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# ON GENERATING SERIES OF COLOURED PLANAR TREES

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Abstract. We generalize and reprove an identity of Parker and Loday. It states that certain pairs of generating series associated to coloured plane rooted trees are mutually reciprocal series.

## 1. Introduction

In [1] Carlitz, Scoville and Vaughan consider finite words over a finite alphabet $\mathcal{A}$ such that all pairs of consecutive letters belong to a fixed subset $\mathcal{L} \subset \mathcal{A} \times \mathcal{A}$. They show (Theorems 6.8 and 7.3 of [1]) that suitably defined pairs of signed generating series counting such words associated to $\mathcal{L} \subset \mathcal{A} \times \mathcal{A}$ and to its complementary set $\overline{\mathcal{L}} = \mathcal{A} \times \mathcal{A} \setminus \mathcal{L}$ are inverses of each other. Their result was generalized in the first part of Parker's thesis [5] who showed an analogous result for suitable classes of finite trees having coloured vertices. Loday in [3], motivated by questions concerning combinatorial realizations of operads, rediscovered Parker's result and gave a new proof based on homological arguments.

This paper presents a combinatorial interpretation and a further generalization of Parker's and Loday's result.

A typical example of our identity can be described as follows: associate to two complex matrices

$$M_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad M_2 = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

formal power series $G_1, G_2, \hat{G}_1, \hat{G}_2$ of the form $X^2 + \ldots$ defined by the algebraic systems of equations

$$\begin{cases} G_1 &= (X + aG_1 + bG_2)(X + \alpha G_1 + \beta G_2) \\ G_2 &= (X + cG_1 + dG_2)(X + \gamma G_1 + \delta G_2) \end{cases}$$

and

$$\begin{cases} \hat{G}_1 &= (X + (1+a)\hat{G}_1 + (1+b)\hat{G}_2)(X + (1+\alpha)\hat{G}_1 + (1+\beta)\hat{G}_2) \\ \hat{G}_2 &= (X + (1+c)\hat{G}_1 + (1+d)\hat{G}_2)(X + (1+\gamma)\hat{G}_1 + (1+\delta)\hat{G}_2) \end{cases} .$$

Our main result states then that the formal power series $\varphi = X - G_1 - G_2$ and $\psi = X + \hat{G}_1 + \hat{G}_2$ satisfy the identity $\varphi \circ \psi = X$. They define thus reciprocal branches of holomorphic algebraic functions in a neighbourhood of the origin.

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A tedious proof of this result can be given by computing a minimal polynomial $P(\varphi, X) = \sum_{i,j} p_{i,j}\varphi^i X^j = 0$ for $\varphi$ and by checking that we have $P(X, \psi) = \sum_{i,j} p_{i,j} X^i \psi^j = 0$. Since this identity is algebraic, the field of complex numbers can be replaced by an arbitrary commutative ring. Our proof is instead based on a simple combinatorial argument showing that the series $G = G_1 + G_2$ and $\hat{G} = \hat{G}_1 + \hat{G}_2$ satisfy the relation $\hat{G} = G \circ (X + \hat{G})$.

The sequel of this paper is organized as follows: the next section states the main result in purely algebraic terms over a not necessarily commutative ring. Parker's and Loday's result is the special case of weight-matrices with coefficients in $\{0, -1\}$ and of colours having a common degree $k$ (corresponding to $k-$regular trees). Parker's thesis contains also a generalization to colours of different degrees (corresponding to trees which are no longer necessarily regular). Our main result removes the restriction on the coefficients. It is also more elementary (at least in a commutative setting) since the statement avoids combinatorial descriptions (although they are crucial to our proof).

It follows from our formulation that all involved generating functions are algebraic in a commutative setting and over a finite alphabet. Section 3 fixes notations concerning trees and proves the main result. The proof avoids homological arguments and is thus more elementary than the proof of [3]. Section 4 contains a combinatorial proof of a related identity. As an application of our main result (Corollary 2.4), we describe formulas for the reversion (or inversion under composition) of formal power series in Section 5. Section 6 describes briefly a further generalization involving several variables associated to trees having vertices of different types. Section 7 contains the computations for example (i) of [3], using notations close to the notations of [3]. We display the defining polynomial of the relevant (algebraic) generating function and discuss briefly its asymptotics.

## 2. Main result

In the sequel, $\mathbb{K}$ denotes a fixed, not necessarily commutative ring. All variables, formal power series, etc., are also non-commutative. This condition can be weakened, especially over a commutative ring $\mathbb{K}$ where one can also work in a totally commutative setting.

Consider a (not necessarily finite) set $\mathcal{A}$, called the *alphabet*, together with a *degree function* $d : \mathcal{A} \longrightarrow \mathbb{N}$. We denote by $Y_\mathcal{A}$ a set of (non-commutative) variables indexed by elements $\alpha \in \mathcal{A}$ and by $X$ a supplementary (non-commutative) variable. We denote by $\mathbb{K}[[X, Y_\mathcal{A}]]$ the obvious ring of non-commutative formal power series. An element of $\mathbb{K}[[X, Y_\mathcal{A}]]$ is a (generally infinite) sum of monomials of the form

$$r_1 Z_1 r_2 Z_2 \cdots r_l Z_l r_{l+1}$$

with $r_i \in \mathbb{K}$ and $Z_i \in Y_\mathcal{A} \cup \{X\}$.

Consider also a sequence

$$w(1), w(2), \dots \subset \mathbb{K}^{\mathcal{A}\times\mathcal{A}}$$

of *weight-matrices* with coefficients $w(k)_{\alpha,\beta} \in \mathbb{K}$ indexed by $(\alpha, \beta) \in \mathcal{A} \times \mathcal{A}$. These data define recursively a set $\{G_\alpha\}_{\alpha\in\mathcal{A}} \subset \mathbb{K}[[X, Y_\mathcal{A}]]$ of formal power series indexed by $\mathcal{A}$ such that

$$G_\alpha = Y_\alpha\Big(X + \sum_{\beta\in\mathcal{A}} w(1)_{\alpha,\beta}G_\beta\Big)\Big(X + \sum_{\beta\in\mathcal{A}} w(2)_{\alpha,\beta}G_\beta\Big)\cdots$$
$$\cdots\Big(X + \sum_{\beta\in\mathcal{A}} w(d(\alpha))_{\alpha,\beta}G_\beta\Big)$$

for all $\alpha \in \mathcal{A}$.

*Remark* 2.1. A coefficient $w(k)_{\alpha,\beta}$ with $k > d(\alpha)$ of the $k$-th weight-matrix $w(k)$ is never used and can thus be left unspecified.

*Remark* 2.2. The formal power series $G_\alpha$ are well-defined: if $d(\alpha) = 0$ we get $G_\alpha = Y_\alpha$ and there is nothing to do. For $d(\alpha) > 0$, we consider all variables $X, Y_\mathcal{A}$ graded of degree 1. The series $G_\alpha$ can now be computed by "bootstrapping": set $G^1_\alpha = Y_\alpha$ as an approximation which is correct up to degree 1. An approximation $\{G^k_\alpha\}_{\alpha\in\mathcal{A}}$ correct up to degree $k$ for all $\alpha \in \mathcal{A}$ produces now an approximation

$$G^{k+1}_\alpha = Y_\alpha\Big(X + \sum_{\beta\in\mathcal{A}} w(1)_{\alpha,\beta}G^k_\beta\Big)\cdots\Big(X + \sum_{\beta\in\mathcal{A}} w(d(\alpha))_{\alpha,\beta}G^k_\beta\Big)$$

which is exact at least up to degree $k + 1$.

Similarly, we consider also the set $\{\hat{G}_\alpha\}_{\alpha\in\mathcal{A}} \subset \mathbb{K}[[X, Y_\mathcal{A}]]$ of formal power series defined by

$$\begin{aligned}\hat{G}_\alpha \ = \ & Y_\alpha\Big(X + \sum_{\beta\in\mathcal{A}}(1 + w(1)_{\alpha,\beta})\hat{G}_\beta\Big)\Big(X + \sum_{\beta\in\mathcal{A}}(1 + w(2)_{\alpha,\beta})\hat{G}_\beta\Big)\cdots \\ & \cdots\Big(X + \sum_{\beta\in\mathcal{A}}(1 + w(d(\alpha))_{\alpha,\beta})\hat{G}_\beta\Big)\ .\end{aligned}$$

Setting $G = \sum_{\alpha\in\mathcal{A}} G_\alpha$ and $\hat{G} = \sum_{\alpha\in\mathcal{A}} \hat{G}_\alpha$, our main result is as follows.

**Theorem 2.3.** *We have*

$$\hat{G} = G \circ_X (X + \hat{G})$$

*where $\circ_X$ indicates composition of formal power series with respect to $X$ (every occurrence of $X$ in $G$ is replaced by the formal power series $X + \hat{G}$).*

**Corollary 2.4.** *We have*

$$(X - G) \circ_X (X + \hat{G}) = X\ .$$

*Proof.* Rewrite the identity of Theorem 2.3 as $\hat{G} - G \circ_X (X + \hat{G}) + X = X$ in order to get

$$\hat{G} - G \circ_X (X + \hat{G}) + X = (X + \hat{G}) - G \circ_X (X + \hat{G}) = (X - G) \circ_X (X + \hat{G})\ .$$

□

*Remark* 2.5. (i) Theorem 2.3 and Corollary 2.4 continue to hold for commutative variables $Y_{\mathcal{A}}$. However, the variable $X$ cannot commute with non-central elements of $\mathbb{K}$.

(ii) All specializations (say to elements of $\mathbb{K}$) of variables $Y_\alpha$ with $\alpha \in \mathcal{A}$ of degree $d(\alpha) \geq 2$ are possible in a setting of formal power series. Specializations of $Y_\alpha$ for $\alpha$ of degree $d(\alpha) = 0$ or 1 lead easily to difficulties and convergence problems.

(iii) The main result of [1] corresponds to $d^{-1}(1) = \mathcal{A}$. In this case, Corollary 2.4 boils down to a simple product in $\mathbb{K}$ since $X - G$ and $X + \hat{G}$ are of the form $cX, \frac{1}{c}X$ with $c \in \mathbb{K}[[Y_{\mathcal{A}}]]$ invertible.

(iv) A straightforward generalization involving variables $X_{\mathcal{VT}}$ associated to *vertex-types* will be presented in Section 6.

*Remark* 2.6. One can also consider the following slightly more general version: let $(\lambda_\alpha)_{\alpha\in\mathcal{A}}$ be an additional vector associating a weight $\lambda_\alpha \in \mathbb{K}$ to each colour $\alpha \in \mathcal{A}$. Theorem 2.3 and Corollary 2.4 (which correspond to the case $\lambda_\alpha = 1$ for all $\alpha \in \mathcal{A}$) remain valid with $G = \sum_{\alpha\in\mathcal{A}} \lambda_\alpha G_\alpha$ and $\hat{G} = \sum_{\alpha\in\mathcal{A}} \lambda_\alpha \hat{G}_\alpha$. The case where there exists a vector $(\mu_\alpha)_{\alpha\in\mathcal{A}}$ such that $\mu_\alpha\lambda_\alpha = 1$ for all $\alpha \in \mathcal{A}$ can be reduced to the case treated in this paper by considering variables $\tilde{Y}_\alpha = \lambda_\alpha Y_\alpha$ and weight-matrices with coefficients $\tilde{w}(k)_{\alpha,\beta} = w(k)_{\alpha,\beta}\mu_\beta$.

*Remark* 2.7. Loday and Parker use slightly different notations. Loday's result in the case where all elements of $\mathcal{A}$ have degree 2 corresponds to $Y_\alpha = -1$ for all $\alpha \in \mathcal{A}$, $t = -X$ with

$$g_\alpha = -G_\alpha = \left(-t + \sum_{\beta\in\mathcal{A}} L_{\alpha,\beta} g_\beta\right)\left(-t + \sum_{\beta\in\mathcal{A}} R_{\alpha,\beta} g_\beta\right)$$

where the matrices $L = -w(1), R = -w(2)$ are in $\{0,1\}^{\sharp(\mathcal{A})\times\sharp(\mathcal{A})}$. By Corollary 2.4, the reciprocal series of $-t + \sum_{\alpha\in\mathcal{A}} g_\alpha$ is then given by $-(X + \hat{G}) = -X + \sum_{\alpha\in\mathcal{A}} \hat{g}_\alpha$ where

$$\begin{aligned}\hat{g}_\alpha = -\hat{G}_\alpha &= \left(X - \sum_{\beta\in\mathcal{A}}(1 - L_{\alpha,\beta})\hat{g}_\alpha\right)\left(X - \sum_{\beta\in\mathcal{A}}(1 - R_{\alpha,\beta})\hat{g}_\alpha\right)\\ &= \left(-X + \sum_{\beta\in\mathcal{A}}(1 - L_{\alpha,\beta})\hat{g}_\alpha\right)\left(-X + \sum_{\beta\in\mathcal{A}}(1 - R_{\alpha,\beta})\hat{g}_\alpha\right).\end{aligned}$$

## 3. Trees

A *rooted tree* $T$ is either given by the empty set $\{\}$ representing the *trivial* rooted tree reduced to its root or is recursively defined by a non-empty set

$$T = \{T_s\}_{s\in\mathcal{S}}$$

where $T_s$ are rooted trees attached to the *sons* (also called children or direct descendants) $\mathcal{S}$ of the root vertex. Such a tree $T$ can be identified with a directed graph $\Gamma(T)$ having a distinguished vertex $r$, called the *root*, by joining the root vertex $r$ to its sons using a directed edge originating in $r$ and terminating at the root $r_s$ of the rooted graph $\Gamma(T_s)$ issued from its son indexed by $s \in \mathcal{S}$. In the sequel, we often identify a vertex $v$ of $T$ (or, more precisely, of $\Gamma(T)$) with the tree $T_v$ rooted in $v$ which consists of $v$ and all its descendants. A rooted tree $T$ is *finite* if $\Gamma(T)$ is a finite graph and locally finite if every vertex has only a finite number of sons.

A *plane rooted tree* is recursively defined by a finite or (enumerably) infinite sequence

$$T = (T_1, T_2, \dots)$$

of plane rooted trees $T_1, T_2, \dots$. A plane rooted tree is thus a rooted tree such that every vertex has a finite or enumerable number of completely ordered sons. The empty sequence () represents the *trivial* plane rooted tree.

In the sequel, a *tree* denotes always a finite plane rooted tree.

3.1. **Coloured trees.** Consider an alphabet $\mathcal{A}$ together with a degree function $d : \mathcal{A} \longrightarrow \mathbb{N}$ inducing a partition

$$\mathcal{A} = \bigcup_{k=0}^{\infty} \mathcal{A}_k$$

defined by $\mathcal{A}_k = d^{-1}(k)$.

An $\mathcal{A}$-*coloured tree* or *coloured tree* is either given by the empty sequence () or is recursively defined as $(\alpha; T_1, \dots, T_{d(\alpha)})$ where $\alpha \in \mathcal{A}$ of degree $d(\alpha)$ is the colour of the root $r$ and where the coloured trees $T_1, \dots, T_{d(\alpha)}$ are attached to the $d(\alpha)$ sons of $r$. We call the coloured tree () represented by the empty sequence *trivial*. A coloured tree $(\alpha)$ reduced to a root with a colour $\alpha \in \mathcal{A}_0$ of degree 0 is not considered as trivial. Every leaf (vertex without sons) of a rooted tree is thus either *trivial* (the subtree issued from the leaf is the trivial tree reduced to its uncoloured root) or coloured by an element in $d^{-1}(0) \subset \mathcal{A}$ of degree 0. A leaf coloured by $\alpha \in \mathcal{A}_0 = d^{-1}(0) \subset \mathcal{A}$ is an *ordinary leaf*.

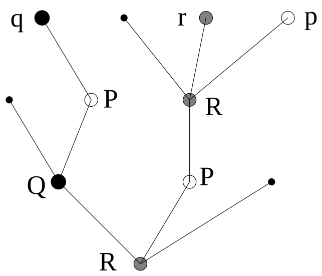


FIGURE 1. A coloured tree

Figure 1 shows the coloured tree

$$(R; (Q; (), (P; (q))), (P; (R; (), (r), (p))), ())$$

having three trivial leaves and three ordinary leaves (coloured $p, q$ and $r$). It has colours in $\mathcal{A} = \{p, q, r, P, Q, R\}$ of degree $d(p) = d(q) = d(r) = 0$, $d(P) = 1,\ d(Q) = 2,\ d(R) = 3$.

Let $w(1), w(2), \dots \in \mathbb{K}^{\mathcal{A}\times\mathcal{A}}$ be a sequence of *weight-matrices* as in Section 2. The *energy* $e(T) \in \mathbb{K}[X, Y_{\mathcal{A}}]$ of a coloured tree $T$ is the monomial defined by $e(T) = X$ if $T$ is trivial and is recursively by

$$e(T) = Y_\alpha e_{1,\alpha}(T_1)e_{2,\alpha}(T_2)\cdots e_{d(\alpha),\alpha}(T_{d(\alpha)})$$

for a non-trivial tree $T = (\alpha; T_1, \dots, T_{d(\alpha)})$, where $e_{k,\alpha}(T_k) = e(T_k) = X$ if $T_k$ is trivial and where $e_{k,\alpha} = w(k)_{\alpha,\beta_k} e(T_k)$ if the tree $T_k = (\beta_k; T_{k,1}, \dots, T_{k,d(\beta_k)})$ (attached to the $k$-th son of the root in $T$) is non-trivial and has root colour $\beta_k$.

*Remark* 3.1. (i) Coefficients $w(k)_{\alpha,\beta}$ with $k > d(\alpha)$ are never used (cf. Remark 2.1).

(ii) Coloured trees as appearing in this paper do in general not correspond to graph-theoretic colourings of the underlying trees: adjacent vertices of a coloured tree can share a common colour. This can of course be avoided: weight-matrices such that $w(k)_{\alpha,\alpha} = 0$ for all $k \in \mathbb{N}$ and for all $\alpha \in \mathcal{A}$ give the energy 0 to all trees with improper colourings in a graph-theoretic sense.

The energy $e(T)$ of the coloured tree $T$ represented in Figure 1 is given by

$$\begin{aligned} &Y_R\ w(1)_{R,Q}Y_Q X w(2)_{Q,P}Y_P w(1)_{P,q}Y_q \\ &\qquad\qquad w(2)_{R,P}Y_P w(1)_{P,R}Y_R X w(2)_{R,r}Y_r w(3)_{R,p}Y_p\ X\ . \end{aligned}$$

The *generating function* (or *partition function*) $G_{\mathcal{CT}}$ for the set $\mathcal{CT}$ of coloured trees is now defined as

$$G_{\mathcal{CT}} = \sum_{T\in\mathcal{CT}} e(T) \in \mathbb{K}[[X, Y_{\mathcal{A}}]]\ .$$

Denoting by

$$\mathcal{CT}_\alpha = \{(\alpha; T_1, \dots, T_{d(\alpha)}) | T_1, \dots, T_{d(\alpha)} \in \mathcal{CT}\} \subset \mathcal{CT}$$

the subset of all non-trivial coloured trees with root colour $\alpha$, we introduce also *restricted generating functions*

$$G_\alpha = \sum_{T\in\mathcal{CT}_\alpha} e(T)\ .$$

The partition

$$\mathcal{CT} = \{()\} \bigcup_{\alpha\in\mathcal{A}} \mathcal{CT}_\alpha$$

shows the identity

$$G_{\mathcal{CT}} = X + \sum_{\alpha\in\mathcal{A}} G_\alpha\ .$$

**Proposition 3.2.** *We have*

$$\begin{aligned} G_\alpha = Y_\alpha &\Big(X + \sum_{\beta\in\mathcal{A}} w(1)_{\alpha,\beta}G_\beta\Big)\Big(X + \sum_{\beta\in\mathcal{A}} w(2)_{\alpha,\beta}G_\beta\Big)\cdots \\ &\qquad\qquad \cdots\Big(X + \sum_{\beta\in\mathcal{A}} w(d(\alpha))_{\alpha,\beta}G_\beta\Big)\ . \end{aligned}$$

*Proof.* The proof is by induction on the total degree of the variables $Y_{\mathcal{A}}$. The formula yields the correct value $G_\alpha = Y_\alpha$ for $\alpha \in \mathcal{A}$ of degree $d(\alpha) = 0$. We suppose thus Proposition 3.2 correct up to degree $n$ in $Y_{\mathcal{A}}$.

The contribution $e(T)$ to $G_\alpha$ of a coloured tree $(\alpha; T_1, \dots, T_{d(\alpha)}) \in \mathcal{CT}_\alpha$ containing $n+1$ coloured vertices factorizes uniquely as

$$e(\alpha; T_1, \dots, T_{d(\alpha)}) = Y_\alpha \tilde{e}_1 \cdots \tilde{e}_{d(\alpha)}$$

where the contribution $\tilde{e}_k$, associated to the $k$-th son $T_k$ of the root, is given by $X$ if $T_k = ()$ is trivial and by $w(k)_{\alpha,\beta_k} e(T_k)$ if $T_k = (\beta_k; T_{k,1}, \dots, T_{k,d(\beta_k)}) \in \mathcal{CT}_{\beta_k}$ is non-trivial with root colour $\beta_k$. The total degree of the product $\tilde{e}_1 \cdots \tilde{e}_{d(\alpha)}$ in $Y_{\mathcal{A}}$ is exactly $n$ and each such product of total degree $n$ in $Y_{\mathcal{A}}$ consisting of $d(\alpha)$ monomials with the $k$-th factor in $X + \sum_{\beta \in \mathcal{A}} w(k)_{\alpha,\beta} G_\beta$ corresponds bijectively to a unique tree in $\mathcal{CT}_\alpha$ and contributes a monomial of degree $n+1$ in $Y_{\mathcal{A}}$ to $G_\alpha$. This proves the result. $\square$

3.2. **Composed coloured trees and proofs.** A *composed coloured tree* is a pair $(T, \mathcal{M})$ where $T$ is a coloured tree, and $\mathcal{M} \subset \mathcal{E}$ a subset of *marked edges* containing all edges with endpoint a trivial leaf of $T$. The set $\mathcal{CCT}$ of composed coloured trees contains the trivial tree $() = ((), \emptyset)$. Every other element $(T, \mathcal{M}) \in \mathcal{CCT}$ can be written uniquely as $(A; B_1, \dots, B_a)$ where $A \in \mathcal{CT}$ is a non-trivial coloured tree having $a$ trivial leaves and where $B_1, \dots, B_a$ are (perhaps trivial) composed coloured trees with the root of $B_k$ glued to the $k$-th trivial leaf of $A$. The set $\mathcal{M}$ of marked edges is the union of all marked edges in $B_1, \dots, B_a$, together with all $a$ edges of $A$ ending at a trivial leaf of $A$.

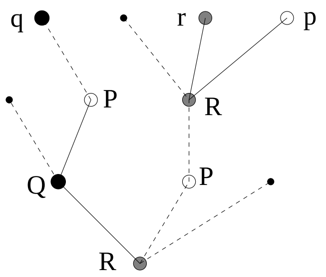


FIGURE 2. A composed coloured tree

Figure 2 shows the composed coloured tree with colours $\{p, q, r, P, Q, R\}$ and 6 marked edges represented by dotted segments which is recursively encoded by

$$(\Big(R; \Big(Q; (), (P; ())\Big), (), ()\Big); (), (q), \Big((P; ()); (\Big(R; (), (r), (p)\Big); ())\Big), ()) \ .$$

We define the energy $e(T)$ of a composed coloured tree by $e(T) = X$ if $T$ is trivial and recursively by

$$e(A; B_1, \dots, B_a) = e(A) \circ_X (e(B_1), \dots, e(B_a))$$

otherwise, where the composition $\circ_X$ with respect to $X$ indicates that the $k$-th occurrence of $X$ in the monomial $e(A)$ (encoding the energy of the ordinary, non-composed coloured tree $A$) has to be replaced by the energy $e(B_k)$ of the composed coloured tree $B_k$ attached to the $k$-th trivial leaf of $A$.

The energy $e(T)$ of the composed coloured tree $T$ represented in Figure 2 is given by

$$Y_R\ w(1)_{R,Q}Y_QXw(2)_{Q,P}Y_PY_q\ Y_PY_RXw(2)_{R,r}Y_rw(3)_{R,p}Y_p\ X$$

and can also be obtained by removing all factors corresponding to marked edges from the energy of the associated non-composed coloured tree depicted in Figure 1.

Denoting by $\hat{G}_\alpha = \sum_{T\in\mathcal{CCT}_\alpha} e(T)$ the partition function associated to the set $\mathcal{CCT}_\alpha$ of all composed coloured trees with a root of colour $\alpha$, we have the following result involving the partition function

$$\hat{G}_{\mathcal{CCT}} = X + \sum_{\alpha\in\mathcal{A}} \hat{G}_\alpha = \sum_{T\in\mathcal{CCT}} e(T)$$

of the set $\mathcal{CCT}$ containing all composed coloured trees.

**Proposition 3.3.** *We have*

$$\hat{G}_\alpha = G_\alpha \circ_X \hat{G}_{\mathcal{CCT}}$$

*where the composition $\circ_X$ indicates that every occurrence of $X$ in $G_\alpha$ has to be replaced by the generating series $\hat{G}_{\mathcal{CCT}}$ associated to all composed coloured trees.*

*Proof.* This reflects simply the recursive definition of the energy of a non-trivial composed coloured tree $T = (A; B_1, \ldots, B_a) \in \mathcal{CCT}_\alpha$ with root colour $\alpha$. □

The following result, analogous to Proposition 3.2, characterizes the formal power series $\hat{G}_\alpha = \sum_{T\in\mathcal{CCT}_\alpha} e(T)$ recursively.

**Proposition 3.4.** *We have*

$$\hat{G}_\alpha = Y_\alpha\left(\hat{G}_{\mathcal{CCT}} + \sum_{\beta\in\mathcal{A}} w(1)_{\alpha,\beta}\hat{G}_\beta\right)\cdots\left(\hat{G}_{\mathcal{CCT}} + \sum_{\beta\in\mathcal{A}} w(d(\alpha))_{\alpha,\beta}\hat{G}_\beta\right)$$

*where $\hat{G}_{\mathcal{CCT}} = X + \sum_{\alpha\in\mathcal{A}} \hat{G}_\alpha$.*

*Proof.* The energy $e(T)$ of a composed coloured tree $(T, \mathcal{M})$ with

$$T = (\alpha; T_1, \ldots, T_{d(\alpha)}) \in \mathcal{CT}_\alpha$$

a coloured tree and $\mathcal{M}$ a suitable subset of marked edges in $T$, is recursively given by

$$Y_\alpha \tilde{w}(1)_{\alpha,\beta_1} e(T_1, \mathcal{M}_1)\cdots \tilde{w}(d(\alpha))_{\alpha,\beta_{d(\alpha)}} e(T_{d(\alpha)}, \mathcal{M}_{d(\alpha)})$$

where $\tilde{w}(k)_{\alpha,\beta_k} = 1$ if the edge relating the root of $T$ to its $k$-th son is marked (and $\beta_k$ may be undefined in this case) and where $\tilde{w}(k)_{\alpha,\beta_k} = w(k)_{\alpha,\beta_k}$ otherwise with $\beta_k$ denoting the colour of the $k$-th son of the root. In the marked case, $e(T_k, \mathcal{M}_k)$ can be an arbitrary monomial of $\hat{G}_{\mathcal{CCT}}$. In the case of an ordinary edge ending at a son of colour $\beta_k$, the energy $e(T_k, \mathcal{M}_k)$ can be an arbitrary monomial of $\hat{G}_{\beta_k}$. Summing over all possibilities, we get

$$\hat{G}_\alpha = Y_\alpha E_1 \dots, E_{d(\alpha)}$$

where $E_k = \hat{G}_{\mathcal{CCT}} + \sum_{\beta\in\mathcal{A}} w(k)_{\alpha,\beta} \hat{G}_\beta$. □

*Proof of Theorem 2.3.* Proposition 3.2 shows that the formal power series $G_\alpha$ appearing in Theorem 2.3 coincide with the generating series $\sum_{T\in\mathcal{CT}_\alpha} e(T)$ (denoted by the same letter $G_\alpha$) associated to $\mathcal{CT}_\alpha$.

The identity $\hat{G}_{\mathcal{CCT}} = X + \sum_{\alpha\in\mathcal{A}} \hat{G}_\alpha$ and Proposition 3.4 show that the formal power series $\hat{G}_\alpha$ involved in Theorem 2.3 coincide with the generating series $\sum_{T\in\mathcal{CCT}_\alpha} e(T)$ associated to $\mathcal{CCT}_\alpha$.

Summing the equality of Proposition 3.3 over $\alpha \in \mathcal{A}$ ends the proof. □

## 4. A combinatorial proof of $G_\alpha = \hat{G}_\alpha \circ_X (X - \sum_{\beta\in\mathcal{A}} G_\beta)$

The equality $\hat{G}_\alpha = G_\alpha \circ_X (X + \sum_{\beta\in\mathcal{A}} \hat{G}_\beta)$ of Proposition 3.3 has a "dual" formulation as follows.

**Proposition 4.1.** *We have*

$$G_\alpha = \hat{G}_\alpha \circ_X (X - \sum_{\beta\in\mathcal{A}} G_\beta) \ .$$

This section is devoted to a direct combinatorial proof of this proposition.

Summing the identity of Proposition 4.1 over $\alpha \in \mathcal{A}$ yields

$$G = \hat{G} \circ_X (X - G)$$

which is the "dual" statement of Theorem 2.3. This identity is of course equivalent to Theorem 2.3 or to Corollary 2.4.

*Proof of Proposition 4.1.* Let $\mathcal{M}$ be the set of marked edges of a non-trivial composed coloured tree $(T, \mathcal{M})$. A marked edge is *trivial* if it contains a trivial (uncoloured) leaf. We denote by $\mathcal{M}^t \subset \mathcal{M}$ the set of all trivial edges and by $\mathcal{M}^o = \mathcal{M} \setminus \mathcal{M}^t$ the complementary set of *ordinary marked edges* in $\mathcal{M}$. The *skeleton* $\mathcal{SK}(T, \mathcal{M})$ of a non-trivial composed coloured tree $(T, \mathcal{M})$ is the plane rooted tree with uncoloured vertices obtained by forgetting all colours and contracting all unmarked edges and all trivial marked edges in $(T, \mathcal{M})$. Edges of $\mathcal{SK}(T, \mathcal{M})$ are thus in bijection with ordinary marked edges of $(T, \mathcal{M})$ and vertices of $\mathcal{SK}(T, \mathcal{M})$ correspond to (non-composed) coloured trees involved in the recursive definition of $(T, \mathcal{M}) = (A; B_1, \dots, B_{d(\alpha)})$. The (non-composed) coloured tree $A$ corresponds thus to the root $\tilde{r}$ of the skeleton $\mathcal{SK}(T, \mathcal{M})$. Sons of $\tilde{r}$ correspond to the ordinary coloured trees involved in the subset $\{B_{i_1}, \dots, B_{i_k}\} \subset \{B_1, \dots, B_{d(\alpha)}\}$ of all composed coloured trees in $\{B_1, \dots, B_{d(\alpha)}\}$ which are non-trivial, etc.

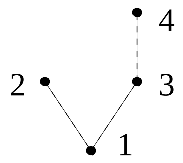


FIGURE 3. A skeleton of a composed coloured tree

Figure 3 depicts the skeleton of the composed coloured tree represented in Figure 2. Its four vertices (labelled $1, \ldots, 4$ for the convenience of the reader) correspond to the following four coloured trees:

| | |
|---|---|
| $(R; (Q; (), (P; ())), (), ())$ | for the root vertex 1 |
| $(q)$ | for vertex 2 |
| $(P; ())$ | for vertex 3 |
| $(R; (), (r), (p))$ | for vertex 4 |

An edge $E$ of the skeleton $\mathcal{SK}(T, \mathcal{M})$ is *trivial* if $E$ contains a leaf. We denote by $\mathcal{TESK}(T, \mathcal{M})$ the set of all trivial edges in the skeleton $\mathcal{SK}(T, \mathcal{M})$ of a non-trivial composed coloured tree $(T, \mathcal{M})$. A *signed composed coloured tree* is a non-trivial composed coloured tree $(T, \mathcal{M})$ together with a sign-function $s : \mathcal{TESK}(T, \mathcal{M}) \longrightarrow \{\pm 1\}$. The set $\mathcal{TESK}$ of trivial edges in the skeleton depicted in Figure 3 for example consists of the two edges $\{1, 2\}$ and $\{3, 4\}$.

We define the energy of a signed composed coloured tree $(T, \mathcal{M}, s)$ as

$$e(T, \mathcal{M}, s) = e(T, \mathcal{M}) \prod_{E \in \mathcal{TESK}(T, \mathcal{M})} s(E)$$

with $e(T, \mathcal{M})$ denoting the energy of the composed coloured tree $(T, \mathcal{M})$ obtained by forgetting the sign-function $s$. The definition of $e(T, \mathcal{M}, s)$ implies that the $2^{\sharp(\mathcal{TESK}(T, \mathcal{M}))}$ energies (corresponding to all possibilities for the sign function $s$) coming from a fixed composed coloured tree $(T, \mathcal{M})$ cancel pairwise out and sum thus up to 0 if $\sharp(\mathcal{TESK}(T, \mathcal{M})) > 0$. If $\sharp(\mathcal{TESK}(T, \mathcal{M})) = 0$, the underlying composed coloured tree is of the form $(T, \mathcal{M}) = (A; (), (), \ldots, ())$ and corresponds to a non-composed coloured tree since all its marked edges are trivial. There is thus only one possibility for the sign function (defined on the empty set) giving rise to a signed energy corresponding to the energy $e(A)$ of the coloured tree $A$. Denoting by $\mathcal{SCCT}_\alpha$ the set of all signed composed coloured trees with root colour $\alpha$ we have thus

$$\sum_{(T, \mathcal{M}, s) \in \mathcal{SCCT}_\alpha} e(T, \mathcal{M}, s) = \sum_{T \in \mathcal{CT}_\alpha} e(T) = G_\alpha \ .$$

On the other hand, the definition of the energy $e(T, \mathcal{M}, s)$ of a signed composed coloured tree shows easily that we have

$$\sum_{(T, \mathcal{M}, s) \in \mathcal{SCCT}_\alpha} e(T, \mathcal{M}, s) = \hat{G}_\alpha \circ_X (X - \sum_{\beta \in \mathcal{A}} G_\beta) \ .$$

This completes the proof of Proposition 4.1. $\square$

## 5. Inversion of power series

We work in this section over a commutative ring $\mathbb{K}$ and set $Y_\alpha = 1$ for all $\alpha \in \mathcal{A}$.

For a natural integer $l \geq 1$, we consider the alphabet

$$\mathcal{A} = \{1, \ldots, l\} \times \{2, 3, 4, \ldots\} \subset \mathbb{N}^2$$

with degree function $d(i, j) = j \geq 2$ and weight-matrices $w(k)$ having coefficients

$$w(k)_{(i,j),(i',j')} = \begin{cases} \alpha(k)_j & \text{if } i = i', \\ 0 & \text{otherwise,} \end{cases}$$

where $\alpha(k)_j \in \mathbb{K}$ are arbitrary. This choice of weight-matrices leads to restricted partition functions $G_{(i,j)}(X)$ and $\hat{G}_{(i,j)}(X)$ which are independent of $i$. Thus the formal power series $G_*(X) = \sum_{j=2}^\infty G_{(i,j)}(X)$ and $\hat{G}_*(X) = \sum_{j=2}^\infty \hat{G}_{(i,j)}(X)$ are well-defined and given by the equations

$$G_*(X) = \sum_{j=2}^\infty \prod_{k=1}^j \left(X + \alpha(k)_j G_*(X)\right),$$

$$\hat{G}_*(X) = \sum_{j=2}^\infty \prod_{k=1}^j \left(X + (l + \alpha(k)_j)\hat{G}_*(X)\right) .$$

Corollary 2.4 shows that we have formally

$$(X - lG_*(X)) \circ (X + l\hat{G}(X)) = (X + l\hat{G}(X)) \circ (X - lG_*(X)) = X .$$

Comparing coefficients of $X^k$ yields algebraic identities and the above identity holds thus for all $l \in \mathbb{K}$. This gives recipes (other than the celebrated Lagrange inversion formula) for computing the reciprocal or reverse series of power series having the form $X + lX^2 + \ldots$ with $l \neq 0$. For an arbitrary formal power series $y(x) = \alpha x + \beta x^2 + \ldots$ (with $\alpha \in \mathbb{K}$ invertible), one can use a homographic transformation and set

$$Y(x) = y\left(\frac{\alpha}{\beta + l\alpha^2} \quad \frac{x}{x + \frac{\alpha^2}{\beta + l\alpha^2}}\right) = x - lx^2 + \ldots$$

for $l \neq 0$ such that $\beta + l\alpha^2 \neq 0$. The reciprocal formal power series (or compositional inverse) $\hat{y}$ of $y$ is then given by

$$\hat{y} = \frac{\alpha}{\beta + l\alpha^2} \quad \frac{\hat{Y}}{\hat{Y} + \frac{\alpha^2}{\beta + l\alpha^2}}$$

where $\hat{Y}$ is the reciprocal formal power series of $Y = x - lx^2 + \ldots$.

*Remark* 5.1. The method outlined above is especially interesting for reversing formal power series of the form $X - lG(X)$ with $G(X) \in X^2\mathbb{K}[[X]]$ a formal power series having a double root at the origin and $l$ a parameter.

*Remark* 5.2. One could of course also work with slightly more general weight-matrices defined by

$$w(k)_{(i,j),(i',j')} = \begin{cases} \alpha(k)_j & \text{if } i = i', \\ \beta(k)_j & \text{otherwise,} \end{cases}$$

where $\alpha(k)_j, \beta(k)_j \in \mathbb{K}$. The restricted partition functions $G_{(i,j)}(X)$ and $\hat{G}_{(i,j)}(X)$ are again independent of $i$ and the associated formal power series $G_*(X) = \sum_{j=2}^{\infty} G_{(i,j)}(X)$ and $\hat{G}_*(X) = \sum_{j=2}^{\infty} \hat{G}_{(i,j)}(X)$ are defined by

$$G_*(X) = \sum_{j=2}^{\infty} \prod_{k=1}^{j} \left(X + (\alpha(k)_j + (l-1)\beta(k)_j)G_*(X)\right),$$
$$\hat{G}_*(X) = \sum_{j=2}^{\infty} \prod_{k=1}^{j} \left(X + (l + \alpha(k)_j + (l-1)\beta(k)_j)\hat{G}_*(X)\right) .$$

The evaluations $\tilde{\alpha}(k)_j = \alpha(k)_j + (l-1)\beta(k)_j$ transform these equations into the former equations.

5.1. **A few examples.** Setting $\alpha(k)_j = 1$ for all $j \geq 2$ and for all $k$, $1 \leq k \leq j$, we get the equation

$$G_* = \frac{(X + G_*)^2}{1 - (X + G_*)}$$

with solution

$$G_* = \frac{1 - 3x - \sqrt{1 - 6x + x^2}}{4} = x^2 + 3x^3 + 11x^4 + 45x^5 + 197x^6 + \dots .$$

The formal power series $Y = X - lG_*$ satisfies thus the algebraic equation $P(X,Y) = 0$ where

$$P(X,Y) = (X - Y)(l - (1+l)X + Y) - ((1+l)X - Y)^2 .$$

Similarly, $\hat{Y} = X + l\hat{G}_*$ (where $\hat{G}_* = \frac{(X+(1+l)\hat{G}_*)^2}{1-(X+(1+l)\hat{G}_*)}$) satisfies $P(\hat{Y}, X) = 0$. The specialization $l = -1$ leads to

$$Y = \frac{1 + X - \sqrt{1 - 6X + X^2}}{4} = X + X^2 + 3X^3 + 11X^4 + \dots$$

(cf. Sequence A1003 in [2]) and

$$\hat{Y} = X - \frac{X^2}{1 - X} = X - X^2 - X^3 - X^4 - \dots$$

The choice $\alpha(k)_j = 1$ if $k = 1$ and $\alpha(k)_j = 0$ otherwise leads to $G_* = \frac{X^2}{1-2X}$ with $Y = X - lG_* = X\frac{1-(l+2)X}{1-2X}$ root of $(X - Y)(1 - X) - ((l+1)X - Y)X$. The specialization $l = -1$ yields $Y = X + X^2 + 2X^3 + 4X^4 + 8X^5 + 16X^6 + \dots$ with reciprocal series

$$\hat{Y} = \frac{1 + 2X - \sqrt{1 + 4X^2}}{2} = X - X^2 + X^4 - 2X^6 + 5X^8 - 14X^{10} + 42X^{12} - \dots$$

closely related to Catalan numbers (cf. A108 in [2]).

Similarly, $\alpha(k)_j = 1$ for $k \leq 2$ and 0 otherwise leads to

$$\begin{aligned} G_* &= \frac{(X+G_*)^2}{1-X} = \frac{1-3X-\sqrt{1-6X+5X}}{2} \\ &= X^2+3X^3+10X^4+36X^5+137X^6+543X^7+\dots \end{aligned}$$

(cf. A2212 of [2]) with $Y = X - lG_*$ root of

$$l(1-X)(X-Y)-((l+1)X-Y)^2 \ .$$

More generally, considering a periodic sequence $\alpha(1)_j, j = 2,3,\dots$ and setting either $\alpha(k)_j = 0$ for $k > 2$ or $\alpha(k)_j = \alpha(1)_j$ yields a formal power series $G_*$ which is algebraic.

## 6. A generalization involving different vertex-types

Consider a set $\mathcal{VT}$ of *vertex-types* and an alphabet $\mathcal{A}$ (with a degree function $d : \mathcal{A} \longrightarrow \mathbb{N}$ as in Section 2) together with an application $vt : \mathcal{A} \longrightarrow \mathcal{VT}$ associating to each element of $\mathcal{A}$ its vertex-type.

A coloured tree with vertex-types in $\mathcal{VT}$ is a coloured tree $T$ together with an application $t : \mathcal{TL}(T) \longrightarrow \mathcal{VT}$ from the set $\mathcal{TL}(T)$ of its trivial leaves into $\mathcal{VT}$. We can thus associate a vertex-type $\tau(v) \in \mathcal{VT}$ to every vertex $v$ of such a tree: if $v$ is a trivial leaf, we set $\tau(v) = t(v)$. If $v$ is an ordinary leaf of colour $\alpha \in \mathcal{A}$, we set $\tau(v) = vt(\alpha)$. For the sake of simplicity, we will henceforth identify the functions $t, vt$ and $\tau$ with $\tau$. (This identification is slightly abusive: the three functions are defined on different sets.)

We denote by $X_{\mathcal{VT}}$ a set of variables indexed by $\mathcal{VT}$ and define the energy $e(T)$ of a coloured tree $T$ with vertex-types in $\mathcal{VT}$ as $X_{\tau(r)}$ if $T$ is a trivial tree reduced to its root $r$ (which is a trivial leaf) of vertex-type $\tau(r)$ and as the usual, recursively defined, product otherwise.

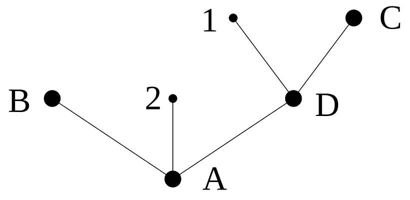


FIGURE 4. A coloured tree with vertex types

The energy of the tree represented in Figure 4 (with vertex types $\{1,2\}$) for example is given by

$$Y_A w(1)_{A,B} Y_B X_2 w(3)_{A,D} Y_D X_1 w(2)_{D,C} Y_C \ .$$

Composed coloured trees with vertex types are defined in the obvious way. Their energy is again the energy of the associated ordinary coloured tree, after removing the contribution of all marked edges.

We denote by $\mathcal{CT}_\tau$ the set of all non-trivial coloured trees with a root $r$ of vertex-type $\tau = \tau(r)$ (and having thus its colour in the subset $vt^{-1}(\tau) \subset \mathcal{A}$).

The set $\mathcal{CCT}_\tau$ is defined analogously. Its elements are all non-trivial composed coloured trees with root type $\tau$ and root colour in $vt^{-1}(\tau)\subset\mathcal{A}$.

Defining $G_\tau=\sum_{T\in\mathcal{CT}_\tau}e(T)$ and $\hat G_\tau=\sum_{T\in\mathcal{CCT}_\tau}e(T)$ we have the following result which is analogous to Theorem 2.3.

**Theorem 6.1.** *We have*

$$\hat G_\tau = G_\tau\circ_{X_{\mathcal{VT}}}(X_\sigma+\hat G_\sigma)_{\sigma\in\mathcal{VT}}$$

*where $\circ_{X_{\mathcal{VT}}}$ indicates that each occurrence of $X_\sigma$ (with $\sigma\in\mathcal{VT}$) in $G_\tau$ has to be replaced by the formal power series $X_\sigma+\hat G_\sigma$.*

It follows easily that the systems of formal power series

$$(X_\tau-G_\tau)_{\tau\in\mathcal{VT}}\quad\text{and}\quad(X_\tau+\hat G_\tau)_{\tau\in\mathcal{VT}}$$

are reciprocal under composition with respect to the variables $X_{\mathcal{VT}}$.

Denoting by $G_\alpha$ the generating function of all non-trivial rooted trees with vertex-type in $\mathcal{VT}$ and a root of colour $\alpha\in\mathcal{A}$ and setting $X=\sum_{\tau\in\mathcal{VT}}X_\tau$ Proposition 3.2 is valid.

Similarly, Proposition 3.4 holds for the similarly defined power series $\hat G_\alpha$ involving composed coloured trees after setting $\hat G_{\mathcal{CCT}}=\sum_{\tau\in\mathcal{VT}}X_\tau+\sum_{\alpha\in\mathcal{A}}\hat G_\alpha$.

## 7. Loday's example (i)

This section contains a partial analysis of example (i) in [3] and was the starting point of this paper. The framework is somewhat simpler than in the previous sections: we work over the commutative ground field $\mathbb{C}$ of complex numbers. We consider the alphabet

$$\mathcal{A}=\{\circ,N,NW,W,SW,S,SE,E,NE\}$$

of nine elements suggesting the graphical notation of [3]. All elements of $\mathcal{A}$ are of degree 2 (and we are thus working with 2-regular trees). We set $X=-t,\ Y_\alpha=-1,\forall\alpha\in\mathcal{A}$ in order to stick to [3], cf. Remark 2.7. We consider weight-matrices $w(1),w(2)$ with coefficients $w(k)_{\alpha,\beta}=-M_k(\alpha,\beta),k=1,2$ where $M_1,M_2$ are the two $9\times 9$ matrices

$$\begin{pmatrix}1&0&0&0&0&0&0&0&0\\1&1&1&0&0&0&0&0&0\\1&1&1&1&0&0&0&0&0\\1&0&1&1&0&0&0&0&0\\1&1&1&1&1&0&0&0&0\\1&1&0&0&0&0&0&0&0\\1&1&1&1&0&1&0&0&0\\1&1&0&1&0&0&0&0&0\\1&1&1&1&0&0&0&0&0\end{pmatrix},\quad\begin{pmatrix}0&1&1&1&0&0&0&0&0\\0&0&0&1&0&0&0&0&0\\0&0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0&0\\0&0&0&0&0&1&0&0&0\\0&0&1&1&1&1&0&0&0\\0&0&0&0&1&0&1&1&1\\0&0&1&0&0&0&0&1&1\\0&0&0&0&0&0&0&1&1\end{pmatrix}$$

Setting $g_\alpha = -G_\alpha$, Proposition 3.2 corresponds to the equations

$$\begin{array}{lcl}
g_\circ &=& (-t+g_\circ)(-t+g_N+g_{NW}+g_W)\\
g_N &=& (-t+g_\circ+g_N+g_{NW})(-t+g_W)\\
g_{NW} &=& (-t+g_\circ+g_N+g_{NW}+g_W)(-t)\\
g_W &=& (-t+g_\circ+g_{NW}+g_W)(-t+g_N)\\
g_{SW} &=& (-t+g_\circ+g_N+g_{NW}+g_W+g_{SW})(-t+g_S)\\
g_S &=& (-t+g_\circ+g_N)(-t+g_{NW}+g_W+g_{SW}+g_S)\\
g_{SE} &=& (-t+g_\circ+g_N+g_{NW}+g_W+g_S)\\
&& \qquad (-t+g_{SW}+g_{SE}+g_E+g_{NE})\\
g_E &=& (-t+g_\circ+g_N+g_W)(-t+g_{NW}+g_E+g_{NE})\\
g_{NE} &=& (-t+g_\circ+g_N+g_{NW}+g_W)(-t+g_E+g_{NE})\ .
\end{array}$$

One sees easily that we have

$$g_o = g_N = g_W\ .$$

Eliminating $g_N, g_W$ we get the simpler equations:

$$\begin{array}{lcl}
g_\circ &=& (-t+g_\circ)(-t+2g_\circ+g_{NW})\\
g_{NW} &=& (-t+3g_\circ+g_{NW})(-t)\\
\hline
g_{SW} &=& (-t+3g_\circ+g_{NW}+g_{SW})(-t+g_S)\\
g_S &=& (-t+2g_\circ)(-t+g_\circ+g_{NW}+g_S+g_{SW})\\
\hline
g_E &=& (-t+3g_\circ)(-t+g_{NW}+g_E+g_{NE})\\
g_{NE} &=& (-t+3g_\circ+g_{NW})(-t+g_E+g_{NE})\\
\hline
g_{SE} &=& (-t+3g_\circ+g_{NW}+g_S)(-t+g_{SW}+g_{SE}+g_E+g_{NE})
\end{array}$$

where functions above a horizontal line are independent of functions below the line.

Computations (done with *Maple*) using Gröbner bases show that

$$y = -t+g_\circ+g_N+g_{NW}+g_W+g_{SW}+g_S+g_{SE}+g_E+g_{NE}$$

(corresponding to the power series $X - G = X - \sum_{\alpha\in\mathcal{A}} G_\alpha$ of Corollary 2.4) satisfies the algebraic equation $P(y(t),t) = 0$ where

$$P(y,t) = c_0 + c_1\ y + c_2\ y^2 + c_3\ y^3 + c_4\ y^4$$

with

$$\begin{array}{lcl}
c_0 &=& t\ \ (288\,t^{31} - 1008\,t^{30} - 17696\,t^{29} + 35124\,t^{28} + 513042\,t^{27}\\
&& -352654\,t^{26} - 8834409\,t^{25} - 2315100\,t^{24} + 94293622\,t^{23}\\
&& +92841847\,t^{22} - 608228325\,t^{21} - 1031578684\,t^{20}\\
&& +2072381165\,t^{19} + 5859780674\,t^{18} - 1127775119\,t^{17}\\
&& -16287829166\,t^{16} - 15833938922\,t^{15} + 9251292427\,t^{14}\\
&& +38652814035\,t^{13} + 44572754075\,t^{12} + 10866248029\,t^{11}\\
&& -40129564125\,t^{10} - 59007425756\,t^{9} - 36829453004\,t^{8}\\
&& -10216139916\,t^{7} - 63849664\,t^{6} + 693364800\,t^{5} + 187804368\,t^{4}\\
&& +24111840\,t^{3} + 1694752\,t^{2} + 63488\,t + 1024)
\end{array}$$

$$\begin{aligned}
c_1 \;=\;& -768\,t^{31}+1488\,t^{30}+44636\,t^{29}-49538\,t^{28}-1198111\,t^{27}\\
&+359773\,t^{26}+19127286\,t^{25}+7602856\,t^{24}-192894854\,t^{23}\\
&-193322898\,t^{22}+1208988418\,t^{21}+1968967542\,t^{20}\\
&-4212884427\,t^{19}-10893520130\,t^{18}+4099837581\,t^{17}\\
&+31343794098\,t^{16}+22508418249\,t^{15}-27437733598\,t^{14}\\
&-67449042813\,t^{13}-62529644946\,t^{12}-3629552721\,t^{11}\\
&+84243589625\,t^{10}+130637976096\,t^{9}+104165077688\,t^{8}\\
&+50704667612\,t^{7}+15902199040\,t^{6}+3290858704\,t^{5}\\
&+451630576\,t^{4}+40498432\,t^{3}+2274080\,t^{2}+72704\,t+1024
\end{aligned}$$

$$\begin{aligned}
c_2 \;=\;& t\;(680\,t^{29}-932\,t^{28}-39270\,t^{27}+33926\,t^{26}+1020385\,t^{25}\\
&-335552\,t^{24}-15580483\,t^{23}-2999586\,t^{22}+151572589\,t^{21}\\
&+104425399\,t^{20}-945694543\,t^{19}-1131146667\,t^{18}\\
&+3558106593\,t^{17}+6544185368\,t^{16}-6226627151\,t^{15}\\
&-20759382401\,t^{14}-4197042728\,t^{13}+29133893401\,t^{12}\\
&+33005986439\,t^{11}+5921959164\,t^{10}-22398414511\,t^{9}\\
&-37477032816\,t^{8}-35648192872\,t^{7}-21631096056\,t^{6}\\
&-8298733544\,t^{5}-1992896768\,t^{4}-295849440\,t^{3}\\
&-26179392\,t^{2}-1255424\,t-24832)
\end{aligned}$$

$$\begin{aligned}
c_3 \;=\;& -t^2\;(t-2)\,(t+1)\,(2\,t^4+6\,t^3-11\,t^2-30\,t-4)\\
&(124\,t^{21}-398\,t^{20}-5146\,t^{19}+14694\,t^{18}+92616\,t^{17}-213234\,t^{16}\\
&-966327\,t^{15}+1518831\,t^{14}+6391763\,t^{13}-5003278\,t^{12}\\
&-26227554\,t^{11}+1248286\,t^{10}+58532080\,t^{9}+36103178\,t^{8}\\
&-41699603\,t^{7}-64544195\,t^{6}-41818519\,t^{5}-21472740\,t^{4}\\
&-8578026\,t^{3}-1961960\,t^{2}-218928\,t-9296)
\end{aligned}$$

and

$$\begin{aligned}
c_4 \;=\;& (t^2-2\,t-2)(t^2-2\,t-7)(t-2)^2(t+1)^2\\
&(2\,t^4+6\,t^3-11\,t^2-30\,t-4)^2t^5\\
&(8\,t^7-10\,t^6-171\,t^5+209\,t^4+948\,t^3-721\,t^2-1892\,t-249).
\end{aligned}$$

The first coefficients of the series $y(t)$ are

$$\begin{aligned}
&-t+9\,t^2-49\,t^3+284\,t^4-1735\,t^5+10955\,t^6-70695\,t^7+463087\,t^8\\
&\qquad -3066450\,t^9+20471641\,t^{10}-137540539\,t^{11}+928791019\,t^{12}\mp\dots
\end{aligned}$$

The reciprocal function

$$\hat{y}=-t+9t^2-113t^3+1724t^4-29309t^5+532896t^6-\dots$$

satisfies the polynomial equation $P(t,\hat{y})=0$ with $P(y,t)=c_0+c_1y+c_2y^2+c_3y^3+c_4y^4$ as above. Corollary 2.4 shows that $\hat{y}$ is also defined by

$$\hat{y}=-t+\hat{g}_\circ+\hat{g}_N+\hat{g}_{NW}+\hat{g}_W+\hat{g}_{SW}+\hat{g}_S+\hat{g}_{SE}+\hat{g}_E+\hat{g}_{NE}$$

where $\hat{g}_\circ, \hat{g}_N, \hat{g}_{NW}, \hat{g}_W, \hat{g}_{SW}, \hat{g}_S, \hat{g}_{SE}, \hat{g}_E, \hat{g}_{NE}$ satisfy the "complementary" equations

$$\begin{aligned}
\hat{g}_\circ &= (-t + \hat{g}_N + \hat{g}_{NW} + \hat{g}_W + \hat{g}_{SW} + \hat{g}_S + \hat{g}_{SE} + \hat{g}_E + \hat{g}_{NE}) \\
&\quad (-t + \hat{g}_\circ + \hat{g}_{SW} + \hat{g}_S + \hat{g}_{SE} + \hat{g}_E + \hat{g}_{NE}) \\
\hat{g}_N &= (-t + \hat{g}_W + \hat{g}_{SW} + \hat{g}_S + \hat{g}_{SE} + \hat{g}_E + \hat{g}_{NE}) \\
&\quad (-t + \hat{g}_\circ + \hat{g}_N + \hat{g}_{NW} + +\hat{g}_{SW} + \hat{g}_S + \hat{g}_{SE} + \hat{g}_E + \hat{g}_{NE}) \\
\hat{g}_{NW} &= (-t + \hat{g}_{SW} + \hat{g}_S + \hat{g}_{SE} + \hat{g}_E + \hat{g}_{NE}) \\
&\quad (-t + \hat{g}_\circ + \hat{g}_N + \hat{g}_{NW} + \hat{g}_W + \hat{g}_{SW} + \hat{g}_S + \hat{g}_{SE} + \hat{g}_E + \hat{g}_{NE}) \\
\hat{g}_W &= (-t + \hat{g}_N + \hat{g}_{SW} + \hat{g}_S + \hat{g}_{SE} + \hat{g}_E + \hat{g}_{NE}) \\
&\quad (-t + \hat{g}_\circ + \hat{g}_{NW} + \hat{g}_W + \hat{g}_{SW} + \hat{g}_S + \hat{g}_{SE} + \hat{g}_E + \hat{g}_{NE}) \\
\hat{g}_{SW} &= (-t + \hat{g}_S + \hat{g}_{SE} + \hat{g}_E + \hat{g}_{NE}) \\
&\quad (-t + \hat{g}_\circ + \hat{g}_N + \hat{g}_{NW} + \hat{g}_W + \hat{g}_{SW} + \hat{g}_{SE} + \hat{g}_E + \hat{g}_{NE}) \\
\hat{g}_S &= (-t + \hat{g}_{NW} + \hat{g}_W + \hat{g}_{SW} + \hat{g}_S + \hat{g}_{SE} + \hat{g}_E + \hat{g}_{NE}) \\
&\quad (-t + \hat{g}_\circ + \hat{g}_N + \hat{g}_{SE} + \hat{g}_E + \hat{g}_{NE}) \\
\hat{g}_{SE} &= (-t + \hat{g}_{SW} + \hat{g}_{SE} + \hat{g}_E + \hat{g}_{NE}) \\
&\quad (-t + \hat{g}_\circ + \hat{g}_N + \hat{g}_{NW} + \hat{g}_W + \hat{g}_S) \\
\hat{g}_E &= (-t + \hat{g}_{NW} + \hat{g}_{SW} + \hat{g}_S + \hat{g}_{SE} + \hat{g}_E + \hat{g}_{NE}) \\
&\quad (-t + \hat{g}_\circ + \hat{g}_N + \hat{g}_W + \hat{g}_{SW} + \hat{g}_S + \hat{g}_{SE}) \\
\hat{g}_{NE} &= (-t + \hat{g}_{SW} + \hat{g}_S + \hat{g}_{SE} + \hat{g}_E + \hat{g}_{NE}) \\
&\quad (-t + \hat{g}_\circ + \hat{g}_N + \hat{g}_{NW} + \hat{g}_W + \hat{g}_{SW} + \hat{g}_S + \hat{g}_{SE})
\end{aligned}$$

*Remark* 7.1. For computations of huge terms in the series expansion of an algebraic function, one can use the following well-known trick: any algebraic function $y(t) = \sum a_n t^n$ of degree $d$ satisfies a linear differential equation

$$\sum_{k=0}^{d} q_k(t) y^{(k)}(t) = 0$$

with polynomial coefficients $q_0, \ldots, q_t \in \mathbb{C}[t]$. This allows a recursive computations of $a_n$ with time and memory requirements linear in $n$. In our case, we get

$$q_0(t)y + q_1(t)y' + q_2(t)y'' + q_3(t)y^{(3)} + q_4(t)y^{(4)} = 0$$

where $q_0, \ldots, q_4 \in \mathbb{Z}[t]$ are polynomials of degrees respectively 150, 151, 155, 156 and 157.

7.1. **Asymptotics.** The asymptotic growth rate of the coefficients of $y(t)$ is governed by the distance of the origin to the first ramification point of the corresponding sheet, see [4]. Ramifications are above the roots of the discriminant $D(t)$ of $P(y,t)$ with respect to $y$. This discriminant is given by

$$D(t) = r_1^2 \; r_2 \; r_3^2 \; r_4^2 \; r_\infty$$

where

$$
\begin{aligned}
r_1 &= t^4-4\,t^3+6\,t^2+8\,t+1\\
r_2 &= t^{13}-3\,t^{12}-16\,t^{11}+100\,t^{10}-86\,t^9-222\,t^8+312\,t^7-544\,t^6\\
&\quad +4845\,t^5+10665\,t^4+9536\,t^3+4084\,t^2+528\,t+16\\
r_3 &= 20\,t^{16}+50\,t^{15}-849\,t^{14}-937\,t^{13}+11563\,t^{12}+7833\,t^{11}\\
&\quad -64177\,t^{10}-58882\,t^9+141152\,t^8+259280\,t^7+82253\,t^6\\
&\quad -366913\,t^5-698955\,t^4-468324\,t^3-122700\,t^2-13720\,t-552\\
r_4 &= 5760\,t^{53}+8864\,t^{52}-425056\,t^{51}+\cdots-7200309248\,t-76152832
\end{aligned}
$$

($r_4$ is not involved in coarse asymptotics of $y(t)$). The roots of the polynomial

$$r_\infty = t^6\ (t-2)^2\ (t+1)^3\ (2\,t^4+6\,t^3-11\,t^2-30\,t-4)^2$$

are critical points for the critical value $\infty$.

Since the coefficients of $y(t)$ have alternating signs, the "smallest" singularity of $y(t)$ is on the negative real halfline. The following table resumes the relevant data for its computation. More precisely, the algebraic function defined by $P$ has a ramification of order 3 with $y=\infty$ above $t=0$. The remaining sheet is unramified above $t=0$ and defines the generating function $y(t)$ under consideration.

The table contains the following informations: the first column shows the argument $t$ considered in the corresponding row. The second column indicates the factor of the discriminant $D(t)$ if $t$ is a root of $D(t)$. The remaining column displays information about the inverse images of $t$ defined by the algebraic equation of $y(t)$.

We have

| | | |
|---|---|---|
| $t_0=0$ | $r_\infty$ | $y_1=y_2=y_3=\infty,\ y_4=0$ |
| $t_0>t>t_1$ | | $y_1<y_2<y_3<0<y_4$ |
| $t_1\sim-0.04355$ | $r_2$ | $y_1\sim-2922<y_2=y_3\sim-1.083<0<y_4\sim0.066$ |
| $t_1>t>t_2$ | | $y_1<0<y_4,\ y_2=\overline{y_3}\in\mathbb{C}\setminus\mathbb{R}$ |
| $t_2\sim-0.1118$ | $r_3$ | $y_1\sim-82.3<y_2=y_3\sim0.2194<y_4\sim0.4499$ |
| $t_2>t>t_3$ | | $y_1<0<y_4,\ y_2=\overline{y_3}\in\mathbb{C}\setminus\mathbb{R}$ |
| $t_3\sim-0.14047$ | | $0<y_4\sim4.113,\ y_1=\infty,\ y_2=\overline{y_3}\sim-2.98\pm8.481i$ |
| $t_3>t>t_4$ | | $0<y_4<y_1,\ y_2=\overline{y_3}\in\mathbb{C}\setminus\mathbb{R}$ |
| $t_4\sim-0.14118$ | $r_\infty$ | $0<y_4\sim8.692<y_1\sim28.07,\ y_2=y_3=\infty$ |
| $t_4>t>t_5$ | | $0<y_4<y_1,\ y_2=\overline{y_3}\in\mathbb{C}\setminus\mathbb{R}$ |
| $t_5\sim-0.14127$ | $r_2$ | $0<y_4=y_1\sim14.89,\ y_2=\overline{y_3}\sim23.95\pm59.72i$ |

The convergence radius of the series for $y(t)$ is of course given by $|t_5|=-t_5$ and the asymptotic growth of the coefficients of $y(t)$ is roughly exponential with argument

$$1/t_5\sim-7.078574585124103038206412527375385868163171 82\,.$$

A slightly more precise asymptotic behaviour of the coefficients of $y(t)$ can be computed as follows:

At the root

$$\rho=t_5\sim-.14127137998962933757540882196178714222253950575630$$

of $r_2$, the ramified sheet corresponds to the double root

$$y_\rho \sim 14.887388086028940552779707880945443 94$$

of $P(y,\rho) \in \mathbb{C}[y]$. (Caution: when computing $y_\rho$ as a root of $P(y,\rho)$ one loses roughly half the digits since an error of order $\epsilon$ on $\rho$ induces an error of order $\sqrt{\epsilon}$ on the corresponding two roots approximating the double root $y_\rho$ of $P(y,\rho)$. A better strategy is of course to compute $y_\rho$ as a (simple) root of the derivative $\frac{d}{dy}P(y,\rho)$ of $P(y,\rho)$).

In a small open neighbourhood of $\rho$ we get now a Puiseux series expansion

$$y(t) = h(t) + g(t)\sqrt{\rho - t}$$

with $h(t)$, $g(t)$ holomorphic. Since the discriminant $D(t)$ contains no other roots of absolute value $|\rho|$, the asymptotics of the generating function $y(t)$ are roughly given by

$$\begin{aligned}
& \gamma_\rho\sqrt{\rho}\sqrt{1-t/\rho} \\
= & \gamma_\rho\sqrt{\rho}\textstyle\sum_{n=0}^\infty \binom{1/2}{n}\left(\frac{-t}{\rho}\right)^n \\
= & \gamma_\rho\sqrt{\rho}\left(1-\textstyle\sum_{n=1}^\infty \frac{1}{2}\,\frac{1/2\cdot 3/2\cdots(n-3/2)}{n!}\left(\frac{t}{\rho}\right)^n\right) \\
= & \gamma_\rho\sqrt{\rho}\left(1-\textstyle\sum_{n=1}^\infty \frac{1\cdot 3\cdots(2n-3)}{2^n\ n!}\left(\frac{t}{\rho}\right)^n\right) \\
= & \gamma_\rho\sqrt{\rho}\left(1-\frac{1}{2}\textstyle\sum_{n=1}^\infty \frac{(2n-2)!}{4^{n-1}\ n!\ (n-1)!\ \rho^n}\ t^n\right)
\end{aligned}$$

where $\gamma(\rho) = g(\rho)$. Since

$$\frac{(2n-2)!}{n!\ (n-1)!} \sim \frac{1}{n}\frac{\sqrt{4\pi(n-1)}\ 4^{n-1}\ (n-1)^{2n-2}\ e^{-2n+2}}{2\pi\ (n-1)\ (n-1)^{2(n-1)}\ e^{-2(n-1)}} \sim \frac{4^{n-1}}{\sqrt{\pi}\ n^{3/2}}$$

we get the asymptotics

$$a_n \sim \frac{\gamma_\rho}{2\ \sqrt{\pi}\ n^{3/2}\ \rho^{n-1/2}}\ .$$

The constant $\gamma_\rho$ can be computed by remarking that

$$\begin{aligned}
0 &= P(h(t)+\sqrt{\rho-t}\ g(t),t) \\
&= P(y_\rho+\gamma_\rho\sqrt{\rho-t}+O((\rho-t)),t) \\
&= \tfrac{\partial^2 P}{\partial y^2}|_{(y_\rho,\rho)}\tfrac{\gamma_\rho^2\ (\rho-t)}{2}+\tfrac{\partial P}{\partial t}|_{(y_\rho,\rho)}(t-\rho)+O((\rho-t)^{3/2}))
\end{aligned}$$

yielding

$$\gamma_\rho\sqrt{\rho} = \sqrt{2\rho\frac{\frac{\partial P}{\partial t}|_{(y_\rho,\rho)}}{\frac{\partial^2 P}{\partial y^2}|_{(y_\rho,\rho)}}} \sim\ 337.171657540870\ .$$

We have thus asymptotically

$$a_n \sim 95.1143685260451189406883 6\ \rho^{-n}\ n^{-3/2}$$

with $\rho \sim -.14127137998962933757540882196178714222253950575630064 18$, (cf. Formula 10.64 in [4]).

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