash N$ be a partition, and let $\lambda\in H(\mathcal{L})$, where

$$H(\mathcal{L})=\{\lambda=(\lambda_1\geq\lambda_2\geq\cdots);\lambda_{m+1}<n+1\}$$

is the hook set of $\mathcal{L}$ (see Subsection 5.4).

The *covariant* modules $\mathbf{V}_\lambda$ are usually defined as the finite dimensional irreducible $\mathbf{gl}(m|n)$-modules that appear in the Berele–Regev complete decomposition of the $\mathbf{gl}(m|n)$-module $V^{\otimes N}$ (see also Sergeev [80]).

We know that the covariant modules $\mathbf{V}_\lambda$ admit a quite explicit description as submodules of a letterplace algebra, and Theorem 12.1 implies that all the $pl(V)$-irreducible submodules of a letterplace algebra are indeed covariant modules (up to isomorphism).

Furthermore, from Section 11 we infer that the covariant modules admit an even more manageable combinatorial description. To be precise, we have the following result.

Proposition 20.2. *Let λ be a partition, and let $\lambda \in H(\mathcal{L})$. The covariant module $\mathbf{V}_\lambda$ is $\boldsymbol{gl}(m|n)$-isomorphic to the module*

$$\mathcal{S}_\lambda = \langle (S \mid D_\lambda); S \in Tab_\lambda(\mathcal{L}) \rangle_{\mathbb{K}} \subseteq Super[\mathcal{L}|\mathcal{P}],$$

where $\mathcal{P} = \mathcal{P}_{\overline{1}} = \{1, 2, \dots, t\}$, $t \geq \lambda_1$ and D_λ is a Deruyts tableau $\in Tab_\lambda(\mathcal{P})$.

Remark 20.6. 1. The covariant module $\mathcal{S}_\lambda$ has a standard basis (see Subsection 8.3):

$$\{(S \mid D_\lambda); S \in Stab_\lambda(\mathcal{L})\}.$$

Since the place tableau D_λ is of Deruyts type, the bitableaux $(S \mid D_\lambda)$ are skew-symmetric in the rows of D_λ. From Subsection 8.3, it follows that the process of expanding the generators $(S \mid D_\lambda)$, $S \in Tab_\lambda(\mathcal{L})$, of $\mathcal{S}_\lambda$ into linear combinations of the standard basis elements is ruled by the following special form of the Straightening Law (see Theorem 8.1):

$$\sum_{(v)} \left(\begin{array}{c|c} uv_{(1)} & x \\ v_{(2)}w & y \end{array} \right) = 0,$$

where $x = 12 \cdots h$, $y = 12 \cdots k$, $h \geq k$, and v is a word of length greater than h. The above identities are known as the "*exchange rules*" (see, e.g., [46]).

2. The covariant module $\mathcal{S}_\lambda$ has a Clebsch–Gordan–Capelli basis (see Subsection 9.4); namely,

$$\left\{ \left(\boxed{S} \,\middle|\, D_\lambda \right); S \in Stab_\lambda(\mathcal{L}) \right\}.$$

Let $S \in Tab_\lambda(\mathcal{L})$. We recall that the content $c(S)$ of the tableau S is the vector

$$c(S) = (c(S, x_1), \dots, c(S, x_m), c(S, x_{m+1}), \dots, c(S, x_{m+n})),$$

where $c(S, x_i)$ is the number of occurrences of the letter x_i in the tableau S, for $i = 1, 2, \dots, m+n$.

Let $\lambda \vdash N$ be a partition, and let $\lambda \in H(\mathcal{L})$. An element

$$\varphi = \sum_{i=1}^{m+n} \varphi_i \, \varepsilon_i, \quad \varphi_i \in \mathbb{Z}$$

of the dual space $\mathbf{h}^*$ is called a *standard weight* of the covariant module $\mathcal{S}_\lambda$ if there exists a standard tableau $S \in Stab_\lambda(\mathcal{L})$ such that

$$(\varphi_1, \varphi_2, \ldots, \varphi_{m+n}) = c(S).$$

From Remark 20.5, it follows that the module $\mathcal{S}_\lambda$ is $\mathbf{h}$-diagonalizable with weight decomposition

$$\mathcal{S}_\lambda = \bigoplus_{\varphi} \mathcal{S}_\lambda(\varphi),$$

where the direct sum is over the set of all standard weights φ of the covariant module $\mathcal{S}_\lambda$.

We come now to the main combinatorial definition of the present subsection.

The standard tableau

$$F_\lambda = (w_1, w_2, \ldots, w_p) \in Tab_\lambda(\mathcal{L})$$

is the tableau defined by the following conditions:

- $w_i = x_i^{\lambda_i}$, for all $i \in \mathbb{Z}^+,\ i \leq m$,
- $w_i = x_{m+1} x_{m+2} \cdots x_{m+\lambda_i}$, for all $i \in \mathbb{Z}^+,\ m < i \leq m+n$.

Note that F_λ is a tableau of Co-Deruyts type if $n = 0$, and F_λ is a tableau of Deruyts type if $m = 0$.

The bitableau $(F_\lambda \,|\, D_\lambda)$ plays a crucial role in the theory of the covariant module $\mathcal{S}_\lambda$. We notice that the bitableau $(F_\lambda \,|\, D_\lambda)$ is (up to a scalar factor) the unique weight vector of $\mathcal{S}_\lambda$ of weight

$$\Lambda_\lambda = \sum_{i=1}^{m} \lambda_i \varepsilon_i \ + \sum_{j=1}^{n} \nu_{m+j} \varepsilon_{m+j},$$

where $\nu_{m+j} = \max\{0, \tilde{\lambda}_j - m\}$, and $\tilde{\lambda} = (\tilde{\lambda}_1 \geq \tilde{\lambda}_2 \geq \cdots)$ denotes the conjugate partition of the partition λ.

We have the following results.

Lemma 20.1. *Consider the set of ordered $(m+n)$-tuples in $\mathbb{N}^{m+n}$*

$$\{c(S);\ S \in Stab_\lambda(\mathcal{L})\}$$

as a linearly ordered set with respect to the lexicographic order. Then

$$c(F_\lambda) = \max\{c(S);\ S \in Stab_\lambda(\mathcal{L})\}.$$

Proof. The assertion immediately follows from Remark 20.5. $\square$

Proposition 20.3. *1. The bitableau $(F_\lambda \,|\, D_\lambda)$ is annihilated by all strictly upper polarizations $\mathcal{D}_{x_i x_j}$, $i < j$, $i, j = 1, 2, \ldots, m + n$.*

2. The elements of the basis $\left\{ \left(\boxed{S} \,\middle|\, D_\lambda \right); S \in Stab_\lambda(\mathcal{L}) \right\}$ of $\mathcal{S}_\lambda$ may be obtained from $(F_\lambda \,|\, D_\lambda)$ by iterated actions of strictly lower polarization $\mathcal{D}_{x_i x_j}$, $i > j$, $i, j = 1, 2, \ldots, m + n$.

3. The bitableau $(F_\lambda \,|\, D_\lambda)$ is (up to a scalar factor) the unique element of $\mathcal{S}_\lambda$ which is annihilated by all strictly upper polarizations $\mathcal{D}_{x_i x_j}$, $i < j$, $i, j = 1, 2, \ldots, m+n$.

Proof. 1) Since $\mathcal{D}_{x_i x_j}(F_\lambda \,|\, D_\lambda)$, $i < j$, is a linear combination of bitableaux $(T \,|\, D_\lambda)$ with $c(T) > c(F_\lambda)$ in the lexicographic order, the assertion follows from Remark 20.5 and Lemma 20.1.

2) We know that the action of the Capelli operator $[\tilde{S} \,|\, \tilde{F}_\lambda]_-$ on the bitableau $(F_\lambda \,|\, D_\lambda)$ yields the following result:

$$[\tilde{S} \,|\, \tilde{F}_\lambda]_- \left((F_\lambda \,|\, D_\lambda) \right) = c_{F_\lambda} \left(\boxed{S} \,\middle|\, D_\lambda \right),$$

where c_{F_λ} is a non-zero integer (see Subsection 8.2).

When we devirtualize the Capelli operator $[\tilde{S} \,|\, \tilde{F}_\lambda]_-$, we may write it as a linear combination of products

$$\mathcal{D}_1 \ \mathcal{D}_2 \ \mathcal{D}_3,$$

where $\mathcal{D}_1$ is a product of strictly lower polarizations $\mathcal{D}_{x_i x_j}$, $i > j$, $i, j = 1, 2, \ldots, m+n$, $\mathcal{D}_2$ is a product of diagonal polarizations $\mathcal{D}_{x_i x_i}$, $i = 1, 2, \ldots, m+n$, and $\mathcal{D}_3$ is a product of strictly upper polarizations $\mathcal{D}_{x_h x_k}$, $h < k$, $h, k = 1, 2, \ldots, m + n$, by the "easy" part of the Poincaré–Birkhoff–Witt theorem (see, e.g., [54], [74], [58]).

Then, by assertion 1), the action of the operator $[\tilde{S} \,|\, \tilde{F}_\lambda]_-$ on $(F_\lambda \,|\, D_\lambda)$ is the same as the action of an operator which may be written as a linear combination of products of strictly lower polarization $\mathcal{D}_{x_i x_j}$, $i > j$, $i, j = 1, 2, \ldots, m + n$.

3) If an element of $\mathcal{F} \in \mathcal{S}_\lambda$ satisfies the conditions of assertion 1), then the cyclic module generated by $\mathcal{F}$ (with respect to the actions of the strictly lower polarization $\mathcal{D}_{x_i x_j}$, $i > j$, $i, j = 1, 2, \ldots, m + n$) equals the $\mathbf{gl}(m|n)$-irreducible module $\mathcal{S}_\lambda$. Thus, the assertion immediately follows from Lemma 20.1. $\square$

Proposition 20.3 implies the following result.

Corollary 20.1. *The vector $(F_\lambda \,|\, D_\lambda)$ is the highest weight vector of the covariant module $\mathcal{S}_\lambda$, and, thus, $\mathcal{S}_\lambda$ is isomorphic to the highest weight module $\mathcal{V}(\Lambda_\lambda)$ with highest weight Λ_λ (with respect to the distinguished set of positive roots defined in Subsection 20.3).*

Covariant modules are not, in general, typical modules.

Proposition 20.4. *The covariant module $\mathcal{S}_\lambda$ is a typical module if and only if $\lambda_m \geq n$.*

Proof. From Remark 20.2, it follows that:

- $$\begin{aligned}(\Lambda_\lambda + \rho, \epsilon_i - \epsilon_j) &= (\Lambda_\lambda, \epsilon_i - \epsilon_j) + (\rho, \epsilon_i - \epsilon_j)\\ &= \begin{cases} \lambda_i - \lambda_j + j - i > 0 & \text{for } 1 \leq i < j \leq m,\\ \lambda_i + \nu_j + 2m + 1 - i - j & \text{for } 1 \leq i \leq m < j \leq m+n,\\ -\nu_i + \nu_j + i - j < 0 & \text{for } m < i < j \leq m+n.\end{cases}\end{aligned}$$

- $$\lambda_i + \nu_j + 2m + 1 - i - j \neq 0, \quad \text{for} \quad 1 \leq i \leq m < j \leq m+n.$$

  In particular, for $i = m$ we have the set of conditions

  $$\begin{aligned} \lambda_m + \nu_{m+1} &\neq 0,\\ \lambda_m + \nu_{m+2} &\neq 1,\\ \lambda_m + \nu_{m+3} &\neq 2,\\ &\vdots\\ \lambda_m + \nu_{m+n} &\neq n-1,\end{aligned}$$

  which is equivalent to the single condition

  $$\lambda_m \geq n.$$

  In turn, this condition implies

  $$\lambda_i + \nu_j + 2m + 1 - i - j \geq n + 0 + 2m + 1 - m - (m+n) = 1, \qquad \text{for} \quad 1 \leq i \leq m < j \leq m+n.$$

□

Remark 20.7.

1. The highest weights Λ_λ were computed by Van der Jeugt et al. in [87] and by Cheng and Wang in [26].
2. The covariant modules are "tame" modules in the sense of Kac and Wakimoto [57].

   Th recent result [66] by Moens and Van der Jeugt has a significant consequence in the theory of supersymmetric Schur polynomials $s_\lambda(x/y)$. These polynomials appeared in the work of Berele and Regev [5] and turn out to be the characters of the covariant modules, regarded as irreducible $\mathbf{gl}(m|n)$-modules. Since the covariant modules are tame, one can apply the character formula of Kac and Wakimoto [57], and a nice determinantal formula holds for the polynomials $s_\lambda(x/y)$ (cf. [66, Formula (1.17)]).

20.7 The basic plethystic superalgebras $\mathbf{S}(\mathbf{S}^k(V))$ and $\bigwedge(\mathbf{S}^k(V))$

In a series of rather recent papers, Cheng and Wang ([26], [27]) and Sergeev ([81], [82]) independently rediscovered our complete decomposition Theorem 12.1 from [11] for the letterplace algebra (regarded as the supersymmetric algebra of the tensor product of a pair of finite dimensional $\mathbb{Z}_2$-graded vector spaces, see Subsection 5.3) and found explicit formulas for the highest weight vectors. Their method is essentially based on the notion of *Howe duality* (see, e.g [49], [50] and Theorem 12.3).

Furthermore, they describe the complete decompositions of the supersymmetric and superexterior algebras $\mathbf{S}(\mathbf{S}^2(V))$ and $\bigwedge(\mathbf{S}^2(V))$ of the supersymmetric square of the natural representations of $\mathbf{gl}(m|n)$, and ask (see [26, Introduction]) "whether the results concerning the decomposition of $\mathbf{S}(\mathbf{S}^2(V))$ (respectively $\bigwedge(\mathbf{S}^2(V))$) and the highest weight vectors in these modules may also be obtained with extra insights from the combinatorial approach in [11] as well."

In this subsection, we briefly describe the main connections between the letterplace algebra approach and the above-mentioned work.

First of all, we remark that the highest weight vectors in $Super[\mathcal{L}|\mathcal{P}]$ can be explicitly described by combining Corollary 20.1 and Theorem 12.1.

With reference to the modules $\mathbf{S}(\mathbf{S}^2(V))$ and $\bigwedge(\mathbf{S}^2(V))$, we mention that they have been systematically studied by Brini, Huang and Teolis [16] and Grosshans [47], in the more general setting of "plethystic superalgebras".

For the convenience of the reader, we recall some basic definitions from [16].

Let $W = W_{\overline{0}} \oplus W_{\overline{1}}$ be a finite dimensional $\mathbb{Z}_2$-graded vector space.

In order to simplify notations, in the remainder of this subsection we write $\mathbf{S}(W)$ for the supersymmetric algebra $Super[W] = Sym(W_{\overline{0}}) \otimes \Lambda(W_{\overline{1}})$ and $\bigwedge(V)$ for the superexterior algebra $Super[W] = \Lambda(W_{\overline{0}}) \otimes Sym(W_{\overline{1}})$.

We recall that $\mathbf{S}^k(W)$ is a $\mathbb{Z}_2$-graded vector space, where

$$\mathbf{S}^k(W)_{\overline{0}} = \bigoplus_m (Sym^{k-2m}(W_{\overline{0}}) \otimes \Lambda^{2m}(W_{\overline{1}})),$$

and

$$\mathbf{S}^k(W)_{\overline{1}} = \bigoplus_m (Sym^{k-2m-1}(W_{\overline{0}}) \otimes \Lambda^{2m+1}(W_{\overline{1}})).$$

Let $V = V_{\overline{0}} \oplus V_{\overline{1}}$ be a finite dimensional $\mathbb{Z}_2$-graded vector space, $\dim(V_{\overline{0}}) = m$, $\dim(V_{\overline{1}}) = n$.

The *basic k-th plethystic (super)symmetric superalgebra* of $V = V_{\overline{0}} \oplus V_{\overline{1}}$ is the superalgebra

$$\mathbf{S}(\mathbf{S}^k(V)) = Sym(\mathbf{S}^k(V)_{\overline{0}}) \otimes \Lambda(\mathbf{S}^k(V)_{\overline{1}}),$$

and the *basic k-th plethystic (super)exterior superalgebra* of $V = V_{\overline{0}} \oplus V_{\overline{1}}$ is the superalgebra

$$\bigwedge(\mathbf{S}^k(V)) = \Lambda(\mathbf{S}^k(V)_{\overline{0}}) \otimes Sym(\mathbf{S}^k(V)_{\overline{1}}).$$

Let $\mathbf{S}^h(\mathbf{S}^k(V))$ be the $\mathbb{Z}$-homogeneous component of degree h of $\mathbf{S}(\mathbf{S}^k(V))$,

$$\mathbf{S}^h(\mathbf{S}^k(V)) = \bigoplus_p (Sym^p(\mathbf{S}^k(V)_{\overline{0}}) \otimes \Lambda^{h-p}(\mathbf{S}^k(V)_{\overline{1}}),$$

and let $\bigwedge^h(\mathbf{S}^k(V))$ be the $\mathbb{Z}$-homogeneous component of degree h of $\bigwedge(\mathbf{S}^k(V)$,

$$\bigwedge^h(\mathbf{S}^k(V)) = \bigoplus_p (\Lambda^p(\mathbf{S}^k(V)_{\overline{0}}) \otimes Sym^{h-p}(\mathbf{S}^k(V)_{\overline{1}}).$$

The $\mathbb{Z}_2$-graded vector spaces $\mathbf{S}^h(\mathbf{S}^k(V))$ and $\bigwedge^h(\mathbf{S}^k(V))$ are in a natural way $pl(V)$-modules.

In [16] and [47], the modules $\mathbf{S}^h(\mathbf{S}^k(V))$ and $\bigwedge^h(\mathbf{S}^k(V))$ are shown to be epimorphic images of suitable letterplace algebras under the so-called *umbral operator* U (see also [46]). The operator U is a $pl(V)$)-equivariant operator.

Since the letterplace algebras are semisimple modules (Theorem 12.1), it follows that the modules $\mathbf{S}^h(\mathbf{S}^k(V))$ and $\bigwedge^h(\mathbf{S}^k(V))$ are semisimple modules and all their irreducible submodules are explicitly constructed; these irreducible submodules are covariant modules. The multiplicities of the covariant modules $\mathcal{S}_\lambda$ in a complete decomposition is described in a rather implicit way; these multiplicities are shown to be equal to the $\mathbb{K}$-linear dimensions of suitable representations of the symmetric group $\mathcal{S}_h$ ([16, Theorem 12], and [47, Theorems 13 and 17]).

In the case $k = 2$ (i.e., the case of *supersymmetric matrices*), the situation is much more satisfactory: the $pl(V)$-modules $\mathbf{S}^h(\mathbf{S}^2(V))$ and $\bigwedge^h(\mathbf{S}^2(V))$ are described in detail in Sections 4 and 6 of [16]. In particular, Straightening Laws for both the modules of type $\mathbf{S}^h(\mathbf{S}^2(V))$ and for the modules of type $\bigwedge^h(\mathbf{S}^2(V))$ are provided; these Straightening Laws are closely related to the work of De Concini and Procesi [32] and Rota and Stein (cf. [70], [71]). Furthermore, two classes of Clebsch–Gordan–Capelli bases are exhibited for both classes of modules in [16, Sections 3 and 4].

We have the following structure theorems.

Theorem 20.3 (see [16], [47], [26], [82])**.** *1. The $pl(V)$-module $\boldsymbol{S}^h(\boldsymbol{S}^2(V))$ is a multiplicity-free module. We have the following complete decomposition result:*

$$\boldsymbol{S}^h(\boldsymbol{S}^2(V)) \cong \bigoplus_\lambda \mathcal{S}_\lambda,$$

where the direct sum ranges over all the partitions $\lambda = (\lambda_1, \lambda_2, \ldots) \vdash 2h$, λ_i even for every i, $\lambda_{m+1} \leq n$.

2. *The* $pl(V)$*-module* $\bigwedge^h(\boldsymbol{S}^2(V))$ *is a multiplicity-free module. We have the following complete decomposition result:*

$$\bigwedge^h(\boldsymbol{S}^2(V)) \cong \bigoplus_{\lambda} \mathcal{S}_\lambda,$$

where the direct sum ranges over all the partitions $\lambda = (\lambda_1, \lambda_2, \ldots) \vdash 2h$ *which are obtained by nesting* $(q+1, q)$*-hooks,* $\lambda_{m+1} \leq n$*.*

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