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MACDONALD REPRESENTATIONS OF COMPLEX REFLECTION GROUPS

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To Alain Lascoux to celebrate his sixtieth birthday and his massive contribution to algebraic combinatorics and representation theory

Abstract. I G Macdonald (1972) introduced a unified approach to give many irreducible representations of Weyl groups in terms of their root systems. This generalised to Weyl groups the earlier well known constructions based on Young tableaux due to W Specht. These were interpreted in terms of positive systems of subsystems of root systems. A M Cohen (1976) extended the idea of root systems to complex reflection groups giving explicitly root systems for all dimensions greater than two. M C Hughes (1980) had further extended his ideas to generalise the concepts of subsystems and positive systems. These are now used to construct some irreducible representations of complex reflection groups.

1. **Introduction.**

In a now classic paper, I G Macdonald [18] gave a unified construction of irreducible representations of Weyl groups in terms of subsystems of their root systems. He further remarked that the same construction could be extended to any finite Coxeter group and its reflection subgroups. Later, G Lusztig and N Spaltenstein [17] generalized his construction in a way which means that it also can be extended to complex reflection groups. In this paper, the main aim will be to extend Macdonald's original construction to complex reflection groups in a more direct way which will use the ideas on root systems defined for these groups as explained below.

G C Shephard and J A Todd [24] have given a complete classification of the irreducible complex reflection groups, they showed that they are either the infinite families, the cyclic groups, the symmetric groups and the groups $G(m, p, n)$, where m, p, n are positive integers such that p divides m, $m \geq 2$ and $p = 1$ if $n = 1$ and 34 exceptional cases which we denote ST_i, $4 \leq i \leq 37$.

In the case of Weyl groups or finite Coxeter groups the theory is well developed and documented, see for example, [2]. For complex reflection groups however, some of the basic ideas are not as well developed with no universally accepted analogues for such fundamental concepts as root systems and their subsystems or a length function; for more recent attempts for some of the classical groups, see [3], [4], [23] and [25]. In addition, the most significant work on complex reflection groups in recent years has originated in the work of M. Broué, G. Malle et al, see for example, [5], [6], which showed how important these groups are in a more general context.

However, for our purpose, the concept of root systems introduced by A M Cohen [10] in his reclassification of the finite irreducible complex reflection groups is far more useful. These root systems have been developed further by M C Hughes [13], [14] and H Can

[8], in particular, their subsystems and positive systems, called primary systems in this context.

Thus, in Section 2 of this paper we introduce all of the preliminary material required on complex reflection groups and their root systems mainly following [10]. In Section 3, we recall the ideas of M C Hughes and H Can on subsystems and primary systems. In particular, in order to obtain the subsystems of these root systems they have developed algorithms which show how to construct extended Cohen-Dynkin diagrams (c.f. extended Dynkin diagrams in the real case) in this context. In Tables 1 and 2 we present in a systematic way the extended Cohen-Dynkin diagrams for all the exceptional groups. In the final section, Section 4, Macdonald modules are presented for complex reflection groups. These are presented not in terms of subsystems but in terms of reflection subgroups. In particular cases, it is seen that most of the irreducible representations can be obtained in terms of subsystems. However, as not all of the reflection subgroups appear via subsystems, the results are presented in this more general context. An additional reason is that some of the components required in the proof have already appeared in the work of R Stanley [26] on determining the relative invariants of complex reflection groups. Absolutely crucial for the proof is Lemma 4.1, which generalizes the result in the real case that if a is a positive root a reflection s_a permutes all the positive roots except a which is mapped onto its negative. The section ends with some applications to the 2-dimensional reflection groups. In the Appendix, further applications are given, where most of the irreducible representations of three further groups are given, ST_{24}, ST_{25} and ST_{26} - this work together with that for the 2-dimensional groups, is mainly the work of the second author, Patrick Mwamba, who has sadly died since completing this work.

2. **Preliminaries.**

In this section, the basic definitions and notation required later are given following the approach in [10], [13].

Let V be a complex vector space of dimension n. A *reflection* in V is a linear transformation of V of finite order with exactly $(n-1)$ eigenvalues equal to 1. A *reflection group* G in V is a group generated by reflections in V. There exists a unitary inner product (,) on V invariant under G. A reflection group G is said to be r-dimensional if the dimension of the subspace V^G of points fixed by G is $n-r$. If $W \subset V$ is a subspace, denote by G_W the subgroup of those elements of G which fix the elements of W; G_W is itself a reflection group. The group G is irreducible if the restriction to a G-invariant complement of V^G in V is irreducible.

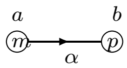
A (unitary) *root* of a reflection in V is an eigenvector (of length 1) corresponding to the unique eigenvalue not equal to 1 of the reflection. A (unitary) *root* of G is a (unitary) root of a reflection in G. Let s be a reflection in V of order $m > 1$. There exists $a \in V, a \neq 0$ and a primitive m-th root of unity ζ such that $s_{a,m}x = x - (1-\zeta)(x,a)(a,a)^{-1}a$ for all $x \in V$, where $s = s_{a,m}$. If t is any unitary transformation of V, we have $ts_{a,m}t^{-1} = s_{ta,m}$. Define $\theta_G : V \to \mathbb{N}$ by $\theta_G(a) = |G_W|$, where $W =< a >^{\perp}, a \in V$. The number $\theta_G(a)$ is called the *order* of a (with respect to G).

Each reflection $s_{a,m}$ fixes a unique reflecting hyperplane

$$H_{s_{a,m}} = \{x \in V \mid (x,a) = 0\} = \{x \in V \mid s_{a,m}x = x\}.$$

Clearly $H_{s_{\zeta^i a,m}} = H^i_{s_{a,m}} = H_{s_{a,m}},\ 0 \leq i \leq m-1$. For each of the reflecting hyperplanes H, choose a functional $\alpha_H \in V^*$, the dual space of V, such that $\ker(\alpha_H) = H$, that is, $H = \{x \in V \mid \alpha_H(v) = (x, a)\}$. For convenience, we sometimes denote a reflection which generates the cyclic group G_H of order e_H by s_H. The collection of all reflecting hyperplanes for the group G is an arrangement $\mathcal{A}$ on which the group G acts in a natural way. Let $\mathcal{C}$ denote the set of G-orbits under this action, thus $\mathcal{A} = \cup_{C\in\mathcal{C}} C$. The order e_H depends on the orbit $C = G.H \in \mathcal{C}$; thus whenever it is convenient we write e_C in place of e_H and $\zeta_H = \zeta_C$ for the corresponding primitive e_C-th root of unity.

Just as for real reflection groups, root systems can be equally useful in the complex case. These are now presented following [10]. (i) A *vector graph* is a pair (B, θ), where B is a non-empty finite subset of $\mathbb{C}^\infty$, the vector space with standard unitary inner product $(\ ,\ )$, such that for all $a, b \in B, |(a, b)| = 1$ if and only if $a = b$ and θ is a map from B to $\mathbb{N} \setminus \{1\}$. We say that B is the set of *vertices* and $\theta(a)$, for $a \in B$, is the *order* of a. A vector graph (B, w) is represented by a directed value graph by assigning to each element $a \in B$ a node a with weight $\theta(a)$ and if $(a, b) \neq 0, 1$ a directed edge from a to b with weight (a, b). For example, if $B = \{a, b\}$, $\theta(a) = m$ and $\theta(b) = p$ and $(a, b) = \alpha$, then the vector graph is



We adopt the following conventions: if $m = 2$ the number 2 is omitted, if $\alpha \in \mathbb{R}$ the arrow is omitted and if $p = m$ and $\alpha = -1/2$, the value $-1/2$ is omitted.

(ii) Let $\pi = (B, \theta)$ be a vector graph. Then, we define dim π to be the dimension of the vector space spanned by B, and $W(\pi)$ to be the group generated by all the reflections $s_{a,\theta(a)}$ for $a \in B$. The vector graph π is called a *root graph* if dim $\pi = |B|$ and $W(\pi)$ is a finite reflection group. Root graphs $\pi = (B, \theta)$ and $\pi' = (B', \theta')$ are *equivalent* if the groups $W(\pi)$ and $W(\pi')$ are conjugate.

(iii) For any root graph $\pi = (B, \theta)$ and for any $w \in W(\pi)$, let $w\pi = (B_w, \theta_w)$, where $B_w = wB$ and $\theta_w(w(a)) = \theta(a)$ with $a \in B$, then $w\pi$ is also a root graph which is equivalent to π since $s_{w(a),\theta_w(w(a))} = ws_{a,\theta(a)}w^{-1}$ for all $a \in B$ it follows that $W(w\pi) = wW(\pi)w^{-1} = W(\pi)$.

(iv) We say that π is *irreducible* if $W(\pi)$ is irreducible (or that π is connected). The vector graph π is said to be *congruent* to the vector graph $\pi' = (B', \theta')$ if there is a $t \in \mathbf{Gl}(\mathbb{C}^\infty)$ such that $\theta'(ta) = \theta(a)$ for $a \in B$ and the elements of B are eigenvectors of t.

(v) A pair (R, f) is called a *pre-root system* if R is a subset of non-zero elements of $\mathbb{C}^\infty$ and $f : R \to \mathbb{N} \setminus \{1\}$ such that for all $a, b \in R, s_{a,f(a)}R = R$ and $f(s_{a,f(a)}a) = f(a)$. To $\Sigma = (R, f)$ is associated the reflection group $W(\Sigma)$ defined by $W(\Sigma) =$
$< s_{a,f(a)}|a \in R >$.

(vi) A pre-root system Σ is called a *root system* if in addition $\alpha a \in R$ if and only if $\alpha a \in W(\Sigma)a$ for all $a \in R, \alpha \in \mathbb{C}$.

We make the following remarks which have been proved in [10].

Remark 2.1. Every root graph defines a pre-root system, for if $\pi = (B, \theta)$ is a root graph, then $\Sigma = (R, f)$, where $R = W(\pi)B,\ \ f : R \to \mathbb{N} \setminus \{1\}$ is induced by the order function $o_{W(\pi)}$ defines a pre-root system with $W(\pi) = W(\Sigma)$.

Remark 2.2. Every finite reflection group yields a pre-root system and every pre-root system contains a root system, that is, if $\Sigma = (R, f)$ is a pre-root system, then there is a root system $\Phi = (S, g)$ with $W(\Phi) = W(\Sigma), S \subset R$ and $g = f|_S$.

Remark 2.3. Every reflection group has a root system, but not every root system is obtained in the above way from a root graph.

Remark 2.4. If a root system Φ is the pre-root system obtained from a root graph π as described in Remark 2.1, then π is called a *simple system* in Φ. Can [8] has shown that if $w \in W$ then $w\pi = (wB, \theta_w)$, where $\theta_w(w(a)) = \theta(a), a \in B$ is a root graph which is equivalent to π which yields the same root system Φ and so $w\pi$ is another simple system in Φ. Hence, the number of simple systems in Φ is equal to the order of $W(\pi)$. If Φ is a root system with simple system π, then the graph associated to π is called a *Cohen-Dynkin diagram* of Φ.

A group G of unitary automorphisms of V is said to be *imprimitive* if V is a direct sum $V = V_1 \oplus \cdots \oplus V_k$ of non-trivial proper subspaces $V_i (1 \leq i \leq k)$ of V such that $\{V_i \mid 1 \leq i \leq k\}$ is invariant under G. If such a direct splitting of V does not exist, G is said to be *primitive*.

Let $\mathcal{S}_n$ be the (symmetric) group of all $n \times n$ permutation matrices and let $A(m,p,n)$ be the set of all diagonal $n \times n$ matrices with $\zeta^{\rho_i}, \rho_i \in \mathbb{Z}$ in the (i,i) position, where ζ is a primitive mth root of unity and $\sum_{i=1}^{n} \rho_i \equiv 0 \pmod{p}$. Define $G(m,p,n) = A(m,p,n) \rtimes \mathcal{S}_n$, then the imprimitive reflection groups in V are of the form $G(m,p,n)$, where $p|m$.

Remark 2.5. (i) $G(m,m,2)$ is conjugate to $W(I_2(m))$ (notation of [2]), the dihedral group of order $2m$.

(ii) $G(1,1,n) = W(A_{n-1}) \cong \mathcal{S}_n$, the Weyl group of type A_{n-1}.

(iii) $G(2,1,n) = W(B_n)$, the Weyl group of type B_n.

(iv) $G(2,2,n) = W(D_n)$, the Weyl group of type D_n.

(v) If $p \neq 1, m$, then $G(m,p,n)$ can be defined with $n+1$ generating reflections, but for $p = 1, m$, then n generating reflections are sufficient.

A root system for $G(m,p,n)$ may be defined as follows. Let ζ be a primitive m-th root of unity and let $\{\epsilon_1, \ldots, \epsilon_n\}$ be the standard basis for the complex vector space $\mathbb{C}^n$. If $d \in \mathbb{N}$, let μ_d be the group of d-th roots of unity. Let

$$\Omega_l(m,n) = \{\epsilon_i - \zeta^a \epsilon_j, 1 \leq i,j \leq n, \ \ i \neq j, \ \ a \in \mathbb{Z}/m\mathbb{Z}\}.$$

and let

$$R_l(m,m,n) = \begin{cases} \mu_m \Omega_l(m,n) & \text{if } m \text{ is even,} \\ \mu_{2m} \Omega_l(m,n) & \text{if } m \text{ is odd.} \end{cases}$$

Let $f_l(r) = 2$ for all $r \in R_l(m,m,n)$. Then $\Phi_l(m,m,n) = (R_l(m,m,n), f_l)$ is a root system for $G(m,m,n)$. Let

$$\Omega_s(n) = \{\epsilon_i, \ \ 1 \leq i \leq n\}$$

and let $R_s(m/p,n) = \mu_{m/p}\Omega_s(n)$. Let $f_s(r) = m$ for all $r \in R_s(m/p,n)$. Let $\Phi_s(m/p,n) = (R_s(m/p,n), f_s)$. Then

$$\Phi(m,p,n) = \Phi_l(m,m,n) \bigcup \Phi_s(m/p,n)$$

is a root system for $G(m,p,n)$ for $p \neq m$.

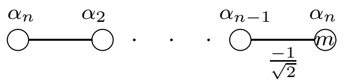
Let

$$\pi(m,1,n) = \{\alpha_1 = \epsilon_1 - \epsilon_2, \ldots, \alpha_{n-1} = \epsilon_{n-1} - \epsilon_n, \alpha_n = \epsilon_n\}$$

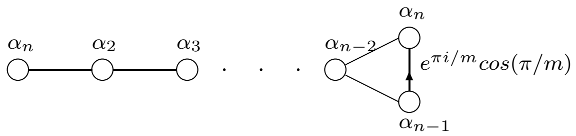
where, $\theta(\alpha_i) = 2,\ \ 1 \leq i \leq n-1,\ \ \theta(\alpha_n) = m$ and

$$\pi(m,m,n) = \{\epsilon_1 - \epsilon_2, \ldots, \epsilon_{n-1} - \epsilon_n, \epsilon_{n-1} - \zeta\epsilon_n\}$$

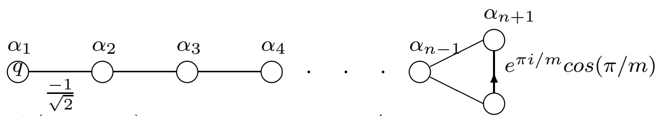
where, $\theta(\alpha_i) = 2,\ \ 1 \leq i \leq n$. Then $\pi(m,1,n)$ and $\pi(m,m,n)$ are vector graphs and so, simple systems for $G(m,1,n)$ and $G(m,m,n)$ respectively.
The corresponding Cohen-Dynkin diagrams for $G(m,1,n)$ and $G(m,m,n)$ are



and



respectively. However, as pointed out by Can [8], based on the work of Popov [22], the vector graph $\pi(m,p,n)$ represented by



is a vector graph for $G(m,p,n)$, where if $q = m/p$

$$\pi(m,p,n) = \{\alpha_1 = -\epsilon_1, \alpha_2 = \epsilon_1 - \epsilon_2, \ldots, \alpha_n = \epsilon_{n-1} - \epsilon_n, \alpha_{n+1} = \epsilon_{n-1} - exp(2\pi i/q)\epsilon_n\}$$

However, it is not a simple system for the corresponding root system $\Phi(m,p,n)$ as this set is clearly not linearly independent over $\mathbb{C}$.

3. **Subsystems and primitive systems.**

Let $\Phi = (R, f)$ be a root system corresponding to the reflection group $W(\Phi)$.If $S \subseteq R$ and $g = f|_S$, then the pair $\Psi = (S, g)$ is called a *subsystem* of Φ if Ψ is itself a root system. The corresponding *reflection subgroup* $W(\Psi)$ is the subgroup of $W(\Phi)$ generated by the $s_{a,g(a)}$ with $a \in S$. Subsystems $\Psi_1 = (S_1, g_1)$ and $\Psi_2 = (S_2, g_2)$ of Φ are *conjugate under* $W(\Phi)$ if $S_2 = wS_1$ and $g_2(w(a)) = g_1(a)$ for some $w \in W(\Phi)$ and for all $a \in S_1$; in which case $W(w\Psi_1) = wW(\Psi_1)w^{-1}$, that is, $W(\Psi_1)$ and $W(\Psi_2)$ are conjugate subgroups in $W(\Phi)$.

Now, as in Hughes [13] and Can [8], *primary systems* for root systems Φ with simple system $\pi = (B, \theta)$ are defined. These play the role of positive systems for real reflection groups. Let $B = \{a_1, \ldots, a_n\}$, and put $r_i = s_{a_i,\theta(a_i)}, 1 \leq i \leq n$, then the corresponding primary system is defined inductively as follows:
(i) Let $\Omega_1 = B$.
(ii) Let $\Omega_2 = \{r_i(a_j) \mid 1 \leq i, j \leq n, i \neq j, a_j \in \Omega_1, r_i(a_j) \notin \Omega_1\}$.
(iii) For $k \geq 3$, let

$$\Omega_k = \{r_i(a) \mid 1 \leq i, j \leq n, i \neq j, a \in \Omega_{k-1}, r_i(a) \neq \mu b \text{ for any } b \in \Omega_l, l < k\},$$

where μ is a root of unity.

Then $\Omega = \bigcup_{k\geq 1} \Omega_k$ is a *primary system* for the simple system π of the root system Φ. The elements of Ω_k are called *primary roots of height k*. The elements of Ω_k for the largest k for which $\Omega_k \neq \emptyset$ are called *highest roots*.

Remark 3.1. The primary system is not unique in that there is an element of choice at each step. However, having fixed a primary system Ω for the simple system π, if the simple system is replaced by $w\pi$, then [8] the corresponding primary system obtained by making the same choices in the above algorithm is the conjugate $w\Omega$ of Ω. Thus, the choice of primary system is of no consequence. In fact, different choices in the above algorithm would result in conjugate primary systems.

Remark 3.2. The roots of maximal height are not in general unique for complex reflection groups. For example, [14], $J_3(4)$ has a unique highest root of height 6, but L_4 has two highest roots of height 9.

Remark 3.3. In the case of real reflection groups, the primary systems are positive systems and the highest roots are the longest roots.

Hughes [13] has shown how subsystems of root systems may be constructed. He has done this by extending the corresponding method for real reflection groups to the complex case. Namely, an *extended Cohen-Dynkin diagram* of a root system Φ is formed by attaching the negative of a highest root to a Cohen-Dynkin diagram for Φ and removing one or more nodes in all possible ways and repeating this process on all the resulting diagrams. At all stages, including the initial Cohen-Dynkin diagram, all equivalent diagrams must also be considered. As there could be more than one highest root in the complex case and since a number of equivalent diagrams must be considered, the algorithm is more difficult to apply in the complex case in comparison with the real case. In [14], a complete list of subsystems is given for spaces of dimension ≥ 3. Can [8] has given an alternative algorithm for obtaining a complete set of subsystems. In particular, he has in [9] obtained subsystems for the groups $G(m, 1, n)$ and $G(m, m, n)$. He proved that the subsystems (up to conjugacy) for the groups $G(m, 1, n)$ and $G(m, m, n)$ are

$$\sum_{i=1}^{m}\sum_{j=1}^{s_i} B^{m_i}_{\lambda_j^{(i)}} + \sum_{j=1}^{s} D^{m}_{\mu_j}, \qquad \sum_{j=1}^{s_1}(\lambda_j^{(1)}+1) + \sum_{i=2}^{m}\sum_{j=1}^{s_1}\lambda_j^{(i)} + \sum_{j=1}^{s}\mu_j = n \tag{3.1}$$

and

$$\sum_{i=1}^{m}\sum_{j=1}^{s_i} D^{m_i}_{\lambda_j^{(i)}}, \qquad \sum_{j=1}^{s_1}(\lambda_j^{(1)}+1) + \sum_{i=2}^{m}\sum_{j=1}^{s_1}\lambda_j^{(i)} = n \tag{3.2}$$

respectively, where $m_1 = 1$ and $m_i = m,\ (i = 2, \ldots, m)$.

In Tables 1 and 2, the results of applying these algorithms to all the exceptional groups $ST_4 - ST_{37}$ are given (excluding the real reflection groups). We note that these results are in line with the relationship between the extended Cohen-Dynkin diagram and the diagrams and presentation for the corresponding irreducible infinite discrete complex reflection groups given by G Malle [19].

From now on, if $\Phi = (R, f)$ is a root system, we will simply write π and Ω for the corresponding simple system and primary system with the corresponding map f being restricted to these sets.

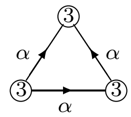
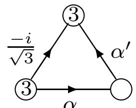
Shephard Todd type	**Cohen-Dynkin diagram**	$i\alpha$	$i\alpha'$	**Extended C-D diagram**
ST_4	$\underset{s}{\textcircled{3}} \xrightarrow[\alpha]{3} \underset{t}{\textcircled{3}}$	$\frac{1}{\sqrt{3}}$	$\frac{\sqrt{2}}{\sqrt{3}}$	
ST_5	$\underset{s}{\textcircled{3}} \xrightarrow[\alpha]{4} \underset{t}{\textcircled{3}}$	$\frac{\sqrt{2}}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	$\textcircled{3} \xrightarrow[\alpha]{} \textcircled{3} \xleftarrow[\alpha']{} \textcircled{3}$
ST_6	$\underset{s}{\textcircled{3}} \xrightarrow[\alpha]{6} \underset{t}{\bigcirc}$	$(\frac{1}{2}(1+\frac{1}{\sqrt{3}}))^{\frac{1}{2}}$	$(\frac{1}{2}(1-\frac{1}{\sqrt{3}}))^{\frac{1}{2}}$	 $\bigcirc \xleftarrow[\alpha']{} \textcircled{3} \xrightarrow[\alpha]{} \bigcirc$
ST_8	$\underset{s}{\textcircled{4}} \xrightarrow[\alpha]{3} \underset{t}{\textcircled{4}}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$\textcircled{4} \xrightarrow[\alpha]{} \textcircled{4} \xleftarrow[\alpha']{} \textcircled{4}$
ST_9	$\underset{s}{\textcircled{4}} \xrightarrow[\alpha]{6} \underset{t}{\bigcirc}$	$(\frac{1}{2}(1+\frac{1}{\sqrt{2}}))^{\frac{1}{2}}$	$(\frac{1}{2}(1-\frac{1}{\sqrt{2}}))^{\frac{1}{2}}$	$\textcircled{4} \xrightarrow[\alpha]{} \bigcirc \xleftarrow[\alpha']{} \textcircled{4}$ $\bigcirc \xleftarrow[\alpha']{} \textcircled{4} \xrightarrow[\alpha]{} \bigcirc$
ST_{10}	$\underset{s}{\textcircled{4}} \xrightarrow[\alpha]{4} \underset{t}{\textcircled{3}}$	$(\frac{1}{2}(1+\frac{1}{\sqrt{3}}))^{\frac{1}{2}}$	$(\frac{1}{2}(1-\frac{1}{\sqrt{3}}))^{\frac{1}{2}}$	$\textcircled{4} \xrightarrow[\alpha]{} \textcircled{3} \xleftarrow[\alpha']{} \textcircled{4}$ $\textcircled{3} \xleftarrow[\alpha']{} \textcircled{4} \xrightarrow[\alpha]{} \textcircled{3}$
ST_{14}	$\underset{s}{\textcircled{3}} \xrightarrow[\alpha]{8} \underset{t}{\bigcirc}$	$(\frac{1}{2}(1+\frac{\sqrt{2}}{\sqrt{3}}))^{\frac{1}{2}}$	$(\frac{1}{2}(1-\frac{\sqrt{2}}{\sqrt{3}}))^{\frac{1}{2}}$	$\textcircled{3} \xrightarrow[\alpha]{} \bigcirc \xleftarrow[\alpha']{} \textcircled{3}$ $\bigcirc \xleftarrow[\alpha']{} \textcircled{3} \xrightarrow[\alpha]{} \bigcirc$
ST_{16}	$\underset{s}{\textcircled{5}} \xrightarrow[\alpha]{3} \underset{t}{\textcircled{5}}$	$(\frac{1}{2}(1+\frac{1}{\sqrt{5}}))^{\frac{1}{2}}$	$(\frac{1}{2}(1-\frac{1}{\sqrt{5}}))^{\frac{1}{2}}$	$\textcircled{5} \xrightarrow[\alpha]{} \textcircled{5} \xleftarrow[\alpha']{} \textcircled{5}$
ST_{17}	$\underset{s}{\textcircled{5}} \xrightarrow[\alpha]{6} \underset{t}{\bigcirc}$	$(\frac{1}{2}(1+\alpha_{16}))^{\frac{1}{2}}$	$(\frac{1}{2}(1-\alpha_{16}))^{\frac{1}{2}}$	$\textcircled{5} \xrightarrow[\alpha]{} \bigcirc \xleftarrow[\alpha']{} \textcircled{5}$ $\bigcirc \xleftarrow[\alpha']{} \textcircled{5} \xrightarrow[\alpha]{} \bigcirc$
ST_{18}	$\underset{s}{\textcircled{5}} \xrightarrow[\alpha]{4} \underset{t}{\textcircled{3}}$	$(\frac{1}{2}(1+\frac{1}{\sqrt{3}}cot\frac{\pi}{5}))^{\frac{1}{2}}$	$(\frac{1}{2}(1-\frac{1}{\sqrt{3}}cot\frac{\pi}{5}))^{\frac{1}{2}}$	$\textcircled{5} \xrightarrow[\alpha]{} \textcircled{3} \xleftarrow[\alpha']{} \textcircled{5}$ $\textcircled{3} \xleftarrow[\alpha']{} \textcircled{5} \xrightarrow[\alpha]{} \textcircled{3}$
ST_{20}	$\underset{s}{\textcircled{3}} \xrightarrow[\alpha]{5} \underset{t}{\textcircled{3}}$	$\frac{1}{2}(\frac{1+\sqrt{5}}{\sqrt{3}})$	$\frac{1}{2}(\frac{1-\sqrt{5}}{\sqrt{3}})$	$\textcircled{3} \xrightarrow[\alpha]{} \textcircled{3} \xleftarrow[\alpha']{} \textcircled{3}$
ST_{21}	$\underset{s}{\textcircled{3}} \xrightarrow[\alpha]{10} \underset{t}{\bigcirc}$	$(\frac{1}{2}(1+\alpha_{20}))^{\frac{1}{2}}$	$(\frac{1}{2}(1-\alpha_{20}))^{\frac{1}{2}}$	$\textcircled{3} \xrightarrow[\alpha]{} \bigcirc \xleftarrow[\alpha']{} \textcircled{3}$ $\bigcirc \xleftarrow[\alpha']{} \textcircled{3} \xrightarrow[\alpha]{} \bigcirc$

TABLE 1. Extended Cohen-Dynkin diagrams in dimension 2

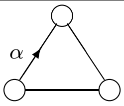
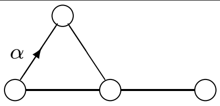
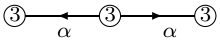
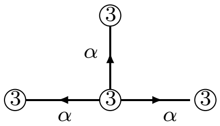
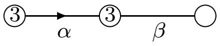
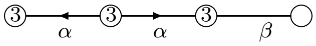
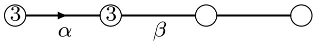
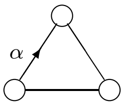
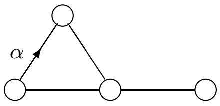
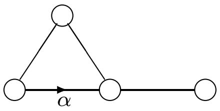
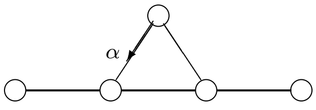
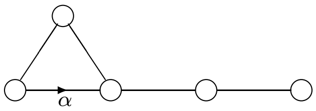
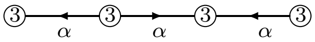
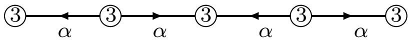
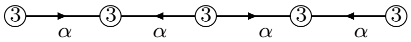
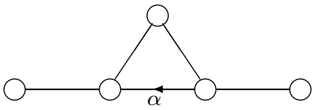
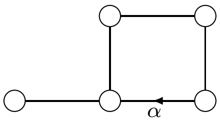
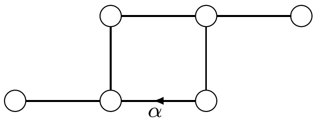
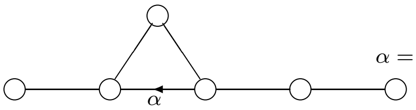
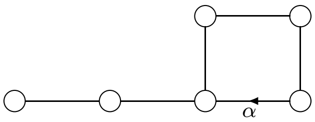
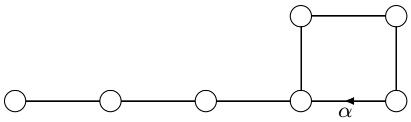
ST | Cohen type	**Cohen-Dynkin diagram**		**Extended C-D diagram**
$ST_{24} \mid J_3(4)$		$\alpha = \frac{(1-\sqrt{7}i)}{4}$	
$ST_{25} \mid L_3$		$\alpha = \frac{-i}{\sqrt{3}}$	
$ST_{26} \mid M_3$		$\alpha = \frac{-i}{\sqrt{3}}$	
		$\beta = \frac{-1}{\sqrt{2}}$	
$ST_{27} \mid J_3(5)$		$\alpha = \frac{1}{2} + \omega cos \frac{\pi}{5}$	
$ST_{29} \mid N_4$		$\alpha = \frac{1-i}{2}$	
$ST_{31} \mid EN_4$		$\alpha = \frac{1-i}{2}$	
$ST_{32} \mid L_4$		$\alpha = \frac{-i}{\sqrt{3}}$	
			
$ST_{33} \mid K_5$		$\alpha = \frac{-\omega}{2}$	
			
$ST_{34} \mid K_6$		$\alpha = \frac{-\omega}{2}$	
			

TABLE 2. Extended Cohen-Dynkin diagrams in dimension > 2

4. **Macdonald representations.**

Let R be the ring of polynomial functions on V which can be identified with the symmetric algebra of V. The reflection group G acts on R as follows

$$(gf)(x) = f(g^{-1}x)$$

for all $f \in R, g \in G, x \in V$. In particular,

$$(g\alpha_H)(x) = \chi_H^{-1}\alpha_H(x)$$

for all $g \in G_H, x \in V$, where $\tilde{G}_H = \{\chi_H^{-i} \mid 0 \leq i \leq e_H - 1\}$ is the character group of G_H. Define $\pi_C = \prod_{H \in C} \alpha_H$ for $C \in \mathcal{C}$ and $\pi_G = \prod_{C \in \mathcal{C}} \pi_C$.

When reflection subgroups are determined by subsystems of root systems, the functions π_G defined earlier will be denoted π_Φ.

We give three examples

Example 4.1. For the real reflection group of type A_{n-1} (symmetric groups of order $n!$), the root system is $\{\epsilon_i - \epsilon_j, 1 \leq i, j \leq n\}$ and a simple system is $\{\epsilon_i - \epsilon_{i+1}, 1 \leq i \leq n-1\}$ and corresponding primary (positive) system $\{\epsilon_i - \epsilon_j, 1 \leq i < j \leq n\}$. Then

$$\pi_{A_{n-1}}(x) = \prod_{1 \leq i < j \leq n} (x_i - x_j)$$

which is the Vandermonde determinant.

Example 4.2. For the complex reflection group of type $G(m,m,n)$ and $G(m,1,n)$ using the primary root systems given above, we obtain

$$\pi_{G(m,m,n)}(x) = \prod_{1 \leq i < j \leq n} \prod_{a=1}^{m} (x_i - \zeta^a x_j) = \prod_{1 \leq i < j \leq n} (x_i^m - x_j^m)$$

and

$$\pi_{G(m,1,n)}(x) = \prod_{1 \leq i < j \leq n} \prod_{a=1}^{m} (x_i - \zeta^a x_j) \prod_{i=1}^{n} x_i = \prod_{1 \leq i < j \leq n} (x_i^m - x_j^m) \prod_{i=1}^{n} x_i.$$

Example 4.3. For the complex reflection group of type ST_4, a simple system is [15] $\{\epsilon_1, -i/\sqrt{3}(\epsilon_1 + \sqrt{2}\epsilon_2)\}$ with primary system $\Omega_{ST4} = \{-i/\sqrt{3}(\epsilon_1 + \omega^k\sqrt{2}\epsilon_2), 0 \leq k \leq 2\}$ and root system $\mu_6\Omega_{ST4}$. Then

$$\pi_{ST4}(x) = x_1(x_1 - \sqrt{2}x_2)(x_1 - \omega\sqrt{2}x_2)(x_1 - \omega^2\sqrt{2}x_2) = x_1(x_1^3 + 2\sqrt{2}x_2^3)/3\sqrt{3}$$

The following lemma is the crucial result which extends the well known result in the real case that the positive roots are permuted by a reflection s_a except that $s_a(a) = -a$, where a is any positive root.

Lemma 4.1.

$$s_H\pi_C = \begin{cases} \pi_C & \textit{if } H \notin C \\ (\chi_H)^{-1}\pi_C & \textit{if } H \in C. \end{cases}$$

Proof. If $H' \notin C$, let $v_0 \in H'$ be a point, thus $\alpha_H(v_0) \neq 0$ for all $H \in C$ which implies that $\pi_C(v_o) = \prod_{H\in C} \alpha_H(v_0) \neq 0$. Hence

$$(s_{H'}\pi_C)(v_0) = \pi_C(s_{H'}^{-1}v_0) = \pi_C(v_0).$$

Furthermore, if $H \in C$, then

$$s_H\pi_C = \prod_{H\in C} s_H\alpha_H = \left(\frac{s_H\alpha_H}{\alpha_H}\right)\pi_C = \chi_H^{-1}\pi_C.$$

Let $\overline{\pi}_C = \prod_{H'\in C, H'\neq H} \alpha_{H'}$ and let $v_0 \in H$ be a point such that $\alpha_{H'}(v_0) \neq 0$ for all $H' \neq H$, then $\overline{\pi}_C(v_0) \neq 0$. Hence

$$(s_H\overline{\pi}_C)(v_0) = \overline{\pi}_C(s_H^{-1}v_0) = \overline{\pi}_C(v_0),$$

which proves the lemma.

Let R^G be the subring of G-invariant elements in R. Then, it is well known [2] that R^G is generated by n algebraically independent homogeneous elements $p_1, \ldots, p_n$. If $d_i = deg(p_i)$, then

$$|G| = \prod_{i=1}^{n} d_i \tag{4.1}$$

$$|\text{complex reflections in } G| = \sum_{i=1}^{n}(d_i - 1) = \sum_{H\in\mathcal{A}}(e_H - 1). \tag{4.2}$$

(In fact, [21] if $m_i = d_i - 1$ are the exponents and n_i are the coexponents of G, then

$$|\text{complex reflections in } G| = \sum_{i=1}^{n} m_i \tag{4.3}$$

$$|\text{reflecting hyperplanes}| = \sum_{i=1}^{n} n_i.) \tag{4.4}$$

If χ is a linear character of G, let R_χ^G be the R^G-module of relative invariants of G introduced by R. Stanley [26], that is,

$$R_\chi^G = \{f \in R \mid gf = \chi(f)f \text{ for all } g \in G.\}$$

Then Stanley has proved, amongst other things, that if $f \in R_\chi^G$, then f is divisible by $f_\chi = \prod_{C\in\mathcal{C}} \pi_C^{l_C}$, where $0 \leq l_C \leq e_C - 1$ and l_C is the least positive integer such that $\chi(s_C) = \zeta_C^{l_C}$, for a fixed generator $s_C \in G_H$ for some $H \in C$. Furthermore, $R_\chi^G = f_\chi R^G$ and $f_\chi \in R_\chi^G$, that is,

$$gf_\chi = \chi(g)f_\chi \text{ for all } g \in G.$$

Now let G' be a reflection subgroup of G and let χ also denote the restriction of the linear character χ to G'. Let f'_χ be defined as above for the subgroup G'. Let $P_{G'}^\chi$ be the subspace of R generated by the polynomial functions gf'_χ for all $g \in G$, then the vector space $P_{G'}^\chi$ is a $\mathbb{C}G$-module. The special case $\chi = 1$ will be denoted by $P_{G'}$. Then we have the following theorem which generalizes the well known result of I G Macdonald [18] to complex reflection groups - in fact, his proof carries over almost verbatim. It can be shown by using some of the above results that (i) below is a direct consequence of a theorem in [17].

Theorem 4.2. *(i) The module $P^{\chi}_{G'}$ is an absolutely irreducible $\mathbb{C}G$-module.*
(ii) If G' and G'' are two non-isomorphic reflection subgroups of G with

$$|\textit{reflecting hyperplanes for } G'| \neq |\textit{reflecting hyperplanes for } G''|,$$

then the modules $P^{\chi}_{G'}$ and $P^{\chi}_{G''}$ are not isomorphic.
(iii) The modules $P^{\chi}_{G'}$ and $\chi \otimes P_{G'}$ are isomorphic.

Proof. (i) Let $\phi : P^{\chi}_{G'} \to P^{\chi}_{G'}$ be a $\mathbb{C}G$-homomorphism, then

$$g\phi(f'_{\chi}) = \phi(gf'_{\chi}) = \phi(\chi(g)f'_{\chi}) = \chi(g)f'_{\chi}$$

for all $g \in G$ and so, by the above result $\phi(f'_{\chi})$ is divisible by f'_{χ}. Since $\phi(f'_{\chi})$ and f'_{χ} are of the same degree and so $\phi(f'_{\chi})$ is a scalar multiple of f'_{χ} and (i) follows.

(ii) The proof is similar to that of (i), but resulting this time in a zero map.

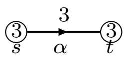
(iii) The map $\phi : P^{\chi}_{G'} \to \chi \otimes P_{G'}$ is $\phi(f_{\chi}) = \chi \otimes f_1$, which is easily seen to be the required isomorphism.

The last result (iii) shows that basically it is only necessary to construct the modules(representations) $P^{\chi}_{G'}$.

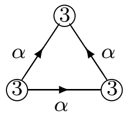
Some examples are now given to illustrate the usefulness of the above approach.

Example 4.4. In the case $G(m, 1, n)$, this construction gives all the irreducible Macdonald modules. It is well known in this case that the irreducible representations are in one-to-one correspondence with the set of m-partitions $(\lambda^{(1)}, \dots, \lambda^{(m)})$ of n; thus we need only take the subsystems of type $\sum_{i=1}^{m} \sum_{j=1}^{s_i} B^{m_i}_{\lambda^{(i)}_j}$ listed in (3.1) - here $\lambda^{(i)}$ is the partition $(\lambda^{(i)}_1, \dots, \lambda^{(i)}_{s_i})$. The representations of the generalized symmetric group $G(m, 1, n)$ have been considered by Can [7] and Hughes [11] from a different point of view (generalizing the concepts of tabloids and polytabloids or the symmetric groups), it is clear that this can be modified to adopt the Macdonald module approach.

Example 4.5. We consider the group ST_4. As was seen in Table 1, this group has Cohen-Dynkin diagram



and extended Cohen-Dynkin diagram



,

where $\alpha = -i/\sqrt{3}$.

Thus, in this case, there are only 3 non-conjugate subsystems

③ $\xrightarrow[\alpha]{3}$ ③ ③ $\emptyset$.

From Table 3, we see that ST_4 has three linear characters χ_1, χ_2 and χ_3 and since $\chi_5 = \chi_4 \otimes \chi_2$ and $\chi_6 = \chi_4 \otimes \chi_3$, thus the only modules required are those corresponding to χ_1, χ_4 and χ_7 of degrees 1,2 and 3 respectively. The first two are obtained from the subsystems ③ $\xrightarrow[\alpha]{3}$ ③ and ③ of ST_4. The third is obtained by using the fact that ST_4 is a subgroup of ST_6 with Cohen-Dynkin diagram ③ $\xrightarrow[\beta]{6}$ ○, where $\beta = -i(\frac{1}{2}(1 + \frac{1}{\sqrt{3}}))^{\frac{1}{2}}$. The representation of degree 3 is the representation of ST_6 corresponding to the subsystem ③ ③ which remains irreducible on restriction to ST_4, indeed this subsystem is also a subsystem of ST_4.

ST_4	cl_1	cl_2	cl_3	cl_4	cl_5	cl_6	cl_7
order	1	2	6	6	3	3	4
class order	1	1	4	4	4	4	6
χ_1	1	1	1	1	1	1	1
χ_2	1	1	ω	ω^2	ω	ω^2	1
χ_3	1	1	ω^2	ω	ω^2	ω	1
χ_4	2	-2	1	1	-1	-1	0
χ_5	2	-2	ω	ω^2	$-\omega$	$-\omega^2$	0
χ_6	2	-2	ω^2	ω	$-\omega^2$	$-\omega$	0
χ_7	3	3	0	0	0	0	-1

TABLE 3. Character Table of ST_4

The irreducible representations obtained from the three subsystems are given in Table 4.

Subsystem	**Basis**	**s**	**t**
$\underset{s}{\textcircled{3}}\xrightarrow[\alpha]{3}\underset{t}{\textcircled{3}}$	$\{s\}$	(ω)	(ω)
$\underset{s}{\textcircled{3}}$	$\{s, \tau_t s\}$	$\begin{pmatrix} 0 & -\omega \\ 1 & -\omega^2 \end{pmatrix}$	$\begin{pmatrix} \omega & \omega \\ 0 & 1 \end{pmatrix}$
$\underset{s}{\textcircled{3}} \qquad \underset{t}{\textcircled{3}}$	$\{s, \tau_t s\}$	$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$
$\emptyset$		(1)	(1)

TABLE 4. Irreducible Representations of ST_4

Example 4.6. Some more general results may be obtained for an arbitrary two-dimensional group. For an arbitrary root system

$$\underset{u}{\textcircled{m}}\xrightarrow[\alpha]{k}\underset{v}{\textcircled{n}}$$

let ζ and η denote primitive m-th and n-th roots of unity respectively. For the subsystem

$$S \;=\; \underset{u}{\textcircled{m}}$$

if $s = \tau_u$ and $t = \tau_v$, then $\pi_S = u$ and $P_S =< u, tu = u + (1-\eta)\alpha v >$, and we see that

$$s \longmapsto \begin{pmatrix} \zeta & -(1-\zeta)(1+(1-\eta)\alpha^2) \\ 0 & 1 \end{pmatrix} \text{ and } t \longmapsto \begin{pmatrix} 0 & -\eta \\ 1 & 1+\eta \end{pmatrix}.$$

It can be shown that the subsystem $\underset{u}{\textcircled{m}}$ gives an equivalent representation.

The explicit results for the separate root systems are listed in Table 5; thus we have the 'basic' representations of degree two for all the two-dimensional reflection groups. In Table 5, ω, i, η and ϵ are primitive cube, fourth, fifth and eigth roots of unity respectively and $\alpha = -i\sqrt{\frac{1}{2}(1+\frac{1}{\sqrt{5}})}$.

ST-type	**s**	**t**	**ST-type**	**s**	**t**
ST_4	$\begin{pmatrix} \omega & -1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & -\omega \\ 1 & -\omega^2 \end{pmatrix}$	ST_{14}	$\begin{pmatrix} \omega & \sqrt{2}i\omega^2 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
ST_5	$\begin{pmatrix} \omega & \omega^2 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & -\omega \\ 1 & -\omega^2 \end{pmatrix}$	ST_{16}	$\begin{pmatrix} \eta & -1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & -\eta \\ 1 & 1+\eta \end{pmatrix}$
ST_6	$\begin{pmatrix} \omega & i\omega^2 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	ST_{17}	$\begin{pmatrix} \eta & (1-\eta)\alpha \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
ST_8	$\begin{pmatrix} i & -1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & -i \\ 1 & 1+i \end{pmatrix}$	ST_{18}	$\begin{pmatrix} \eta & \omega^2 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & -\omega \\ 1 & -\omega^2 \end{pmatrix}$
ST_9	$\begin{pmatrix} i & -\epsilon^3 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	ST_{20}	$\begin{pmatrix} \omega & \omega^2-\omega(\eta+\eta^4) \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & -\omega \\ 1 & -\omega^2 \end{pmatrix}$
ST_{10}	$\begin{pmatrix} i & \omega^2 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & -\omega \\ 1 & -\omega^2 \end{pmatrix}$	ST_{21}	$\begin{pmatrix} \omega & i\omega^2(1+\eta+\eta^4) \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

TABLE 5. Irreducible Representations of degree 2

ST-type	**s**	**t**
ST_4	$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$
ST_5	$\begin{pmatrix} \omega & 0 & -1 \\ 0 & 0 & -\omega^2 \\ 0 & 1 & -\omega \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$
ST_6	$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
ST_8	$\begin{pmatrix} i & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & i \end{pmatrix}$
ST_9	$\begin{pmatrix} i & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$
	$\begin{pmatrix} 0 & 0 & -i \\ 1 & 0 & 1 \\ 0 & 1 & i \end{pmatrix}$	$\begin{pmatrix} -1 & -i & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

TABLE 6. Irreducible Representations of degree 3

Example 4.7. The irreducible representations of degree three are obtained by taking the subsystems $m \quad m$ and $n \quad n$ of the root system $m \longrightarrow n$. In Table 6, as examples, the irreducible representations of degree three of ST_4, ST_5, ST_6, ST_8 and ST_9 are given, the two representations of ST_9 are clearly not equivalent.

Further examples are given in the Appendix below.

5. ACKNOWLEGEMENTS

The first author is grateful to Eric Opdam for a helpful discussion concerning this work, in particular for drawing his attention to the paper by Richard Stanley.

Subsystem	**s**	**t**	**u**
s	$\begin{pmatrix} -1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & -\beta \\ 1 & 0 & \beta \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & -\bar{\alpha} & 1 \\ 0 & 1 & 0 \\ 1 & \bar{\alpha} & 0 \end{pmatrix}$
$s \longrightarrow t \qquad \sigma$	$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$
$s \qquad \sigma$	$\begin{pmatrix} -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$
$s \text{ --- } u$	$\begin{pmatrix} -1 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

TABLE 7. Irreducible Representations of ST_{24}

Appendix. The group ST_{24} has irreducible representations of degrees

$$\underbrace{1,1},\underbrace{3,3},\underbrace{\overline{3,3}},\underbrace{6,6},\underbrace{7,7},\underbrace{8,8}.$$

Information concerning the irreducible representations and characters of the complex reflection groups may be found in M. Benard [1].

Thus the representations given in Table 7 lead to all the irreducible representations of ST_{24}.

The group ST_{25} has irreducible representations of degrees

$$\underbrace{1,1,1},\underbrace{2,2,2},\underbrace{3,3,3},\underbrace{\overline{3,3,3}},\underbrace{6,6,6},\underbrace{\overline{6,6,6}},\underbrace{8,8,8},3,9,\overline{9}.$$

Thus the representations in Table 8 lead to all the irreducible representations except the final ones of degrees 8, 3 and 9.

The group ST_{26} has irreducible representations of degrees

$$\underbrace{1,1,1,1,1,1},\underbrace{2,2,2,2,2,2},\underbrace{3,3,3,3,3,3},\underbrace{\overline{3,3,3,3,3,3}},\underbrace{6,6,6,6,6,6},\underbrace{\overline{6,6,6,6,6,6}},\underbrace{8,8,8,8,8,8},$$

$$\underbrace{3,3},\underbrace{9,9},\underbrace{\overline{9,9}}.$$

Thus the representations given in Table 9 lead to all the irreducible representations except the final ones of degrees 3 and 9.

Subsystem	**s**	**t**	**u**
$\underset{s}{\textcircled{3}}$	$\begin{pmatrix} \omega & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & -\omega & 0 \\ 1 & -\omega^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -\omega \\ 0 & 1 & -\omega^2 \end{pmatrix}$
$\underset{s}{\textcircled{3}} \quad \underset{u}{\textcircled{3}} \quad \underset{\sigma}{\textcircled{3}}$	$\begin{pmatrix} \omega & -1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & -\omega \\ 1 & -\omega^2 \end{pmatrix}$	$\begin{pmatrix} \omega & -1 \\ 0 & 1 \end{pmatrix}$
$\underset{s}{\textcircled{3}} \quad \underset{u}{\textcircled{3}}$	$\begin{pmatrix} \omega & 0 & 0 & -1 & 0 & \omega^2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -\omega & 0 & -\omega \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \omega & 0 & \omega \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 1 & -\omega^2 & -\omega \\ 1 & 0 & 0 & 0 & \omega^2 & \omega \\ 0 & 0 & \omega & 0 & \omega & -\omega \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \omega & \omega & 0 & 0 & -\omega^2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & \omega \\ 0 & \omega^2 & \omega^2 & 0 & \omega^2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$
$\underset{s}{\textcircled{3}} \xrightarrow[\alpha]{3} \underset{t}{\textcircled{3}}$	$\begin{pmatrix} \omega & 0 & 0 & 0 & -1 & -\omega^2 \\ 0 & \omega & 0 & 0 & \omega^2 & 0 \\ 0 & 0 & 0 & 0 & -\omega^2 & -1 \\ 0 & 0 & 0 & \omega & \omega & \omega^2 \\ 0 & 0 & 1 & 0 & -\omega & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \omega & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -\omega & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \omega & \omega & 0 \\ 0 & 0 & 1 & \omega & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & -\omega^2 \\ 1 & 0 & 0 & 0 & 0 & \omega^2 \\ 0 & 0 & \omega & 0 & 0 & \omega \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \omega & -\omega \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

Table 8. Irreducible Representations of ST_{25}

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Subsystem	**s**	**t**	**u**
s	$\begin{pmatrix} -1 & \omega^2 & \omega^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & -\omega & 0 \\ 1 & -\omega^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -\omega \\ 0 & 1 & -\omega^2 \end{pmatrix}$
(3) t	$\begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \omega & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & -\omega \\ 0 & 1 & 0 \\ 1 & 1 & -\omega^2 \end{pmatrix}$
$\sigma_1 - s \to t$ (3)	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} \omega & -1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & -\omega \\ 1 & -\omega^2 \end{pmatrix}$
(3) $t \xrightarrow{4} u$ (3), α	$\begin{pmatrix} 0 & 1 & 0 & 0 & -\omega^2 & \omega \\ 1 & 0 & 0 & 0 & \omega^2 & -\omega \\ 0 & 0 & 0 & 1 & \omega & \omega \\ 0 & 0 & 1 & 0 & -\omega & -\omega \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} a & & & & & \\ \omega & 0 & 0 & 0 & 0 & \omega \\ 0 & 0 & 0 & 0 & 1 & -\omega \\ 0 & 1 & 0 & 0 & 0 & \omega \\ 0 & 0 & 0 & \omega & 0 & -\omega \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \omega & 0 & -\omega & 0 & 0 & 0 \\ 0 & \omega & \omega & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & \omega \\ 0 & 0 & \omega^2 & \omega^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$
$s \xrightarrow{4} t$ (3), α	$\begin{pmatrix} -1 & 0 & -\omega & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\omega^2 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} \omega & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -\omega & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \omega & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 1 & -\omega^2 & 0 \\ 1 & 0 & 0 & 0 & \omega^2 & 1 \\ 0 & 0 & \omega & 0 & \omega & \omega \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$
$s \to t$ $\quad\sigma$	$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

TABLE 9. Irreducible Representations of ST_{26}

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