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On the formula of Goulden and Rattan for Kerov polynomials

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Abstract. We give a simple proof of an explicit formula for Kerov polynomials. This explicit formula is closely related to a recent formula of Goulden and Rattan.

1. Kerov polynomials

Kerov polynomials are universal polynomials which express the characters of symmetric groups evaluated on cycles, in terms of quantities known as the free cumulants of a Young diagram. We now explain these notions.

Let $\lambda = \lambda_1 \geq \lambda_2 \geq \cdots$ be a Young diagram, to which we associate a piecewise affine function $\omega : \mathbb{R} \to \mathbb{R}$, with slopes ± 1, such that $\omega(x) = |x|$ for $|x|$ large enough, as in Figure 1 below, which corresponds to the partition $8 = 4 + 3 + 1$. We can encode the Young diagram using the local minima and local maxima of the function ω, denoted by $x_1, \ldots, x_m$ and $y_1, \ldots, y_{m-1}$ respectively, which form two interlacing sequences of integers. These are $(-3, -1, 2, 4)$ and $(-2, 1, 3)$ respectively in the picture. To the Young diagram we associate the rational fraction

$$G_\lambda(z) = \frac{\prod_{i=1}^{m-1}(z - y_i)}{\prod_{i=1}^{m}(z - x_i)}$$

and the formal power series K_λ, inverse to G for composition,

$$K_\lambda(z) = G_\lambda^{(-1)}(z) = z^{-1} + \sum_{k=2}^{\infty} R_k(\lambda) z^{k-1}.$$

The quantities $R_k(\lambda)$, $k = 2, 3, \ldots$, are called the free cumulants of the diagram λ. Note that $R_1(\lambda) = 0$ for any Young diagram, so we do not include it in the series of free cumulants. These quantities arise in the asymptotic study of representations of symmetric groups, see [**B1**]. It turns out that there exist universal polynomials $\Sigma_2, \Sigma_3, \ldots$ in the

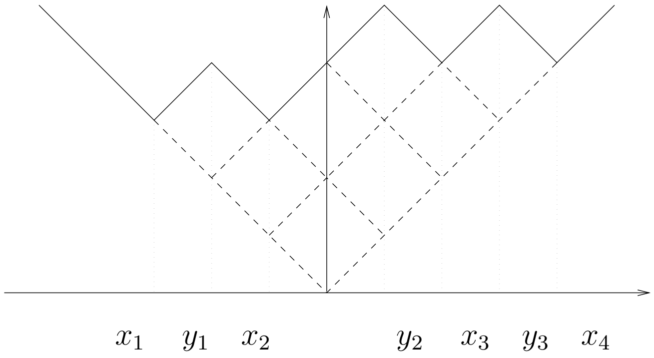

FIGURE 1

variables $R_2, R_3, \ldots$ such that for any Young diagram λ the normalized character χ_λ evaluated at a cycle c_k of length k is given by

$$(n)_k \chi_\lambda(c_k) = \Sigma_k(R_2(\lambda), R_3(\lambda), \ldots).$$

The remarkable fact here is that these polynomials do not depend on the size of the symmetric group. We list the first such polynomials below.

$$\begin{aligned}
\Sigma_1 &= R_2, \\
\Sigma_2 &= R_3, \\
\Sigma_3 &= R_4 + R_2, \\
\Sigma_4 &= R_5 + 5R_3, \\
\Sigma_5 &= R_6 + 15R_4 + 5R_2^2 + 8R_2.
\end{aligned}$$

We refer to [**B2**] and [**GR**] for more information about results and conjectures on the coefficients of these polynomials. We take from [**B2**], Section 5, the following expression for Kerov polynomials. Here $[z^{-k}]\, f(z)$ denotes the coefficient of z^{-k} (the residue if $k = 1$) of a Laurent series $f(z)$.

PROPOSITION 1.1. Consider the formal power series

$$H(z) = z - \sum_{j=2}^{\infty} B_j z^{1-j}.$$

Define

$$\Sigma_k = -\frac{1}{k}\, [z^{-1}]\, H(z)H(z-1)\cdots H(z-k+1) \tag{1}$$

and

$$R_{k+1} = -\frac{1}{k}\, [z^{-1}]\, H(z)^k, \tag{2}$$

then the expression of Σ_k in terms of the R_k's is given by Kerov's polynomials.

Recently I. P. Goulden and A. Rattan [**GR**] have given an explicit expression for Kerov polynomials, from which they have deduced a certain number of positivity properties of the coefficients of these polynomials. Their proof uses the Lagrange inversion formula. In the next section we use the invariance of the residue under change of variables to derive in a simple way a closely related formula, and show how to recover Goulden and Rattan's formula.

2. Explicit expression for Kerov polynomials

We use the notations of Proposition 1.1 above. Let us introduce the power series

$$L(z) = z + \sum_{j=2}^{\infty} R_j z^{1-j}.$$

From Equation (2) one gets $H \circ L(z) = z$, by Lagrange inversion formula. We use the invariance of the residue under change of variables, namely if u, f are Laurent series, and if u is invertible for composition, then

$$[z^{-1}]\, f(z) = [\zeta^{-1}]\, u'(\zeta) f \circ u(\zeta).$$

Using Taylor's formula about $j = 0$ for $H(z-j)$ as well as the change of variables $z = L(\zeta)$ in the residue, one gets from (1) that

$$\begin{aligned}
\Sigma_k &= -\frac{1}{k}\,[z^{-1}] \prod_{j=0}^{k-1} \left(\sum_{r=0}^{\infty} \frac{(-j)^r}{r!} H^{(r)}(z) \right) \\
&= -\frac{1}{k}\,[\zeta^{-1}]\, L'(\zeta) \prod_{j=0}^{k-1} \left(\sum_{r=0}^{\infty} \frac{(-j)^r}{r!} H^{(r)} \circ L(\zeta) \right).
\end{aligned}$$

Using $H' \circ L(\zeta) = \frac{1}{L'(\zeta)}$, one gets

$$H^{(r)} \circ L(\zeta) = \left(\frac{1}{L'(\zeta)} \frac{d}{d\zeta} \right)^{r-1} \frac{1}{L'(\zeta)}.$$

Therefore,

$$\Sigma_k = -\frac{1}{k}\,[\zeta^{-1}]\, L'(\zeta) \prod_{j=0}^{k-1} \left(\zeta + \sum_{r=1}^{\infty} \frac{(-j)^r}{r!} \left(\frac{1}{L'(\zeta)} \frac{d}{d\zeta} \right)^{r-1} \frac{1}{L'(\zeta)} \right).$$

Putting $F(\zeta) = \frac{1}{L'(\zeta)}$, we obtain the following proposition.

PROPOSITION 2.1. Let

$$F(\zeta) = \frac{1}{L'(\zeta)} = \frac{1}{1 - \sum_{k=2}^{\infty}(k-1)R_k\zeta^{-k}},$$

then Kerov's polynomials are given by the expression

$$\Sigma_k = -\frac{1}{k}\,[\zeta^{-1}]\,\frac{1}{F(\zeta)}\prod_{j=0}^{k-1}\left(\zeta + \sum_{r=1}^{\infty}\frac{(-j)^r}{r!}\left(F(\zeta)\frac{d}{d\zeta}\right)^{r-1}F(\zeta)\right). \quad (3)$$

3. The formula of Goulden and Rattan

Goulden and Rattan give various equivalent formulas for Σ_k. They introduce the series $C(\zeta) = F(\zeta^{-1})$, and define polynomials

$$P_m(z) = -\frac{1}{m!}C(z)(D + (m-2)I)\,[C(z)\cdots(D+I)\,[C(z)DC(z)]\cdots]\,,$$

where $D = z\frac{d}{dz}$. The generating function form of their formula now reads

$$\Sigma_k = -\frac{1}{k}[z^{k+1}]\frac{1}{C(z)}\prod_{j=1}^{k-1}(1 + \sum_{i=1}^{\infty} j^i P_i(z)z^i).$$

We recover this formula using (3). For this, we factor out ζ^k in the expression in the right-hand side of (3), and use the change of variable $z = \zeta^{-1}$. This gives

$$\begin{aligned}
\Sigma_k &= -\frac{1}{k}\,[\zeta^{-1-k}]\,\frac{1}{F(\zeta)}\prod_{j=1}^{k-1}\left(1 + \sum_{r=1}^{\infty}\frac{(-j)^r}{r!}\zeta^{-1}\left(F(\zeta)\frac{d}{d\zeta}\right)^{r-1}F(\zeta)\right)\\
&= -\frac{1}{k}\,[z^{k+1}]\,\frac{1}{C(z)}\prod_{j=1}^{k-1}\left(1 + \sum_{r=1}^{\infty}\frac{(-j)^r}{r!}z\left(-C(z)z^2\frac{d}{dz}\right)^{r-1}C(z)\right)\\
&= -\frac{1}{k}\,[z^{k+1}]\,\frac{1}{C(z)}\prod_{j=1}^{k-1}\left(1 - \sum_{r=1}^{\infty}\frac{j^r z}{r!}(C(z)zD)^{r-1}C(z)\right).
\end{aligned}$$

Now observe that $z^{-j}Dz^j = D + jI$, to get

$$\begin{aligned}
&z(C(z)zD)^{r-1}C(z)\\
&\qquad = z^r C(z)(D + (r-2)I)\,[C(z)\cdots(D+I)\,[C(z)DC(z)]\cdots]\\
&\qquad = -r!P_r(z).
\end{aligned}$$

References

[B1] P. Biane, *Representations of symmetric groups and free probability.* Adv. Math. **138** (1998), no. 1, 126–181.

[B2] P. Biane, *Characters of symmetric groups and free cumulants.* Asymptotic combinatorics with applications to mathematical physics (St. Petersburg, 2001), 185–200, Lecture Notes in Math., 1815, Springer, Berlin, 2003.

[GR] I. P. Goulden, A. Rattan, *An explicit form for Kerov's character polynomials.* `math.CO/0505317`.

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