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NON-SYMMETRIC HALL–LITTLEWOOD POLYNOMIALS

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À Adriano Garsia, en toute amitié

Abstract. Using the action of the Yang–Baxter elements of the Hecke algebra on polynomials, we define two bases of polynomials in n variables. The Hall–Littlewood polynomials are a subfamily of one of them. For $q = 0$, these bases specialize to the two families of classical Key polynomials (i.e., Demazure characters for type A). We give a scalar product for which the two bases are adjoint to each other.

1. Introduction

We define two linear bases of the ring of polynomials in $x_1, \ldots, x_n$, with coefficients in q.

These polynomials, which we call *q-Key polynomials*, and denote by $U_v, \widehat{U}_v$, $v \in \mathbb{N}^n$, specialize at $q = 0$ into key polynomials $K_v, \widehat{K}_v$. The polynomials U_v are symmetric polynomials for v such that $v_1 \le \cdots \le v_n$. In that case, U_v is equal to the Hall–Littlewood polynomial P_λ, λ being the partition $[v_n, \ldots, v_1]$.

Our main tool is the Hecke algebra $\mathcal{H}_n(q)$ of the symmetric group, acting on polynomials by deformation of divided differences. This algebra contains two adjoint bases of Yang–Baxter elements (Theorem 2.1). The q-Key polynomials are the images of dominant monomials under these Yang–Baxter elements (Def. 3.1). These polynomials are clearly two linear bases of polynomials, since the transition matrix to monomials is uni-triangular. We show in the last section that $\{U_v\}$ and $\{\widehat{U}_v\}$ are two adjoint bases with respect to a certain scalar product reminiscent of Weyl's scalar product on symmetric functions. We have intensively used MuPAD (package `MuPAD-Combinat` [13]) and Maple (package `ACE` [12]).

When this article was written, the authors were not aware of the work of Bogdan Ion [3, 4], who shows how to obtain, from nonsymmetric Macdonald polynomials, Demazure characters and their adjoint basis for an affine Kac–Moody algebra. Hence our own work should be considered as the part of the theory of nonsymmetric Macdonald polynomials for type A which is accessible using only divided differences, and not requiring double affine Hecke algebras.

2. The Hecke algebra $\mathcal{H}_n(q)$

Let $\mathcal{H}_n(q)$ be the Hecke algebra of the symmetric group $\mathfrak{S}_n$, with coefficients the rational functions in a parameter q. It has generators $T_1, \ldots, T_{n-1}$ satisfying the braid relations

$$\begin{cases} T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \\ T_i T_j = T_j T_i \ (|j-i| > 1), \end{cases} \tag{1}$$

and the Hecke relations

$$(T_i+1)(T_i-q)=0\ ,\ 1\le i\le n-1 \tag{2}$$

For a permutation σ in $\mathfrak{S}_n$, we denote by T_σ the element $T_\sigma = T_{i_1}\dots T_{i_p}$ where $(i_1,\dots,i_p)$ is any reduced decomposition of σ. The set $\{T_\sigma : \sigma\in\mathfrak{S}_n\}$ is a linear basis of $\mathcal{H}_n(q)$.

2.1. **Yang–Baxter bases.** Let $s_1,\dots,s_{n-1}$ denote the simple transpositions, $\ell(\sigma)$ denote the length of $\sigma\in\mathfrak{S}_n$, and let ω be the permutation of maximal length.

Given any set of indeterminates $\mathbf{u}=(u_1,\dots,u_n)$, let

$$\mathcal{H}_n(q)[u_1,\dots,u_n]=\mathcal{H}_n(q)\otimes\mathbb{C}[u_1,\dots,u_n].$$

One defines recursively a *Yang–Baxter basis* $(Y^{\mathbf{u}}_\sigma)_{\sigma\in\mathfrak{S}_n}$, depending on $\mathbf{u}$, by

$$Y^{\mathbf{u}}_{\sigma s_i}=Y^{\mathbf{u}}_\sigma\left(T_i+\frac{1-q}{1-u_{\sigma_{i+1}}/u_{\sigma_i}}\right),\quad \text{when } \ell(\sigma s_i)>l(\sigma)\,, \tag{3}$$

starting with $Y^{\mathbf{u}}_{id}=1$.

Let φ be the anti-automorphism of $\mathcal{H}_n(q)[u_1,\dots,u_n]$ such that

$$\begin{cases}\varphi(T_\sigma)=T_{\sigma^{-1}},\\ \varphi(u_i)=u_{n-i+1}.\end{cases}$$

We define a bilinear form $<\,,\,>$ on $\mathcal{H}_n(q)[u_1,\dots,u_n]$ by

$$<h_1\,,\,h_2>:=\ \text{coefficient of } T_\omega \text{ in } h_1\cdot\varphi(h_2)\ . \tag{4}$$

The main result of [8, Th. 5.1] is the following duality property of Yang–Baxter bases.

Theorem 2.1. *For any set of parameters $\mathbf{u}=(u_1,\dots,u_n)$, the basis adjoint to $(Y^{\mathbf{u}}_\sigma)_{\sigma\in\mathfrak{S}_n}$ with respect to $<\,,\,>$ is the basis $(\widehat{Y}^{\mathbf{u}}_\sigma)_{\sigma\in\mathfrak{S}_n}=(Y^{\varphi(\mathbf{u})}_\sigma)_{\sigma\in\mathfrak{S}_n}$. More precisely, one has*

$$<Y^{\mathbf{u}}_\sigma\,,\,\widehat{Y}^{\mathbf{u}}_\nu>=\delta_{\lambda,\ \nu\omega}\quad \textit{for all } \sigma,\nu\in\mathfrak{S}_n.$$

Let us fix from now on the parameters u to be $\mathbf{u}=(1,q,q^2,\dots,q^{n-1})$. Write $\mathcal{H}_n$ for $\mathcal{H}_n(q)[1,q,\dots,q^{n-1}]$.

In that case, the Yang–Baxter basis $(Y_\sigma)_{\sigma\in\mathfrak{S}_n}$ and its adjoint basis $(\widehat{Y}_\sigma)_{\sigma\in\mathfrak{S}_n}$ are defined recursively, starting with $Y_{id}=1=\widehat{Y}_{id}$, by

$$Y_{\sigma s_i}=Y_\sigma\ (T_i+1/[k]_q)\quad \text{and } \widehat{Y}_{\sigma s_i}=\widehat{Y}_\sigma\ \left(T_i+q^{k-1}/[k]_q\right)\,)\ ,\ \ell(\sigma s_i)>\ell(\sigma)\,, \tag{5}$$

with $k=\sigma_{i+1}-\sigma_i$ and $[k]_q=(1-q^k)/(1-q)$.

Notice that the maximal Yang–Baxter elements have another expression [2]:

$$Y_\omega=\sum_{\sigma\in\mathfrak{S}_n}T_\sigma\ \ \text{and}\ \ \widehat{Y}_\omega=\sum_{\sigma\in\mathfrak{S}_n}(-q)^{\ell(\sigma\omega)}\,T_\sigma\ .$$

Example 2.2. *For $\mathcal{H}_3$, the transition matrix between $\{Y_\sigma\}_{\sigma\in\mathfrak{S}_3}$ and $\{T_\sigma\}_{\sigma\in\mathfrak{S}_3}$ is*

$$\begin{array}{c} 123\\ 132\\ 213\\ 231\\ 312\\ 321 \end{array}\left|\begin{array}{cccccc} 1 & 1 & 1 & \frac{1}{q+1} & \frac{1}{q+1} & 1\\ \cdot & 1 & \cdot & 1 & \frac{1}{q+1} & 1\\ \cdot & \cdot & 1 & \frac{1}{q+1} & 1 & 1\\ \cdot & \cdot & \cdot & 1 & \cdot & 1\\ \cdot & \cdot & \cdot & \cdot & 1 & 1\\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{array}\right| ,$$

writing '$\cdot$' for 0. Each column represents the expansion of some element Y_σ.

2.2. **Action of $\mathcal{H}_n$ on polynomials.** Let $\mathfrak{Pol}$ be the ring of polynomials in the variables $x_1,\dots,x_n$ with coefficients the rational functions in q. We write monomials exponentially: $x^v = x_1^{v_1}\dots x_n^{v_n}$, $v=(v_1,\dots,v_n)\in\mathbb{Z}^n$. A monomial x^v is *dominant* if $v_1\geq\dots\geq v_n$.

We extend the natural order on partitions to elements of $\mathbb{Z}^n$ by

$$u\leq v \quad \text{if and only if} \quad \sum_{i=k}^{n}(v_i-u_i)\geq 0 \quad \text{for all } k>0.$$

For any polynomial P in $\mathfrak{Pol}$, we call *leading term* of P all the monomials (multiplied by their coefficients) which are maximal with respect to this partial order. This order is compatible with the right-to-left lexicographic order, that we shall also use. We also use the classiccal notation $\mathfrak{n}(v)=0v_1+1v_2+2v_3+\dots+(n-1)v_n$.

Let i be an integer such that $1\leq i\leq n-1$. As an operator on $\mathfrak{Pol}$, the simple transposition s_i acts by switching x_i and x_{i+1}, and we denote this action by $f\to f^{s_i}$. The i-th *divided difference* ∂_i and the i-th *isobaric divided difference* π_i, written on the right of the operand, are the following operators:

$$\partial_i:\ f\longmapsto f\,\partial_i := \frac{f-f^{s_i}}{x_i-x_{i+1}} \qquad , \qquad \pi_i:\ f\longmapsto f\,\pi_i := \frac{x_if-x_{i+1}f^{s_i}}{x_i-x_{i+1}} \ .$$

The Hecke algebra $\mathcal{H}_n$ has a faithful representation as an algebra of operators on $\mathfrak{Pol}$ given by the following equivalent formulas [2, 10]:

$$\left\{\begin{array}{lllllll} T_i & = & \Box_i-1 & = & (x_i-qx_{i+1})\,\partial_i-1 & = & (1-qx_{i+1}/x_i)\pi_i-1\,,\\ Y_{s_i} & = & \Box_i & = & (x_i-qx_{i+1})\,\partial_i & = & (1-qx_{i+1}/x_i)\pi_i\,,\\ \hat{Y}_{s_i} & = & \nabla_i & = & \Box_i-(1+q) & = & \partial_i\,(x_{i+1}-qx_i)\,. \end{array}\right.$$

The Hecke relations imply that

$$\Box_i^2=(1+q)\Box_i\ ,\ \nabla_i^2=-(1+q)\nabla_i \text{ and } \Box_i\nabla_i=\nabla_i\Box_i=0\ .$$

One easily checks that the operators $R_i(a,b)$ and $S_i(a,b)$ defined by

$$R_i(a,b)=\Box_i-q\frac{[b-a-1]_q}{[b-a]_q} \quad \text{and} \quad S_i(a,b)=\nabla_i+q\frac{[b-a-1]_q}{[b-a]_q}$$

satisfy the Yang–Baxter equation

$$R_i(a,b)\ R_{i+1}(a,c)\ R_i(b,c)=R_{i+1}(c,b)\ R_i(a,c)\ R_{i+1}(a,b)\,. \tag{6}$$

We have implicitly used these equations in the recursive definition of Yang–Baxter elements (5).

This realization comes from geometry [5], where the maximal Yang–Baxter elements are interpreted as Euler–Poincaré characteristic for the flag variety of $GL_n(\mathbb{C})$. This gives in particular another expression for the maximal Yang–Baxter elements:

$$Y_\omega = \prod_{1\le i<j\le n} (x_i - qx_j)\,\partial_\omega \qquad , \qquad \widehat{Y}_\omega = \partial_\omega \prod_{1\le i<j\le n} (x_j - qx_i)\,. \tag{7}$$

Example 2.3. *Let $\sigma = (3412) = s_2s_3s_1s_2$. The elements Y_{3412} and $\widehat{Y}_{3412}$ can be written as*

$$\begin{aligned} Y_{3412} &= \square_2\left(\square_3 - \frac{q}{1+q}\right)\left(\square_1 - \frac{q}{1+q}\right)\left(\square_2 - \frac{q+q^2}{1+q+q^2}\right)\,, \\ \widehat{Y}_{3412} &= \nabla_2\left(\nabla_3 + \frac{q}{1+q}\right)\left(\nabla_1 + \frac{q}{1+q}\right)\left(\nabla_2 + \frac{q+q^2}{1+q+q^2}\right)\,. \end{aligned}$$

We shall now identify the images of dominant monomials under the maximal Yang–Baxter operators with Hall–Littlewood polynomials. Recall that there are two proportional families $\{P_\lambda\}$ and $\{Q_\lambda\}$ of Hall–Littlewood polynomials. Given a partition $\lambda = [\lambda_1, \lambda_2, \dots, \lambda_r] = (0^{m_0}, 1^{m_1}, \dots, n^{m_n})$, with $m_0 = n - r = n - m_1 - \cdots - m_n$, then

$$Q_\lambda = \prod_{1\le i\le n} \prod_{j=1}^{m_i} (1-q^j)\, P_\lambda\,.$$

Let moreover $d_\lambda(q) = \prod_{0\le i\le n} \prod_{j=1}^{m_i} [j]_q$. The definition of Hall–Littlewood polynomials with raising operators [9], [11, III.2] can be rewritten, thanks to (7), as follows.

Proposition 2.4. *Let λ be a partition of n. Then one has*

$$x^\lambda\, Y_\omega\, d_\lambda(q)^{-1} = P_\lambda(x_1, \dots, x_n; q). \tag{8}$$

The family of the Hall–Littlewood functions $\{Q_\lambda\}$ indexed by partitions can be extended to a family $\{Q_v : v \in \mathbb{Z}^n\}$, using the following relations due to Littlewood ([9], [11, III.2.Ex. 2]):

$$Q_{(\dots,u_i,u_{i+1},\dots)} = -Q_{(\dots,u_{i+1}-1,u_i+1,\dots)} + q\, Q_{(\dots,u_{i+1},u_i,\dots)} + q\, Q_{(\dots,u_i+1,u_{i+1}-1,\dots)} \quad \text{if} \quad u_i < u_{i+1}, \tag{9}$$

$$Q_{(u_1,\dots,u_n)} = 0 \quad \text{if} \quad u_n < 0 \quad . \tag{10}$$

By iteration of the first relation, one can write any Q_u in terms of Hall–Littlewood functions indexed by decreasing vectors v such that $|v| = |u|$. Consequently, for any u with $|u| = 0$, Q_u must be proportional to $Q_{0\dots0} = 1$, i.e., Q_u is a constant that one can obtain as the specialization $Q_u(0)$ (i.e., the specialization of Q_u at $x_1 = \cdots = x_n = 0$).

The final expansion of Q_u, after iterating (9) many times, is not easy to predict. In particular, one needs to know whether $Q_u \neq 0$. For that purpose, we shall isolate a distinguished term in the expansion of Q_u. Given a sum $\sum_{\lambda\in\mathfrak{Part}} c_\lambda(t) Q_\lambda$, call *top term* the image of the leading term $\sum c_\mu(t) Q_\mu$ after restricting each coefficient $c_\mu(t)$ to its term in highest degree in t.

Given $u \in \mathbb{Z}^n$, define recursively $\mathfrak{p}(u) \in \mathfrak{Part} \cup \{-\infty\}$ by

- if $u \not\geq [0, \dots, 0]$ then $\mathfrak{p}(u) = -\infty$;
- if $u_2 \geq u_3 \geq \cdots \geq u_n > 0$ then $\mathfrak{p}(u)$ is the maximal partition of length $\leq n$, of weight $|u|$ (eventual zero terminal parts are suppressed);

- $\mathfrak{p}(u) = \mathfrak{p}\big(u\, \mathfrak{p}([u_2, \ldots, u_n])\big)$.

Lemma 2.5. *Let $u \in \mathbb{Z}^n$. Then*

- *if $u \not\geq [0, \ldots, 0]$ then $Q_u = 0$,*
- *if $u \geq [0, \ldots, 0]$, let $v = \mathfrak{p}(u)$. Then $Q_u \neq 0$ and its leading term is $q^{\mathfrak{n}(u)-\mathfrak{n}(v)} Q_v$.*

Proof. Given any decomposition $u = u'.u''$, one can apply (9) to u'' and write Q_u as a linear combination of terms $Q_{u'v}$ with v decreasing, with $|v| = |u''|$. Therefore, if $|u''| = 0$, the last components of such v are negative, all $Q_{u'v}$ are 0, and $Q_u = 0$.

If $u \geq [0, \ldots, 0]$ and u is not a partition, write $u = [\ldots, a, b, \ldots]$, with a, b the rightmost increase in u. We apply relation (9), assuming the validity of the lemma for the three terms on the right-hand side:

$$Q_{\ldots,a,b,\ldots} = -Q_{\ldots,b-1,a+1,\ldots} + qQ_{\ldots,b,a,\ldots} + qQ_{\ldots,a+1,b-1,\ldots}$$

Notice that the first two terms have not necessarily an index $\geq [0, \ldots, 0]$, but that $[\ldots, a+1, b-1, \ldots] \geq [0, \ldots, 0]$.

In any case, it is clear that $\mathfrak{p}([\ldots, b-1, a+1, \ldots]) = p_1 \leq v$, $\mathfrak{p}([\ldots, b, a, \ldots]) = p_2 \leq v$, and $\mathfrak{p}([\ldots, a+1, b-1, \ldots]) = v$.

Restricted to top terms, the expansion of the right-hand side in the basis Q_λ becomes

$$-\Big((q^{\mathfrak{n}(u)+a+1-b-\mathfrak{n}(v)} + \cdots)\, Q_v\Big) + q\Big((q^{\mathfrak{n}(u)+a-b-\mathfrak{n}(v)} + \cdots)\, Q_v\Big) + q\Big((q^{\mathfrak{n}(u)-1-\mathfrak{n}(v)} + \cdots)\, Q_v\Big),$$

where one or two of the first two terms may be replaced by 0, depending on the value of p_1, or p_2. Finally, the top term of the right-hand side is $q^{\mathfrak{n}(u)-\mathfrak{n}(v)} Q_v$, as desired. □

Example 2.6. *For $v = [-2, 3, 2]$, we have*

$$Q_{-2,3,2} = (q^3 - q^2)Q_3 + (q^5 + q^4 - q^3 - 2q^2 + 1)Q_{21} + (q^4 - q^3 - q^2 + q)Q_{111}\,,$$

and the top term is $q^4 Q_{111}$, since $4 = (0(-2) + 1(3) + 2(2)) - (0(1) + 1(1) + 2(1))$ and $[1, 1, 1] > [2, 1]$, $[1, 1, 1] > [3]$. Notice that the coefficient of Q_{21} is of higher degree.

3. q-Key Polynomials

In this section, we show that the images of dominant monomials under the Yang–Baxter elements Y_σ (respectively $\widehat{Y}_\sigma$), $\sigma \in \mathfrak{S}_n$ constitute two bases of $\mathfrak{Pol}$, which specialize to the two families of Demazure characters.

We have already identified in the preceding section the images of dominant monomials under Y_ω as Hall–Littlewood polynomial, using the relation between Y_ω and ∂_ω. The other polynomials are new.

3.1. **Two bases.** The dimension of the linear span of the image of a monomial x^v under all permutations depends upon the stabilizer of v. We meet the same phenomenon when taking the images of a monomial under Yang–Baxter elements.

Let $\lambda = [\lambda_1, \ldots, \lambda_n]$, $\lambda_1 \geq \ldots \geq \lambda_n \geq 0$, be a partition (adding eventual parts equal to 0). Denote its orbit under permutations of components by $\mathcal{O}(\lambda)$. Given any v in $\mathcal{O}(\lambda)$, let $\zeta(v)$ be the permutation of maximal length such that $\lambda\, \zeta(v) = v$, and let $\eta(v)$ be the permutation of minimal length such that $\lambda\, \eta(v) = v$. These two permutations are representatives of the same coset of $\mathfrak{S}_n$ modulo the stabilizer of λ.

$$\begin{tikzcd}
& U_{210} = x^{210} \arrow[dl, "\square_1"'] \arrow[dr, "\square_2"] & \\
U_{120} = x^{120} + x^{210} \arrow[d, "\square_2 - q/(1+q)"] & & U_{201} = x^{201} + x^{210} \arrow[d, "\square_1 - q/(1+q)"] \\
U_{102} = x^{102} + (1-q)x^{111} + \frac{1}{1+q}x^{120} + x^{201} + \frac{1}{1+q}x^{210} \arrow[dr, "\square_1"] & & U_{021} = x^{021} + (1-q)x^{111} + \frac{1}{1+q}x^{201} + x^{120} + \frac{1}{1+q}x^{210} \arrow[dl, "\square_2"'] \\
& U_{012} = +x^{102} + (-q-q+2)x^{111} + x^{120} + x^{201} + x^{210} &
\end{tikzcd}$$

FIGURE 1. q-Key polynomials generated from x^{210}.

Definition 3.1. *For all v in $\mathbb{N}^n$, the q-*Key *polynomials U_v and $\widehat{U}_v$ are the following polynomials:*

$$U_v(x;q) = \left(\frac{1}{d_\lambda(q)} x^\lambda\right) Y_{\zeta(v)} \qquad , \qquad \widehat{U}_v(x;q) = x^\lambda \widehat{Y}_{\eta(v)} \,,$$

where λ is the dominant reordering of v.

In particular, if v is (weakly) increasing, then $\zeta(v) = \omega$ and U_v is a Hall–Littlewood polynomial.

Lemma 3.2. *The leading term of U_v and $\widehat{U}_v$ is x^v. Consequently, the transition matrix between the U_v (respectively the $\widehat{U}_v$) and the monomials is upper unitriangular with respect to the right-to-left lexicographic order.*

Proof. Let k be an integer and u be a weight such that $u_k > u_{k+1}$. Suppose by induction that x^u is the leading term of U_u. Recall the the explicit action of $\square_k$ is (concentrating only on the two variables x_k, x_{k+1})

$$\begin{aligned}
x^{\beta\alpha}\,\square_k &= x^{\beta\alpha} + (1-t)(x^{\beta-1,\alpha+1} + \cdots + x^{\alpha+1,\beta-1}) + x^{\alpha\beta}\,,\ \beta > \alpha \\
x^{\beta\beta}\,\square_k &= (1+t)x^{\beta\beta} \\
x^{\alpha\beta}\,\square_k &= tx^{\beta\alpha} + (t-1)(x^{\beta-1,\alpha+1} + \cdots + x^{\alpha+1,\beta-1}) + tx^{\alpha\beta}\,,\ \alpha < \beta\,.
\end{aligned}$$

From these formulas, it is clear that for any constant c, the leading term of $x^u\,(\square_k + c)$ is $(x^u)^{s_k}$, and, for any v such that $v < u$, all the monomials in $x^v\,(\square_k + c)$ are strictly less (with respect to the partial order) than $(x^u)^{s_k}$. $\square$

Example 3.3. *For $n = 3$, Figures 1 and 2 show the case of a regular dominant weight x^{210}, and Figures 3 and 4 correspond to a case, x^{200}, where the stabilizer is not trivial. In this*

$$\begin{tikzcd}
& \widehat{U}_{210} = x^{210} \arrow[dl, "\nabla_1"'] \arrow[dr, "\nabla_2"] & \\
\widehat{U}_{120} = x^{120} - qx^{210} \arrow[d, "\nabla_2 + \frac{q}{1+q}"] & & \widehat{U}_{201} = x^{201} - qx^{210} \arrow[d, "\nabla_1 + \frac{q}{1+q}"] \\
\widehat{U}_{102} = x^{102} + (1-q)x^{111} - qx^{201} - \frac{q^2}{q+1}x^{120} + \frac{q^3}{q+1}x^{210} \arrow[dr, "\nabla_1"] & & \widehat{U}_{021} = x^{021} + (1-q)x^{111} - qx^{120} - \frac{q^2}{q+1}x^{201} + \frac{q^3}{q+1}x^{210} \arrow[dl, "\nabla_2"'] \\
& \widehat{U}_{012} = (1+q)B - q\widehat{K}_{120} &
\end{tikzcd}$$

FIGURE 2. Dual q-Key polynomials generated from x^{210}.

$$\begin{tikzcd}
& x^{200}/(1+q) \arrow[dl, "\Box_1"'] \arrow[dr, "\Box_2"] & \\
A \arrow[d, "\Box_2 - q/(1+q)"] & & \boxed{U_{200} = x^{200}} \arrow[d, "\Box_1 - q/(1+q)"] \\
B \arrow[dr, "\Box_1"] & & \boxed{U_{020} = x^{020} - qx^{200} + (1-q)x^{110}} \arrow[dl, "\Box_2"'] \\
& \boxed{U_{002} = (1-q)x^{011} + (1-q)x^{101} + x^{002} + \dfrac{q^2(q-1)}{q+1}x^{110} - \dfrac{q^2}{q+1}x^{020} - \dfrac{q^2}{1+q}x^{200}} &
\end{tikzcd}$$

FIGURE 3. q-Key polynomials generated from $x^{200}/(1+q)$.

$$
\begin{tikzcd}
 & \boxed{\widehat{U}_{200} = x^{200}} \arrow[dl, "\nabla_1"'] \arrow[dr, "\nabla_2"] & \\
\boxed{\widehat{U}_{020} = (1-q)x^{110} + x^{020} + \frac{1}{q+1}x^{200}} \arrow[d, "\nabla_2 + q/(1+q)"] & & 0 \arrow[d, "\nabla_1 + q/(1+q)"] \\
\boxed{\widehat{U}_{002} = (1-q)x^{011} + (1-q)x^{101} + x^{002} + (1-q)x^{110} + x^{020} + x^{200}} \arrow[dr, "\nabla_1"] & & 0 \arrow[dl, "\nabla_2"] \\
 & 0 &
\end{tikzcd}
$$

FIGURE 4. Dual q-Key polynomials generated from x^{200}.

last case, the polynomials belonging to the family are framed, the extra polynomials denoted A, B do not belong to the basis.

3.2. **Specialization at $q = 0$.** The specialization at $q = 0$ of the Hecke algebra is called the 0-*Hecke algebra*. The elementary Yang–Baxter elements specialize in that case to

$$
Y_{s_i} = T_i + 1 = \Box_i \quad \rightarrow \quad x_i \partial_i = \pi_i \ , \tag{11}
$$

$$
\hat{Y}_{s_i} = T_i = \nabla_i \quad \rightarrow \quad \partial_i x_{i+1} = \widehat{\pi}_i . \tag{12}
$$

Definition 3.4 (Key polynomials)**.** *Let $v \in \mathbb{N}^n$. The* Key polynomials *K_v and $\widehat{K}_v$ are defined recursively, starting with $K_v = x^v = \widehat{K}_v$ if x^v dominant, by*

$$
K_{vs_i} = K_v \, \pi_i \qquad , \qquad \widehat{K}_{vs_i} = \widehat{K}_v \, \widehat{\pi}_i \ , \quad \textit{for } i \textit{ such that } v_i > v_{i+1} \, .
$$

In particular, the subfamily (K_v) for v increasing is the family of Schur functions in $x_1, \ldots, x_n$. Demazure [1] defined Key polynomials (using another terminology) for all the classical groups, and not only for the type A_{n-1} which is our case.

Lemma 3.2 specializes to the following lemma.

Lemma 3.5. *The transition matrix between the U_v and the K_v (respectively from $\widehat{U}_v$ to $\widehat{K}_v$) is upper unitriangular with respect to the lexicographic order.*

Example 3.6. *For $n = 3$, the transition matrix between $\{U_v\}$ and $\{K_v\}$ in weight 3 is (reading a column as the expansion of some U_v)*

$$\begin{array}{c} 300 \\ 210 \\ 201 \\ 120 \\ 111 \\ 102 \\ 030 \\ 021 \\ 012 \\ 003 \end{array}
\begin{vmatrix}
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{-q}{(q+1)} & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \frac{-q}{(q+1)} & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \frac{-q}{(q+1)} & -q & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & -q & \cdot & -q & -q\,(q+1) & q^2 \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & -q \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{vmatrix},$$

and the transition matrix between $\{\widehat{U}_v\}$ and $\{\widehat{K}_v\}$ is

$$\begin{array}{c} 300 \\ 210 \\ 201 \\ 120 \\ 111 \\ 102 \\ 030 \\ 021 \\ 012 \\ 003 \end{array}
\begin{vmatrix}
1 & \cdot & \cdot & \cdot & \cdot & \cdot & -q & \cdot & \cdot & \frac{-q^2}{(q+1)} \\
\cdot & 1 & -q & -q & \cdot & \frac{q^3}{(q+1)} & -q & \frac{q^3}{(q+1)} & -q^3 & \frac{q^3}{(q+1)} \\
\cdot & \cdot & 1 & \cdot & \cdot & -q & \cdot & \frac{-q^2}{(q+1)} & q^2 & -q \\
\cdot & \cdot & \cdot & 1 & \cdot & \frac{-q^2}{(q+1)} & -q & -q & q^2 & \frac{q^3}{(q+1)} \\
\cdot & \cdot & \cdot & \cdot & 1 & -q & \cdot & -q & q\,(q+1) & q^2 \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & -q & -q \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \frac{-q^2}{(q+1)} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & -q & -q \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & -q \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{vmatrix}.$$

4. Orthogonality properties for the q-Key polynomials

We show in this section that the q-Key polynomials U_v and $\widehat{U}_v$ are two adjoint bases with respect to a certain scalar product.

4.1. **A scalar product on $\mathfrak{Pol}$.** For any Laurent series $f = \sum_{i=k}^{\infty} f_i x^i$, we denote by $CT_x(f)$ the coefficient f_0.

Let

$$\Theta := \prod_{1 \le i < j \le n} \frac{1 - x_i/x_j}{1 - q x_i/x_j} .$$

Therefore, for any Laurent polynomial $f(x_1, \ldots, x_n)$, the expression

$$CT(f\,\Theta) := CT_{x_n}\left(CT_{x_{n-1}}\left(\ldots\left(CT_{x_1}\left(f\,\Theta\right)\right)\ldots\right)\right)$$

is well defined. Let us use it to define a bilinear form $(\ ,\ )_q$ on $\mathfrak{Pol}$ by

$$(f\ ,\ g)_q = CT\left(f\,g^{\clubsuit} \prod_{1 \le i < j \le n} \frac{1 - x_i/x_j}{1 - q x_i/x_j}\right), \tag{13}$$

where $\clubsuit$ is the automorphism defined by $x_i \longmapsto 1/x_{n+1-i}$ for $1 \le i \le n$.

Since Θ is invariant under $♣$, the form $(\,,\,)_q$ is symmetric. Under the specialization $q=0$, the previous scalar product becomes

$$(f\,,\,g) := (f\,,\,g)\big|_{q=0} = CT\left(f\,g^{♣}\prod_{1\le i<j\le n}(1-x_i/x_j)\right). \tag{14}$$

We can also write $(f\,,\,g)_q = (f\,,\,g\Omega)$ with $\Omega = \prod_{1\le i<j\le n}(1-qx_i/x_j)^{-1}$.

Notice that, interpreting Schur functions as characters of unitary groups, Weyl defined the scalar product of two symmetric functions f,g in n variables as the constant term of

$$\frac{1}{n!}\,f\,g^{♣}\prod_{i,j:\,i\neq j}(1-x_i/x_j)\,.$$

Essentially, Weyl takes the square of the Vandermonde, while we are taking the quotient of the Vandermonde by the q-Vandermonde.

We now examine the compatibility of $\Box_i$ and ∇_i with the scalar product.

Lemma 4.1. *For i such that $1\le i\le n-1$, $\Box_i$ (respectively ∇_i) is adjoint to $\Box_{n-i}$ (respectively ∇_{n-i}) with respect to $(\,,\,)_q$.*

Proof. Since π_i (respectively $\widehat{\pi}_i$) is adjoint to π_{n-i} (respectively $\widehat{\pi}_{n-i}$) with respect to $(\,,\,)$ (see [7] for more details), we have

$$\begin{aligned}(f\Box_i\,,\,g)_q &= (f\,,\,g\;\Omega\;\pi_{n-i}(1-qx_{n-i+1}/x_{n-i}))\\ &= (f\,,\,g\;\frac{(1-qx_{n-i+1}/x_{n-i})}{(1-qx_{n-i+1}/x_{n-i})}\;\Omega\;\pi_{n-i}(1-qx_{n-i+1}/x_{n-i})).\end{aligned}$$

Since the polynomial $\Omega/(1-qx_{n-i+1}/x_{n-i})$ is symmetric in the indeterminates x_{n-i} and x_{n-i+1}, it commutes with the action of π_{n-i}. Therefore

$$(f\Box_i\,,\,g)_q = (f\,,\,g\;(1-qx_{n-i+1}/x_{n-i})\;\pi_{n-i}\;\Omega) = (f\,,\,g\;\Box_{n-i})_q\;.$$

This proves that $\Box_i$ is adjoint to $\Box_{n-i}$, and, equivalently, that ∇_i is adjoint to ∇_{n-i}. $\Box$

We shall need to characterize whether the scalar product of two monomials vanishes or not. Notice that, by definition,

$$(x^u\,,\,x^v) = (x^{u-v\omega}\,,\,1)\,,$$

so that one of the two monomials can be taken equal to 1.

Lemma 4.2. *For any $u\in\mathbb{Z}^n$, we have $(x^u,\,1)_q\neq 0$ if and only if $|u|=0$ and $u\ge[0,\ldots 0]$. In that case, $(x^u,\,1)_q = Q_u(0)$.*

Proof. Let us first show that the scalar products $(x^u\,,\,1)_q$ satisfy the same relations (9) as the Hall–Littlewood functions Q_u.

Let k be a positive integer less than n. Write $x_k=y$, $x_{k+1}=z$. Any monomial x^v can be written as $x^t\,y^a z^b$, with x^t of degree 0 in x_k,x_{k+1}. The product

$$x^t(y^az^b+y^bz^a)(z-qy)\prod_{1\le i<j\le n}\frac{1-x_i/x_j}{1-qx_i/x_j}$$

is equal to

$$(y^a z^b + y^b z^a)(z - qy)\frac{1 - y/z}{1 - qy/z}\, F_1 = (y^a z^b + y^b z^a)(z - y)F_1,$$

with F_1 symmetric in y, z. The constant term $CT_{x_{k-1}} \dots CT_{x_1}(x^t(y^a z^b + y^b z^a)F_1) = F_2$ is still symmetric in x_k, x_{k+1}. Therefore

$$CT_y\Big(CT_z\big((z - y)F_2\big)\Big)$$

is null, and finally

$$CT\left(x^t(y^a z^b + y^b z^a)(z - qy)\prod_{1 \le i < j \le n} \frac{1 - x_i/x_j}{1 - qx_i/x_j}\right) = 0\,.$$

This relation can be rewritten as

$$(y^a z^{b+1} x^t, 1)_q + (y^{b+1} z^{a+1} x^t, 1)_q - q(y^{b+1} z^a x^t, 1)_q - q(y^{a+1} z^b x^t, 1)_q = 0\ ,$$

which is indeed relation (9).

On the other hand, if $u_n < 0$, then there is no term of degree 0 in x_n in

$$x^u \prod_{1 \le i < j \le n} (1 - x_i/x_j)(1 - qx_i/x_j)^{-1},$$

and $(x^u, 1) = 0$, so that rule (10) is also satisfied.

As a consequence, the function $u \in \mathbb{Z}^n \to (x^u, 1)$ is determined by the values $(x^\lambda, 1)$, λ a partition, as the function $u \in \mathbb{Z}^n \to Q_u$ is determined by its restriction to partitions. However, for degree reasons, $(x^\lambda, 1) = 0$ if $\lambda \neq 0$. Since $(x^0, 1) = 1$, one has finally that $(x^u, 1) = Q_u(0)$. $\square$

Example 4.3. *For $u = [1, 0, 3]$ and $v = [0, 1, 3]$,*

$$(x^{103}\, ,\, x^{013})_q = (x^{-2,-1,3}\, ,\, 1)_q = Q_{-2,-1,3}(0) = q^2(1 - q)(1 - q^2)\ .$$

4.2. **Duality between $(U_v)_{v \in \mathbb{N}^n}$ and $(\widehat{U}_v)_{v \in \mathbb{N}^n}$.** Using that $\square_i$ is adjoint to $\square_{n-i}$, we are going to prove in this section that U_v and $\widehat{U}_v$ are two adjoint bases of $\mathfrak{Pol}$ with respect to the scalar product $(\ ,\ )_q$.

We first need some technical lemmas, to allow an induction on the q-Key polynomials, starting with dominant weights.

Lemma 4.4. *Let i be an integer such that $1 \le i \le n - 1$, let f_1, f_2, g_1 be three polynomials and b be a constant such that*

$$f_2 = f_1\,(\square_i + b)\ ,\ (f_1\, ,\, g_1)_q = 0 \quad \text{and} \quad (f_2\, ,\, g_1)_q = 1\,.$$

Then the polynomial $g_2 = g_1(\nabla_{n-i} - b)$ is such that

$$(f_1\, ,\, g_2)_q = 1\ ,\ (f_2\, ,\, g_2)_q = 0\,.$$

Proof. Using that ∇_{n-i} is adjoint to $\square_i$ and that $\square_i \nabla_i = 0$, one has

$$\begin{aligned}(f_2\, ,\, g_2)_q = (f_1\,(\square_i + b)\, ,\, g_1\,(\nabla_{n-i} - b))_q &= (f_1\,(\square_i + b)\,(\nabla_i - b)\, ,\, g_1)_q \\ &= (f_1(-b(1+q) - b^2), g_1)_q = 0\,.\end{aligned}$$

Similarly, we have

$$\begin{aligned}(f_1\,,\,g_2)_q &= (f_1\,,\,g_1\,(\nabla_{n-i}-b))_q\\ &= (f_1\,,\,g_1\,(\Box_{n-i}-1-q-b))_q\\ &= (f_1\,(\Box_i+b-1-q-2b)\,,\,g_1)_q = (f_2\,,\,g_1)_q = 1.\end{aligned}$$

□

Corollary 4.5. *Let i be an integer such that $1 \leq i \leq n-1$, let V be a vector space such that $V = V^{'} \oplus < f_1, f_2 >$ with $f_2 = f_1(\Box_i + b)$ and $V^{'}$ stable under $\Box_i$, and let g_1 be an element with*

$$(f_1\,,\,g_1)_q = 0 \quad \textit{and} \quad (f_2\,,\,g_1)_q = 1 \quad \textit{and} \quad (v\,,\,g_1)_q = 0, \quad \textit{for all } v \in V^{'}.$$

Then the element $g_2 = g_1(\nabla_{n-i} - b)$ satisfies

$$(f_2\,,\,g_2)_q = 0 \quad \textit{and} \quad (f_1\,,\,g_2)_q = 1 \quad \textit{and} \quad (v\,,\,g_2)_q = 0, \quad \textit{for all } v \in V^{'}.$$

Lemma 4.6. *Let u and λ be two dominant weights, and let v and μ be two permutations of u and λ, respectively. If $(x^v\,,\,x^\lambda) \neq 0$ and $(x^u\,,\,x^\mu) \neq 0$ then*

$$u = \lambda \quad , \quad v = \lambda\omega \quad \textit{and} \quad \mu = u\omega\,.$$

Proof. Using Lemma 4.2, the conditions $(x^v, x^\lambda)_q \neq 0$ and $(x^u, x^\mu)_q \neq 0$ imply two systems of inequalities:

$$\left\{\begin{array}{ccc} v_n & \geq & \lambda_1\,, \\ v_n + v_{n-1} & \geq & \lambda_1 + \lambda_2\,, \\ \vdots & \vdots & \vdots \\ v_n + \ldots + v_1 & \geq & \lambda_1 + \ldots + \lambda_n\,. \end{array}\right. \quad \text{and} \quad \left\{\begin{array}{ccc} \mu_n & \geq & u_1\,, \\ \mu_n + \mu_{n-1} & \geq & u_1 + u_2\,, \\ \vdots & \vdots & \vdots \\ \mu_n + \ldots + \mu_1 & \geq & u_1 + \ldots + u_n\,. \end{array}\right.$$

The first inequalities of the systems give $v_n \geq \lambda_1 \geq \mu_n \geq u_1 \geq v_n$. Consequently $u_1 = \lambda_1 = v_n = u_n$. By induction, using the other inequalities, one gets the lemma. □

Corollary 4.7. *Let v be a weight and λ be a dominant weight. Then,*

$$(U_v, x^\lambda)_q = \delta_{v,\lambda\omega}\ .$$

Proof. Let u be the decreasing reordering of v, and let σ be the permutation such that $U_v = x^u Y_\sigma$. By Lemma 4.6 and the fact that the leading term of U_v is x^v, the condition $(x^u Y_\sigma\,, x^\lambda)_q \neq 0$ implies $(x^v\,,\,x^\lambda)_q \neq 0$. By writing $\triangle_\sigma$ for the adjoint of Y_σ with respect to $(\,,\,)_q$, we have $(x^u Y_\sigma\,,\,x^\lambda)_q = (x^u\,,\,x^\lambda \triangle_\sigma)_q \neq 0$. As the leading term of $x^\lambda\triangle_\sigma$ is $x^{\lambda\sigma^{'}}$, where $\lambda\sigma^{'}$ is a permutation of λ, we obtain that $(x^u\,,\,x^{\lambda\sigma^{'}})_q \neq 0$. Using Lemma 4.6 we conclude that $v = \lambda\omega$. □

Our main result is the following duality property between U_v and $\widehat{U}_v$.

Theorem 4.8. *The two sets of polynomials $(U_v)_{v\in\mathbb{N}^n}$ and $(\widehat{U}_v)_{v\in\mathbb{N}^n}$ are two adjoint bases of $\mathfrak{Pol}$ with respect to the scalar product $(\,,\,)_q$. More precisely, they satisfy*

$$(U_v\,,\,\widehat{U}_{u\omega})_q = \delta_{v,u}\ .$$

Proof. Let λ be a dominant weight, and let V be the vector space spanned by the U_v for v in $\mathcal{O}(\lambda)$. The idea of the proof is to build by iteration the elements $(\widehat{U}_v)_{v\in\mathcal{O}(\lambda)}$, starting with $x^\lambda = \widehat{U}_\lambda$. By definition of the q-Key polynomials, there exists a constant b such that $U_{\lambda\omega} = U_{\lambda\omega s_1}\ (\square_1 + b)$. One can write the decomposition $V = V' \oplus < U_{\lambda\omega}\ ,\ U_{\lambda\omega s_1} >$, with V' invariant under the action of $\square_1$. Using the previous lemma, we have that $(U_{\lambda\omega}\ ,\ x^\lambda)_q = (U_{\lambda\omega}\ ,\ \widehat{U}_\lambda)_q = 1$ and $(U_{\lambda\omega\sigma_1}\ ,\ x^\lambda)_q = (U_{\lambda\omega\sigma_1}\ ,\ \widehat{U}_\lambda)_q = 0$. Consequently, by Lemma 4.5, the function $x^\lambda(\nabla_{n-1} - b) = \widehat{U}_{\lambda s_1}$ satisfies the duality conditions

$$(U_{\lambda\omega}\ ,\ \widehat{U}_{\lambda s_1})_q = 0 \quad , \quad (U_{\lambda\omega\sigma_1}\ ,\ \widehat{U}_{\lambda s_1})_q = 1 \quad \text{and} \quad (v\ ,\ \widehat{U}_{\lambda s_1})_q = 0 \quad \text{for all } v \in V'.$$

By iteration, this proves that for all u, v, one has $(U_v\ ,\ \widehat{U}_{u\omega})_q = \delta_{v,u}$. □

This theorem implies that the space of symmetric functions and the linear span of dominant monomials are dual of each other, the Hall–Littlewood functions being the basis dual to dominant monomials.

We finally mention that in the case $q = 0$, one has a reproducing kernel, as stated by the following theorem of [6], which gives another implicit definition of the scalar product (,).

Theorem 4.9. *The two families of polynomials $(K_v)_{v\in\mathbb{N}^n}$ and $(\widehat{K}_v)_{v\in\mathbb{N}^n}$ satisfy the Cauchy formula*

$$\sum_{u\in\mathbb{N}^n} K_u(x)\widehat{K}_{u\omega}(y) = \prod_{i+j\leq n+1} \frac{1}{1 - x_i y_j} \ . \tag{15}$$

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