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IDEALS AND QUOTIENTS OF B-QUASISYMMETRIC POLYNOMIALS

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Abstract. The space $QSym_n(B)$ of B-quasisymmetric polynomials in 2 sets of n variables was recently studied by Baumann and Hohlweg [*Trans. Amer. Math. Soc.*, to appear]. The aim of this work is a study of the ideal $\langle QSym_n(B)^+\rangle$ generated by B-quasisymmetric polynomials without constant term. In the case of the space $QSym_n$ of quasisymmetric polynomials in 1 set of n variables, Aval, Bergeron and Bergeron [*Proc. Amer. Math. Soc.* **131** (2003), 1053–1062; *Adv. in Math.* **181** (2004), 353–367] proved that the dimension of the quotient of the space of polynomials by the ideal $\langle QSym_n^+\rangle$ is given by Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$. In the case of B-quasisymmetric polynomials, our main result is that the dimension of the analogous quotient is equal to $\frac{1}{2n+1}\binom{3n}{n}$, the numbers of ternary trees with n nodes. The construction of a Gröbner basis for the ideal, as well as of a linear basis for the quotient are interpreted by a bijection with lattice paths. These results are finally extended to p sets of variables, and the dimension is in this case $\frac{1}{pn+1}\binom{(p+1)n}{n}$, the number of p-ary trees with n nodes.

Résumé. L'espace $QSym_n(B)$ des polynômes B-quasisymétriques en deux ensembles de n variables a été récemment étudié par Baumann et Hohlweg [*Trans. Amer. Math. Soc.*, à paraître]. Nous considérons ici l'idéal $\langle QSym_n(B)^+\rangle$ engendré par les polynômes B-quasisymétriques sans terme constant. Dans le cas de l'espace $QSym_n$ des polynômes quasisymétriques en 1 ensemble de n variables, Aval, Bergeron et Bergeron [*Proc. Amer. Math. Soc.* **131** (2003), 1053–1062; *Adv. in Math.* **181** (2004), 353–367] ont montré que la dimension du quotient de l'espace des polynômes par l'idéal $\langle QSym_n^+\rangle$ est donnée par les nombres de Catalan $C_n = \frac{1}{n+1}\binom{2n}{n}$. Dans le cas des polynômes B-quasisymétriques, notre principal résultat est que la dimension du quotient analogue est ici $\frac{1}{2n+1}\binom{3n}{n}$, à savoir le nombre d'arbres ternaires à n nœuds. Nous construisons une base de Gröbner pour l'idéal, de même qu'une base du quotient, toutes deux explicites et en bijection avec des chemins. Nous étendons enfin ces résultats à p ensembles de variables, et montrons que dans ce cas la dimension est $\frac{1}{pn+1}\binom{(p+1)n}{n}$, le nombre d'arbres p-aires à n nœuds.

1. Introduction

To start with, we recall (a part of) the story of the study of ideals and quotients related to symmetric or quasisymmetric polynomials. The root of this work is a result of Artin [1]. Let us consider the set of variables $X_n = x_1, x_2, \ldots, x_n$. The space of

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polynomials in the variables X_n with rational coefficients is denoted by $\mathbb{Q}[X_n]$. The subspace of symmetric polynomials is denoted by Sym_n. Symmetric polynomials may be seen (cf. [19]) as invariants of the symmetric group $\mathcal{S}_n$ under the action defined as follows: for $\sigma \in \mathcal{S}_n$ anf $P \in \mathbb{Q}[X_n]$,

$$\sigma \cdot P(X_n) = P(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}).$$

Let $\mathbf{V}$ be a subset of the polynomial ring. We denote by $\langle \mathbf{V}^+ \rangle$ the ideal generated by elements of a $\mathbf{V}$ with no constant term. Artin's result is given by:

$$\dim \mathbb{Q}[X_n]/\langle Sym_n^+ \rangle = n!\,. \tag{1.1}$$

Another, more recent, part of the story deals with quasisymmetric polynomials. The space $QSym_n \subset \mathbb{Q}[X_n]$ of quasisymmetric polynomials was introduced by Gessel [16] as generating functions for Stanley's P-partitions [24]. This is the starting point of many recent works in several areas of combinatorics [10, 20, 15, 25]. Quasisymmetric polynomials may also be seen as $\mathcal{S}_n$-invariants under Hivert's quasisymmetrizing action ([18]), defined as follows.

Let $I = \{i_1, \ldots, i_k\}$ be a subset of $\{1, \ldots, n\}$ and $a = (a_1, \ldots, a_k)$ a sequence of positive (> 0) integers, of the same cardinality. We define $X_I^a = x_{i_1}^{a_1} \cdots x_{i_k}^{a_k}$, where the elements of I are listed in increasing order. Hivert's action is then defined on monomials by

$$\sigma * X_I^a = X_{\sigma(I)}^a$$

where $\sigma(I)$ is the set $\{\sigma(i_1), \ldots, \sigma(i_k)\}$ arranged in increasing order.

In [2, 3], Aval et al. study the problem analogous to Artin's work in the case of quasisymmetric polynomials. Their main result is that the dimension of the quotient is given by Catalan numbers:

$$\dim \mathbb{Q}[X_n]/\langle QSym_n^+ \rangle = C_n = \frac{1}{n+1}\binom{2n}{n}. \tag{1.2}$$

An interesting axis of research is the extension of these results to 2 (or $p \geq 2$) sets of variables. In the case of two sets of variables, let $\mathcal{A}_n = \mathcal{A}_n^2$ denote the alphabet

$$\mathcal{A}_n = x_1, y_1, x_2, y_2, \ldots, x_n, y_n.$$

The diagonal action of $\mathcal{S}_n$ on $\mathbb{Q}[\mathcal{A}_n]$ is defined as simultaneous permutation of variables x's and y's:

$$\sigma \cdot P(\mathcal{A}_n) = P(x_{\sigma(1)}, y_{\sigma(1)}, \ldots, x_{\sigma(n)}, y_{\sigma(n)}).$$

Invariants associated to this action are called diagonally symmetric polynomials. Their set is denoted by $DSym_n$.The diagonal coinvariant space $\mathbb{Q}[\mathcal{A}_n]/\langle DSym_n^+ \rangle$ has been studied extensively in the last 15 years by several authors [8, 9, 13, 14, 17]. A great achievment in this area is Haiman's proof of the following equality (cf. [17]):

$$\dim \mathbb{Q}[\mathcal{A}_n]/\langle DSym_n^+ \rangle = (n+1)^{n-1}\,.$$

In [5], the space $DQSym_n$ of diagonally quasisymmetric polynomials is defined as the invariant space of the diagonal extension of Hivert's action. This space was originally introduced by Poirier [22], and, with generalizations, has been recently studied in [21] and [7].

The coinvariant space $\mathbb{Q}[\mathcal{A}_n]/\langle DQSym_n^+\rangle$ is investigated in [5], and conjectures are stated. In particular, a conjectural basis for this quotient is presented.

To end this presentation, we introduce the space $QSym_n(B)$ of B-quasisymmetric polynomials, which is the focus of this article. This space, whose definition appears implicitly in [22], is studied with more details in [7]. A precise definition will be given in the next section, and we only mention here that $QSym_n(B)$ is a subspace (and in fact a subalgebra, cf. [7]) of $DQSym_n$.

We now state the main result of this work, which appears as a generalization of equation (1.2).

Theorem 1.1.

$$\dim \mathbb{Q}[\mathcal{A}_n]/\langle QSym_n(B)^+\rangle = \frac{1}{2n+1}\binom{3n}{n}. \tag{1.3}$$

Observe that in equations (1.2) and (1.3), the dimensions $\frac{1}{n+1}\binom{2n}{n}$ and $\frac{1}{2n+1}\binom{3n}{n}$ are respectively the numbers of binary and ternary trees (cf. [23]). This will be generalized in the last section of this paper.

The content of this paper is divided into 5 main sections. After this introduction, the Section 2 defines the central objects of this work, the B-quasisymmetric polynomials. Sections 3 and 4 are the proof of the Theorem 1.1. In Section 3 is introduced a set $\mathcal{G}$ of polynomials, which is proved in Section 4 to be a Gröbner basis for $\langle QSym_n(B)^+\rangle$. The Gröbner basis $\mathcal{G}$, as well as the basis of the quotient "deduced" from it, are interpreted in terms of plane paths. Finally, Section 5 gives a generalization of this work for p sets of variables, where the equation analogous to (1.3) replaces $\frac{1}{2n+1}\binom{3n}{n}$ by $\frac{1}{pn+1}\binom{(p+1)n}{n}$, the number of p-ary trees.

2. QSym(B): definitions and notations

For these definitions, we follow [7], with some minor differences, for the sake of simplicity of the computations we will have to make.

Let $\mathbb{N}$ and $\bar{\mathbb{N}}$ denote two occurrrences of the set of nonnegative integers. We shall write $\bar{\mathbb{N}} = \{\bar{0}, \bar{1}, \bar{2}, \dots\}$ and make no difference between the elements of $\mathbb{N}$ and $\bar{\mathbb{N}}$ in any arithmetical expression. We distinguish $\mathbb{N}$ and $\bar{\mathbb{N}}$ for the ease of reading.

A *bivector* is a vector $v = (v_1, v_2, \dots, v_{2k-1}, v_{2k})$ such that the odd entries $\{v_{2i-1},\ i = 1,\dots,k\}$ are in $\mathbb{N}$, and the even entries $\{v_{2i},\ i = 1,\dots,k\}$ are in $\bar{\mathbb{N}}$.

A *bicomposition* is a bivector in which there is no consecutive zeros, i.e., no pattern $0\bar{0}$ or $\bar{0}0$.

The integer k is called the *size* of v. The *weight* of v is by definition the couple $(|v|_{\mathbb{N}}, |v|_{\bar{\mathbb{N}}}) = (\sum_{i=1}^k v_{2i-1}, \sum_{i=1}^k v_{2i})$. We also set $|v| = |v|_{\mathbb{N}} + |v|_{\bar{\mathbb{N}}}$.

For example $(1,\bar{0},2,\bar{1},0,\bar{2},3,\bar{0})$ is a bicomposition of size 4, and of weight $(6,3)$.

To make notations lighter, we shall sometimes write bivectors or bicomposition as words, for example $1\bar{0}2\bar{1}0\bar{2}3\bar{0}$ stands for $(1,\bar{0},2,\bar{1},0,\bar{2},3,\bar{0})$ (see also the following definition).

The *fundamental B-quasisymmetric polynomials*, indexed by bicompositions, are defined as follows

$$F_{c_1c_2\ldots c_{2k-1}c_{2k}}(\mathcal{A}_n) = \sum x_{i_1}\cdots x_{i_{|c|_{\mathbb{N}}}}\, y_{j_1}\cdots y_{j_{|c|_{\bar{\mathbb{N}}}}} \in \mathbb{Q}[\mathcal{A}_n]$$

where the sum is taken over indices i's and j's such that

$$i_1 \leq \cdots i_{c_1} \leq j_1 \leq \cdots j_{c_2} < i_{c_1+1} \leq \cdots i_{c_1+c_3} \leq j_{c_2+1} \leq \cdots \leq j_{c_2+c_4} < i_{c_1+c_3+1} \leq \cdots$$

We give some examples:
$F_{1\bar{2}} = \sum_{i\leq j\leq k} x_i y_j y_k$,
$F_{0\bar{2}1\bar{0}} = \sum_{i\leq j<k} y_i y_j x_k$.

It is clear from the definition that the bidegree (i.e., the couple (degree in x, degree in y)) of F_c in $\mathbb{Q}[\mathcal{A}_n]$ is the weight of c. If the size of c is greater than n, we shall set $F_c(\mathcal{A}_n) = 0$.

The space of B-quasisymmetric polynomials, denoted by $QSym_n(B)$ is the vector subspace of $\mathbb{Q}[\mathcal{A}_n]$ generated by the $F_c(\mathcal{A}_n)$, for all bicompositions c.

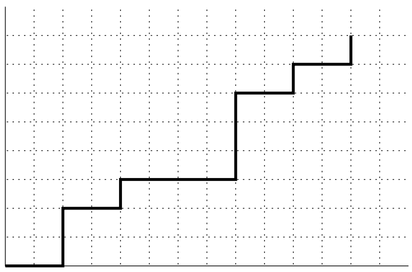
Let us denote by $\mathcal{I}_n^2$ the ideal $\langle QSym_n(B)^+\rangle$ generated by B-quasisymmetric polynomials with zero constant term.

3. Paths and $\mathcal{G}$-set

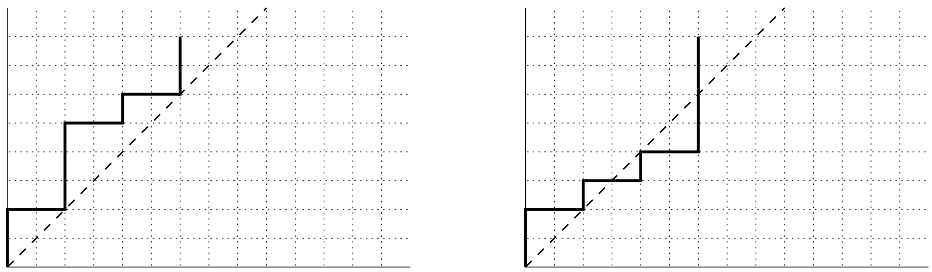
The aim of this section is to construct a set $\mathcal{G}$ of polynomials, which will be proved in the next section to be a Gröbner basis of $\mathcal{I}_n^2$. These two sections are greatly inspired from [2, 3].

Let $v = (v_1, v_2, \ldots, v_{2k-1}, v_{2k})$ be a bivector of size n. We associate to v a path $\pi(v)$ in the plane $\mathbb{N}\times\mathbb{N}$, with steps (0,1) or (2,0). We start from (0,0) and add for each entry v_i (read from left to right): v_i steps (2,0), followed by one step (0,1).

As an example, the path associated to $(1, \bar{0}, 1, \bar{2}, 0, \bar{0}, 1, \bar{1})$ is



We have two kinds of path, regarding their position to the diagonal $x = y$. If a path always remains above this line, we call it a *2-Dyck path*, and say that the corresponding vector is *2-Dyck*. Conversely, if the path enters the region $x < y$, we call both the path and the vector *transdiagonal*. For example, $v = (0, \bar{0}, 1, \bar{0}, 0, \bar{1}, 1, \bar{0})$ is 2-Dyck, whereas $w = (0, \bar{0}, 1, \bar{1}, 1, \bar{0}, 0, \bar{0})$ is transdiagonal.



π (v) π (w)

A simple but important observation is that a vector $v = (v_1, v_2, \ldots, v_{2k-1}, v_{2k})$ is transdiagonal if and only if there exists $1 \le l \le k$ such that

$$v_1 + v_2 + \cdots + v_{2l-1} + v_{2l} \ge l. \tag{3.1}$$

Our next task is to construct a set $\mathcal{G}$ of polynomials, mentionned above. From now on, unless otherwise indicated, vectors are of size n. For w a vector of size $k < n$, $w0^*$ denotes the vector (of size n) obtained by adding the desired number of $0\bar{0}$ patterns. We shall define the *length* $\ell(v)$ of a vector v as the integer k such that $v = v_1\, v_2\, \ldots\, v_{2k-1}\, v_{2k}\, 0^*$ with $v_{2k-1}\, v_{2k} \neq 0\bar{0}$. In the case of bicompositions, the notions of size and length coincide.

For v a vector (of length n), we denote by $\mathcal{A}_n^v$ the monomial

$$\mathcal{A}_n^v = x_1^{v_1} y_1^{v_2} \cdots x_n^{v_{2n-1}} y_n^{v_{2n}}.$$

To deal with leading terms of polynomials, we will use the lexicographic order induced by the ordering of the variables:

$$x_1 > y_1 > x_2 > y_2 > \cdots > x_n > y_n.$$

The lexicographic order is defined on monomials as follows: $\mathcal{A}_n^v >_{\text{lex}} \mathcal{A}_n^w$ if and only if the first non-zero entry of $v - w$ (componentwise) is positive.

The set

$$\mathcal{G} = \{G_v\} \subset \mathcal{I}_n^2$$

is indexed by transdiagonal vectors. Let v be a transdiagonal vector.

For $v = c0^*$ with c a non-zero bicomposition of length $\ge n$ (which implies that v is transdiagonal), we define

$$G_v = F_c.$$

If v cannot be written as $c0^*$, the polynomial G_v is defined recursively. We look at the rightmost occurrence of two consecutive zeros (on the left of a non-zero entry: we do not consider the subword 0^*). Two cases are to be distinguished according to the parity of the position of this pattern:

- if $v = w0\bar{0}\alpha\beta c0^*$, with w a vector of size $k-1$, $\alpha \in \mathbb{N}$ (by definition non-zero), $\beta \in \bar{\mathbb{N}}$, c a bicomposition, we define

$$G_{w0\bar{0}\alpha\beta c0^*} = G_{w\alpha\beta c0^*} - x_k\, G_{w(\alpha-1)\beta c0^*}; \tag{3.2}$$

- if $v = w\alpha\bar{0}0\beta c0^*$, with w a vector of size $k-1$, $\alpha \in \mathbb{N}$, $\beta \in \bar{\mathbb{N}}$ (by definition non-zero), c a bicomposition, we define

$$G_{w\alpha\bar{0}0\beta c0^*} = G_{w\alpha\beta c0^*} - y_k\, G_{w\alpha(\beta-1)c0^*}. \tag{3.3}$$

We easily check that both terms on the right of (3.2) and (3.3) are indexed by vectors that are transdiagonal as soon as v is transdiagonal. We do it for (3.2) : let us denote $v' = w\alpha\beta c0^*$ and $v'' = w(\alpha-1)\beta c0^*$. Let l be the smallest integer such that (3.1) holds for v. If $l \geq k-1$ then w is transdiagonal thus so are v' and v'', and if not:

$$v'_1 + v'_2 + \cdots + v'_{2l-3} + v'_{2l-2} \geq l \quad \text{and} \quad v''_1 + v''_2 + \cdots + v''_{2l-3} + v''_{2l-2} \geq l-1.$$

Since v' and v'' are of length equal to $\ell(v)-1$, this defines any G_v for v transdiagonal by induction on $\ell(v)$.

It is interesting to develop an example, where we take $n = 3$.

$$\begin{aligned}
G_{0\bar{0}1\bar{0}0\bar{2}} &= G_{0\bar{0}1\bar{2}0\bar{0}} - y_2\, G_{0\bar{0}1\bar{1}0\bar{0}} \\
&= (G_{1\bar{2}0\bar{0}0\bar{0}} - x_1\, G_{0\bar{2}0\bar{0}0\bar{0}}) - y_2\,(G_{1\bar{1}0\bar{0}0\bar{0}} - x_1\, G_{0\bar{1}0\bar{0}0\bar{0}}) \\
&= (F_{1\bar{2}} - x_1\, F_{0\bar{2}}) - y_2(F_{1\bar{1}} - x_1\, F_{0\bar{1}}) \\
&= (x_1y_1^2 + x_1y_1y_2 + x_1y_1y_3 + x_1y_2^2 + x_1y_2y_3 + x_1y_3^2 + x_2y_2^2 + x_2y_2y_3 \\
&\quad\ + x_2y_3^2 + x_3y_3^2 - x_1(y_1^2 + y_1y_2 + y_1y_3 + y_2^2 + y_2y_3 + y_3^2)) \\
&\quad - y_2(x_1y_1 + x_1y_2 + x_1y_3 + x_2y_2 + x_2y_3 + x_3y_3 - x_1(y_1+y_2+y_3)) \\
&= x_2y_3^2 - y_2x_3y_3 + x_3y_3^2
\end{aligned}$$

The monomials of the result are ordered with respect to the lexicographic order and we observe that the leading monomial (denoted LM) of $G_{0\bar{0}1\bar{0}0\bar{2}}$ is $\mathcal{A}_3^{0\bar{0}1\bar{0}0\bar{2}}$. The following proposition shows that this fact holds in general for the family $\mathcal{G}$.

Proposition 3.1. *Let v be a transdiagonal vector. The leading monomial of G_v is*

$$LM(G_v) = \mathcal{A}_n^v. \tag{3.4}$$

The proof of this proposition is, as the definition of the G_v polynomials, inductive on the length of v. First observe that the definitions of the F_c and of the lexicographic order imply (3.4) when $v = c0^*$ with c a bicomposition. Now Proposition 3.1 is a consequence of the following lemma.

We shall write $\mathcal{A}_{n\backslash k} = x_{k+1}, y_{k+1}, \ldots, x_n, y_n$.

Lemma 3.2. *Let w be a vector of size k, and c a bicomposition, then we have*

$$G_{wc0^*}(\mathcal{A}_n) = \mathcal{A}_k^w F_c(\mathcal{A}_{n\backslash k}) + (\text{terms} < \mathcal{A}_k^w). \tag{3.5}$$

Proof. If w is a bicomposition, then (3.5) is a consequence of the definition of the polynomials F's. If not this is readily done by induction on $\ell(w)$, by using the recursive definition of the G's.

We suppose that we are in the case of recursion (3.2), i.e., w can be written $w = u0\bar{0}\alpha\beta d$ with u a bivector of size l, $\alpha \in \mathbb{N}$, $\alpha > 0$, $\beta \in \bar{\mathbb{N}}$, and d a bicomposition.

We first observe that the F polynomials obey to recursive relations. We suppose we have a bicomposition $\gamma\delta g$, with $\gamma > 0$. Then the definition of the fundamental quasisymmetric polynomials implies:

$$F_{\gamma\delta g}(\mathcal{A}_n) = F_{\gamma\delta g}(\mathcal{A}_{n\backslash 1}) + x_1 F_{(\gamma-1)\delta g}(\mathcal{A}_n) \tag{3.6}$$

if $\gamma\delta \neq 1\bar{0}$, and

$$F_{1\bar{0}g}(\mathcal{A}_n) = F_{1\bar{0}g}(\mathcal{A}_{n\backslash 1}) + x_1 F_g(\mathcal{A}_{n\backslash 1}). \tag{3.7}$$

The same kind of equalities holds when $\gamma = 0$, with recursive terms multiple of y_1.

We now prove (3.5). We have to distinguish two cases. We first suppose $\alpha\beta \neq 1\bar{0}$, and use (3.2) and (3.6) to write:

$$\begin{array}{rcl} G_{u0\bar{0}\alpha\beta dc0^*} & = & G_{u\alpha\beta dc0^*} - x_{l+1}\, G_{u(\alpha-1)\beta dc0^*} \\ & = & \mathcal{A}_l^u\,(F_{\alpha\beta dc}(\mathcal{A}_{n\backslash l}) - x_{l+1} F_{(\alpha-1)\beta dc}(\mathcal{A}_{n\backslash l})) + (\text{terms} < \mathcal{A}_l^u) \\ & = & \mathcal{A}_l^u\,(F_{\alpha\beta dc}(\mathcal{A}_{n\backslash (l+1)})) + (\text{terms} < \mathcal{A}_l^u) \\ & = & A_n^{u0\bar{0}\alpha\beta dc0^*}\,(F_c(\mathcal{A}_{n\backslash k})) + (\text{terms} < \mathcal{A}_n^w). \end{array}$$

Now if $\alpha\beta = 1\bar{0}$, the computation is almost the same:

$$\begin{array}{rcl} G_{u0\bar{0}1\bar{0}dc0^*} & = & G_{u1\bar{0}dc0^*} - x_{l+1}\, G_{u0\bar{0}dc0^*} \\ & = & \mathcal{A}_l^u\, F_{1\bar{0}dc}(\mathcal{A}_{n\backslash l}) + (\text{terms} < \mathcal{A}_l^u) - x_{l+1} F_{dc}(\mathcal{A}_{n\backslash (l+1)}) + (\text{terms} < \mathcal{A}_{l+1}^{u0\bar{0}}) \\ & = & \mathcal{A}_l^u\, F_{\alpha\beta dc}(\mathcal{A}_{n\backslash (l+1)}) + (\text{terms} < \mathcal{A}_l^u) \\ & = & A_n^{u0\bar{0}1\bar{0}dc0^*}\,(F_c(\mathcal{A}_{n\backslash k})) + (\text{terms} < \mathcal{A}_n^w). \end{array}$$

All this process can be done in the case of recurrence (3.3), and this completes the proof. □

4. Proof of the main theorem

The aim of this section is to prove Theorem 1.1, by showing that the set $\mathcal{G}$ constructed in the previous section is a Gröbner basis for $\mathcal{I}_n^2$. This will be achieved in several steps.

We introduce the notation $\mathcal{Q}_n = \mathbb{Q}[\mathcal{A}_n]/\mathcal{I}_n^2$ and define

$$\mathcal{B}_n = \{\mathcal{A}_n^v \;/\; \pi(v) \textit{ is a } 2{-}\textit{Dyck path}\}.$$

Lemma 4.1. *Any polynomial $P \in \mathbb{Q}[\mathcal{A}_n]$ is in the span of $\mathcal{B}_n$ modulo $\mathcal{I}_n^2$. That is*

$$P(\mathcal{A}_n) \equiv \sum_{\mathcal{A}_n^v \in \mathcal{B}_n} c_v \mathcal{A}_n^v. \tag{4.1}$$

Proof. It clearly suffices to show that (4.1) holds for any monomial $\mathcal{A}_n^v$, with v transdiagonal. We assume that there exists $\mathcal{A}_n^v$ not reducible of the form (4.1) and we choose $\mathcal{A}_n^w$ to be the smallest amongst them with respect to the lexicographic order. Let us write

$$\begin{array}{rcl} \mathcal{A}_n^w & = & LM(G_w) \\ & = & (\mathcal{A}_n^w - G_w) + G_w \\ & \equiv & \mathcal{A}_n^w - G_w \pmod{\mathcal{I}_n^2}. \end{array}$$

All monomials in $(\mathcal{A}_n^w - G_w)$ are lexicographically smaller than $\mathcal{A}_n^w$, thus they are reducible. This contradicts our assumption and completes our proof. □

This lemma implies that $\mathcal{B}_n$ spans the quotient $\mathcal{Q}_n$. We will now prove its linear independence. The next lemma is a crucial step.

Lemma 4.2. *If we denote by $\mathcal{L}[S]$ the linear span of a set S, then*

$$\mathbb{Q}[\mathcal{A}_n] = \mathcal{L}[\mathcal{A}_n^v F_c \ / \ \mathcal{A}_n^v \in \mathcal{B}_n, \ |c| \geq 0]. \tag{4.2}$$

Proof. We have already obtained the following reduction for any monomial $\mathcal{A}_n^w$ in $\mathbb{Q}[\mathcal{A}_n]$:

$$\mathcal{A}_n^w \equiv \sum_{\mathcal{A}_n^v \in \mathcal{B}_n} c_v \mathcal{A}_n^v \pmod{\mathcal{I}_n^2},$$

which is equivalent to

$$\mathcal{A}_n^w = \sum_{\mathcal{A}_n^v \in \mathcal{B}_n} c_v \mathcal{A}_n^v + \sum_{|c|>0} Q_c F_c. \tag{4.3}$$

We then apply the reduction (4.1) to each monomial of the Q_c's. Now we use the algebra structure of QSym(B) (cf. Proposition 37 of [7]) to reduce products of fundamental B-quasisymmetric polynomials as linear combinations of F_c's. We obtain (4.2) in a finite number of operations since degrees strictly decrease at each operation, because $|c| > 0$ implies $\deg Q_c < |w|$. □

Now we come to the final step in the proof. Before stating this lemma, we introduce some notations, and make an observation.

For $v = (v_1, v_2, v \ldots, v_{2k-1}, v_{2k})$ a bivector, let $r(v)$ denote the reverse bivector: $r(v) = (v_{2k}, v_{2k-1}, \ldots, v_2, v_1)$. In the same way, let $R(\mathcal{A}_n)$ denote the reverse alphabet of $\mathcal{A}_n$: $R(\mathcal{A}_n) = y_n, x_n, \ldots, y_1, x_1$. Then one has for any bicomposition c:

$$F_c(R(\mathcal{A}_n)) = F_{r(c)}(\mathcal{A}_n). \tag{4.4}$$

Lemma 4.3. *The set $\mathcal{G}$ is a linear basis of $\mathcal{I}_n^2$, i.e.*

$$\mathcal{I}_n^2 = \mathcal{L}[G_w \ / \ w \ \textit{transdiagonal}]. \tag{4.5}$$

Proof. The proof will be achieved in several steps. The first one is to use Lemma 4.2 and observation (4.4) to obtain:

$$\mathbb{Q}[\mathcal{A}_n] = \mathcal{L}[R(\mathcal{A}_n)^v F_c \ / \ \mathcal{A}_n^v \in \mathcal{B}_n, \ |c| \geq 0]. \tag{4.6}$$

We shall denote $\mathcal{C}_n = \{R(\mathcal{A}_n)^v \ / \ \mathcal{A}_n^v \in \mathcal{B}_n\}$.

Now we reduce the problem, using (4.6) and the algebra structure of $QSym_n(B)$ to write:

$$\mathcal{I}_n^2 = \langle F_c, \ |c| > 0 \rangle_{\mathbb{Q}[\mathcal{A}_n]} = \mathcal{L}[\mathcal{A}_n^v \, F_c \, F_{c'} \ / \ \mathcal{A}_n^v \in \mathcal{C}_n, \ |c| > 0, \ |c'| \geq 0]$$

$$= \mathcal{L}[(\mathcal{A}_n)^v \, F_{c''} / \ \mathcal{A}_n^v \in \mathcal{C}_n, \ |c''| > 0].$$

Now we have to prove that for any monomial $\mathcal{A}_n^v \in \mathcal{C}_n$ and any non-zero bicomposition c:

$$\mathcal{A}_n^v \, F_c \in \mathcal{L}[G_w \ /w \ \textit{transdiagonal}]. \tag{4.7}$$

Thanks to Lemma reflem1, any monomial of degree at least equal to n is in $\mathcal{I}_n^2$, thus we can restrict to $|v| + |c| < n$.

We consider the product

$$y_n^{v_{2n}}(x_n^{v_{2n-1}}(\cdots(y_1^{v_2}(x_1^{v_1} F_c)))). \tag{4.8}$$

To reduce (4.8), we use the following relations, where w denotes a bivector, d a bicomposition, α and β elements of $\mathbb{N}$ and $\bar{\mathbb{N}}$, not simultaneously zero:

$$x_k \, G_{w\alpha\beta d0^*} = G_{w(\alpha+1)\beta d0^*} - G_{w0\bar{0}(\alpha+1)\beta d0^*} \tag{4.9}$$

or

$$x_k \, G_{w0^*00^*} = G_{w0^*10^*} - G_{w0^*0\bar{0}10^*} \tag{4.10}$$

for the x_k factors and:

$$y_k \, G_{w\alpha\beta d0^*} = G_{w\alpha(\beta+1)d0^*} - G_{w0\bar{0}\alpha(\beta+1)d0^*} \tag{4.11}$$

or

$$y_k \, G_{w0^*\bar{0}0^*} = G_{w0^*\bar{1}0^*} - G_{w0^*\bar{0}0\bar{1}0^*} \tag{4.12}$$

for the y_k factors. All these equations are direct consequences of the recursive definition of the G polynomials.

The reduction of the product (4.8) is made possible because of the order of the multiplications: the successive "shifts" are processed from left to right.

Our final task is to show that all vectors u generated in this process are transdiagonal and that their length never exceeds n.

Let us first check that the generated vectors are all transdiagonal. In the case of relations (4.10) and (4.12), this is obvious. Now, let us consider, for example relation (4.9). Let us denote $u = w\alpha\beta d0^*$, $u' = w(\alpha+1)\beta d0^*$ and $u'' = w0\bar{0}(\alpha+1)\beta d0^*$. Since u is transdiagonal, there exists $1 \geq l \geq \ell(u)$ such that

$$u_1 + u_2 + \cdots + u_{2l-1} + u_{2l} \geq l.$$

If $l > \ell(w)$, u' and u'' are clearly transdiagonal, and if not:

$$u'_1 + u'_2 + \cdots + u'_{2l-1} + u'_{2l} \geq l+1 \geq l \qquad \text{and} \qquad u''_1 + u''_2 + \cdots + u''_{2l+1} + u''_{2l+2} \geq l+1$$

whence u' and u'' are transdiagonal.

Let us now check that the length of the generated vectors never exceeds n. We keep track of the couple $e = u_{2\ell(u)-1}, u_{2\ell(u)}$. We distinguish two cases.

(1) e comes from $c_{2\ell(c)-1}, c_{2\ell(c)}$ that is shifted to the right by relations (4.9) and/or (4.11). It may be shifted at most $|v|$ steps to the right, thus:

$$\ell(u) \leq \ell(c) + |v| \leq |c| + |v| \leq n.$$

(2) e comes from a $1\bar{0}$ or $0\bar{1}$ generated by relation (4.10) or (4.12), then shifted to the right (by any relations). We suppose it is created by a multiplication by x_k or y_k, and we consider the vector

$$t = v_{2n}v_{2n-1}\ldots v_{2k}v_{2k-1}0^*.$$

Since $\mathcal{A}_n^v$ is in $\mathcal{C}_n$, the word t is 2-Dyck. Thus:

$$|t| < \ell(t) = n - k + 1.$$

This implies that the term $1\bar{0}$ or $0\bar{1}$ can be shifted at most to position

$$k + |t| \leq k + n - k = n.$$

□

To illustrate the recursive reduction of a product of the form (4.8), we give the following example, where $n = 5$:

$$\begin{aligned} x_1\, y_2\, F_{1\bar{0}0\bar{1}} &= y_2(x_1\, F_{1\bar{0}0\bar{1}}) \\ &= y_2(G_{2\bar{0}0\bar{1}0\bar{0}0\bar{0}0\bar{0}} - G_{0\bar{0}2\bar{0}0\bar{1}0\bar{0}0\bar{0}}) \\ &= y_2\, G_{2\bar{0}0\bar{1}0\bar{0}0\bar{0}0\bar{0}} - y_2\, G_{0\bar{0}2\bar{0}0\bar{1}0\bar{0}0\bar{0}} \\ &= G_{2\bar{0}0\bar{2}0\bar{0}0\bar{0}0\bar{0}} - G_{2\bar{0}0\bar{0}0\bar{2}} - G_{0\bar{0}2\bar{1}0\bar{1}0\bar{0}0\bar{0}} + G_{0\bar{0}2\bar{0}0\bar{1}0\bar{1}0\bar{0}}. \end{aligned}$$

Now we are able to complete the proof of Theorem 1.1. We can even state a more precise result.

Theorem 4.4. *A basis of the quotient $\mathcal{Q}_n$ is given by the set*

$$\mathcal{B}_n = \{\mathcal{A}_n^v \ / \ \pi(v) \text{ is a } 2\text{−Dyck path}\},$$

which implies

$$\dim \mathcal{Q}_n = \frac{1}{2n+1}\binom{3n}{n}. \tag{4.13}$$

Since $\mathcal{I}_n^2$ is bihomogeneous, the quotient $\mathcal{Q}_n$ is bigraded and we can consider $\mathbf{H}_{k,l}(\mathcal{Q}_n)$ the subspace of $\mathcal{Q}_n$ consisting of polynomials of bidegree (k,l), then

$$\dim \mathbf{H}_{k,l}(\mathcal{Q}_n) = \binom{n+k-1}{k}\binom{n+l-1}{l}\frac{n-k-l}{n}. \tag{4.14}$$

Proof. By Lemma 4.1, the set $\mathcal{B}_n$ spans $\mathcal{Q}_n$. Assume we have a linear dependence:

$$P = \sum_{\mathcal{A}_n^v \in \mathcal{B}_n} a_v\, \mathcal{A}_n^v \ \in \ \mathcal{I}_n^2.$$

By Lemma 4.3, the set $\mathcal{G}$ spans $\mathcal{I}_n^2$, thus

$$P = \sum_{u \text{ transdiagonal}} b_u\, G_u.$$

This implies $LM(P) = \mathcal{A}_n^u$, with u transdiagonal, which is absurd. Hence $\mathcal{B}_n$ is a basis of the quotient $\mathcal{Q}_n$.

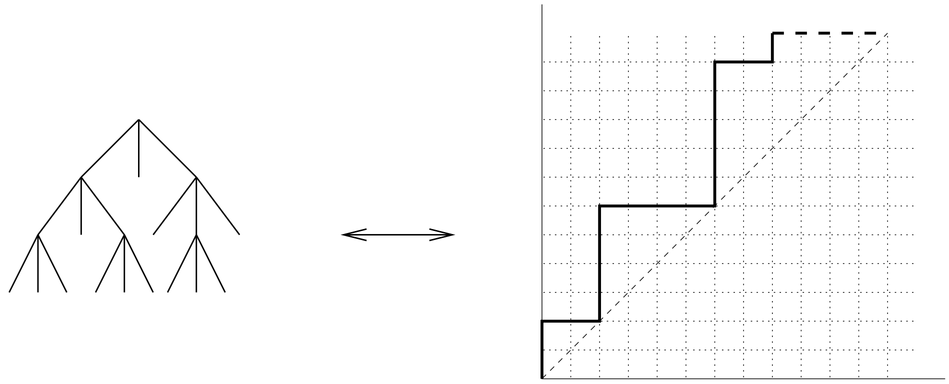
For the combinatorial part, we refer to [6], but we give a short proof of (4.13).

A ternary tree is a tree in which every internal node has exactly 3 sons. Ternary trees are known [23] to be enumerated by

$$C_3(n) = \frac{1}{2n+1}\binom{3n}{n}.$$

To conclude we observe that the depth-first search [11] of a tree gives a bijection between ternary trees and 2-Dyck paths ; we recall that we search recursively the left son, the middle son, the right son, and finally the root, and associate to each external node (except the leftmost one) a $(0,1)$ step, and to each internal node a $(2,0)$ step.

Below is given an illustration of this bijection, where we put in dashed lines the final horizontal sequence, to coincide with our definition of 2-Dyck paths.



Now if we denote by $\mathcal{D}_{n,k,l}$ the set of 2-Dyck paths with k horizontal steps at even height (corresponding to x terms) and l horizontal steps at odd height (corresponding to y terms), then the following equality is proven in [6]:

$$\#\,\mathcal{D}_{n,k,l} = \binom{n+k-1}{k}\binom{n+l-1}{l}\frac{n-k-l}{n}.$$

$\square$

5. Quotient of polynomials by QSym(G^p)

Every results of this paper can be extended to p sets of variables. Because of the great similarity to the previous sections, we shall only present here the result and a short sketch of the proof.

We denote by $\mathcal{A}_n^p$ the alphabet of $p \times n$ variables:

$$\mathcal{A}_n^p = x_1^{(1)} x_1^{(2)} \dots x_1^{(p)} x_2^{(1)} \dots x_2^{(p)} \dots x_n^{(1)} \dots x_n^{(p)}.$$

We define p-vectors of size k as vectors of $p \times k$ integers. For the ease of reading, we can write for example when $p = 3$: $v = 0\dot{1}\ddot{2}1\dot{0}\ddot{0}2\dot{0}\ddot{1}$. A p-composition is a p-vector avoiding 3 consecutive zeros.

The set $QSym(G^p)$ of G^p-quasisymmetric polynomials is the vector subspace of $\mathbb{Q}[\mathcal{A}_n^p]$ spanned by fundamental G^p-quasisymmetric polynomials, defined for a p-composition c by:

$$F_c = \sum \prod x_{i^{(1)}}^{(1)} \cdots \prod x_{i^{(p)}}^{(p)}$$

with

$$i_1^{(1)} \leq \cdots \leq i_{c_1}^{(1)} \leq i_{c_1+1}^{(2)} \leq \cdots \leq i_{c_1+c_2}^{(2)} \leq \cdots \leq \cdots \leq i_{c_1+\cdots+c_p}^{(p)} < i_{c_1+\cdots+c_p+1}^{(1)} \leq \cdots .$$

We give an example (here $p = 3$ and we use letters x, y, z for the alphabets $x^{(1)}$, $x^{(2)}$, $x^{(3)}$):

$$F_{0\dot{1}\ddot{0}2\dot{0}\ddot{1}} = \sum_{i<j\leq k\leq l} y_i\, x_j\, x_k\, z_l.$$

We define the ideal $\mathcal{I}_n^p = \langle QSym(G^p)^+ \rangle$ and the quotient $\mathcal{Q}_n^p = \mathbb{Q}[\mathcal{A}_n^p]/\mathcal{I}_n^p$. The result which generalizes Theorem 4.4 is

Theorem 5.1. *For $p \geq 1$,*

$$\dim \mathcal{Q}_n^p = \frac{1}{pn+1}\binom{(p+1)n}{n}. \tag{5.1}$$

Proof. We shall only give a brief description of the proof, which is very similar to the one of Theorem 4.4.

We first associate to any monomial a plane path, as in Section 3, with the difference that horizontal steps are of length p. Paths (and associated monomials) are said to be p-Dyck if they stay above the diagonal, and transdiagonal if not.

The construction of the set $\mathcal{G}$ indexed by transdiagonal path is the same, with p cases of recurrence. We prove that $\mathcal{G}$ is a Gröbner basis of $\mathcal{I}_n^p$ as in Section 4, and conclude that a basis of the quotient $\mathcal{Q}_n^p$ is given by the monomials associated to p-Dyck paths.

To conclude, we observe that the depth-first search [11] gives a bijection between p-ary trees, enumerated by the right-hand side of (5.1), and p-Dyck paths. □

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References

[1] E. Artin, *Galois Theory*, Notre Dame Mathematical Lecture **2** (1944), Notre Dame, IN.

[2] J.-C. Aval and N. Bergeron, *Catalan Paths and Quasi-Symmetric Functions*, Proc. Amer. Math. Soc. **131** (2003), 1053–1062.

[3] J.-C. Aval, F. Bergeron and N. Bergeron, *Ideals of Quasi-Symmetric Functions and Super-Coinvariant Polynomials for S_n*, Adv. in Math. **181** (2004), 353–367.

[4] J.-C. Aval and N. Bergeron, *Quasi-symmetric polynomials and Temperley-Lieb algebra of type B*, in preparation.

[5] J.-C. Aval, F. Bergeron and N. Bergeron, *Diagonal Temperley-Lieb invariants and harmonics*, Séminaire Lotharingien Combin. **54A** (to appear).

[6] J.-C. Aval, *Multivariate Fuss-Catalan numbers*, preprint.

[7] P. Baumann and C. Hohlweg, *A Solomon theory for the wreath product $G \wr \mathfrak{S}_n$*, Trans. Amer. Math. Soc. (to appear).

[8] F. Bergeron, N. Bergeron, A. Garsia, M. Haiman and G. Tesler, *Lattice Diagram Polynomials and Extended Pieri Rules*, Adv. in Math. **142** (1999), 244–334.

[9] F. Bergeron, A. Garsia, M. Haiman and G. Tesler, *Identities and Positivity Conjectures for Some Remarkable Operators in the Theory of Symmetric Functions*, Methods and Applications of Analysis **6** (1999), 363–420.

[10] N. Bergeron, S. Mykytiuk, F. Sottile, and S. van Willigenburg, *Pieri Operations on Posets*, J. Combin. Theory Ser. A **91** (2000), 84–110.

[11] T. H. Cormen, C. E. Leiserson, R. L. Rivest, and C. Stein, *Introduction to Algorithms*, second edition, MIT Press and McGraw-Hill, 2001.

[12] D. Cox, J. Little and D. O'Shea, *Ideals, Varieties, and Algorithms*, Springer-Verlag, New-York, 1992.

[13] A. M. Garsia and M. Haiman, *A graded representation model for Macdonald's polynomials*, Proc. Nat. Acad. Sci. U.S.A. **90** (1993), 3607–3610.

[14] A. Garsia, M. Haiman, *Orbit Harmonics and Graded Representations*, Éditions du Lacim, to appear.

[15] I. Gelfand, D. Krob, A. Lascoux, B. Leclerc, V. Retakh, and J.-Y. Thibon, *Noncommutative symmetric functions*, Adv. in Math. **112** (1995), 218–348.

[16] I. Gessel, *Multipartite P-partitions and products of skew Schur functions*, in Combinatorics and Algebra (Boulder, Colo., 1983), C. Greene, ed., vol. 34 of Contemp. Math., AMS, 1984, pp. 289–317.

[17] M. Haiman, *Vanishing theorems and character formulas for the Hilbert scheme of points in the plane*, Invent. Math. **149** (2002), 371–407.

[18] F. Hivert, *Hecke algebras, difference operators, and quasi-symmetric functions*, Adv. in Math. **155** (2000), 181–238.

[19] I. Macdonald, *Symmetric Functions and Hall Polynomials*, second edition, Oxford Univ. Press, 1995.

[20] C. Malvenuto and C. Reutenauer, *Duality between quasi-symmetric functions and the Solomon descent algebra*, J. Algebra **177** (1995), 967–982.

[21] J.-C. Novelli and J.-Y. Thibon, *Free quasi-symmetric functions of arbitrary level*, preprint.

[22] S. Poirier, *Cycle type and descent set in wreath products*, Proceedings of the 7th Conference on Formal Power Series and Algebraic Combinatorics (Noisy-le-Grand, 1995). Discrete Math. **180** (1998), 315–343.

[23] N. J. A. Sloane, *A Handbook of Integer Sequences*, Academic Press, New York and London, 1973.

[24] R. Stanley, *Enumerative Combinatorics, Vol. 1*, Wadsworth and Brooks/Cole, 1986.

[25] R. Stanley, *Enumerative Combinatorics Vol. 2*, no. 62 in Cambridge Studies in Advanced Mathematics, Cambridge University Press, 1999. Appendix 1 by Sergey Fomin.

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