Séminaire Lotharingien de Combinatoire **52** (2007), Article B52i

ASYMPTOTICS FOR RANDOM WALKS IN ALCOVES OF AFFINE WEYL GROUPS

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Dédié à Alain Lascoux

Abstract. Asymptotic results are derived for the number of random walks in alcoves of affine Weyl groups (which are certain regions in n-dimensional Euclidean space bounded by hyperplanes), thus solving problems posed by Grabiner [J. Combin. Theory Ser. A **97** (2002), 285–306]. These results include asymptotic expressions for the number of vicious walkers on a circle, as well as for the number of vicious walkers in an interval. The proofs depart from the exact results of Grabiner [loc. cit.], and require as diverse means as results from symmetric function theory and the saddle point method, among others.

1. Introduction

The enumeration of random walks in lattice regions bounded by hyperplanes is a classical and frequently studied subject in combinatorics and related fields. Its attractiveness stems from the fact that this problem has implications to many other, often seemingly unrelated problems, and, thus, to several different fields. To mention some examples, random walk interpretations exist for ballot problems (see e.g. [7, 34, 45]), standard Young tableaux (see e.g. [45]), semistandard tableaux and plane partitions (see e.g. [22, 25, 28] and [38, Sec. 8]), symplectic tableaux (see e.g. [12, 28]), oscillating tableaux (see e.g. [2, 26]), cylindric partitions (see [15]), non-intersecting lattice paths and vicious walkers (see e.g. [8, 19, 22, 28, 29]), and are therefore used for the solution of problems in these areas (see e.g. [17, 21, 38] for applications in representation theory, and e.g. [8] for applications in statistical physics), as well as in the analysis

2000 *Mathematics Subject Classification.* Primary 05A16; Secondary 05E05 05E15 60G50 82B23 82B41.

Key words and phrases. random walks, reflection principle, alcoves of affine Weyl groups, vicious walkers, non-intersecting lattice paths, cylindric partitions, Schur functions, symplectic characters, orthogonal characters.

† Research partially supported by the Austrian Science Foundation FWF, grant P13190-MAT, as well as grant S9607-N13, in the framework of the National Research Network "Analytic Combinatorics and Probabilistic Number Theory," and by EC's IHRP Programme, grant HPRN-CT-2001-00272, "Algebraic Combinatorics in Europe."

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of non-parametric statistics in probability theory (see [34] for an introduction to that area).

Clearly, at the very beginning stands the problem of enumerating all lattice paths in the plane integer lattice between two given points, which consist of positive unit steps and do not cross a given diagonal line. (In fact, the original formulation is in terms of a two-candidate ballot problem.) As is well-known, a solution to this problem is provided by the famous *reflection principle*, which is usually attributed to André [1] (see e.g. [6, p. 22]). It is more than a hundred years later, when Gessel and Zeilberger showed in [16] how far one can go by using the reflection principle. Their main result gives (under certain conditions) the number of random walks in regions of n-dimensional Euclidean space which are bounded by hyperplanes. Their formula involves the elements of the reflection group which is generated by the reflections with respect to the hyperplanes which bound these regions. (The same formula for the case of finite Weyl groups has been independently discovered by Biane [4]. We refer the reader to [24] for an introduction to reflection groups and Weyl groups.) It covers numerous formulae that occurred in the literature earlier (and even afterwards ...).[1]

Recently, Grabiner [19] has revisited the problem of enumerating random walks in *alcoves* of *affine* (i.e., infinite) *Weyl groups*. (See the next section for precise definitions.) To be precise, he considered three types of random walks in these regions: (1) lattice walks consisting of *positive unit steps* e_j (with e_j denoting the j-th standard unit vector), (2) lattice walks consisting of positive *and negative* unit steps $\pm e_j$, and (3) lattice walks consisting of steps of the form $\pm\frac{1}{2}e_1\pm\frac{1}{2}e_2\pm\cdots\pm\frac{1}{2}e_n$ (where any sign pattern is allowed). We will refer to these three types of walks as *walks with standard steps in the positive direction* (or *walks with positive standard steps*, for short), *walks with standard steps*, and *walks with diagonal steps*, respectively. Starting from the result of Gessel and Zeilberger, Grabiner derived interesting determinantal formulae for the enumeration of these three types of walks in alcoves of types $\tilde{A}_{n-1}$, $\tilde{B}_n$, $\tilde{C}_n$, and $\tilde{D}_n$. From the $\tilde{A}_{n-1}$ results he was also able to derive determinantal formulae for the enumeration of walks *on the circle* (see the next section for the precise definition), which includes the enumeration of n non-colliding particles on a circle.

All of Grabiner's formulae are *exact* results. Hence, as an afterthought, he posed the problem of determining the asymptotic behaviour of the number of these walks if the number of steps becomes large. It happens that this had already been done independently in [29] for walks with diagonal steps in the alcove of type $\tilde{C}_n$, albeit in a completely different language, the language of vicious walkers.

The purpose of this paper is to carry out the asymptotic analysis of the number of random walks in alcoves of affine Weyl groups in all the other cases, and also for the number of random walks on the circle. To be precise, we determine the asymptotic behaviour of the number of random walks in an alcove as the number of steps tends to infinity for the case that starting and end point are held fixed, as well as for the case

[1] There are in fact only very few known results on the enumeration of walks in regions bounded by hyperplanes that are not covered by this result. For earlier results see [34, Ch. 1 and 2]. More recent results include for example [3, 23, 30, 35, 36, 37, 42, 43, 44], where, from a conceptual point of view, the papers [3] and [35, 36, 37] have to be emphasized most: in [3] the so-called *kernel method* is exploited (which seems to be especially well-suited for this type of problem), whereas in [35, 36, 37] it is the *umbral calculus* which is systematically applied to solve lattice path enumeration problems.

where the end point can be arbitrary. Frequently, the results depend heavily on the parities of the involved parameters, a phenomenon which distinguishes the discrete case from the continuous case. (This phenomenon does also not occur for the corresponding problem for the walks in chambers of finite Weyl groups; see the last paragraph of the Introduction.) While, from an analytic point of view, the order of magnitude is always rather straight-forward to determine, for which very basic tools (if at all), such as Stirling's formula, or, in one case, a rather standard application of the saddle point method, suffice, the determination of the multiplicative constant poses quite frequently a substantial challenge. Interestingly, carrying out the latter task requires quite often some advanced facts from symmetric function theory (see the proofs in Sections 4–7). In particular, identities for classical group characters from [27] come in handy at many places. It should be observed that the proofs show that the errors are always exponentially small, with the exception of Theorem 15, where the error is dictated by the Stirling approximation of the binomial coefficient in (3.10), and of Theorem 16, where the error is dominated by those coming from the saddle point approximation given in Lemma A in Appendix A.

In the next section we provide the basic definitions, in particular, the definitions of the alcoves to which our walks are confined, and we summarize all the exact results that exist for the enumeration of the three types of walks in these alcoves. These will be the starting points for our asymptotic calculations, which we carry out in the subsequent sections. The results for the alcove of type $\tilde{A}_{n-1}$ are given in Section 3, the results for the enumeration of walks on the circle are the subject of Section 4, we give the results for the alcove of type $\tilde{C}_n$ in Section 5, in Section 6 there follow the results for the alcove of type $\tilde{B}_n$, and, finally, we present the results for the alcove of type $\tilde{D}_n$ in Section 7. Auxiliary results that are needed in the proofs of the theorems are collected in three appendices.

In concluding the introduction, it is probably useful to review the state of affairs for *finite* Weyl groups, i.e., the known results on the asymptotic behaviour of walks in chambers of finite Weyl groups as the number of steps of the walks becomes large. (We refer the reader again to the book [24] for definitions and more information on finite Weyl groups.) In fact, since, as we already indicated, the random walk problems considered in this paper can be seen from various different angles, numerous results can be found scattered in the combinatorics, probability, physics, and even representation theory literature. If starting and end point are fixed, the asymptotics of walks with diagonal steps in Weyl chambers of types A_{n-1} and C_n were determined (in the language of vicious walkers) by Rubey [40, Ch. 2, Sections 3 and 4], [41, Sections 3 and 4], with previous results in special cases given in [28, Sections 2, 4, 7]. For the case that the starting point is the origin and the end point is fixed, a result of Biane [5] on the asymptotics of multiplicities of irreducible representations in tensor powers of irreducible representations of semisimple Lie groups, combined with an observation due to Grabiner and Magyar [21, Sec. 3.3] that, under mild restrictions, the number of random walks in Weyl chambers is equal to such multiplicities, implies a uniform asymptotic formula for random walks in Weyl chambers of any type, with the exception of walks with standard steps in a Weyl chamber of type A_{n-1}. For the case that the starting point is arbitrary but fixed and the end point is not fixed, Grabiner [20] has recently shown that, by combining a result of Kuperberg [31, Theorem 1.2.1] on the approximation of sums of random variables defined on lattices by the corresponding Brownian motion, and of himself [18] on Brownian motion in Weyl chambers, one obtains the dominating term of the asymptotic behaviour for all types and for all possible step sets at once. We want to remark that walks with standard steps in the positive direction in a chamber of the Weyl group of type A_{n-1} are equivalent to skew standard Young tableaux with at most n rows, the shape depending on starting and end point of the walk. The asymptotic behaviour of the number of non-skew standard Young tableaux with at most n rows is covered by the celebrated earlier (and more general) result of Regev [39]. More precise results than that of Grabiner's (i.e., with bounds on the errors also) in the case of walks with diagonal steps in a Weyl chamber of type A_{n-1} were found (again in the language of vicious walkers) by Rubey [40, Ch. 2, Sec. 3], [41, Sec. 3], with previous results in special cases given in [28, Sections 2, 3, 6]. In addition, Rubey [40, Ch. 2, Sec. 4], [41, Sec. 4] also provides more precise results in the case of walks with diagonal steps in a Weyl chamber of type C_n. (Again, results in special cases can already be found in [28, Sec. 4, 5, 7].)

2. A summary of exact results of random walks in alcoves of affine Weyl groups

In this section we summarize the exact results for random walks in alcoves of affine Weyl groups, also including two results for random walks on a circle, which are the starting points for our asymptotic calculations which are to follow in the later sections. We also use the opportunity to point out, in each case, equivalent formulations of the walk problems (in case they exist).

Before we state the results, let us recall the definitions of these alcoves. Let m be some given positive integer or half-integer. (By definition, a *half-integer* is an odd number divided by 2.) We define the alcove $\mathcal{A}_m^{\tilde A_{n-1}}$ of type $\tilde A_{n-1}$ to be the region

$$\mathcal{A}_m^{\tilde A_{n-1}} := \{(x_1,x_2,\dots,x_n) : x_1 > x_2 > \dots > x_n > x_1 - m\}. \tag{2.1}$$

(Strictly speaking, this is a *scaled* alcove.) The (scaled) alcove of type $\tilde C_n$ is defined by

$$\mathcal{A}_m^{\tilde C_n} := \{(x_1,x_2,\dots,x_n) : m > x_1 > x_2 > \dots > x_n > 0\}. \tag{2.2}$$

The (scaled) alcove of type $\tilde B_n$ is defined by

$$\mathcal{A}_m^{\tilde B_n} := \{(x_1,x_2,\dots,x_n) : x_1 > x_2 > \dots > x_n > 0 \text{ and } x_1 + x_2 < 2m\}. \tag{2.3}$$

Finally, the (scaled) alcove of type $\tilde D_n$ is the region

$$\mathcal{A}_m^{\tilde D_n} := \{(x_1,x_2,\dots,x_n) : x_1 > x_2 > \dots > x_{n-1} > |x_n|, \text{ and } x_1 + x_2 < 2m\}. \tag{2.4}$$

We begin with results for the enumeration of walks in $\mathcal{A}_m^{\tilde A_{n-1}}$. The first result is originally due to Filaseta [7]. It is however covered by the general result [16]. In the statement of the theorem, and also subsequently, given a vector $\eta = (\eta_1,\eta_2,\dots,\eta_n)$ we use the symbol $|\eta|$ to denote the sum of its components, i.e., $|\eta| := \eta_1 + \eta_2 + \dots + \eta_n$.

Theorem 1 ([7]). *Let m be a positive integer. Furthermore, let $\eta = (\eta_1,\eta_2,\dots,\eta_n)$ and $\lambda = (\lambda_1,\lambda_2,\dots,\lambda_n)$ be vectors of integers in the alcove $\mathcal{A}_m^{\tilde A_{n-1}}$ of type $\tilde A_{n-1}$ (defined in*

(2.1)). *Then the number of random walks from* η *to* λ*, which consist entirely of standard steps in the positive direction, and which stay in the alcove* $\mathcal{A}_m^{\tilde{A}_{n-1}}$*, is given by*

$$\big(|\lambda|-|\eta|\big)!\sum_{k_1+\cdots+k_n=0}\det_{1\le h,t\le n}\left(\frac{1}{(\lambda_t-\eta_h+mk_h)!}\right). \tag{2.5}$$

The corresponding result for positive *and* negative standard steps is also a direct consequence of the general result [16], and is stated explicitly in [19].

Theorem 2 ([19, Eq. (34)])**.** *Let* m *be a positive integer. Furthermore, let* $\eta=(\eta_1,\eta_2,\dots,\eta_n)$ *and* $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_n)$ *be vectors of integers in the alcove* $\mathcal{A}_m^{\tilde{A}_{n-1}}$ *of type* $\tilde{A}_{n-1}$ *(defined in* (2.1)*). Then the number of random walks from* η *to* λ *with exactly* k *standard steps, which stay in the alcove* $\mathcal{A}_m^{\tilde{A}_{n-1}}$*, is given by the coefficient of* $x^k/k!$ *in*

$$\sum_{k_1+\cdots+k_n=0}\det_{1\le h,t\le n}\left(I_{\lambda_t-\eta_h+mk_h}(2x)\right), \tag{2.6}$$

where $I_\alpha(x)$ *is the modified Bessel function of the first kind*

$$I_\alpha(x)=\sum_{j=0}^{\infty}\frac{(x/2)^{2j+\alpha}}{j!\,(j+\alpha)!}.$$

This result has also a different interpretation: by considering each of the n coordinates as a separate walk, the walks with standard steps in $\mathcal{A}_m^{\tilde{A}_{n-1}}$ can be seen as n separate particles on the integer line, where at each tick of the clock *exactly one* particle moves to the right or to the left by one unit (a move to the right, respectively to the left, of the j-th particle corresponding to a step $+e_j$, respectively $-e_j$), under the constraint that at no time two particles occupy the same lattice site, and such that in addition a shift by m of any of the particles never collides with any of the other particles. Thus we obtain a sub-model of Fisher's [8] *random turns vicious walker model.*[2]

Similarly, random walks with diagonal steps in $\mathcal{A}_m^{\tilde{A}_{n-1}}$ with given starting and end point can be seen in several ways: by considering each of the n coordinates as a separate walk, such random walks can be seen as n separate particles on the integer line, where at each tick of the clock *each* particle moves one unit step to the right (corresponding to a change of $+\frac{1}{2}$ in the corresponding coordinate) or to the left (corresponding to a change of $-\frac{1}{2}$ in the corresponding coordinate), such that they never collide, and such that in addition a shift by $2m$ of any of the particles never collides with any of the other particles. Thus we obtain a sub-model of Fisher's [8] *lock step vicious walker model.*[3] An alternative, two-dimensional picture arises if we convert the movements in each coordinate of the random walk to a separate path in the plane integer lattice, identifying a change by $+\frac{1}{2}$ in a coordinate with an up-step $(1,1)$ and a change by $-\frac{1}{2}$ with a down-step $(1,-1)$ of the corresponding path. Thus, such random walks can be

[2]In statistical physics, a model of n walkers on the integer line where at each tick of the clock exactly one walker moves to the right or to the left, under the constraint that at no time two walkers occupy the same lattice site, is called the *random turns vicious walker model.*

[3]In statistical physics, a model of n walkers on the integer line where at each tick of the clock each walker moves to the right or to the left, under the constraint that at no time two walkers occupy the same lattice site, is called the *lock step vicious walker model.*

seen to be equivalent to families of *non-intersecting*[4] *lattice paths* in the plane integer lattice with steps $(1,1)$ and $(1,-1)$ (the starting points of which being aligned along a vertical line, as well as the end points) where a shift of the bottom-most path dominates the top-most path. The latter objects are in turn in bijection with (special) *cylindric partitions* (as defined in [15]) of rectangular shape (see [15, Sec. 3] for that translation; to obtain the presentation of the lattice paths in [15], the above described picture has to be rotated by 45°).

The following result is at the same time a direct consequence of the general result in [16] and of Theorem 3 in [15]. It is stated explicitly in [19]. It is however important to note that it is only true for *integral* m (as well as the "m-circle result" Theorem 5 for diagonal steps which it implies, as opposed to the companion results Theorems 7, 9, and 11 for the types $\tilde C_n$, $\tilde B_n$, and $\tilde D_n$). This is because the reflection argument from [16] (repeated in [19], and in an equivalent form in [15]) only guarantees that (using the vicious walker picture) particles never occupy the same site, respectively a particle shifted by $2m$ never occupies the same site as another particle. If m is a half-integer, this does not exclude that a shifted particle changes sides with another particle, and thus the formula (2.7) below would also include walks which violate the condition $x_n > x_1 - m$ which is contained in the definition (2.1) of the alcove $\mathcal{A}_m^{\tilde A_{n-1}}$.

Theorem 3 ([19, Eq. (35)])**.** *Let m be a positive integer. Furthermore, let $\eta = (\eta_1, \eta_2, \dots, \eta_n)$ and $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ be vectors of integers or of half-integers in the alcove $\mathcal{A}_m^{\tilde A_{n-1}}$ of type $\tilde A_{n-1}$ (defined in (2.1)). Then the number of random walks from η to λ with exactly k diagonal steps, which stay in the alcove $\mathcal{A}_m^{\tilde A_{n-1}}$, is given by*

$$\sum_{k_1+\dots+k_n=0} \det_{1\le h,t\le n}\left(\binom{k}{\frac{k}{2}+\lambda_t-\eta_h+mk_h}\right). \tag{2.7}$$

As observed by Grabiner in [19], the above results can be used to derive results on the enumeration of random walks on the m-circle, where by "*random walks on the m-circle*" we mean random walks in n-dimensional Euclidean space, where each coordinate is reduced modulo m (i.e., a point $(x_1, x_2, \dots, x_n)$ is identified with $(x_1 + k_1 m, x_2 + k_2 m, \dots, x_n + k_n m)$ for any integers $k_1, k_2, \dots, k_n$).

Whereas in the case of standard steps in the positive direction this does not define a different model, it does for standard steps in the positive *and* negative direction, and also for diagonal steps. The result from [19] for standard steps reads as follows.

Theorem 4 ([19, Eq. (32)])**.** *Let m be a positive integer. Furthermore, let $\eta = (\eta_1, \eta_2, \dots, \eta_n)$ be a vector of integers with $m > \eta_1 > \eta_2 > \dots > \eta_n \ge 0$, and let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ be a vector of integers with $m > \lambda_{s+1} > \dots > \lambda_n > \lambda_1 > \dots > \lambda_s \ge 0$, for some s. Then the number of random walks on the m-circle from η to λ with exactly k standard steps, such that at no time two coordinates of a point on the random walk are equal, is given by*

$$\frac{1}{n}\sum_{u=0}^{n-1} e^{-2\pi i u s/n} \det_{1\le h,t\le n}\left(\frac{1}{m}\sum_{r=0}^{m-1} e^{-2\pi i(u+nr)(\lambda_t-\eta_h)/mn} \exp(2x\cos(2\pi(u+nr)/mn))\right). \tag{2.8}$$

[4]A family of paths is called *non-intersecting* if no two paths from the family have any common points.

In the same way as explained above for random walks in $\mathcal{A}_m^{\tilde{A}_{n-1}}$, this result can also seen as counting n non-colliding particles moving on the integer circle of length m (the interval $[0,m]$ with 0 and m identified), where at each tick of the clock *exactly one* particle moves to the right or to the left by one unit.

Similarly, random walks with diagonal steps on the circle, with the property that at no time two coordinates of a point on the walk are equal can be equivalently seen as the movements of n non-colliding particles on a circle, where at each tick of the clock each particle moves one unit step to the right or to the left. This version of the lock step vicious walker model had been first considered by Forrester [10]. He solved the problem of counting the number of ways n such particles in this model may move from given starting points to given end points in the case that n is odd, however, where the particles may reach the end points in *any* (cyclic) order (see [10, Sec. 2.2]). An analogous formula for the case that n is even has been recently found by Fulmek [11]. Thus, the result from Grabiner's paper [19], which we state below, constitutes a refinement of Forrester's and Fulmek's formulae, as in Grabiner's formula the order in which the particles arrive at the end points is fixed. In the statement below, a small typo from [19] has been corrected (in the determinant in Eq. (33) in [19] the term $\zeta^{-(u+nr)(\lambda_j-\eta_i)}$ has to be replaced by $\zeta^{-2(u+nr)(\lambda_j-\eta_i)}$).

Theorem 5 ([19, Eq. (33)]). *Let m be a positive integer or half-integer. Furthermore, let $\eta=(\eta_1,\eta_2,\dots,\eta_n)$ be a vector of integers or of half-integers with $m>\eta_1>\eta_2>\dots>\eta_n\ge 0$, and let $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_n)$ be a vector of integers or of half-integers with $m>\lambda_{s+1}>\dots>\lambda_n>\lambda_1>\dots>\lambda_s\ge 0$, for some s. Then the number of random walks on the m-circle from η to λ with exactly k diagonal steps, such that at no time two coordinates of a point on the random walk are equal, is given by*

$$\frac{1}{n}\sum_{u=0}^{n-1} e^{-2\pi ius/n} \det_{1\le h,t\le n}\left(\frac{2^{k-1}}{m}\sum_{r=0}^{2m-1} e^{-2\pi i(u+nr)(\lambda_t-\eta_h)/mn}\cos^k(\pi(u+nr)/mn)\right). \tag{2.9}$$

Next we quote the two results from [19] on the enumeration of random walks in alcoves of type $\tilde{C}_n$. In this case, there is no separate result for positive standard steps, since for such walks the condition $m>x_1$, which appears in the definition (2.2) of the alcove $\mathcal{A}_m^{\tilde{C}_n}$, is without meaning, as well as the condition $x_n>0$, so that the problem of enumerating random walks with positive standard steps between two given points which stay in $\mathcal{A}_m^{\tilde{C}_n}$ is equivalent to counting random walks with positive standard steps which stay in the *Weyl chamber of type A_{n-1}*, the latter being defined by

$$\{(x_1,x_2,\dots,x_n): x_1>x_2>\dots>x_n\}. \tag{2.10}$$

As we mentioned in the Introduction, this problem has been dealt with in [20, 28, 40, 41]. On the other hand, random walks in $\mathcal{A}_m^{\tilde{C}_n}$ from η to λ with standard steps in the positive *and* negative direction are equivalent to oscillating tableaux from $(\eta_1-n,\eta_2-(n-1),\dots,\eta_n-1)$ to $(\lambda_1-n,\lambda_2-(n-1),\dots,\lambda_n-1)$ with at most n rows and at most $m-n$ columns, as is easily seen by identifying a step $+e_j$ with the augmentation of the i-th row of the Ferrers diagram by a box, respectively identifying a step $-e_j$ with the deletion of a box from the i-th row of the Ferrers diagram. (The reader should recall that an oscillating tableau is a sequence of Ferrers diagrams where successive diagrams in the sequence differ by exactly one box.) The corresponding result from [19] reads as follows.

Theorem 6 ([19, Eq. (23)])**.** *Let m be a positive integer. Furthermore, let $\eta = (\eta_1, \eta_2, \dots, \eta_n)$ and $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ be vectors of integers in the alcove $\mathcal{A}_m^{\tilde{C}_n}$ of type $\tilde{C}_n$ (defined in (2.2)). Then the number of random walks from η to λ with exactly k standard steps, which stay in the alcove $\mathcal{A}_m^{\tilde{C}_n}$, is given by the coefficient of $x^k/k!$ in*

$$\det_{1\le h,t\le n}\left(\frac{1}{m}\sum_{r=0}^{2m-1}\sin\frac{\pi r\lambda_t}{m}\cdot\sin\frac{\pi r\eta_h}{m}\cdot\exp\left(2x\cos\frac{\pi r}{m}\right)\right). \tag{2.11}$$

On the other hand, random walks with diagonal steps in $\mathcal{A}_m^{\tilde{C}_n}$ are equivalent (by means of the translation explained earlier for random walks in $\mathcal{A}_m^{\tilde{A}_{n-1}}$) to the movements of n non-colliding particles in an interval, where at each tick of the clock each particle moves one unit step to the right or to the left (see also [19, Sec. 5]). Equivalently, these may seen as families of non-intersecting lattice paths consisting of up- and down-steps between two horizontal boundaries (see [19, Sec. 5] and [29]). The corresponding result from [19] is the following.

Theorem 7 ([19, Eq. (18)])**.** *Let m be a positive integer or half-integer. Furthermore, let $\eta = (\eta_1, \eta_2, \dots, \eta_n)$ and $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ be vectors of integers or of half-integers in the alcove $\mathcal{A}_m^{\tilde{C}_n}$ of type $\tilde{C}_n$ (defined in (2.2)). Then the number of random walks from η to λ with exactly k diagonal steps, which stay in the alcove $\mathcal{A}_m^{\tilde{C}_n}$, is given by*

$$\det_{1\le h,t\le n}\left(\frac{2^{k-1}}{m}\sum_{r=0}^{4m-1}\sin\frac{\pi r\lambda_t}{m}\cdot\sin\frac{\pi r\eta_h}{m}\cdot\cos^k\frac{\pi r}{2m}\right). \tag{2.12}$$

The next two results concern the enumeration of random walks in alcoves of type $\tilde{B}_n$. While the first result, the result for standard steps, is stated explicitly in [19], the second, the result for diagonal steps, is not made explicit there, although it is made clear how to derive it. We state it here for the sake of completeness. Again, there is no separate result for positive standard steps, since for such walks the condition $2m > x_1 + x_2$, which appears in the definition (2.3) of the alcove $\mathcal{A}_m^{\tilde{B}_n}$, is without meaning, as well as the condition $x_n > 0$, so that the problem of enumerating random walks with positive standard steps between two given points which stay in $\mathcal{A}_m^{\tilde{B}_n}$ is again equivalent to counting random walks with positive standard steps which stay in the Weyl chamber of type A_{n-1}, the latter being defined by (2.10).

Theorem 8 ([19, Eq. (43)])**.** *Let m be a positive integer or half-integer. Furthermore, let $\eta = (\eta_1, \eta_2, \dots, \eta_n)$ and $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ be vectors of integers in the alcove $\mathcal{A}_m^{\tilde{B}_n}$ of type $\tilde{B}_n$ (defined in (2.3)). Then the number of random walks from η to λ with exactly k standard steps, which stay in the alcove $\mathcal{A}_m^{\tilde{B}_n}$, is given by the coefficient of $x^k/k!$ in*

$$\frac{1}{2}\det_{1\le h,t\le n}\left(\frac{1}{m}\sum_{r=0}^{2m-1}\sin\frac{\pi r\lambda_t}{m}\cdot\sin\frac{\pi r\eta_h}{m}\cdot\exp\left(2x\cos\frac{\pi r}{m}\right)\right)$$

$$+\frac{1}{2}\det_{1\le h,t\le n}\left(\frac{1}{m}\sum_{r=0}^{2m-1}\sin\frac{\pi(2r+1)\lambda_t}{2m}\cdot\sin\frac{\pi(2r+1)\eta_h}{2m}\cdot\exp\left(2x\cos\frac{\pi(2r+1)}{2m}\right)\right). \tag{2.13}$$

Theorem 9 ([19]). *Let m be a positive integer or half-integer. Furthermore, let $\eta=(\eta_1,\eta_2,\dots,\eta_n)$ and $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_n)$ be vectors of integers or of half-integers in the alcove $\mathcal{A}_m^{\tilde B_n}$ of type $\tilde B_n$ (defined in (2.3)). Then the number of random walks from η to λ with exactly k diagonal steps, which stay in the alcove $\mathcal{A}_m^{\tilde B_n}$, is given by*

$$\frac{1}{2}\det_{1\le h,t\le n}\left(\frac{2^{k-1}}{m}\sum_{r=0}^{4m-1}\sin\frac{\pi r\lambda_t}{m}\cdot\sin\frac{\pi r\eta_h}{m}\cdot\cos^k\frac{\pi r}{2m}\right)$$
$$+\frac{1}{2}\det_{1\le h,t\le n}\left(\frac{2^{k-1}}{m}\sum_{r=0}^{4m-1}\sin\frac{\pi(2r+1)\lambda_t}{2m}\cdot\sin\frac{\pi(2r+1)\eta_h}{2m}\cdot\cos^k\frac{\pi(2r+1)}{4m}\right). \tag{2.14}$$

The final two results concern the enumeration of random walks in alcoves of type $\tilde D_n$. Again, the first result, the result for standard steps, is stated explicitly in [19], while the second, the result for diagonal steps, is not made explicit there, although, again, it is clearly described how to derive it. We state it here for the sake of completeness. Also here, there is no separate result for positive standard steps, since for such walks the condition $2m>x_1+x_2$, which appears in the definition (2.4) of the alcove $\mathcal{A}_m^{\tilde D_n}$, is without meaning, as well as the condition $x_{n-1}>|x_n|$, so that the problem of enumerating random walks with positive standard steps between two given points which stay in $\mathcal{A}_m^{\tilde D_n}$ is again equivalent to counting random walks with positive standard steps which stay in the Weyl chamber of type A_{n-1}, the latter being defined by (2.10).

Theorem 10 ([19, Eq. (46)]). *Let m be a positive integer or half-integer. Furthermore, let $\eta=(\eta_1,\eta_2,\dots,\eta_n)$ and $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_n)$ be vectors of integers in the alcove $\mathcal{A}_m^{\tilde D_n}$ of type $\tilde D_n$ (defined in (2.4)). Then the number of random walks from η to λ with exactly k standard steps, which stay in the alcove $\mathcal{A}_m^{\tilde D_n}$, is given by the coefficient of $x^k/k!$ in*

$$\frac{1}{4}\det_{1\le h,t\le n}\left(\frac{1}{m}\sum_{r=0}^{2m-1}\sin\frac{\pi r\lambda_t}{m}\cdot\sin\frac{\pi r\eta_h}{m}\cdot\exp\left(2x\cos\frac{\pi r}{m}\right)\right)$$
$$+\frac{1}{4}\det_{1\le h,t\le n}\left(\frac{1}{m}\sum_{r=0}^{2m-1}\sin\frac{\pi(2r+1)\lambda_t}{2m}\cdot\sin\frac{\pi(2r+1)\eta_h}{2m}\cdot\exp\left(2x\cos\frac{\pi(2r+1)}{2m}\right)\right)$$
$$+\frac{1}{4}\det_{1\le h,t\le n}\left(\frac{1}{m}\sum_{r=0}^{2m-1}\cos\frac{\pi r\lambda_t}{m}\cdot\cos\frac{\pi r\eta_h}{m}\cdot\exp\left(2x\cos\frac{\pi r}{m}\right)\right)$$
$$+\frac{1}{4}\det_{1\le h,t\le n}\left(\frac{1}{m}\sum_{r=0}^{2m-1}\cos\frac{\pi(2r+1)\lambda_t}{2m}\cdot\cos\frac{\pi(2r+1)\eta_h}{2m}\cdot\exp\left(2x\cos\frac{\pi(2r+1)}{2m}\right)\right). \tag{2.15}$$

Theorem 11 ([19]). *Let m be a positive integer or half-integer. Furthermore, let $\eta=(\eta_1,\eta_2,\dots,\eta_n)$ and $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_n)$ be vectors of integers or of half-integers in the* *alcove $\mathcal{A}_m^{\tilde D_n}$ of type $\tilde D_n$ (defined in (2.4)). Then the number of random walks from η to λ with exactly k diagonal steps, which stay in the alcove $\mathcal{A}_m^{\tilde D_n}$, is given by*

$$\frac{1}{4}\det_{1\le h,t\le n}\left(\frac{2^{k-1}}{m}\sum_{r=0}^{4m-1}\sin\frac{\pi r\lambda_t}{m}\cdot\sin\frac{\pi r\eta_h}{m}\cdot\cos^k\frac{\pi r}{2m}\right)$$
$$+\frac{1}{4}\det_{1\le h,t\le n}\left(\frac{2^{k-1}}{m}\sum_{r=0}^{4m-1}\sin\frac{\pi(2r+1)\lambda_t}{2m}\cdot\sin\frac{\pi(2r+1)\eta_h}{2m}\cdot\cos^k\frac{\pi(2r+1)}{4m}\right)$$
$$+\frac{1}{4}\det_{1\le h,t\le n}\left(\frac{2^{k-1}}{m}\sum_{r=0}^{4m-1}\cos\frac{\pi r\lambda_t}{m}\cdot\cos\frac{\pi r\eta_h}{m}\cdot\cos^k\frac{\pi r}{2m}\right)$$
$$+\frac{1}{4}\det_{1\le h,t\le n}\left(\frac{2^{k-1}}{m}\sum_{r=0}^{4m-1}\cos\frac{\pi(2r+1)\lambda_t}{2m}\cdot\cos\frac{\pi(2r+1)\eta_h}{2m}\cdot\cos^k\frac{\pi(2r+1)}{4m}\right). \tag{2.16}$$

3. Asymptotics for random walks in alcoves of type $\tilde A$

This section is devoted to finding the asymptotic behaviour of the number of walks from a given starting point to a given end point which stay in the alcove $\mathcal{A}_m^{\tilde A_{n-1}}$ of type $\tilde A_{n-1}$ as the number of steps becomes large, as well as the asymptotic behaviour of the number of those walks which start at a given point but may terminate anywhere. In technical terms, we determine the asymptotic behaviour of the expressions given by Theorems 1–3 as k becomes large (in the case of Theorem 1 the role of k is played by $|\lambda|-|\eta|$), and as well if these expressions are summed over all possible end points of the walks.

We begin with the walks with standard steps in the positive direction.

Theorem 12. *Let m be a positive integer. Furthermore, let $\eta=(\eta_1,\eta_2,\dots,\eta_n)$ and $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_n)$ be vectors of integers in the alcove $\mathcal{A}_m^{\tilde A_{n-1}}$ of type $\tilde A_{n-1}$ (defined in (2.1)). Then, for large $|\lambda|$, the number of random walks from η to λ, which consist entirely of standard steps in the positive direction, and which stay in the alcove $\mathcal{A}_m^{\tilde A_{n-1}}$, is asymptotically*

$$\frac{2^{n^2-n}}{m^{n-1}}\left(\frac{\sin\frac{n\pi}{m}}{\sin\frac{\pi}{m}}\right)^{|\lambda|-|\eta|}\prod_{1\le h<t\le n}\left(\sin\frac{\pi(\eta_h-\eta_t)}{m}\cdot\sin\frac{\pi(\lambda_h-\lambda_t)}{m}\right). \tag{3.1}$$

Proof. We want to estimate the expression (2.5) for $|\lambda|$ large. In order to accomplish that, we write it in the form

$$\left\langle z^0\right\rangle(|\lambda|-|\eta|)!\sum_{k_1,\dots,k_n=-\infty}^{\infty}z^{m\sum_{j=1}^n k_j}\det_{1\le h,t\le n}\left(\frac{1}{(\lambda_t-\eta_h+mk_h)!}\right)$$
$$=\left\langle z^0\right\rangle(|\lambda|-|\eta|)!\det_{1\le h,t\le n}\left(\sum_{k_h=-\infty}^{\infty}\frac{z^{mk_h}}{(\lambda_t-\eta_h+mk_h)!}\right),$$

where here, and in the sequel, the notation $\langle f\rangle F$ stands for the coefficient of f in (an appropriate expansion of) F. (I.e., here, $\langle z^0\rangle\, g(z)$ denotes the constant coefficient in the Laurent series $g(z)$.) With $\omega = e^{2\pi i/m}$, we can rewrite this expression as

$$\left\langle z^0\right\rangle (|\lambda|-|\eta|)! \det_{1\le h,t\le n}\left(\frac{1}{m}\sum_{r_h=0}^{m-1}\sum_{k_h=-\infty}^{\infty}\frac{(z\omega^{r_h})^{k_h}}{(\lambda_t-\eta_h+k_h)!}\right),$$

because $\sum_{r=0}^{m-1}\omega^{rk}$ is equal to m if k is divisible by m, and it vanishes otherwise. Evaluating the sum over k_h, we obtain the expression

$$\begin{aligned}\left\langle z^0\right\rangle (|\lambda|-|\eta|)! &\det_{1\le h,t\le n}\left(\frac{1}{m}\sum_{r_h=0}^{m-1}(z\omega^{r_h})^{\eta_h-\lambda_t}\exp(z\omega^{r_h})\right)\\ &= \left\langle z^{|\lambda|-|\eta|}\right\rangle \frac{(|\lambda|-|\eta|)!}{m^n}\sum_{r_1,\dots,r_n=0}^{m-1}\exp\left(z\sum_{j=1}^{n}\omega^{r_j}\right)\det_{1\le h,t\le n}\left(\omega^{r_h(\eta_h-\lambda_t)}\right)\\ &= \frac{1}{m^n}\sum_{r_1,\dots,r_n=0}^{m-1}W(\mathbf{r})^{|\lambda|-|\eta|}\det_{1\le h,t\le n}\left(\omega^{r_h(\eta_h-\lambda_t)}\right), \end{aligned} \tag{3.2}$$

where in the last line $W(\mathbf{r})$ is an abbreviation for $\sum_{j=1}^{n}\omega^{r_j}$.

The sum in (3.2) is a finite sum of the form $\sum_\ell c_\ell b_\ell^k$, with $k=|\lambda|-|\eta|$, where the b_ℓ's are of the form $W(\mathbf{r})$, and the c_ℓ's are bounded. Thus, the asymptotic behaviour of this sum as $k\to\infty$ is dominated by the terms $c_\ell b_\ell^k$ for which $|b_\ell|$ is maximal (and $c_\ell\ne 0$ of course).

Now, if two summation indices r_h and r_t, for $h\ne t$, should be equal, then the determinant in the summand in (3.2) vanishes. Therefore we may restrict the sum in (3.2) to indices $r_1,r_2,\dots,r_n$ which are pairwise distinct. Among the latter, the sets of indices $\{r_1,r_2,\dots,r_n\}$ for which $W(\mathbf{r})$ has largest modulus are those for which the r_j's are as "close" together as possible, i.e., the sets

$$\{r_1,r_2,\dots,r_n\}=\{\ell,\ell+1,\dots,\ell+n-1\}, \tag{3.3}$$

for some ℓ between 0 and $m-1$. (On the right-hand side, the elements must be reduced modulo m.) Hence, let $r_j=\ell+\sigma(j)-1$, $j=1,2,\dots,n$, for some permutation $\sigma\in S_n$, with S_n denoting the symmetric group of order n. For this choice of indices, we have

$$\begin{aligned} W(\mathbf{r})^{|\lambda|-|\eta|}&\det_{1\le h,t\le n}\left(\omega^{r_h(\eta_h-\lambda_t)}\right)\\ &= \left|\sum_{j=1}^{n}\omega^{j-1}\right|^{|\lambda|-|\eta|}\omega^{\frac{n-1}{2}(|\lambda|-|\eta|)}\det_{1\le h,t\le n}\left(\omega^{(\sigma(h)-1)(\eta_h-\lambda_t)}\right)\\ &= \left(\frac{\sin\frac{n\pi}{m}}{\sin\frac{\pi}{m}}\right)^{|\lambda|-|\eta|}\omega^{\frac{n-1}{2}(|\lambda|-|\eta|)}\omega^{\sum_{j=1}^{n}(\sigma(j)-1)\eta_j}\det_{1\le h,t\le n}\left(\omega^{-(\sigma(h)-1)\lambda_t}\right). \end{aligned}$$

Thus, if we combine all our findings, we obtain that, as $(|\lambda|-|\eta|)\to\infty$, the expression (3.2) is asymptotically

$$\frac{1}{m^n}\left(\frac{\sin\frac{n\pi}{m}}{\sin\frac{\pi}{m}}\right)^{|\lambda|-|\eta|}\sum_{\ell=0}^{m-1}\sum_{\sigma\in S_n}\omega^{\frac{n-1}{2}(|\lambda|-|\eta|)}\omega^{\sum_{j=1}^{n}(\sigma(j)-1)\eta_j}(\operatorname{sgn}\sigma)\det_{1\le h,t\le n}\left(\omega^{-(h-1)\lambda_t}\right)$$

$$= \frac{1}{m^{n-1}} \left(\frac{\sin \frac{n\pi}{m}}{\sin \frac{\pi}{m}} \right)^{|\lambda|-|\eta|} \omega^{\frac{n-1}{2}(|\lambda|-|\eta|)} \det_{1\le h,t\le n} \left(\omega^{(h-1)\eta_t}\right) \det_{1\le h,t\le n} \left(\omega^{-(h-1)\lambda_t}\right).$$

Both determinants are Vandermonde determinants, and are therefore easily evaluated. The resulting expression is exactly (3.1). □

Next we address the question of determining the asymptotic behaviour of the number of *all* walks which start in a given point and proceed for k standard steps in the positive direction, always staying in the alcove $\mathcal{A}_m^{\tilde{A}_{n-1}}$ of type $\tilde{A}_{n-1}$. Clearly, this amounts to summing the expression (3.1) over all λ with $|\lambda| = |\eta| + k$. A moment's reflection shows that this is in fact a finite sum, with the number of terms bounded by $(2m)^m$ (to give a very crude bound), a quantity which is independent of k. Thus, it is obvious that the order of magnitude of the number of all these walks is $\left(\sin \frac{n\pi}{m} / \sin \frac{\pi}{m}\right)^k$. However, determining the multiplicative constant poses a formidable challenge, in particular if one attempts to do it directly from summing up the expression (3.1). An elegant way to bypass (some of this) difficulty is to set up a relationship between the enumeration of walks with positive standard steps and the enumeration of walks with arbitrary standard steps, which we do in Lemma 14.

Theorem 13. *Let m be a positive integer. Furthermore, let $\eta = (\eta_1, \eta_2, \dots, \eta_n)$ be a vector of integers in the alcove $\mathcal{A}_m^{\tilde{A}_{n-1}}$ of type $\tilde{A}_{n-1}$ (defined in (2.1)). Then, as k tends to infinity, the number of random walks which start at η and proceed for exactly k standard steps in the positive direction, which stay in the alcove $\mathcal{A}_m^{\tilde{A}_{n-1}}$, is asymptotically*

$$\frac{2^{\binom{n}{2}}}{m^{n/2}} \left(\frac{\sin \frac{n\pi}{m}}{\sin \frac{\pi}{m}} \right)^k \prod_{1\le h<t\le n} \left(\sin \frac{\pi(\eta_h - \eta_t)}{m} \right) \times \left(\prod_{h=1}^{n/2} \cot \frac{(2h-1)\pi}{2m} + (-1)^{|\eta|+k+\frac{n}{2}} \prod_{h=1}^{n/2} \tan \frac{(2h-1)\pi}{2m} \right) \tag{3.4}$$

if both n and m are even, it is asymptotically

$$\frac{2^{\binom{n}{2}}}{m^{n/2}} \left(\frac{\sin \frac{n\pi}{m}}{\sin \frac{\pi}{m}} \right)^k \prod_{1\le h<t\le n} \left(\sin \frac{\pi(\eta_h - \eta_t)}{m} \right) \prod_{h=1}^{n/2} \cot \frac{(2h-1)\pi}{2m} \tag{3.5}$$

if n is even and m is odd, and it is asymptotically

$$\frac{2^{\binom{n}{2}}}{m^{(n-1)/2}} \left(\frac{\sin \frac{n\pi}{m}}{\sin \frac{\pi}{m}} \right)^k \prod_{1\le h<t\le n} \left(\sin \frac{\pi(\eta_h - \eta_t)}{m} \right) \prod_{h=1}^{(n-1)/2} \cot \frac{h\pi}{m} \tag{3.6}$$

if n is odd (regardless of m).

Proof. Our original proof proceeded as indicated in the paragraph before the statement of the theorem, namely by summing the expression (3.1) over all possible λ with $|\lambda| = |\eta| + k$. Although feasible, this path turned out to be a thorny one. After the result had been obtained, the surprising observation was that the asymptotic behaviour of walks *in $\mathcal{A}_m^{\tilde{A}_{n-1}}$ with positive standard steps* is, up to a factor of 2^k, identical with the asymptotic behaviour of walks *on the m-circle with arbitrary* (i.e., positive *and* negative) *standard steps* (cf. Theorem 18). As we show in Lemma 14 below, this is even true *non-asymptotically.* (This is indeed the assertion of Lemma 14 since it does not matter whether we are in the alcove $\mathcal{A}_m^{\tilde{A}_{n-1}}$ or on the m-circle if we let the end point of the walks be arbitrary.) The theorem now follows by the (independent) proof of Theorem 18 given in Section 4. □

Lemma 14. *The number of random walks which start at η, proceed for exactly k standard steps in the positive direction, and stay in the alcove $\mathcal{A}_m^{\tilde{A}_{n-1}}$, is equal to 2^{-k} times the number of random walks which start at η, proceed for exactly k standard steps, and stay in the alcove $\mathcal{A}_m^{\tilde{A}_{n-1}}$.*

Proof. We are going to show this inductively. Let $\eta = (\eta_1, \eta_2, \dots, \eta_n)$ be given. We decompose η into its maximal *circular* subsequences of consecutive elements. The meaning of "circular" is that η_n and η_1 are considered to be "consecutive" if $\eta_n = \eta_1 - m + 1$. In the sequel, we shall use the short hand *maximal circular subsequences* for these subsequences. For example, if $m = 11$ there are 3 maximal circular subsequences in $(9, 8, 5, 4, 3, 1)$, namely

$$(9, 8),\ (5, 4, 3),\ (1). \tag{3.7}$$

On the other hand, if $m = 9$, then the maximal circular subsequences are

$$(1, 9, 8),\ (5, 4, 3). \tag{3.8}$$

Let the maximal circular subsequences in the decomposition of η have respectively the lengths $a_1, a_2, \dots, a_\ell$, with gaps $b_1, b_2, \dots, b_\ell$. More precisely, the gap b_j is the difference between the smallest element in the j-th sub-sequence and the largest element in the $(j+1)$-st sub-sequence, reduced modulo m. Here, $(j+1)$ has to be interpreted as 1 if $j = \ell$, Thus, in the example (3.7) we have $a_1 = 2$, $a_2 = 3$, $a_3 = 1$, and $b_1 = 3$, $b_2 = 2$, $b_3 = 3$, while in the example (3.8) we have $a_1 = 3$, $a_2 = 3$, and $b_1 = 3$, $b_2 = 2$.

It is obvious that, starting from such an η, there are exactly ℓ ways to move by a positive standard step. To be precise, one would increase the largest element of a maximal circular sub-sequence by 1. Similarly, there are exactly 2ℓ ways to move by an *arbitrary* standard step, namely the ℓ possibilities of *positive* standard steps described above, together with the ℓ possibilities of decreasing a *smallest* element of a maximal circular sub-sequence by 1. Thus, for one step, i.e., for $k = 1$, our claim is true.

The question is whether this persists. As we have seen, the number of possibilities to walk is (for *positive* standard steps, as well as for *arbitrary* standard steps) a multiple of the number of maximal circular subsequences. Thus, if we are able to show that the total number of maximal circular sub-sequences in the set of all possible points that we reached from η by walking one positive standard step is exactly one half of the corresponding number of maximal circular sub-sequences in the set of all possible points that we reached from η by walking one arbitrary standard step, then we are sure that the ratio of $1:2$ will continue to hold for each step.

The latter claim is easy to establish: if we increase the largest element of the j-th maximal circular sub-sequence of η by 1, then we obtain a point whose decomposition has

$$\ell + 1 - \chi(a_j = 1) - \chi(b_{j-1} = 2)$$

maximal circular subsequences, where we used the notation $\chi(\mathcal{A})$=1 if $\mathcal{A}$ is true and $\chi(\mathcal{A})$=0 otherwise, and where b_0 has to be interpreted as b_ℓ. On the other hand, if we decrease the smallest element of the j-th maximal circular sub-sequence of η by 1, then we obtain a point whose decomposition has

$$\ell + 1 - \chi(a_j = 1) - \chi(b_j = 2)$$

maximal circular subsequences. Thus, the total number of maximal circular subsequences in the set of points reached from η by walking one *positive* standard step is

$$\ell(\ell+1) - \sum_{j=1}^{\ell} \big(\chi(a_j = 1) + \chi(b_j = 1)\big),$$

while the total number of maximal circular subsequences in the set of points reached from η by walking one *arbitrary* standard step is exactly twice of that. This finishes the proof of the lemma. □

Now we turn our attention to walks with positive *and* negative standard steps. The theorem below gives the asymptotic behaviour of the walks in $\mathcal{A}_m^{\tilde{A}_{n-1}}$ with fixed starting and end point. If the end point is allowed to be arbitrary, then the enumeration of the corresponding walks is equivalent to the enumeration of walks with standard steps on the m-circle, and, thus, its asymptotic behaviour is given by Theorem 18 in Section 4.

Theorem 15. *Let m be a positive integer. Furthermore, let $\eta = (\eta_1, \eta_2, \dots, \eta_n)$ and $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ be vectors of integers in the alcove $\mathcal{A}_m^{\tilde{A}_{n-1}}$ of type $\tilde{A}_{n-1}$ (defined in (2.1)). Then, as k tends to infinity such that $k \equiv |\eta| + |\lambda|$ mod 2, the number of random walks from η to λ with exactly k standard steps, which stay in the alcove $\mathcal{A}_m^{\tilde{A}_{n-1}}$, is asymptotically*

$$\frac{2^{n^2-n}}{m^{n-1}} \sqrt{\frac{2}{\pi k}} \left(2\frac{\sin\frac{n\pi}{m}}{\sin\frac{\pi}{m}}\right)^k \prod_{1\le h<t\le n} \left(\sin\frac{\pi(\eta_h - \eta_t)}{m} \cdot \sin\frac{\pi(\lambda_h - \lambda_t)}{m}\right). \tag{3.9}$$

Proof. We have to determine the asymptotics of the coefficient of $x^k/k!$ in (2.6) as $k \to \infty$. To begin with, we rewrite this coefficient as

$$\left\langle z^0 \frac{x^k}{k!}\right\rangle \sum_{k_1,\dots,k_n=-\infty}^{\infty} z^{m(k_1+\dots+k_n)} \det_{1\le h,t\le n} \left(I_{\lambda_t - \eta_h + mk_h}(2x)\right)$$
$$= \left\langle z^0 \frac{x^k}{k!}\right\rangle \det_{1\le h,t\le n} \left(\sum_{k_h=-\infty}^{\infty} I_{\lambda_t - \eta_h + mk_h}(2x)\, z^{mk_h}\right),$$

Here, $\langle z^0 \frac{x^k}{k!}\rangle g(x,z)$ denotes the coefficient of $z^0 x^k/k!$ in $g(x,z)$, which is in accordance with our earlier general definition of the coefficient notation. As in the proof of Theorem 12, with $\omega = e^{2\pi i/m}$ we may rewrite this expression as

$$\left\langle z^0 \frac{x^k}{k!}\right\rangle \det_{1\le h,t\le n} \left(\frac{1}{m}\sum_{r_h=0}^{m-1} \sum_{k_h=-\infty}^{\infty} I_{\lambda_t - \eta_h + k_h}(2x)\, (\omega^{r_h} z)^{k_h}\right).$$

Now using the easily verified fact that

$$\sum_{j=-\infty}^{\infty} I_j(2x)\, z^j = \exp\left(x\left(z + z^{-1}\right)\right),$$

we obtain

$$\left\langle z^0\,\frac{x^k}{k!}\right\rangle \det_{1\le h,t\le n}\left(\frac{1}{m}\left(\omega^{r_h}z\right)^{\eta_h-\lambda_t}\sum_{r_h=0}^{m-1}\exp\left(x\left(\omega^{r_h}z+(\omega^{r_h}z)^{-1}\right)\right)\right)$$
$$=\left\langle z^{|\lambda|-|\eta|}\,\frac{x^k}{k!}\right\rangle\frac{1}{m^n}\sum_{r_1,\dots,r_n=0}^{m-1}\det_{1\le h,t\le n}\left(\omega^{r_h(\eta_h-\lambda_t)}\right)$$
$$\cdot\exp\left(x\left(z\textstyle\sum_{j=1}^n\omega^{r_j}+z^{-1}\sum_{j=1}^n\omega^{-r_j}\right)\right)$$
$$=\left\langle z^{|\lambda|-|\eta|}\right\rangle\frac{1}{m^n}\sum_{r_1,\dots,r_n=0}^{m-1}\det_{1\le h,t\le n}\left(\omega^{r_h(\eta_h-\lambda_t)}\right)\cdot\left(z\textstyle\sum_{j=1}^n\omega^{r_j}+z^{-1}\sum_{j=1}^n\omega^{-r_j}\right)^k$$
$$=\frac{1}{m^n}\binom{k}{\frac12\big(k+|\lambda|-|\eta|\big)}\sum_{r_1,\dots,r_n=0}^{m-1}\det_{1\le h,t\le n}\left(\omega^{r_h(\eta_h-\lambda_t)}\right)$$
$$\cdot\left(\sum_{j=1}^n\omega^{r_j}\right)^{\frac12(k+|\lambda|-|\eta|)}\left(\sum_{j=1}^n\omega^{-r_j}\right)^{\frac12(k-|\lambda|+|\eta|)}$$

for the coefficient of $x^k/k!$ in (2.6). Writing again $W(\mathbf{r})$ for $\sum_{j=1}^n\omega^{r_j}$, we have obtained that the coefficient of $x^k/k!$ in (2.6) is equal to

$$\frac{1}{m^n}\binom{k}{\frac12\big(k+|\lambda|-|\eta|\big)}$$
$$\times\sum_{r_1,\dots,r_n=0}^{m-1}|W(\mathbf{r})|^k\left(W(\mathbf{r})\Big/\overline{W(\mathbf{r})}\right)^{\frac12(|\lambda|-|\eta|)}\det_{1\le h,t\le n}\left(\omega^{r_h(\eta_h-\lambda_t)}\right).\quad(3.10)$$

Stirling's formula implies that the binomial coefficient in this expression is asymptotically $2^k\sqrt{\frac{2}{k\pi}}$ as $k\to\infty$.

The sum in (3.10), on the other hand, is again a finite sum of the form $\sum_\ell c_\ell b_\ell^k$, with the b_ℓ's of the form $|W(\mathbf{r})|$, and both the c_ℓ's and b_ℓ's independent of k. Thus, the asymptotic behaviour of this sum as $k\to\infty$ is dominated by the terms $c_\ell b_\ell^k$ for which $|b_\ell|$ is maximal (and $c_\ell\ne0$ of course).

If we compare the expression (3.10) that we have obtained so far with the expression (3.2), then we see that we are in a very similar situation here as at the analogous place in the proof of Theorem 12. Therefore, if we apply the arguments given there to our situation, we obtain that, as $k\to\infty$, the expression (3.10) is asymptotically

$$\frac{1}{m^n}\sqrt{\frac{2}{\pi k}}\left(2\frac{\sin\frac{n\pi}{m}}{\sin\frac{\pi}{m}}\right)^k\sum_{\ell=0}^{m-1}\sum_{\sigma\in S_n}\omega^{\frac{n-1}{2}(|\lambda|-|\eta|)}\omega^{\sum_{j=1}^n(\sigma(j)-1)\eta_j}(\operatorname{sgn}\sigma)\det_{1\le h,t\le n}\left(\omega^{-(\sigma(h)-1)\lambda_t}\right)$$
$$=\frac{1}{m^{n-1}}\sqrt{\frac{2}{\pi k}}\left(2\frac{\sin\frac{n\pi}{m}}{\sin\frac{\pi}{m}}\right)^k\omega^{\frac{n-1}{2}(|\lambda|-|\eta|)}\det_{1\le h,t\le n}\left(\omega^{(h-1)\eta_t}\right)\det_{1\le h,t\le n}\left(\omega^{-(h-1)\lambda_t}\right).$$

Both determinants are Vandermonde determinants, and are therefore easily evaluated. The resulting expression is exactly (3.9). □

As the final issue in this section, we consider the asymptotic behaviour of walks in the alcove $\mathcal{A}_m^{\tilde A_{n-1}}$ consisting of diagonal steps. The theorem below provides the solution of the problem if the starting and end point are fixed. Exceptionally, to resolve this problem, we need to apply a more advanced asymptotic method, the saddle point method (although a rather basic instance of it). Should the end point be allowed to be arbitrary, then the corresponding enumeration problem is equivalent to the enumeration of walks with diagonal steps on the m-circle, and, thus, its asymptotic behaviour is given by Theorem 20 in Section 4.

Theorem 16. *Let m be a positive integer or half-integer. Furthermore, let $\eta = (\eta_1, \eta_2, \dots, \eta_n)$ and $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ be vectors of integers or of half-integers in the alcove $\mathcal{A}_m^{\tilde A_{n-1}}$ of type $\tilde A_{n-1}$ (defined in (2.1)). Then, as k tends to infinity such that $k \equiv 2\eta_j + 2\lambda_j \bmod 2$, the number of random walks from η to λ with exactly k diagonal steps, which stay in the alcove $\mathcal{A}_m^{\tilde A_{n-1}}$, is asymptotically*

$$\frac{1}{m^{n-1}} \frac{2^{n^2-n}}{\sqrt{2\pi c_0 k}} \left(2^n \prod_{j=1}^{n} \cos \frac{\pi\left(j - \frac{n+1}{2}\right)}{m} \right)^k \prod_{1\le h<t\le n} \left(\sin \frac{\pi(\eta_h-\eta_t)}{m} \cdot \sin \frac{\pi(\lambda_h-\lambda_t)}{m} \right), \tag{3.11}$$

where

$$c_0 = \sum_{j=1}^{n} \left(2\cos \frac{\pi\left(j-\frac{n+1}{2}\right)}{m} \right)^{-2}.$$

Proof. We have to determine the asymptotics of the expression (2.7) as $k \to \infty$. To begin with, we write (2.7) in the form

$$\begin{aligned} \left\langle z^0 \right\rangle \sum_{k_1,\dots,k_n=-\infty}^{\infty} z^{m(k_1+\dots+k_n)} \det_{1\le h,t\le n} \left(\binom{k}{\frac{k}{2}+\lambda_t-\eta_h+mk_h} \right) \\ = \left\langle z^0 \right\rangle \det_{1\le h,t\le n} \left(\sum_{k_h=-\infty}^{\infty} \binom{k}{\frac{k}{2}+\lambda_t-\eta_h+mk_h} z^{mk_h} \right), \end{aligned}$$

where, again, the notation $\langle z^0\rangle g(z)$ denotes the coefficient of z^0 in $g(z)$. Arguing in the same way as before in the proofs of Theorems 12 and 15, with $\omega = e^{2\pi i/m}$ we can rewrite this expression as

$$\begin{aligned} &\left\langle z^0 \right\rangle \det_{1\le h,t\le n} \left(\frac{1}{m} \sum_{r_h=0}^{m-1} \sum_{k_h=-\infty}^{\infty} \binom{k}{\frac{k}{2}+\lambda_t-\eta_h+k_h} (\omega^{r_h} z)^{k_h} \right) \\ &= \left\langle z^0 \right\rangle \frac{1}{m^n} \det_{1\le h,t\le n} \left(\sum_{r_h=0}^{m-1} (\omega^{r_h} z)^{-\frac{k}{2}-\lambda_t+\eta_h} (1+\omega^{r_h} z)^k \right) \\ &= \frac{1}{m^n} \sum_{r_1,\dots,r_n=0}^{m-1} \omega^{-\frac{k}{2}|\mathbf{r}|} \det_{1\le h,t\le n} \left(\omega^{r_h(\eta_h-\lambda_t)} \right) \left\langle z^{\frac{nk}{2}+|\lambda|-|\eta|} \right\rangle \prod_{j=1}^{n} (1+\omega^{r_j} z)^k, \end{aligned} \tag{3.12}$$

where in the next-to-last line we used the binomial theorem.

We have again obtained a finite sum. Therefore the task now is to isolate the summands which are asymptotically largest as $k \to \infty$. First of all, if two summation indices

r_h and r_t, for $h \neq t$, should be equal, then the determinant in the summand in (3.12) vanishes. Therefore we may restrict the sum in (3.12) to the summands corresponding to indices $r_1, r_2, \dots, r_n$ which are pairwise distinct.

According to Lemma A in Appendix A, among the latter, those will be asymptotically largest for which

$$\prod_{j=1}^{n} \left| \cos \frac{\pi(\theta_0 + r_j)}{m} \right| \tag{3.13}$$

is largest (where θ_0 is a solution of (A.4)). As is not difficult to see, these are those sets of indices $\{r_1, r_2, \dots, r_n\}$ for which the r_j's are as "close" together as possible, i.e., again the sets as given in (3.3), for some ℓ between 0 and $m-1$.

Let ℓ be fixed. Let $\{r_1, r_2, \dots, r_n\}$ be a set of indices as in (3.3), i.e., $r_j = \ell + \sigma(j) - 1$, $j = 1, 2, \dots, n$, for some permutation $\sigma \in S_n$. For this set, there is a unique θ_0 such that (3.13) is maximal, namely $\theta_0 = -\ell - \frac{n-1}{2}$. Thus, using Lemma A in (3.12), we obtain that the expression (3.12) is asymptotically

$$\begin{aligned}\frac{1}{m^n} \sum_{\ell=0}^{m-1} \sum_{\sigma \in S_n} \det_{1 \le h,t \le n} \left(\omega^{(\sigma(h)-1)(\eta_h - \lambda_t)} \right) \frac{\omega^{\frac{n-1}{2}(|\lambda| - |\eta|)}}{\sqrt{2\pi c_0 k}} \prod_{j=1}^{n} \left(2 \cos \frac{\pi \left(j - \frac{n+1}{2} \right)}{m} \right)^k \\ = \frac{1}{m^{n-1}} \frac{\omega^{\frac{n-1}{2}(|\lambda| - |\eta|)}}{\sqrt{2\pi c_0 k}} \prod_{j=1}^{n} \left(2 \cos \frac{\pi \left(j - \frac{n+1}{2} \right)}{m} \right)^k \\ \times \det_{1 \le h,t \le n} \left(\omega^{(h-1)\eta_t} \right) \det_{1 \le h,t \le n} \left(\omega^{-(h-1)\lambda_t} \right),\end{aligned}$$

where c_0 is given as in the statement of the theorem. Again, both determinants are Vandermonde determinants, and are therefore easily evaluated. The resulting expression is exactly (3.11). □

4. Asymptotics for random walks on the circle

In this section we find the asymptotic behaviour of the number of walks from a given starting point to a given end point on the m-circle as the number of steps becomes large, as well as the asymptotic behaviour of the number of those walks which start at a given point but may terminate anywhere. In technical terms, we determine the asymptotic behaviour of the expressions given by Theorems 4 and 5 as k becomes large, and as well if these expressions are summed over all possible end points of the walks.

Before we state the next theorem, which gives the asymptotic behaviour of walks with standard steps between two fixed points on the m-circle, we need to discuss under which conditions such walks can exist. In the theorem below we consider walks from $\eta = (\eta_1, \eta_2, \dots, \eta_n)$ to $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ where $\lambda_{s+1} > \cdots > \lambda_n > \lambda_1 > \cdots > \lambda_s$, which means that, when we interpret such walks as the movements of n separate particles, the first s particles (the particles which start at $\eta_1, \eta_2, \dots, \eta_s$) wind themselves once more around the circle than the other particles. Thus, if we want to get from η to λ in k steps, we must have

$$k \equiv |\lambda| - |\eta| + Nnm + sm \pmod 2, \tag{4.1}$$

where the integer N is the number of times the latter particles wind around the circle. If m is even, then this condition reduces to the familiar $k \equiv |\lambda| + |\eta| \bmod 2$. However, if m is odd, then there are two possibilities. If in addition n is even, then, depending on whether k is even or odd, there are walks only for every second s. If both m and n are odd, then there is no restriction for k.

We are now ready to state the theorem. As we discussed in Section 2, this theorem gives at the same time the asymptotic behaviour of n non-colliding particles on the circle in the random turns vicious walker model.

Theorem 17. *Let m be a positive integer. Furthermore, let $\eta = (\eta_1, \eta_2, \dots, \eta_n)$ be a vector of integers with $m > \eta_1 > \eta_2 > \dots > \eta_n \ge 0$, and let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ be a vector of integers with $m > \lambda_{s+1} > \dots > \lambda_n > \lambda_1 > \dots > \lambda_s \ge 0$, for some s. Then, as k tends to infinity such that* (4.1) *holds for some integer N, the number of random walks on the m-circle from η to λ with exactly k standard steps, such that at no time two coordinates of a point on the random walk are equal, is asymptotically*

$$\frac{2^{n^2-n+1}}{nm^n} \left(2\frac{\sin\frac{n\pi}{m}}{\sin\frac{\pi}{m}}\right)^k \prod_{1\le h<t\le n} \left(\sin\frac{\pi(\eta_h-\eta_t)}{m} \cdot \left|\sin\frac{\pi(\lambda_h-\lambda_t)}{m}\right|\right) \tag{4.2}$$

if n is even, and as well if n is odd and m is even, and it is asymptotically

$$\frac{2^{n^2-n}}{nm^n} \left(2\frac{\sin\frac{n\pi}{m}}{\sin\frac{\pi}{m}}\right)^k \prod_{1\le h<t\le n} \left(\sin\frac{\pi(\eta_h-\eta_t)}{m} \cdot \left|\sin\frac{\pi(\lambda_h-\lambda_t)}{m}\right|\right) \tag{4.3}$$

if both n and m are odd.

Proof. We have to determine the asymptotic behaviour of the coefficient of $x^k/k!$ in (2.8). We expand the determinant by linearity in the rows and obtain

$$\frac{1}{nm^n}\sum_{u=0}^{n-1}\sum_{r_1,\dots,r_n=0}^{m-1} \exp\left(2x\sum_{j=1}^{n}\cos(2\pi(u+nr_j)/mn)\right) \\ \cdot e^{-2\pi ius/n} \det_{1\le h,t\le n}\left(e^{-2\pi i(u+nr_h)(\lambda_t-\eta_h)/mn}\right). \tag{4.4}$$

In this expression, we have to extract the coefficient of $x^k/k!$ to obtain the number of walks with exactly k steps. The expression that we obtain is a finite sum of the form $\sum_\ell c_\ell b_\ell^k$, with the b_ℓ's of the form $2\sum_{j=1}^n \cos(2\pi(u+nr_j)/mn)$, and both the c_ℓ's and b_ℓ's independent of k. Also in (4.4) there is the constraint that we must have $r_h \ne r_t$ for $h \ne t$ in order to obtain a non-vanishing summand, because otherwise the determinant is zero. It is not difficult to see that, because of that, the maximal modulus of such a b_ℓ is equal to

$$2\sum_{j=-n/2}^{n/2-1}\cos\frac{(2j+1)\pi}{m} = 4\sum_{j=0}^{n/2-1}\cos\frac{(2j+1)\pi}{m}$$

if n is even, and is equal to

$$2\sum_{j=-(n-1)/2}^{(n-1)/2}\cos\frac{2j\pi}{m} = 2+4\sum_{j=1}^{(n-1)/2}\cos\frac{2j\pi}{m}$$

if n is odd. Since we have

$$2\sum_{j=0}^{n/2-1}\cos\frac{(2j+1)\pi}{m}=\frac{\sin\frac{n\pi}{m}}{\sin\frac{\pi}{m}}$$

and also

$$1+2\sum_{j=1}^{(n-1)/2}\cos\frac{2j\pi}{m}=\frac{\sin\frac{n\pi}{m}}{\sin\frac{\pi}{m}},$$

this maximal modulus is $2\sin(n\pi/m)/\sin(\pi/m)$ in both cases.

If n is even, then the maximal modulus is attained by choosing

(e1) $u=n/2$ and $\{r_1,r_2,\dots,r_n\}=\{0,1,\dots,\frac{n}{2}-1,m-\frac{n}{2},\dots,m-2,m-1\}$, regardless of m;

(e2) $u=n/2$ and $\{r_1,r_2,\dots,r_n\}=\{\frac{m}{2}-\frac{n}{2},\frac{m}{2}-\frac{n}{2}+1,\dots,\frac{m}{2}+\frac{n}{2}-1\}$ if m is even;

(e3) $u=0$ and $\{r_1,r_2,\dots,r_n\}=\{\frac{m+1}{2}-\frac{n}{2},\frac{m+1}{2}-\frac{n}{2}+1,\dots,\frac{m+1}{2}+\frac{n}{2}-1\}$ if m is odd.

If n is odd, then the maximal modulus is attained by choosing

(o1) $u=0$ and $\{r_1,r_2,\dots,r_n\}=\{0,1,\dots,\frac{n-1}{2},m-\frac{n-1}{2},\dots,m-2,m-1\}$, regardless of m;

(o2) $u=0$ and $\{r_1,r_2,\dots,r_n\}=\{\frac{m}{2}-\frac{n-1}{2},\frac{m}{2}-\frac{n-1}{2}+1,\dots,\frac{m}{2}+\frac{n-1}{2}\}$ if m is even.

(If both n and m are odd, then there is no additional choice beyond (o1).)

Case (e1) yields the contribution

$$\frac{(-1)^s}{nm^n}\left(2\frac{\sin\frac{n\pi}{m}}{\sin\frac{\pi}{m}}\right)^k\det_{1\le h,t\le n}\left(e^{-2\pi i((n+1)/2-h)\lambda_t/m}\right)\det_{1\le h,t\le n}\left(e^{2\pi i((n+1)/2-t)\eta_h/m}\right)$$

to the asymptotics of (2.8). The two determinants are again Vandermonde-type determinants. They are therefore easily evaluated. Thus we obtain

$$\frac{2^{n^2-n}}{nm^n}\left(2\frac{\sin\frac{n\pi}{m}}{\sin\frac{\pi}{m}}\right)^k\prod_{1\le h<t\le n}\left(\sin\frac{\pi(\eta_t-\eta_h)}{m}\cdot\left|\sin\frac{\pi(\lambda_t-\lambda_h)}{m}\right|\right)\tag{4.5}$$

as the contribution of Case (e1) to the asymptotics. As similar computations show, the contributions of Cases (e2), (e3), (o1) and (o2) are also equal to (4.5). The claims (4.2) and (4.3) follow now upon adding up the corresponding terms in each case. $\square$

Now we are in the position to address the question of determining the asymptotic behaviour of the number of *all* walks on the m-circle which start in a given point and proceed for k standard steps (and, thus, at the same time for the number of n non-colliding particles on the circle in the random turns vicious walker model). Clearly, this amounts to summing the expressions given in Theorem 17 over all possible λ. Again, this is just a finite sum, with the number of terms bounded by nm^m, a quantity which is independent of k. Thus, it is obvious that the order of magnitude of the number of all these walks is $\left(2\sin\frac{n\pi}{m}/\sin\frac{\pi}{m}\right)^k$. In order to determine the multiplicative constant, we have to make use of identities featuring Schur functions and odd orthogonal characters.

Theorem 18. *Let m be a positive integer. Furthermore, let $\eta=(\eta_1,\eta_2,\dots,\eta_n)$ be a vector of integers with $m>\eta_1>\eta_2>\cdots>\eta_n\ge 0$. Then, as k tends to infinity, the number of random walks on the m-circle which start at η and proceed for exactly k*

standard steps, such that at no time two coordinates of a point on the random walk are equal, is asymptotically

$$\frac{2^{\binom n2}}{m^{n/2}}\left(2\frac{\sin\frac{n\pi}{m}}{\sin\frac{\pi}{m}}\right)^k\prod_{1\le h<t\le n}\left(\sin\frac{\pi(\eta_h-\eta_t)}{m}\right)\\ \times\left(\prod_{h=1}^{n/2}\cot\frac{(2h-1)\pi}{2m}+(-1)^{|\eta|+k+\frac n2}\prod_{h=1}^{n/2}\tan\frac{(2h-1)\pi}{2m}\right) \tag{4.6}$$

if both n and m are even, it is asymptotically

$$\frac{2^{\binom n2}}{m^{n/2}}\left(2\frac{\sin\frac{n\pi}{m}}{\sin\frac{\pi}{m}}\right)^k\prod_{1\le h<t\le n}\left(\sin\frac{\pi(\eta_h-\eta_t)}{m}\right)\prod_{h=1}^{n/2}\cot\frac{(2h-1)\pi}{2m} \tag{4.7}$$

if n is even and m is odd, and it is asymptotically

$$\frac{2^{\binom n2}}{m^{(n-1)/2}}\left(2\frac{\sin\frac{n\pi}{m}}{\sin\frac{\pi}{m}}\right)^k\prod_{1\le h<t\le n}\left(\sin\frac{\pi(\eta_h-\eta_t)}{m}\right)\prod_{h=1}^{(n-1)/2}\cot\frac{h\pi}{m} \tag{4.8}$$

if n is odd (regardless of m).

Proof. As we already explained, in view of Theorem 17, we have to compute the sum of (4.2), respectively of (4.3), over all possible choices $(\lambda_1,\lambda_2,\dots,\lambda_n)$. That is, we have to sum these expressions over all integers $\lambda_1,\lambda_2,\dots,\lambda_n$ with $m>\lambda_1>\lambda_2>\dots>\lambda_n\ge0$, and its cyclic permutations, *such that walks are possible from η to $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_n)$*, respectively to its cyclic permutations, *in k steps*. What the latter means, was discussed in the paragraph containing (4.1). Thus, we have to distinguish between several cases.

Before we list these cases, the reader should observe that the expressions (4.2) and (4.3) are invariant under cyclic permutations of $\lambda_1,\lambda_2,\dots,\lambda_n$. Thus, as the first step, we will multiply them by n, respectively by $n/2$, depending on whether *any* cyclic permutation of λ can be reached from η in k steps, or only *every second*.

Now, if both m and n are odd, then *any* λ, and *any* cyclic permutation of it can be reached from η if k is large enough. Thus, in this case, we have to multiply the expression (4.3) by n, and subsequently sum it over all integers $\lambda_1,\lambda_2,\dots,\lambda_n$ with $m>\lambda_1>\lambda_2>\dots>\lambda_n\ge0$.

If m is odd but n is even, then only *every second* cyclic permutation of a given λ can be reached from η for a given k which is large enough. Thus, in this case, we have to multiply the expression (4.2) by $n/2$, and subsequently sum it over all integers $\lambda_1,\lambda_2,\dots,\lambda_n$ with $m>\lambda_1>\lambda_2>\dots>\lambda_n\ge0$.

On the other hand, if m is even, then (4.1) implies that $|\lambda|\equiv|\eta|+k \bmod 2$. In particular, given a λ satisfying this condition, *every* cyclic permutation of λ can be reached from η for a given k which is large enough. Thus, in this case, we have to multiply the expression (4.2) by n, and subsequently sum it over all integers $\lambda_1,\lambda_2,\dots,\lambda_n$ with $m>\lambda_1>\lambda_2>\dots>\lambda_n\ge0$ and $|\lambda|\equiv|\eta|+k \bmod 2$.

In summary, what we need is, on the one hand, the sum

$$S_1=\sum_{m>\lambda_1>\dots>\lambda_n\ge0}\ \prod_{1\le h<t\le n}\sin\frac{\pi(\lambda_h-\lambda_t)}{m}, \tag{4.9}$$

and, on the other hand, the same sum (4.9), but restricted to those λ for which $|\lambda| \equiv |\eta| + k \mod 2$. The latter sum is equal to $\frac{1}{2}(S_1 + (-1)^{|\eta|+k} S_2)$, with

$$S_2 = \sum_{m>\lambda_1>\cdots>\lambda_n\ge 0} (-1)^{|\lambda|} \prod_{1\le h<t\le n} \sin\frac{\pi(\lambda_h-\lambda_t)}{m}. \tag{4.10}$$

We may again write the summand in the sums S_1 and S_2 as a Vandermonde-type determinant, so that (4.9) becomes

$$(2i)^{-\binom{n}{2}} \sum_{m>\lambda_1>\cdots>\lambda_n\ge 0} \det_{1\le h,t\le n}\left(e^{2\pi i((n+1)/2-h)\lambda_t/m}\right),$$

and the expression (4.10) becomes

$$(2i)^{-\binom{n}{2}} \sum_{m>\lambda_1>\cdots>\lambda_n\ge 0} \det_{1\le h,t\le n}\left(\left(-e^{2\pi i((n+1)/2-h)/m}\right)^{\lambda_t}\right).$$

Upon replacing λ_j by $\lambda_j + n - j$, these expressions are transformed to

$$(2i)^{-\binom{n}{2}} \sum_{m-n\ge\lambda_1\ge\cdots\ge\lambda_n\ge 0} \det_{1\le h,t\le n}\left(\left(e^{2\pi i((n+1)/2-h)/m}\right)^{\lambda_t+n-t}\right), \tag{4.11}$$

respectively

$$(2i)^{-\binom{n}{2}} \sum_{m-n\ge\lambda_1\ge\cdots\ge\lambda_n\ge 0} \det_{1\le h,t\le n}\left(\left(-e^{2\pi i((n+1)/2-h)/m}\right)^{\lambda_t+n-t}\right). \tag{4.12}$$

We may rewrite the latter two determinants using Schur functions (see (C.31) for the definition). Using this notation, we may write (4.11) and (4.12) in the form

$$(2i)^{-\binom{n}{2}} \det_{1\le h,t\le n}\left((\varepsilon q^{n+1-2h})^{n-t}\right) \sum_{m-n\ge\lambda_1\ge\cdots\ge\lambda_n\ge 0} s_\lambda(\varepsilon q^{n-1}, \varepsilon q^{n-3}, \dots, \varepsilon q^{-n+3}, \varepsilon q^{-n+1}), \tag{4.13}$$

where $q = e^{\pi i/m}$, with $\varepsilon = 1$ to yield equality with (4.11), and $\varepsilon = -1$ to yield equality with (4.12). The determinant in this expression is a Vandermonde determinant, and is therefore easily evaluated. The sum over Schur functions, on the other hand, can be evaluated by means of (see [28, proof of Theorem 2] for a discussion of this identity, with references to various proofs)

$$\sum_{p\ge\lambda_1\ge\cdots\ge\lambda_n\ge 0} s_\lambda(x_1,x_2,\dots,x_n) = (x_1x_2\cdots x_n)^{p/2}\, so^{odd}_{\left((p/2)^n\right)}(x_1,x_2,\dots,x_n), \tag{4.14}$$

where $so^{odd}_\lambda(x_1,x_2,\dots,x_n)$ is an *odd orthogonal character*. (See (C.1) for the definition. The notation $\left((p/2)^n\right)$ in (4.14) means a vector of n components, all of them equal to $p/2$.) Thus, the expression (4.13) becomes

$$\left(\varepsilon^{\binom{n}{2}} \prod_{1\le h<t\le n} \sin\frac{\pi(t-h)}{m}\right) \varepsilon^{n(m-n)/2}\, so^{odd}_{\left(((m-n)/2)^n\right)}(\varepsilon q^{n-1}, \varepsilon q^{n-3}, \dots, \varepsilon q^{-n+3}, \varepsilon q^{-n+1}). \tag{4.15}$$

If $\varepsilon = 1$, the odd orthogonal character, specialized in this manner, is evaluated in Lemma C1, while for $\varepsilon = -1$ this is done in Lemma C2. If the results are substituted in (4.15), which, as we argued above, is in fact equal to the sum (4.9), respectively to

(4.10), the sums that we wanted to evaluate, then the claims (4.6)–(4.8) follow after some further straight-forward (but tedious) calculations. □

The next two results address the asymptotic behaviour of the number of walks on the m-circle consisting of diagonal steps. As we discussed in Section 2, these theorems give at the same time the asymptotic behaviour of n non-colliding particles on the circle in the lock-step vicious walker model. As before, we begin with the result which described the asymptotic behaviour of the number of walks with fixed starting and end point.

Theorem 19. *Let m be a positive integer. Furthermore, let $\eta = (\eta_1, \eta_2, \dots, \eta_n)$ be a vector of integers or of half-integers with $m > \eta_1 > \eta_2 > \cdots > \eta_n \ge 0$, and let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ be a vector of integers or of half-integers with $m > \lambda_{s+1} > \cdots > \lambda_n > \lambda_1 > \cdots > \lambda_s \ge 0$, for some s. Then, as k tends to infinity such that $k \equiv 2\eta_j + 2\lambda_j \mod 2$, the number of random walks on the m-circle from η to λ with exactly k diagonal steps, such that at no time two coordinates of a point on the random walk are equal, is asymptotically*

$$\frac{2^{n^2-n}}{nm^n}\left(2^n\prod_{j=1}^{n}\cos\frac{\pi\left(j-\frac{n+1}{2}\right)}{m}\right)^k\prod_{1\le h<t\le n}\left(\sin\frac{\pi(\eta_h-\eta_t)}{m}\cdot\left|\sin\frac{\pi(\lambda_h-\lambda_t)}{m}\right|\right). \tag{4.16}$$

Proof. Clearly, this time we want to determine the asymptotic behaviour of the expression (2.9). First of all, if $k \equiv 2\eta_j + \lambda_j \mod 2$, i.e., in the case where k, the η_j's and the λ_j's are chosen so that walks exist, then we may replace the sum over r in (2.9) by twice the sum of the same summand, but where r runs from 0 to $m-1$ (instead of $2m-1$; here it is important that m is an integer). Then we expand again the determinant by linearity, and obtain

$$\frac{1}{nm^n}\sum_{u=0}^{n-1}\sum_{r_1,\dots,r_n=0}^{m-1}\prod_{j=1}^{n}\left(2\cos(\pi(u+nr_j)/mn)\right)^k \\ \cdot e^{-2\pi ius/n}\det_{1\le h,t\le n}\left(e^{-2\pi i(u+nr_h)(\lambda_t-\eta_h)/mn}\right). \tag{4.17}$$

This is again a finite sum of the form $\sum_\ell c_\ell b_\ell^k$, with the b_ℓ's of the form $2^n\prod_{j=1}^{n}\cos(\pi(u+nr_j)/mn)$, and both the c_ℓ's and b_ℓ's independent of k. Also in (4.17) there is the constraint that we must have $r_h \ne r_t$ for $h \ne t$ in order to obtain a non-vanishing summand, because otherwise the determinant is zero. Because of that, one discovers that in order to have b_ℓ with maximal modulus we must have $u = n/2$ and

$$\{r_1, r_2, \dots, r_n\} = \{0, 1, \dots, \tfrac{n}{2}-1, m-\tfrac{n}{2}, \dots, m-2, m-1\}$$

if n is even, and we must have $u = 0$ and

$$\{r_1, r_2, \dots, r_n\} = \{0, 1, \dots, \tfrac{n-1}{2}, m-\tfrac{n-1}{2}, \dots, m-2, m-1\}$$

if n is odd.

Let first n be even. Then we obtain for the asymptotics of (4.17) the expression

$$\frac{(-1)^s}{nm^n}\left(2^n\prod_{j=1}^{n}\cos\frac{\pi\left(j-\frac{n+1}{2}\right)}{m}\right)^k \\ \times\det_{1\le h,t\le n}\left(e^{-2\pi i((n+1)/2-h)\lambda_t/m}\right)\det_{1\le h,t\le n}\left(e^{2\pi i((n+1)/2-t)\eta_h/m}\right).$$

Both determinants are essentially Vandermonde determinants and are therefore easily evaluated. The result is exactly (4.16).

If n is odd then we obtain for the asymptotics of (4.17) the expression

$$\frac{1}{nm^n}\left(2^n\prod_{j=1}^{n}\cos\frac{\pi\left(j-\frac{n+1}{2}\right)}{m}\right)^k \det_{1\le h,t\le n}\left(e^{-2\pi i((n+1)/2-h)\lambda_t/m}\right)\det_{1\le h,t\le n}\left(e^{2\pi i((n+1)/2-t)\eta_h/m}\right),$$

which becomes again (4.16) if the Vandermonde-type determinants are evaluated. □

By summation of the corresponding expressions, the previous result allows us now to derive the asymptotic behaviour of walks with a fixed starting point but with arbitrary end point. The sums that need to be carried out are equivalent to some of those that we already evaluated in the proof of Theorem 18.

Theorem 20. *Let m be a positive integer. Furthermore, let $\eta=(\eta_1,\eta_2,\dots,\eta_n)$ be a vector of integers or of half-integers with $m>\eta_1>\eta_2>\cdots>\eta_n\ge 0$. Then, as k tends to infinity, the number of random walks on the m-circle which start at η and proceed for exactly k diagonal steps, such that at no time two coordinates of a point on the random walk are equal, is asymptotically*

$$\frac{2^{\binom{n}{2}}}{m^{n/2}}\left(2^n\prod_{j=1}^{n}\cos\frac{\pi\left(j-\frac{n+1}{2}\right)}{m}\right)^k\prod_{1\le h<t\le n}\left(\sin\frac{\pi(\eta_h-\eta_t)}{m}\right)\prod_{h=1}^{n/2}\cot\frac{(2h-1)\pi}{2m} \tag{4.18}$$

if n is even, and it is asymptotically

$$\frac{2^{\binom{n}{2}}}{m^{(n-1)/2}}\left(2^n\prod_{j=1}^{n}\cos\frac{\pi\left(j-\frac{n+1}{2}\right)}{m}\right)^k\prod_{1\le h<t\le n}\left(\sin\frac{\pi(\eta_h-\eta_t)}{m}\right)\prod_{h=1}^{(n-1)/2}\cot\frac{h\pi}{m} \tag{4.19}$$

if n is odd.

Proof. Clearly, we have to sum (4.16) over all possible choices of λ. Depending on the parity of $k+\eta_j$, this means to take the sum over all integers $\lambda_1,\lambda_2,\dots,\lambda_n$ with $m>\lambda_1>\lambda_2>\cdots>\lambda_n\ge 0$, or over all half-integers with the same property, and over all their cyclic permutations. Since the expression (4.16) is independent of s, every cyclic permutation yields the same value. Therefore we have to multiply this expression by n, and subsequently sum it over all integers, respectively half-integers, $\lambda_1,\lambda_2,\dots,\lambda_n$ with $m>\lambda_1>\lambda_2>\cdots>\lambda_n\ge 0$.

So, what we need is the sum

$$\sum_{m>\lambda_1>\cdots>\lambda_n\ge 0}{}' \prod_{1\le h<t\le n}\sin\frac{\pi(\lambda_h-\lambda_t)}{m} = (2i)^{-\binom{n}{2}}\sum_{m>\lambda_1>\cdots>\lambda_n\ge 0}{}' \det_{1\le h,t\le n}\left(e^{2\pi i((n+1)/2-h)\lambda_t/m}\right), \tag{4.20}$$

where the sum $\sum'$ is over all integral $\lambda_1,\lambda_2,\dots,\lambda_n$, but also the sum (4.20) where $\sum'$ is restricted to *half-integral* $\lambda_1,\lambda_2,\dots,\lambda_n$. (The equality of the two expressions in (4.20) follows again from the Vandermonde determinant evaluation.)

The sum over all integers $\lambda_1,\lambda_2,\dots,\lambda_n$ has already been evaluated in the proof of Theorem 18, when we evaluated S_1. If we want to form the sum (4.20) over all half-integers $\lambda_1,\lambda_2,\dots,\lambda_n$, then we may replace λ_j by $\lambda_j+n-j+\frac{1}{2}$, and rewrite it as

$$(2i)^{-\binom n2}\sum_{m-n\ge\lambda_1\ge\dots\ge\lambda_n\ge0}\det_{1\le h,t\le n}\left(e^{2\pi i((n+1)/2-h)(\lambda_t+n-t+1/2)/m}\right)\\
=(2i)^{-\binom n2}\sum_{m-n\ge\lambda_1\ge\dots\ge\lambda_n\ge0}\det_{1\le h,t\le n}\left(e^{2\pi i((n+1)/2-h)(\lambda_t+n-t)/m}\right),$$

where the sums are now over *integral* $\lambda_1,\lambda_2,\dots,\lambda_n$. The sum in the last line is exactly the same sum as (4.11), which is in turn equal to S_1 (and, thus, also to the sum (4.20) when $\sum'$ is taken over all integral $\lambda_1,\lambda_2,\dots,\lambda_n$). Therefore, regardless of the parity of k, the result is the same. Since the evaluation of S_1 in the proof of Theorem 18 yielded two different expressions depending on whether n is even or odd, we obtain the two cases in the statement of the theorem. □

5. Asymptotics for random walks in alcoves of type $\tilde C$

The subject of this section is the determination of the asymptotic behaviour of the number of walks from a given starting point to a given end point which stay in the alcove $\mathcal A_m^{\tilde C_n}$ of type $\tilde C_n$ as the number of steps becomes large, as well as the asymptotic behaviour of the number of those walks which start at a given point but may terminate anywhere. In technical terms, we determine the asymptotic behaviour of the expressions given by Theorems 6 and 7 as k becomes large, and as well if these expressions are summed over all possible end points of the walks. In fact, for Theorem 7, i.e., for the case of diagonal steps, this had already been carried out in [29], so that we only copy the corresponding results for the sake of completeness; see Theorems 23 and 24 below.

Before, however, we address the case of standard steps. As we discussed in Section 2, this case is also equivalent to the movements of n non-colliding particles in an interval according to the random turns vicious walker model. We begin, as usual, with the corresponding results when starting and end point are fixed.

Theorem 21. *Let m be a positive integer. Furthermore, let $\eta=(\eta_1,\eta_2,\dots,\eta_n)$ and $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_n)$ be vectors of integers in the alcove $\mathcal A_m^{\tilde C_n}$ of type $\tilde C_n$ (defined in (2.2)). Then, as k tends to infinity such that $k\equiv|\eta|+|\lambda|$ mod 2, the number of random walks from η to λ with exactly k standard steps, which stay in the alcove $\mathcal A_m^{\tilde C_n}$, is asymptotically*

$$\frac{2^{2n^2-n+1}}{m^n}\left(\frac{2\sin\frac{n\pi}{2m}\,\cos\frac{(n+1)\pi}{2m}}{\sin\frac{\pi}{2m}}\right)^k\prod_{1\le h<t\le n}\left(\sin\frac{\pi(\eta_h-\eta_t)}{2m}\cdot\sin\frac{\pi(\lambda_h-\lambda_t)}{2m}\right)\\
\times\prod_{1\le h\le t\le n}\left(\sin\frac{\pi(\eta_h+\eta_t)}{2m}\cdot\sin\frac{\pi(\lambda_h+\lambda_t)}{2m}\right).\quad(5.1)$$

Proof. We have to determine the asymptotic behaviour of the coefficient of $x^k/k!$ in (2.11). If we expand expression (2.11), i.e., if we use linearity of the determinant in the rows, then we obtain the expression

$$\frac{1}{m^n}\sum_{r_1,\dots,r_n=1}^{2m-1}\exp\left(2x\textstyle\sum_{j=1}^n\cos(\pi r_j/m)\right)\left(\prod_{j=1}^n\sin(\pi r_j\eta_j/m)\right)\\ \cdot\det_{1\le h,t\le n}\left(\sin(\pi r_h\lambda_t/m)\right). \quad (5.2)$$

As we said, in this expression we have to extract the coefficient of $x^k/k!$ to obtain the number of walks with exactly k steps. The expression that we obtain is a finite sum of the form $\sum_\ell c_\ell b_\ell^k$, with the b_ℓ's of the form $2\sum_{j=1}^n\cos(\pi r_j/m)$, and both the c_ℓ's and b_ℓ's independent of k.

Finding the asymptotics of (5.2) means to find the b_ℓ's with largest modulus. This, in turn, means to choose the parameters r_h either close to the lower limit of the summation, 1, respectively close to the upper limit, $2m-1$, or close to $m-1$, respectively close to $m+1$. (In the first case, all the cosines $\cos(\pi r_h/m)$ will be close to 1, whereas in the second case all of them will be close to -1. If some r_j is equal to m, then the corresponding term vanishes because of the expression $\sin(\pi r_j\eta_j/m)$ occurring in the summand.) There are again restrictions however: if $r_h=r_t$ for $h\ne t$ then the determinant in (5.2) vanishes, as well as if $r_h=2m-r_t$ for some h and t. Therefore we may restrict ourselves to the cases where $r_h\ne r_t$ and $r_h\ne 2m-r_t$ for all h and t.

Hence, we will choose the set $\{r_1,r_2,\dots,r_n\}$ either from

$$\{1,2,\dots,n,2m-n,\dots,2m-2,2m-1\}$$

or from

$$\{m-n,\dots,m-2,m-1,m+1,m+2,\dots,m+n\},$$

in such a way that the r_j's are distinct and $r_h\ne 2m-r_t$ for all h and t. Clearly, there are 2^n sets of the first type, and 2^n sets as well of the second type. As is not difficult to see, for each fixed set, the sum of the corresponding terms $c_\ell b_\ell^k$ is equal to

$$\frac{1}{m^n}\left(2\sum_{j=1}^n\cos(\pi j/m)\right)^k\cdot\det_{1\le h,t\le n}\left(\sin(\pi t\eta_h/m)\right)\cdot\det_{1\le h,t\le n}\left(\sin(\pi h\lambda_t/m)\right) \quad (5.3)$$

in both cases.

Now we have

$$\sum_{j=1}^n\cos(\pi j/m)=\frac{\sin(n\pi/2m)\,\cos((n+1)\pi/2m)}{\sin(\pi/2m)}$$

and

$$\det_{1\le h,t\le n}\left(\sin(\pi h\lambda_t/m)\right)=2^{n^2-n}\prod_{1\le h<t\le n}\sin\frac{\pi(\lambda_h-\lambda_t)}{2m}\prod_{1\le h\le t\le n}\sin\frac{\pi(\lambda_h+\lambda_t)}{2m}. \quad (5.4)$$

The latter identity follows from writing the determinant as

$$\det_{1\le h,t\le n}\left(\sin(\pi h\lambda_t/m)\right)=(2i)^{-n}\det_{1\le h,t\le n}\left(e^{\pi ih\lambda_t/m}-e^{-\pi ih\lambda_t/m}\right),$$

and evaluating it by means of (B.4). Substituting this in (5.3), and multiplying the resulting expression by $2\cdot 2^n=2^{n+1}$ (the number of these sets $\{r_1,r_2,\dots,r_n\}$), we obtain (5.1). ☐

Having accomplished the asymptotic analysis of the walks with fixed starting and end point, we can now turn to the analysis of the walks with fixed starting point but arbitrary end point. Again, this amounts to a summation problem, namely summing expression (5.1) over all possible λ. To carry out this task, we make use of identities featuring Schur functions and symplectic characters.

Theorem 22. *Let m be a positive integer. Furthermore, let $\eta = (\eta_1, \eta_2, \dots, \eta_n)$ be a vector of integers in the alcove $\mathcal{A}_m^{\tilde C_n}$ of type $\tilde C_n$ (defined in (2.2)). Then, as k tends to infinity, the number of random walks which start at η and proceed for exactly k standard steps, which stay in the alcove $\mathcal{A}_m^{\tilde C_n}$, is asymptotically*

$$\frac{2^{2n^2-n}}{m^n}\left(\frac{2\sin\frac{n\pi}{2m}\,\cos\frac{(n+1)\pi}{2m}}{\sin\frac{\pi}{2m}}\right)^k\prod_{1\le h<t\le n}\left(\sin\frac{\pi(\eta_h-\eta_t)}{2m}\cdot\sin\frac{\pi(t-h)}{2m}\right)$$
$$\times\prod_{1\le h\le t\le n}\left(\sin\frac{\pi(\eta_t+\eta_h)}{2m}\cdot\sin\frac{\pi(t+h)}{2m}\right)$$
$$\times\left(\prod_{h=0}^{n}\prod_{t=1}^{n}\frac{\sin\frac{\pi(t-h+m-1)}{2m}}{\sin\frac{\pi(t-h+n)}{2m}}+(-1)^{|\eta|+k+\frac{n}{2}}2^{-n^2}\frac{\prod_{h=1}^{n/2}\tan^2\frac{(2h-1)\pi}{2m}}{\prod_{h=1}^{n+1}\prod_{t=1}^{n}\left|\sin\frac{(2t-2h+1)\pi}{2m}\right|}\right)\tag{5.5}$$

if both m and n are even, it is asymptotically

$$\frac{2^{2n^2-n}}{m^n}\left(\frac{2\sin\frac{n\pi}{2m}\,\cos\frac{(n+1)\pi}{2m}}{\sin\frac{\pi}{2m}}\right)^k\prod_{1\le h<t\le n}\left(\sin\frac{\pi(\eta_h-\eta_t)}{2m}\cdot\sin\frac{\pi(t-h)}{2m}\right)$$
$$\times\prod_{1\le h\le t\le n}\left(\sin\frac{\pi(\eta_t+\eta_h)}{2m}\cdot\sin\frac{\pi(t+h)}{2m}\right)$$
$$\times\left(\prod_{h=0}^{n}\prod_{t=1}^{n}\frac{\sin\frac{\pi(t-h+m-1)}{2m}}{\sin\frac{\pi(t-h+n)}{2m}}+(-1)^{|\eta|+k+\frac{n+1}{2}}2^{-n^2}\frac{\prod_{h=1}^{(n+1)/2}\tan^2\frac{(2h-1)\pi}{2m}}{\prod_{h=1}^{n+1}\prod_{t=1}^{n}\left|\sin\frac{(2t-2h+1)\pi}{2m}\right|}\right)\tag{5.6}$$

if m is even and n is odd, it is asymptotically

$$\frac{2^{2n^2-n}}{m^n}\left(\frac{2\sin\frac{n\pi}{2m}\,\cos\frac{(n+1)\pi}{2m}}{\sin\frac{\pi}{2m}}\right)^k\prod_{1\le h<t\le n}\left(\sin\frac{\pi(\eta_h-\eta_t)}{2m}\cdot\sin\frac{\pi(t-h)}{2m}\right)$$
$$\times\prod_{1\le h\le t\le n}\left(\sin\frac{\pi(\eta_t+\eta_h)}{2m}\cdot\sin\frac{\pi(t+h)}{2m}\right)$$
$$\times\left(\prod_{h=0}^{n}\prod_{t=1}^{n}\frac{\sin\frac{\pi(t-h+m-1)}{2m}}{\sin\frac{\pi(t-h+n)}{2m}}+(-1)^{|\eta|+k+\frac{n}{2}}2^{-n^2}\prod_{h=1}^{n/2}\frac{\sin^2\frac{(2h-1)\pi}{2m}}{\cos^2\frac{h\pi}{m}}\frac{1}{\prod_{h=1}^{n+1}\prod_{t=1}^{n}\left|\sin\frac{(2t-2h+1)\pi}{2m}\right|}\right)\tag{5.7}$$

if m is odd and n is even, and it is asymptotically

$$\frac{2^{2n^2-n}}{m^n}\left(\frac{2\sin\frac{n\pi}{2m}\cos\frac{(n+1)\pi}{2m}}{\sin\frac{\pi}{2m}}\right)^k\prod_{1\le h<t\le n}\left(\sin\frac{\pi(\eta_h-\eta_t)}{2m}\cdot\sin\frac{\pi(t-h)}{2m}\right)$$
$$\times\prod_{1\le h\le t\le n}\left(\sin\frac{\pi(\eta_t+\eta_h)}{2m}\cdot\sin\frac{\pi(t+h)}{2m}\right)\left(\prod_{h=0}^{n}\prod_{t=1}^{n}\frac{\sin\frac{\pi(t-h+m-1)}{2m}}{\sin\frac{\pi(t-h+n)}{2m}}\right)\tag{5.8}$$

if both m and n are odd.

Proof. As we already observed, in view of Theorem 21, we have to carry out the sum of (5.1) over all possible λ, i.e., over all $m>\lambda_1>\lambda_2>\cdots>\lambda_n>0$, where $|\lambda|\equiv k+|\eta|$ mod 2. Leaving away factors which are independent of λ, the problem is to compute the sum

$$\sum_{m>\lambda_1>\cdots>\lambda_n>0}{}'\prod_{1\le h<t\le n}\left(\sin\frac{\pi(\lambda_h-\lambda_t)}{2m}\right)\prod_{1\le h\le t\le n}\left(\sin\frac{\pi(\lambda_h+\lambda_t)}{2m}\right),\tag{5.9}$$

where the sum $\sum{}'$ is either restricted to those λ for which $|\lambda|$ is even, or to those for which $|\lambda|$ is odd, depending on the parity of $k+|\eta|$. This task will be accomplished if we are able to evaluate the (complete) sum

$$T_1=\sum_{m>\lambda_1>\cdots>\lambda_n>0}\prod_{1\le h<t\le n}\left(\sin\frac{\pi(\lambda_h-\lambda_t)}{2m}\right)\prod_{1\le h\le t\le n}\left(\sin\frac{\pi(\lambda_h+\lambda_t)}{2m}\right),$$

and its “signed variant”

$$T_2=\sum_{m>\lambda_1>\cdots>\lambda_n>0}(-1)^{|\lambda|}\prod_{1\le h<t\le n}\left(\sin\frac{\pi(\lambda_h-\lambda_t)}{2m}\right)\prod_{1\le h\le t\le n}\left(\sin\frac{\pi(\lambda_h+\lambda_t)}{2m}\right).$$

The sum (5.9) is then equal to $\frac12(T_1+T_2)$ if the sum $\sum{}'$ is restricted to the λ's for which $|\lambda|$ is even, and it is equal to $\frac12(T_1-T_2)$ if the sum $\sum{}'$ is restricted to the λ's for which $|\lambda|$ is odd.

A sum equivalent to T_1 had already been evaluated in [29, first part of the proof of Theorem 6]. The result is

$$T_1=\prod_{1\le h<t\le n}\sin\frac{\pi(t-h)}{2m}\prod_{1\le h\le t\le n}\sin\frac{\pi(t+h)}{2m}\prod_{h=0}^{n}\prod_{t=1}^{n}\frac{\sin\frac{\pi(t-h+m-1)}{2m}}{\sin\frac{\pi(t-h+n)}{2m}}.$$

In order to evaluate T_2, we proceed in a manner similar to the evaluation in [29]. By means of (5.4), we may rewrite T_2 as

$$\sum_{m>\lambda_1>\cdots>\lambda_n>0}(-1)^{|\lambda|}2^{-n^2+n}\det_{1\le h,t\le n}\big(\sin(\pi h\lambda_t/m)\big)$$
$$=\sum_{m>\lambda_1>\cdots>\lambda_n>0}\frac{1}{2^{n^2}i^n}\det_{1\le h,t\le n}\big((-e^{\pi ih/m})^{\lambda_t}-(-e^{\pi ih/m})^{-\lambda_t}\big).$$

Replacing λ_j by $\lambda_j+n-j+1$, $j=1,2,\dots,n$, we obtain the expression

$$\sum_{m-n-1\ge\lambda_1\ge\cdots\ge\lambda_n\ge0}\frac{1}{2^{n^2}i^n}\det_{1\le h,t\le n}\big((-e^{\pi ih/m})^{\lambda_t+n-t+1}-(-e^{\pi ih/m})^{-(\lambda_t+n-t+1)}\big).\tag{5.10}$$

This determinant can be expressed in terms of a *symplectic character*. Given a partition $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_n)$ (i.e., a non-increasing sequence of non-negative integers), the symplectic character $sp_\lambda(x_1,x_2,\dots,x_n)$ is defined by (see [13, (24.18)])

$$sp_\lambda(x_1,x_2,\dots,x_n)=\frac{\det\limits_{1\le h,t\le n}\left(x_h^{\lambda_t+n-t+1}-x_h^{-(\lambda_t+n-t+1)}\right)}{\det\limits_{1\le h,t\le n}\left(x_h^{n-t+1}-x_h^{-(n-t+1)}\right)}. \tag{5.11}$$

Therefore, writing again q for $e^{\pi i/m}$, the sum in (5.10) equals

$$\frac{1}{2^{n^2}i^n}\det_{1\le h,t\le n}\left((-q^h)^{n-t+1}-(-q^h)^{-(n-t+1)}\right)\sum_{m-n-1\ge\lambda_1\ge\dots\ge\lambda_n\ge0}sp_\lambda(-q,-q^2,\dots,-q^n). \tag{5.12}$$

Now we appeal to the formula (see [27, (3.4)]),

$$s_{(c^r)}(x_1,x_1^{-1},\dots,x_n,x_n^{-1},1)=\sum_{c\ge\nu_1\ge\nu_2\ge\dots\ge\nu_r\ge0}sp_\nu(x_1,\dots,x_n), \tag{5.13}$$

which is valid for $r\le n$, where on the left-hand side we have again a Schur function (cf. (C.31) for the definition). The notation (c^r) is short for the vector in which the first r components are equal to c, followed by $2n+1-r$ components all of which are 0. Use of this formula in (5.12) yields the equivalent expression

$$\begin{aligned}&\frac{1}{2^{n^2}i^n}\det_{1\le h,t\le n}\left((-q^h)^{n-t+1}-(-q^h)^{-(n-t+1)}\right)\\&\qquad\qquad\times s_{\left((m-n-1)^n\right)}(-q^n,-q^{n-1},\dots,-q,1,-q^{-1},\dots,-q^{-n+1},-q^n)\\&\quad=(-1)^{(m-n-1)n}\frac{1}{2^{n^2}i^n}\det_{1\le h,t\le n}\left((-q^h)^{n-t+1}-(-q^h)^{-(n-t+1)}\right)\\&\qquad\qquad\qquad\qquad\times s_{\left((m-n-1)^n\right)}(q^n,q^{n-1},\dots,q,-1,q^{-1},\dots,q^{-n+1},q^n).\end{aligned}$$

Clearly, the determinant is easily evaluated by means of (B.4). The specialized Schur function is evaluated in Lemma C3. If everything is combined and simplified, the claimed formulae (5.5)–(5.8) are eventually obtained. □

We conclude this section by reporting the results from [29] on the asymptotic behaviour of walks in the alcove $\mathcal{A}_m^{\tilde C_n}$ which consist entirely of diagonal steps. These results have been stated there in an equivalent form, namely in the language of walkers in the lock-step vicious walkers model, which are bounded by two walls.

Theorem 23 ([29, Theorem 4])**.** *Let m be a positive integer or half-integer. Furthermore, let $\eta=(\eta_1,\eta_2,\dots,\eta_n)$ and $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_n)$ be vectors of integers or of half-integers in the alcove $\mathcal{A}_m^{\tilde C_n}$ of type $\tilde C_n$ (defined in (2.2)). Then, as k tends to infinity such that $k\equiv 2\eta_j+2\lambda_j$ mod 2, the number of random walks from η to λ with exactly k diagonal steps, which stay in the alcove $\mathcal{A}_m^{\tilde C_n}$, is asymptotically*

$$\frac{4^{n^2}}{(2m)^n}\left(2^n\prod_{j=1}^n\cos\frac{j\pi}{2m}\right)^k\prod_{1\le h<t\le n}\left(\sin\frac{\pi(\eta_h-\eta_t)}{2m}\cdot\sin\frac{\pi(\lambda_h-\lambda_t)}{2m}\right)$$

$$\times \prod_{1\le h\le t\le n} \left(\sin\frac{\pi(\eta_h+\eta_t)}{2m}\cdot\sin\frac{\pi(\lambda_h+\lambda_t)}{2m}\right). \quad (5.14)$$

Theorem 24 ([29, Theorem 6])**.** *Let m be a positive integer or half-integer. Furthermore, let $\eta=(\eta_1,\eta_2,\dots,\eta_n)$ be a vector of integers or of half-integers in the alcove $\mathcal{A}_m^{\tilde C_n}$ of type $\tilde C_n$ (defined in (2.2)). Then, as k tends to infinity, the number of random walks which start at η and proceed for exactly k diagonal steps, which stay in the alcove $\mathcal{A}_m^{\tilde C_n}$, is asymptotically*

$$\frac{4^{n^2}}{(2m)^n}\left(2^n\prod_{j=1}^n\cos\frac{j\pi}{2m}\right)^k\prod_{1\le h<t\le n}\left(\sin\frac{\pi(\eta_h-\eta_t)}{2m}\cdot\sin\frac{\pi(t-h)}{2m}\right)$$
$$\times\prod_{1\le h\le t\le n}\left(\sin\frac{\pi(\eta_h+\eta_t)}{2m}\cdot\sin\frac{\pi(t+h)}{2m}\right)\prod_{h=0}^n\prod_{t=1}^n\frac{\sin\frac{\pi(t-h+\lfloor m\rfloor)}{2m}}{\sin\frac{\pi(t-h+n)}{2m}}$$
$$\times\prod_{h=1}^n\frac{\sin\frac{\pi(h+\lfloor m\rfloor-n)}{2m}}{\sin\frac{\pi(2h+\lfloor m\rfloor-n)}{2m}}, \quad (5.15)$$

if $k+2\eta_j$ is odd, and

$$\frac{4^{n^2}}{(2m)^n}\left(2^n\prod_{j=1}^n\cos\frac{j\pi}{2m}\right)^k\prod_{1\le h<t\le n}\left(\sin\frac{\pi(\eta_h-\eta_t)}{2m}\cdot\sin\frac{\pi(t-h)}{2m}\right)$$
$$\times\prod_{1\le h\le t\le n}\left(\sin\frac{\pi(\eta_h+\eta_t)}{2m}\cdot\sin\frac{\pi(t+h)}{2m}\right)\prod_{h=0}^n\prod_{t=1}^n\frac{\sin\frac{\pi(t-h+\lceil m-1\rceil)}{2m}}{\sin\frac{\pi(t-h+n)}{2m}}, \quad (5.16)$$

if $k+2\eta_j$ is even.

6. Asymptotics for random walks in alcoves of type $\tilde B$

This section is devoted to finding the asymptotic behaviour of the number of walks from a given starting point to a given end point which stay in the alcove $\mathcal{A}_m^{\tilde B_n}$ of type $\tilde B_n$ as the number of steps becomes large, as well as the asymptotic behaviour of the number of those walks which start at a given point but may terminate anywhere. In technical terms, we determine the asymptotic behaviour of the expressions given by Theorems 8 and 9 as k becomes large, and as well if these expressions are summed over all possible end points of the walks.

The following two theorems address the case of walks with standard steps. The result for fixed starting *and* end point is the subject of the first of the two.

Theorem 25. *Let m be a positive integer or half-integer. Furthermore, let $\eta=(\eta_1,\eta_2,\dots,\eta_n)$ and $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_n)$ be vectors of integers in the alcove $\mathcal{A}_m^{\tilde B_n}$ of type $\tilde B_n$ (defined in (2.3)). Then, as k tends to infinity such that $k\equiv|\eta|+|\lambda|$ mod 2, the number of random walks from η to λ with exactly k standard steps, which stay in the alcove $\mathcal{A}_m^{\tilde B_n}$, is asymptotically*

$$\frac{4^{n^2}}{(2m)^n}\left(\frac{\sin\frac{n\pi}{m}}{\sin\frac{\pi}{2m}}\right)^k\prod_{1\le h<t\le n}\left(\sin\frac{\pi(\eta_h-\eta_t)}{2m}\cdot\sin\frac{\pi(\lambda_h-\lambda_t)}{2m}\right.$$

$$\cdot \sin \frac{\pi(\eta_h+\eta_t)}{2m} \cdot \sin \frac{\pi(\lambda_h+\lambda_t)}{2m} \Bigg) \prod_{h=1}^{n} \left(\sin \frac{\pi \eta_h}{2m} \cdot \sin \frac{\pi \lambda_h}{2m} \right). \tag{6.1}$$

Proof. The analysis is analogous to the one in the proof of Theorem 21. Here, we have to estimate the coefficient of $x^k/k!$ in (2.13). Expanding the two determinants in (2.13) by linearity in the rows, we obtain

$$\frac{1}{2m^n} \sum_{r_1,\dots,r_n=1}^{2m-1} \left(2 \sum_{j=1}^{n} \cos \frac{\pi r_j}{m} \right)^k \left(\sum_{j=1}^{n} \sin \frac{\pi r_j \eta_j}{m} \right) \det_{1\le h,t\le n} \left(\sin \frac{\pi r_h \lambda_t}{m} \right)$$
$$+ \frac{1}{2m^n} \sum_{r_1,\dots,r_n=0}^{2m-1} \left(2 \sum_{j=1}^{n} \cos \frac{\pi(2r_j+1)}{2m} \right)^k$$
$$\cdot \left(\prod_{j=1}^{n} \sin \frac{\pi(2r_j+1)\eta_j}{2m} \right) \det_{1\le h,t\le n} \left(\sin \frac{\pi(2r_h+1)\lambda_t}{2m} \right).$$

Again, this is a finite sum of the form $\sum_\ell c_\ell b_\ell^k$, with both the c_ℓ's and b_ℓ's independent of k.

We have to locate the b_ℓ's with largest modulus. As can be seen in a manner similar to the considerations in the proof of Theorem 21, the b_ℓ's with largest modulus come from the expansion of the *second* determinant, when we choose distinct $r_1, r_2, \dots, r_n$ from

$$\{0, 1, \dots, n-1, 2m-n, 2m-n+1, \dots, 2m-1\},$$

or from

$$\{m-n, m-n+1, \dots, m-1, m, \dots, m+n-1\},$$

such that $r_h \ne 2m-1-r_t$ for all h and t. Clearly, there are $2\cdot 2^n$ such sets $\{r_1, r_2, \dots, r_n\}$. For each fixed set, the sum of the corresponding terms $c_\ell b_\ell^k$ is equal to

$$\frac{1}{2m^n} \left(2 \sum_{j=1}^{n} \cos \frac{\pi(2j-1)}{2m} \right)^k \det_{1\le h,t\le n} \left(\sin \frac{\pi(2h-1)\eta_t}{2m} \right) \det_{1\le h,t\le n} \left(\sin \frac{\pi(2h-1)\lambda_t}{2m} \right). \tag{6.2}$$

We have on the one hand

$$\sum_{j=1}^{n} \cos \frac{\pi(2j-1)}{2m} = \frac{\sin \frac{n\pi}{m}}{2 \sin \frac{\pi}{2m}}.$$

On the other hand, the first determinant in (6.2) can be rewritten in the form

$$(2i)^{-n} \det_{1\le h,t\le n} \left(e^{\frac{\pi i(2h-1)\eta_t}{2m}} - e^{-\frac{\pi i(2h-1)\eta_t}{2m}} \right),$$

and can thus be evaluated by means of (B.2). After some simplification, the result is

$$\det_{1\le h,t\le n} \left(\sin \frac{\pi(2h-1)\eta_t}{2m} \right)$$
$$= 2^{n^2-n} \prod_{1\le h<t\le n} \left(\sin \frac{\pi(\eta_h-\eta_t)}{2m} \cdot \sin \frac{\pi(\eta_h+\eta_t)}{2m} \right) \prod_{h=1}^{n} \left(\sin \frac{\pi \eta_h}{2m} \right). \tag{6.3}$$

Clearly, the second determinant in (6.2) is equal to the same expression with η_j replaced by λ_j, $j=1,2,\dots,n$. If all this is substituted in (6.2), and if the result is multiplied by $2\cdot 2^n$, then we obtain exactly (6.1). $\square$

If the starting point is fixed but the end point is not, we have the following result.

Theorem 26. *Let m be a positive integer or half-integer. Furthermore, let $\eta=(\eta_1,\eta_2,\dots,\eta_n)$ be a vector of integers in the alcove $\mathcal{A}_m^{\tilde B_n}$ of type $\tilde B_n$ (defined in (2.3)). Then, as k tends to infinity, the number of random walks which start at η and proceed for exactly k standard steps, which stay in the alcove $\mathcal{A}_m^{\tilde B_n}$, is asymptotically*

$$\frac{4^{n^2}}{2(2m)^n}\left(\frac{\sin\frac{n\pi}{m}}{\sin\frac{\pi}{2m}}\right)^k\prod_{1\le h<t\le n}\left(\sin\frac{\pi(\eta_h-\eta_t)}{2m}\cdot\sin\frac{\pi(t-h)}{2m}\right.$$
$$\left.\cdot\sin\frac{\pi(\eta_h+\eta_t)}{2m}\cdot\sin\frac{\pi(t+h)}{2m}\right)\prod_{h=1}^{n}\left(\sin\frac{\pi\eta_h}{2m}\cdot\sin\frac{\pi h}{2m}\right)$$
$$\times\left(\prod_{h=1}^{2n}\frac{\sin\frac{\pi(m-n+h)}{4m}}{\sin\frac{\pi h}{4m}}\prod_{h=1}^{n-1}\prod_{t=1}^{n}\frac{\sin\frac{\pi(m+t-h)}{2m}}{\sin\frac{\pi(n+t-h)}{2m}}+\prod_{h=1}^{2n}\frac{\sin\frac{\pi(m-n+h-1)}{4m}}{\sin\frac{\pi h}{4m}}\prod_{h=1}^{n-1}\prod_{t=1}^{n}\frac{\sin\frac{\pi(m+t-h-1)}{2m}}{\sin\frac{\pi(n+t-h)}{2m}}\right.$$
$$+(-1)^{|\eta|+k+\binom{n+1}{2}}\prod_{h=1}^{n}\frac{\cos\frac{\pi(m-n+2h-1)}{4m}\cdot\sin\frac{\pi(m-n+2h)}{4m}}{\cos\frac{\pi(2h-1)}{4m}\cdot\sin\frac{\pi h}{2m}}\prod_{h=1}^{n-1}\prod_{t=1}^{n}\frac{\sin\frac{\pi(m+t-h)}{2m}}{\sin\frac{\pi(n+t-h)}{2m}}$$
$$\left.+(-1)^{|\eta|+k+\binom{n}{2}}\prod_{h=1}^{n}\frac{\sin\frac{\pi(m-n+2h-2)}{4m}\cdot\cos\frac{\pi(m-n+2h-1)}{4m}}{\cos\frac{\pi(2h-1)}{4m}\cdot\sin\frac{\pi h}{2m}}\prod_{h=1}^{n-1}\prod_{t=1}^{n}\frac{\sin\frac{\pi(m+t-h-1)}{2m}}{\sin\frac{\pi(n+t-h)}{2m}}\right),\quad(6.4)$$

if m is an integer with parity equal to that of n, it is asymptotically

$$\frac{4^{n^2}}{2(2m)^n}\left(\frac{\sin\frac{n\pi}{m}}{\sin\frac{\pi}{2m}}\right)^k\prod_{1\le h<t\le n}\left(\sin\frac{\pi(\eta_h-\eta_t)}{2m}\cdot\sin\frac{\pi(t-h)}{2m}\right.$$
$$\left.\cdot\sin\frac{\pi(\eta_h+\eta_t)}{2m}\cdot\sin\frac{\pi(t+h)}{2m}\right)\prod_{h=1}^{n}\left(\sin\frac{\pi\eta_h}{2m}\cdot\sin\frac{\pi h}{2m}\right)$$
$$\times\left(\prod_{h=1}^{2n}\frac{\sin\frac{\pi(m-n+h)}{4m}}{\sin\frac{\pi h}{4m}}\prod_{h=1}^{n-1}\prod_{t=1}^{n}\frac{\sin\frac{\pi(m+t-h)}{2m}}{\sin\frac{\pi(n+t-h)}{2m}}+\prod_{h=1}^{2n}\frac{\sin\frac{\pi(m-n+h-1)}{4m}}{\sin\frac{\pi h}{4m}}\prod_{h=1}^{n-1}\prod_{t=1}^{n}\frac{\sin\frac{\pi(m+t-h-1)}{2m}}{\sin\frac{\pi(n+t-h)}{2m}}\right.$$
$$+(-1)^{|\eta|+k+\binom{n}{2}}\prod_{h=1}^{n}\frac{\sin\frac{\pi(m-n+2h-1)}{4m}\cdot\cos\frac{\pi(m-n+2h)}{4m}}{\cos\frac{\pi(2h-1)}{4m}\cdot\sin\frac{\pi h}{2m}}\prod_{h=1}^{n-1}\prod_{t=1}^{n}\frac{\sin\frac{\pi(m+t-h)}{2m}}{\sin\frac{\pi(n+t-h)}{2m}}$$
$$\left.+(-1)^{|\eta|+k+\binom{n+1}{2}}\prod_{h=1}^{n}\frac{\cos\frac{\pi(m-n+2h-2)}{4m}\cdot\sin\frac{\pi(m-n+2h-1)}{4m}}{\cos\frac{\pi(2h-1)}{4m}\cdot\sin\frac{\pi h}{2m}}\prod_{h=1}^{n-1}\prod_{t=1}^{n}\frac{\sin\frac{\pi(m+t-h-1)}{2m}}{\sin\frac{\pi(n+t-h)}{2m}}\right),\quad(6.5)$$

if m is an integer with parity different from that of n, and it is asymptotically

$$\frac{4^{n^2}}{(2m)^n}\left(\frac{\sin\frac{n\pi}{m}}{\sin\frac{\pi}{2m}}\right)^k\prod_{1\le h<t\le n}\left(\sin\frac{\pi(\eta_h-\eta_t)}{2m}\cdot\sin\frac{\pi(t-h)}{2m}\right.$$

$$\cdot \sin\frac{\pi(\eta_h+\eta_t)}{2m}\cdot \sin\frac{\pi(t+h)}{2m}\Bigg)\prod_{h=1}^{n}\left(\sin\frac{\pi\eta_h}{2m}\cdot\sin\frac{\pi h}{2m}\right)$$

$$\times\prod_{h=1}^{2n}\frac{\sin\frac{\pi(\lfloor m\rfloor-n+h)}{4m}}{\sin\frac{\pi h}{4m}}\prod_{h=1}^{n-1}\prod_{t=1}^{n}\frac{\sin\frac{\pi(\lfloor m\rfloor+t-h)}{2m}}{\sin\frac{\pi(n+t-h)}{2m}},\quad (6.6)$$

if m is a half-integer.

Proof. Here, in view of Theorem 25, we have to carry out the sum of (6.1) over all possible λ, i.e., over all $\lambda_1>\lambda_2>\dots>\lambda_n>0$ with $\lambda_1+\lambda_2<2m$, where $|\lambda|\equiv k+|\eta|$ mod 2. Leaving away factors which are independent of λ, the problem is to compute the sum

$$\sideset{}{'}\sum_{\substack{\lambda_1>\dots>\lambda_n>0\\ \lambda_1+\lambda_2<2m}}\prod_{1\le h<t\le n}\left(\sin\frac{\pi(\lambda_h-\lambda_t)}{2m}\cdot\sin\frac{\pi(\lambda_h+\lambda_t)}{2m}\right)\prod_{h=1}^{n}\sin\frac{\pi\lambda_h}{2m},\qquad(6.7)$$

where the sum $\sum'$ is either restricted to those λ for which $|\lambda|$ is even, or to those for which $|\lambda|$ is odd, depending on the parity of $k+|\eta|$.

Let first m be an integer. In order to get rid of the restriction $\lambda_1+\lambda_2<2m$ in the sum in (6.7), we observe that the summand remains unchanged if we replace λ_1 by $2m-\lambda_1$ and, moreover, that $(\lambda_1,\lambda_2,\dots,\lambda_n)$ satisfies the conditions on the summation indices if and only if $(2m-\lambda_1,\lambda_2,\dots,\lambda_n)$ does. Hence, we may rewrite the sum in (6.7) as

$$\sideset{}{'}\sum_{m\ge\lambda_1>\dots>\lambda_n>0}\prod_{1\le h<t\le n}\left(\sin\frac{\pi(\lambda_h-\lambda_t)}{2m}\cdot\sin\frac{\pi(\lambda_h+\lambda_t)}{2m}\right)\prod_{h=1}^{n}\sin\frac{\pi\lambda_h}{2m}$$

$$+\sideset{}{'}\sum_{m-1\ge\lambda_1>\dots>\lambda_n>0}\prod_{1\le h<t\le n}\left(\sin\frac{\pi(\lambda_h-\lambda_t)}{2m}\cdot\sin\frac{\pi(\lambda_h+\lambda_t)}{2m}\right)\prod_{h=1}^{n}\sin\frac{\pi\lambda_h}{2m},\quad(6.8)$$

where, again, the sums $\sum'$ are either restricted to those λ for which $|\lambda|$ is even, or to those for which $|\lambda|$ is odd, depending on the parity of $k+|\eta|$.

The task of evaluating the sums in (6.8) will be accomplished if we are able to evaluate the (complete) sum

$$U_1(c)=\sum_{c\ge\lambda_1>\dots>\lambda_n>0}\prod_{1\le h<t\le n}\left(\sin\frac{\pi(\lambda_h-\lambda_t)}{2m}\cdot\sin\frac{\pi(\lambda_h+\lambda_t)}{2m}\right)\prod_{h=1}^{n}\sin\frac{\pi\lambda_h}{2m},$$

and its "signed variant"

$$U_2(c)=\sum_{c\ge\lambda_1>\dots>\lambda_n>0}(-1)^{|\lambda|}\prod_{1\le h<t\le n}\left(\sin\frac{\pi(\lambda_h-\lambda_t)}{2m}\cdot\sin\frac{\pi(\lambda_h+\lambda_t)}{2m}\right)\prod_{h=1}^{n}\sin\frac{\pi\lambda_h}{2m},$$

for $c=m$ and $c=m-1$. The expression (6.8) is then equal to

$$\frac{1}{2}\Big(U_1(m)+U_1(m-1)+U_2(m)+U_2(m-1)\Big)$$

if the sum $\sum'$ is restricted to the λ's for which $|\lambda|$ is even, and it is equal to

$$\frac{1}{2}\Big(U_1(m)+U_1(m-1)-U_2(m)-U_2(m-1)\Big)$$

if the sum $\sum'$ is restricted to the λ's for which $|\lambda|$ is odd.

In order to evaluate $U_1(c)$ and $U_2(c)$, we use (6.3) to rewrite them uniformly as

$$\sum_{c\ge\lambda_1>\dots>\lambda_n>0} \varepsilon^{|\lambda|}2^{-n^2+n} \det_{1\le h,t\le n}\big(\sin(\pi(2h-1)\lambda_t/2m)\big)\\ = \sum_{c\ge\lambda_1>\dots>\lambda_n>0} \frac{1}{2^{n^2}i^n}\det_{1\le h,t\le n}\big((\varepsilon e^{\pi i(2h-1)/2m})^{\lambda_t}-(\varepsilon e^{\pi i(2h-1)/2m})^{-\lambda_t}\big),$$

with $\varepsilon=1$ or $\varepsilon=-1$ depending on whether we want to express $U_1(c)$ or $U_2(c)$. Replacing λ_t by $\lambda_t+n-t+1$, $t=1,2,\dots,n$, we obtain the expression

$$\sum_{c-n\ge\lambda_1\ge\dots\ge\lambda_n\ge0} \frac{1}{2^{n^2}i^n}\det_{1\le h,t\le n}\big((\varepsilon e^{\pi i(2h-1)/2m})^{\lambda_t+n-t+1}-(\varepsilon e^{\pi i(2h-1)/2m})^{-(\lambda_t+n-t+1)}\big). \tag{6.9}$$

As in the proof of Theorem 22, this determinant can be expressed in terms of a *symplectic character* as defined in (5.11). Specifically, writing q for $e^{\pi i/2m}$, the sum in (6.9) equals

$$\frac{1}{2^{n^2}i^n}\det_{1\le h,t\le n}\big((\varepsilon q^{2h-1})^{n-t+1}-(\varepsilon q^{2h-1})^{-(n-t+1)}\big)\sum_{c-n\ge\lambda_1\ge\dots\ge\lambda_n\ge0} sp_\lambda(\varepsilon q,\varepsilon q^3,\dots,\varepsilon q^{2n-1}). \tag{6.10}$$

Use of Formula (5.13) in (6.10) yields the equivalent expression

$$\frac{1}{2^{n^2}i^n}\det_{1\le h,t\le n}\big((\varepsilon q^{2h-1})^{n-t+1}-(\varepsilon q^{2h-1})^{-(n-t+1)}\big)\\ \times s_{\left((c-n)^n\right)}(\varepsilon q^{2n-1},\varepsilon q^{2n-3},\dots,\varepsilon q^3,\varepsilon q,1,\varepsilon q^{-1},\varepsilon q^{-3},\dots,\varepsilon q^{-2n+3},\varepsilon q^{-2n+1})\\ =\varepsilon^{(c-n)n}\frac{1}{2^{n^2}i^n}\det_{1\le h,t\le n}\big((\varepsilon q^{2h-1})^{n-t+1}-(\varepsilon q^{2h-1})^{-(n-t+1)}\big)\\ \times s_{\left((c-n)^n\right)}(q^{2n-1},q^{2n-3},\dots,q^3,q,\varepsilon,q^{-1},q^{-3},\dots,q^{-2n+3},q^{-2n+1}). \tag{6.11}$$

Clearly, the determinant is easily evaluated by means of (B.2). The specialized Schur function is evaluated in Lemma C5 in the case that $\varepsilon=1$, and in Lemma C6 in the case that $\varepsilon=-1$. If everything is combined and simplified, the claimed formulae (6.4) and (6.5) are eventually obtained.

In the case that m is a half-integer, an adaption of the above argument of replacement of λ_1 by $2m-\lambda_1$ in (6.7) shows that (6.7) equals

$$\sum_{\lfloor m\rfloor\ge\lambda_1>\dots>\lambda_n>0}{}' \prod_{1\le h<t\le n}\left(\sin\frac{\pi(\lambda_h-\lambda_t)}{2m}\cdot\sin\frac{\pi(\lambda_h+\lambda_t)}{2m}\right)\prod_{h=1}^n\sin\frac{\pi\lambda_h}{2m}\\ +\sum_{\lfloor m\rfloor\ge\lambda_1>\dots>\lambda_n>0}{}'' \prod_{1\le h<t\le n}\left(\sin\frac{\pi(\lambda_h-\lambda_t)}{2m}\cdot\sin\frac{\pi(\lambda_h+\lambda_t)}{2m}\right)\prod_{h=1}^n\sin\frac{\pi\lambda_h}{2m},$$

where one of the sums $\sum'$ and $\sum''$ is restricted to those λ for which $|\lambda|$ is even while the other to those for which $|\lambda|$ is odd, depending on the parity of $k+|\eta|$. Hence, it equals $U_1(\lfloor m\rfloor)$, regardless of the parity of $k+|\eta|$. On the other hand, we have seen above that $U_1(\lfloor m\rfloor)$ can be rewritten as (6.11) with $\varepsilon=1$ and $c=\lfloor m\rfloor$. Application of Lemma C5 with $c=\lfloor m\rfloor-n$ and some simplification then lead to (6.6). $\square$

We proceed with the corresponding theorems for the case of walks with diagonal steps.

Theorem 27. *Let m be a positive integer or half-integer. Furthermore, let $\eta=(\eta_1,\eta_2,\dots,\eta_n)$ and $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_n)$ be vectors of integers or of half-integers in the alcove $\mathcal{A}_m^{\tilde B_n}$ of type $\tilde B_n$ (defined in (2.3)). Then, as k tends to infinity such that $k\equiv 2\eta_j+2\lambda_j$ mod 2, the number of random walks from η to λ with exactly k diagonal steps, which stay in the alcove $\mathcal{A}_m^{\tilde B_n}$, is asymptotically*

$$\frac{4^{n^2}}{2(2m)^n}\left(2^n\prod_{j=1}^{n}\cos\frac{(2j-1)\pi}{4m}\right)^k\prod_{1\le h<t\le n}\left(\sin\frac{\pi(\eta_h-\eta_t)}{2m}\cdot\sin\frac{\pi(\lambda_h-\lambda_t)}{2m}\right.$$
$$\left.\cdot\sin\frac{\pi(\eta_h+\eta_t)}{2m}\cdot\sin\frac{\pi(\lambda_h+\lambda_t)}{2m}\right)\prod_{h=1}^{n}\left(\sin\frac{\pi\eta_h}{2m}\cdot\sin\frac{\pi\lambda_h}{2m}\right).\tag{6.12}$$

Proof. We proceed in the same way as in the proof of Theorem 25. This time we have to estimate the two determinants in (2.14). As a preparatory step, we replace the two sums over r in the determinants by 2 times the same sums, but restricted to r from 0 to $2m-1$, which can be safely done when η, λ and k are chosen so that paths from η to λ in k steps exist. (We already did an analogous transformation in the proof of Theorem 19). Then, by expanding the two determinants, we obtain

$$\frac{1}{2m^n}\sum_{r_1,\dots,r_n=1}^{2m-1}\left(2^n\prod_{j=1}^{n}\cos\frac{\pi r_j}{2m}\right)^k\left(\prod_{j=1}^{n}\sin\frac{\pi r_j\eta_j}{m}\right)\det_{1\le h,t\le n}\left(\sin\frac{\pi r_h\lambda_t}{m}\right)$$
$$+\frac{1}{2m^n}\sum_{r_1,\dots,r_n=0}^{2m-1}\left(2^n\prod_{j=1}^{n}\cos\frac{\pi(2r_j+1)}{4m}\right)^k$$
$$\cdot\left(\prod_{j=1}^{n}\sin\frac{\pi(2r_j+1)\eta_j}{2m}\right)\det_{1\le h,t\le n}\left(\sin\frac{\pi(2r_h+1)\lambda_t}{2m}\right),$$

which is again a finite sum of the form $\sum_\ell c_\ell b_\ell^k$, with both the c_ℓ's and b_ℓ's independent of k.

As it turns out, also here the dominating terms (the terms for which b_ℓ has largest modulus) come from the second determinant. More precisely, these are the terms corresponding to the subsets $\{r_1,r_2,\dots,r_n\}$ of

$$\{0,1,\dots,n,2m-n,2m-n+1,\dots,2m-1\},$$

with the property that all r_h's are distinct and $r_h\ne 2m-1-r_t$ for all h and t. Clearly, there are 2^n such sets $\{r_1,r_2,\dots,r_n\}$. For each fixed set, the sum of the corresponding terms $c_\ell b_\ell^k$ is equal to

$$\frac{1}{2m^n}\left(2^n\prod_{j=1}^{n}\cos\frac{\pi(2j-1)}{4m}\right)^k\det_{1\le h,t\le n}\left(\sin\frac{\pi(2h-1)\lambda_t}{2m}\right)\det_{1\le h,t\le n}\left(\sin\frac{\pi(2h-1)\eta_t}{2m}\right).\tag{6.13}$$

The two determinants in this expression have already been evaluated in (6.3). If this is substituted in (6.13), and if the result is multiplied by 2^n, then we obtain exactly (6.12). $\square$

Theorem 28. *Let m be a positive integer or half-integer. Furthermore, let $\eta = (\eta_1, \eta_2, \dots, \eta_n)$ be a vector of integers or of half-integers in the alcove $\mathcal{A}_m^{\tilde{B}_n}$ of type $\tilde{B}_n$ (defined in (2.3)). Then, as k tends to infinity the number of random walks which start at η and proceed for exactly k diagonal steps, which stay in the alcove $\mathcal{A}_m^{\tilde{B}_n}$, is asymptotically*

$$\frac{4^{n^2}}{2(2m)^n}\left(2^n\prod_{j=1}^n\cos\frac{(2j-1)\pi}{4m}\right)^k\prod_{1\le h<t\le n}\left(\sin\frac{\pi(\eta_h-\eta_t)}{2m}\cdot\sin\frac{\pi(t-h)}{2m}\right.$$
$$\left.\cdot\sin\frac{\pi(\eta_h+\eta_t)}{2m}\cdot\sin\frac{\pi(t+h)}{2m}\right)\prod_{h=1}^n\left(\sin\frac{\pi\eta_h}{2m}\cdot\sin\frac{\pi h}{2m}\right)$$
$$\times\left(\prod_{h=1}^{2n}\frac{\sin\frac{\pi(m-n+h)}{4m}}{\sin\frac{\pi h}{4m}}\prod_{h=1}^{n-1}\prod_{t=1}^{n}\frac{\sin\frac{\pi(m+t-h)}{2m}}{\sin\frac{\pi(n+t-h)}{2m}}+\prod_{h=1}^{2n}\frac{\sin\frac{\pi(m-n+h-1)}{4m}}{\sin\frac{\pi h}{4m}}\prod_{h=1}^{n-1}\prod_{t=1}^{n}\frac{\sin\frac{\pi(m+t-h-1)}{2m}}{\sin\frac{\pi(n+t-h)}{2m}}\right), \tag{6.14}$$

if m is an integer and $k+2\eta_j$ is even, it is asymptotically

$$\frac{4^{n^2}}{(2m)^n}\left(2^n\prod_{j=1}^n\cos\frac{(2j-1)\pi}{4m}\right)^k\prod_{1\le h<t\le n}\left(\sin\frac{\pi(\eta_h-\eta_t)}{2m}\cdot\sin\frac{\pi(t-h)}{2m}\right.$$
$$\left.\cdot\sin\frac{\pi(\eta_h+\eta_t)}{2m}\cdot\sin\frac{\pi(t+h)}{2m}\right)\prod_{h=1}^n\left(\sin\frac{\pi\eta_h}{2m}\cdot\sin\frac{\pi h}{2m}\right)$$
$$\times\prod_{h=1}^{2n}\frac{\sin\frac{\pi(\lfloor m\rfloor-n+h)}{4m}}{\sin\frac{\pi h}{4m}}\prod_{h=1}^{n-1}\prod_{t=1}^{n}\frac{\sin\frac{\pi(\lfloor m\rfloor+t-h)}{2m}}{\sin\frac{\pi(n+t-h)}{2m}}, \tag{6.15}$$

if m is a half-integer and $k+2\eta_j$ is even, it is asymptotically

$$\frac{4^{n^2}}{(2m)^n}\left(2^n\prod_{j=1}^n\cos\frac{(2j-1)\pi}{4m}\right)^k\prod_{1\le h<t\le n}\left(\sin\frac{\pi(\eta_h-\eta_t)}{2m}\cdot\sin\frac{\pi(t-h)}{2m}\right.$$
$$\left.\cdot\sin\frac{\pi(\eta_h+\eta_t)}{2m}\cdot\sin\frac{\pi(t+h-1)}{2m}\right)\prod_{h=1}^n\left(\sin\frac{\pi\eta_h}{2m}\cdot\sin\frac{\pi(2h-1)}{4m}\right)$$
$$\times\prod_{h=1}^{2n}\frac{\sin\frac{\pi(m-n+h)}{4m}}{\sin\frac{\pi h}{4m}}\prod_{h=1}^{n}\prod_{t=1}^{n}\frac{\sin\frac{\pi(m+t-h)}{2m}}{\sin\frac{\pi(n+t-h)}{2m}}\prod_{h=1}^{n}\frac{\sin\frac{\pi h}{2m}}{\sin\frac{\pi(m-n+2h)}{4m}}\frac{\cos\frac{\pi(2h-1)}{4m}}{\cos\frac{\pi(m-n+2h-1)}{4m}}, \tag{6.16}$$

if m is an integer and $k+2\eta_j$ is odd, and it is asymptotically

$$\frac{4^{n^2}}{2(2m)^n}\left(2^n\prod_{j=1}^n\cos\frac{(2j-1)\pi}{4m}\right)^k\prod_{1\le h<t\le n}\left(\sin\frac{\pi(\eta_h-\eta_t)}{2m}\cdot\sin\frac{\pi(t-h)}{2m}\right.$$
$$\left.\cdot\sin\frac{\pi(\eta_h+\eta_t)}{2m}\cdot\sin\frac{\pi(t+h-1)}{2m}\right)\prod_{h=1}^n\left(\sin\frac{\pi\eta_h}{2m}\cdot\sin\frac{\pi(2h-1)}{4m}\right)$$

$$\times\left(\prod_{h=1}^{2n}\frac{\sin\frac{\pi(\lceil m\rceil-n+h)}{4m}}{\sin\frac{\pi h}{4m}}\prod_{h=1}^{n}\prod_{t=1}^{n}\frac{\sin\frac{\pi(\lceil m\rceil+t-h)}{2m}}{\sin\frac{\pi(n+t-h)}{2m}}\prod_{h=1}^{n}\frac{\sin\frac{\pi h}{2m}}{\sin\frac{\pi(\lceil m\rceil-n+2h)}{4m}}\frac{\cos\frac{\pi(2h-1)}{4m}}{\cos\frac{\pi(\lceil m\rceil-n+2h-1)}{4m}}\right.$$
$$\left.+\prod_{h=1}^{2n}\frac{\sin\frac{\pi(\lfloor m\rfloor-n+h)}{4m}}{\sin\frac{\pi h}{4m}}\prod_{h=1}^{n}\prod_{t=1}^{n}\frac{\sin\frac{\pi(\lfloor m\rfloor+t-h)}{2m}}{\sin\frac{\pi(n+t-h)}{2m}}\prod_{h=1}^{n}\frac{\sin\frac{\pi h}{2m}}{\sin\frac{\pi(\lfloor m\rfloor-n+2h)}{4m}}\frac{\cos\frac{\pi(2h-1)}{4m}}{\cos\frac{\pi(\lfloor m\rfloor-n+2h-1)}{4m}}\right),\tag{6.17}$$

if m is a half-integer and $k+2\eta_j$ is odd.

Proof. Here, in view of Theorem 27, we have to carry out the sum of (6.12) over all possible λ, i.e., over all $\lambda_1>\lambda_2>\cdots>\lambda_n>0$ with $\lambda_1+\lambda_2<2m$, where $k\equiv 2\eta_j+2\lambda_j$ mod 2. Leaving away factors which are independent of λ, the problem is to compute the sum

$$\sideset{}{'}\sum_{\substack{\lambda_1>\cdots>\lambda_n>0\\ \lambda_1+\lambda_2<2m}}\prod_{1\le h<t\le n}\left(\sin\frac{\pi(\lambda_h-\lambda_t)}{2m}\cdot\sin\frac{\pi(\lambda_h+\lambda_t)}{2m}\right)\prod_{h=1}^{n}\sin\frac{\pi\lambda_h}{2m},\tag{6.18}$$

where the sum $\sum'$ is restricted to integral, respectively to half-integral $\lambda_1,\lambda_2,\dots,\lambda_n$, depending on whether $k+2\eta_j$ is even or odd.

As in previous proofs, we have to distinguish between several cases. Let us first restrict the sum (6.18) to integral $\lambda_1,\lambda_2,\dots,\lambda_n$. Then, by repeating the argument from the proof of Theorem 26 of replacement of λ_1 by $2m-\lambda_1$, and using notation from that proof, we obtain that the sum (6.18) equals $U_1(m)+U_1(m-1)$ if m is an integer, and it equals $2U_1(\lfloor m\rfloor)$ if m is a half-integer. In the proof of Theorem 26 it was shown how to evaluate the sum $U_1(c)$ by means of Lemma C5. If the result is substituted, we arrive at (6.14) and (6.15).

If we restrict the sum (6.18) to half-integral $\lambda_1,\lambda_2,\dots,\lambda_n$, then we are faced with the problem of evaluating the sum

$$V(c)=\sideset{}{'}\sum_{c\ge\lambda_1>\cdots>\lambda_n>0}\prod_{1\le h<t\le n}\left(\sin\frac{\pi(\lambda_h-\lambda_t)}{2m}\cdot\sin\frac{\pi(\lambda_h+\lambda_t)}{2m}\right)\prod_{h=1}^{n}\sin\frac{\pi\lambda_h}{2m},$$

where the sum $\sum'$ is over all *half-integral* $\lambda_1,\lambda_2,\dots,\lambda_n$. More specifically, using the replacement of λ_1 by $2m-\lambda_1$ another time, one sees readily that the sum (6.18) equals $2V(m-\frac{1}{2})$ if m is an integer, and it equals $V(m)+V(m-1)$ if m is a half-integer.

In order to evaluate $V(c)$, where c is a half-integer, we use (6.3) to rewrite it as

$$\sideset{}{'}\sum_{c\ge\lambda_1>\cdots>\lambda_n>0}2^{-n^2+n}\det_{1\le h,t\le n}\big(\sin(\pi(2h-1)\lambda_t/2m)\big)$$
$$=\sideset{}{'}\sum_{c\ge\lambda_1>\cdots>\lambda_n>0}\frac{1}{2^{n^2}i^n}\det_{1\le h,t\le n}\big((e^{\pi i(2h-1)/2m})^{\lambda_t}-(e^{\pi i(2h-1)/2m})^{-\lambda_t}\big).$$

Replacing λ_t by $\lambda_t+n-t+\frac{1}{2}$, $t=1,2,\dots,n$, we obtain the expression

$$\sum_{\lceil c-n\rceil\ge\lambda_1\ge\cdots\ge\lambda_n\ge0}\frac{1}{2^{n^2}i^n}\det_{1\le h,t\le n}\big((e^{\pi i(2h-1)/2m})^{\lambda_t+n-t+1/2}-(e^{\pi i(2h-1)/2m})^{-(\lambda_t+n-t+1/2)}\big).\tag{6.19}$$

(Note that the last sum is over *integral* $\lambda_1, \lambda_2, \dots, \lambda_n$.) This determinant can be expressed in terms of an *odd orthogonal character*. (See (C.1) for the definition.) Writing again q for $e^{\pi i/2m}$, the sum in (6.19) equals

$$\frac{1}{2^{n^2} i^n} \det_{1\le h,t\le n} \left((q^{2h-1})^{n-t+\frac12} - (q^{2h-1})^{-(n-t+\frac12)}\right) \sum_{\lceil c-n\rceil \ge \lambda_1 \ge \dots \ge \lambda_n \ge 0} so^{odd}_\lambda(q^{2n-1}, q^{2n-3}, \dots, q). \tag{6.20}$$

Again there is a formula which allows us to evaluate the sum in the last line (see [27, (3.2)]),

$$s_{((a^{n-p},(a-1)^p)}(x_1, x_1^{-1}, \dots, x_p, x_p^{-1}, 1) = \sum_{\substack{a\ge\nu_1\ge\dots\ge\nu_n\ge0\\ \operatorname{oddrows}\left((a^r)/\nu\right)=p}} so^{odd}_\nu(x_1, \dots, x_p),$$

where the notation $\left(a^{n-p}, (a-1)^p\right)$ is a short notation for the vector in which the first $n-p$ components are a, the next p components are $a-1$, followed by $n+1$ components all of which are 0, and where $\operatorname{oddrows}\left((a^n)/\nu\right) = p$ means that the number of rows of odd length in the skew shape $(a^n)/\nu$ equals exactly p. Use of this formula in (6.20) gives

$$\frac{1}{2^{n^2} i^n} \det_{1\le h,t\le n} \left((q^{2h-1})^{n-t+\frac12} - (q^{2h-1})^{-(n-t+\frac12)}\right)$$
$$\times \sum_{p=0}^{n} s_{\left(\lceil c-n\rceil^{n-p}, \lceil c-n-1\rceil^p\right)}(q^{2n-1}, q^{2n-3}, \dots, q^3, q, 1, q^{-1}, q^{-3}, \dots, q^{-2n+3}, q^{-2n+1}).$$

We now use Lemma C5 to evaluate the Schur function in the last line to obtain the expression

$$\frac{1}{2^{n^2} i^n} \det_{1\le h,t\le n} \left((q^{2h-1})^{n-t+\frac12} - (q^{2h-1})^{-(n-t+\frac12)}\right)$$
$$\times \prod_{h=1}^{2n} \frac{\left(q^{\frac{\lceil c\rceil-n+h}{2}} - q^{-\frac{\lceil c\rceil-n+h}{2}}\right)}{\left(q^{\frac h2} - q^{-\frac h2}\right)} \prod_{h=1}^{n}\prod_{t=1}^{n} \frac{\left(q^{\lceil c\rceil+t-h} - q^{-\lceil c\rceil-t+h}\right)}{\left(q^{n+t-h} - q^{-n-t+h}\right)} \prod_{h=1}^{n} \left(q^h - q^{-h}\right)^2$$
$$\times \sum_{p=0}^{n} \prod_{h=1}^{n} \frac{1}{\left(q^{\lceil c\rceil-n+p+h} - q^{-\lceil c\rceil+n-p-h}\right)} \frac{1}{\prod_{h=1}^{p}\left(q^h - q^{-h}\right) \prod_{h=1}^{n-p}\left(q^h - q^{-h}\right)}$$
$$\cdot \frac{\left(q^{\frac{\lceil c\rceil-n}{2}} - q^{-\frac{\lceil c\rceil-n}{2}}\right)\left(q^{\frac{\lceil c\rceil-n}{2}+p} + q^{-\frac{\lceil c\rceil-n}{2}-p}\right)}{\left(q^{\lceil c\rceil-n+p} - q^{-\lceil c\rceil+n-p}\right)}.$$

In terms of the standard basic hypergeometric notation

$${}_r\phi_s\left[\begin{matrix}a_1, \dots, a_r\\ b_1, \dots, b_s\end{matrix}; q, z\right] = \sum_{\ell=0}^{\infty} \frac{(a_1;q)_\ell \cdots (a_r;q)_\ell}{(q;q)_\ell (b_1;q)_\ell \cdots (b_s;q)_\ell} \left((-1)^\ell q^{\binom{\ell}{2}}\right)^{s-r+1} z^\ell , \tag{6.21}$$

where the shifted q-factorials $(a;q)_\ell$ are defined by $(a;q)_\ell := (1-a)(1-aq)\cdots(1-aq^{\ell-1})$, $\ell \ge 1$, $(a;q)_0 := 1$, this can be written in the form

$$\frac{1}{2^{n^2} i^n} \det_{1\le h,t\le n} \left((q^{2h-1})^{n-t+\frac{1}{2}} - (q^{2h-1})^{-(n-t+\frac{1}{2})} \right)$$
$$\times \prod_{h=1}^{2n} \frac{\left(q^{\frac{\lceil c\rceil -n+h}{2}} - q^{-\frac{\lceil c\rceil -n+h}{2}}\right)}{\left(q^{\frac{h}{2}} - q^{-\frac{h}{2}}\right)} \prod_{h=1}^{n}\prod_{t=1}^{n} \frac{\left(q^{\lceil c\rceil +t-h} - q^{-\lceil c\rceil -t+h}\right)}{\left(q^{n+t-h} - q^{-n-t+h}\right)} \prod_{h=1}^{n} \left(q^h - q^{-h}\right)^2$$
$$\times \frac{q^{\lceil c\rceil n+n}}{(q^2;q^2)_n\,(q^{2\lceil c\rceil -2n+2};q^2)_n} {}_3\phi_2\left[\begin{matrix} q^{2\lceil c\rceil -2n}, -q^{\lceil c\rceil -n+2}, q^{-2n} \\ -q^{\lceil c\rceil -n}, q^{2\lceil c\rceil +2}\end{matrix}; q^2, -q^{2n+1}\right]. \quad (6.22)$$

The determinant is again easily evaluated by means of (B.2). On the other hand, the ${}_3\phi_2$-series in the last line can be evaluated by a limit case of Jackson's very-well-poised ${}_8\phi_7$-summation (see [14, (2.6.2); Appendix (II.22)])

$${}_8\phi_7\left[\begin{matrix} A, \sqrt{A}q, -\sqrt{A}q, B, C, D, A^2q^{1+N}/BCD, q^{-N} \\ \sqrt{A}, -\sqrt{A}, Aq/B, Aq/C, Aq/D, BCD/Aq^N, Aq^{1+N}\end{matrix}; q, q\right]$$
$$= \frac{(Aq;q)_N (Aq/BC;q)_N (Aq/BD;q)_N (Aq/CD;q)_N}{(Aq/B;q)_N (Aq/C;q)_N (Aq/D;q)_N (Aq/BCD;q)_N}, \quad (6.23)$$

where N is a nonnegative integer. Namely, if in (6.23) we let $N \to \infty$, put $D = \sqrt{A}$ and $C = -\sqrt{A}$, and finally replace q by q^2, we are left with

$${}_3\phi_2\left[\begin{matrix} A, -\sqrt{A}q^2, B \\ -\sqrt{A}, Aq^2/B\end{matrix}; q^2, -\frac{q}{B}\right] = \frac{(-q;q^2)_\infty\,(-\sqrt{A}q/B;q^2)_\infty\,(Aq^2;q^2)_\infty\,(\sqrt{A}q^2/B;q^2)_\infty}{(-\sqrt{A}q;q^2)_\infty\,(-q/B;q^2)_\infty\,(\sqrt{A}q^2;q^2)_\infty\,(Aq^2/B;q^2)_\infty}. \quad (6.24)$$

The ${}_3\phi_2$-series in (6.22) is a special case of the above ${}_3\phi_2$-series in which $A = q^{2\lceil c\rceil -2n}$ and $B = q^{-2n}$. If we substitute the corresponding right-hand side of (6.24) in (6.22), we have evaluated the sum $V(c)$. This, in its turn, leads to the expressions (6.16) and (6.17). □

7. Asymptotics for random walks in alcoves of type $\tilde{D}$

In this final section we turn our attention to the walks in the alcove $\mathcal{A}_m^{\tilde{D}_n}$ of type $\tilde{D}_n$. We determine the asymptotic behaviour of the number of walks from a given starting point to a given end point which stay in $\mathcal{A}_m^{\tilde{D}_n}$ as the number of steps becomes large, as well as the asymptotic behaviour of the number of those walks which start at a given point but may terminate anywhere. In technical terms, we determine the asymptotic behaviour of the expressions given by Theorems 10 and 11 as k becomes large, and as well if these expressions are summed over all possible end points of the walks.

The first two theorems in this section give our results for the case of walks with standard steps.

Theorem 29. *Let m be a positive integer. Furthermore, let $\eta = (\eta_1, \eta_2, \dots, \eta_n)$ and $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ be vectors of integers in the alcove $\mathcal{A}_m^{\tilde{D}_n}$ of type $\tilde{D}_n$ (defined in* (2.4)*). Then, as k tends to infinity such that $k \equiv |\eta| + |\lambda| \bmod 2$, the number of random walks from η to λ with exactly k standard steps, which stay in the alcove $\mathcal{A}_m^{\tilde{D}_n}$, is asymptotically*

$$\frac{4^{n^2}}{(8m)^n}\left(2\frac{\sin\frac{n\pi}{2m}\cos\frac{(n-1)\pi}{2m}}{\sin\frac{\pi}{2m}}\right)^k$$

$$\times \prod_{1\le h<t\le n} \left(\sin \frac{\pi(\eta_h-\eta_t)}{2m} \cdot \sin \frac{\pi(\lambda_h-\lambda_t)}{2m} \cdot \sin \frac{\pi(\eta_h+\eta_t)}{2m} \cdot \sin \frac{\pi(\lambda_h+\lambda_t)}{2m} \right). \quad (7.1)$$

Proof. We proceed as in the proof of Theorem 25. Since we have done similar calculations already several times, we shall be brief here.

We have to find the asymptotic behaviour of (2.15) (instead of the rather similar (2.13), with which we dealt in the proof of Theorem 25). By applying arguments very similar to those in the proof of Theorem 25, we infer that the dominating terms in the expansion of (2.15) come from the expansion of the third determinant. To be precise, these dominating terms add up to $2\cdot 2^{n-1}$ times

$$\frac{1}{4m^n} \left(2\sum_{j=0}^{n-1} \cos \frac{\pi j}{m} \right)^k \det_{1\le h,t\le n} \left(\cos \frac{\pi(h-1)\eta_t}{m} \right) \det_{1\le h,t\le n} \left(\cos \frac{\pi(h-1)\lambda_t}{m} \right). \quad (7.2)$$

We have

$$\sum_{j=0}^{n-1} \cos \frac{\pi j}{m} = \frac{\sin \frac{n\pi}{2m} \cdot \cos \frac{(n-1)\pi}{2m}}{\sin \frac{\pi}{2m}}.$$

On the other hand, the first determinant in (7.2) can be rewritten in the form

$$2^{-n} \det_{1\le h,t\le n} \left(e^{\frac{\pi i (h-1)\eta_t}{m}} + e^{-\frac{\pi i (h-1)\eta_t}{m}} \right),$$

and can thus be evaluated by means of (B.3). The result is that

$$\det_{1\le h,t\le n} \left(\cos \frac{\pi(h-1)\eta_t}{m} \right) = 2^{n^2-2n+1} \prod_{1\le h<t\le n} \left(\sin \frac{\pi(\eta_h-\eta_t)}{2m} \cdot \sin \frac{\pi(\eta_h+\eta_t)}{2m} \right). \quad (7.3)$$

If this is substituted in (7.2) (and as well the analogous evaluation of the second determinant in (7.2)), and if the result is multiplied by $2\cdot 2^{n-1} = 2^n$, then we obtain exactly (7.1). ☐

If the starting point is fixed but the end point is not, we have the following result.

Theorem 30. *Let m be a positive integer or half-integer. Furthermore, let $\eta = (\eta_1, \eta_2, \dots, \eta_n)$ be a vector of integers in the alcove $\mathcal{A}_m^{\tilde D_n}$ of type $\tilde D_n$ (defined in (2.4)). Then, as k tends to infinity, the number of random walks which start at η and proceed for exactly k standard steps, which stay in the alcove $\mathcal{A}_m^{\tilde D_n}$, is asymptotically*

$$\frac{4^{n^2}}{2^n(8m)^n} \left(2\frac{\sin \frac{n\pi}{2m} \cos \frac{(n-1)\pi}{2m}}{\sin \frac{\pi}{2m}} \right)^k$$
$$\times \prod_{1\le h<t\le n} \left(\sin \frac{\pi(\eta_h-\eta_t)}{2m} \cdot \sin \frac{\pi(t-h)}{2m} \cdot \sin \frac{\pi(\eta_h+\eta_t)}{2m} \cdot \sin \frac{\pi(t+h-2)}{2m} \right)$$
$$\times \left(\frac{1}{2^{n-1}} \frac{\prod_{h=1}^n \sin \frac{\pi(m-n+2h)}{2m}}{\prod_{h=1}^{n-1} \sin \frac{\pi h}{2m}} \prod_{1\le h<t\le n} \frac{\sin \frac{\pi(m-n+h+t-1)}{2m} \cdot \sin \frac{\pi(m-n+t+h)}{2m}}{\sin^2 \frac{\pi(t+h-2)}{2m}} \right.$$
$$\cdot \sum_{k=1}^n \frac{(-1)^{n-k}(m-n+2k)}{\prod_{h=1}^n \sin \frac{\pi(m-n+k+h)}{2m} \prod_{h=1}^{k-1} \sin \frac{\pi h}{2m} \prod_{h=1}^{n-k} \sin \frac{\pi h}{2m}}$$

$$+\frac{1}{2^{n-1}}\frac{\prod_{h=1}^{n}\sin\frac{\pi(m-n+2h-1)}{2m}}{\prod_{h=1}^{n-1}\sin\frac{\pi h}{2m}}\prod_{1\le h<t\le n}\frac{\sin\frac{\pi(m-n+h+t-2)}{2m}\cdot\sin\frac{\pi(m-n+t+h-1)}{2m}}{\sin^2\frac{\pi(t+h-2)}{2m}}$$
$$\cdot\sum_{k=1}^{n}\frac{(-1)^{n-k}(m-n+2k-2)}{\prod_{h=1}^{n}\sin\frac{\pi(m-n+k+h-1)}{2m}\prod_{h=1}^{k-1}\sin\frac{\pi h}{2m}\prod_{h=1}^{n-k}\sin\frac{\pi h}{2m}}$$
$$+\frac{1}{2^{n-1}}\frac{\prod_{h=1}^{n}\sin\frac{\pi(m-n+2h-1)}{2m}}{\prod_{h=1}^{n-1}\sin\frac{\pi h}{2m}}\prod_{1\le h<t\le n}\frac{\sin\frac{\pi(m-n+h+t)}{2m}\cdot\sin\frac{\pi(m-n+t+h-1)}{2m}}{\sin^2\frac{\pi(t+h-2)}{2m}}$$
$$\cdot\sum_{k=1}^{n}\frac{(-1)^{n-k}(m-n+2k-1)}{\prod_{h=1}^{n}\sin\frac{\pi(m-n+k+h-1)}{2m}\prod_{h=1}^{k-1}\sin\frac{\pi h}{2m}\prod_{h=1}^{n-k}\sin\frac{\pi h}{2m}}$$
$$+\frac{1}{2^{n-1}}\frac{\prod_{h=1}^{n}\sin\frac{\pi(m-n+2h-2)}{2m}}{\prod_{h=1}^{n-1}\sin\frac{\pi h}{2m}}\prod_{1\le h<t\le n}\frac{\sin\frac{\pi(m-n+h+t-1)}{2m}\cdot\sin\frac{\pi(m-n+t+h-2)}{2m}}{\sin^2\frac{\pi(t+h-2)}{2m}}$$
$$\cdot\sum_{k=1}^{n}\frac{(-1)^{n-k}(m-n+2k-3)}{\prod_{h=1}^{n}\sin\frac{\pi(m-n+k+h-2)}{2m}\prod_{h=1}^{k-1}\sin\frac{\pi h}{2m}\prod_{h=1}^{n-k}\sin\frac{\pi h}{2m}}$$
$$+(-1)^{mn+k+|\eta|}\prod_{h=1}^{n-1}\frac{1}{\cos\frac{\pi(h-1)}{2m}}\prod_{1\le h<t\le n}\frac{\sin\frac{\pi(m-n+h+t-1)}{2m}\cdot\sin\frac{\pi(m-n+t+h)}{2m}}{\sin^2\frac{\pi(t+h-2)}{2m}}$$
$$+(-1)^{mn+k+|\eta|}\prod_{h=1}^{n-1}\frac{1}{\cos\frac{\pi(h-1)}{2m}}\prod_{1\le h<t\le n}\frac{\sin\frac{\pi(m-n+h+t-2)}{2m}\cdot\sin\frac{\pi(m-n+t+h-1)}{2m}}{\sin^2\frac{\pi(t+h-2)}{2m}}$$
$$+(-1)^{mn+k+|\eta|}\prod_{h=1}^{n-1}\frac{1}{\cos\frac{\pi(h-1)}{2m}}\prod_{1\le h<t\le n}\frac{\sin\frac{\pi(m-n+h+t)}{2m}\cdot\sin\frac{\pi(m-n+t+h-1)}{2m}}{\sin^2\frac{\pi(t+h-2)}{2m}}$$
$$+(-1)^{mn+k+|\eta|}\prod_{h=1}^{n-1}\frac{1}{\cos\frac{\pi(h-1)}{2m}}\prod_{1\le h<t\le n}\frac{\sin\frac{\pi(m-n+h+t-1)}{2m}\cdot\sin\frac{\pi(m-n+t+h-2)}{2m}}{\sin^2\frac{\pi(t+h-2)}{2m}}\Bigg) \tag{7.4}$$

if m is an integer, and it is asymptotically

$$\frac{4^{n^2-n+1}}{(8m)^n}\left(2\frac{\sin\frac{n\pi}{2m}\cos\frac{(n-1)\pi}{2m}}{\sin\frac{\pi}{2m}}\right)^k$$
$$\times\prod_{1\le h<t\le n}\left(\sin\frac{\pi(\eta_h-\eta_t)}{2m}\cdot\sin\frac{\pi(t-h)}{2m}\cdot\sin\frac{\pi(\eta_h+\eta_t)}{2m}\cdot\sin\frac{\pi(t+h-2)}{2m}\right)$$
$$\Bigg(\frac{\prod_{h=1}^{n}\sin\frac{\pi(\lfloor m\rfloor-n+2h)}{2m}}{\prod_{h=1}^{n-1}\sin\frac{\pi h}{2m}}\prod_{1\le h<t\le n}\frac{\sin\frac{\pi(\lfloor m\rfloor-n+h+t-1)}{2m}\cdot\sin\frac{\pi(\lfloor m\rfloor-n+t+h)}{2m}}{\sin^2\frac{\pi(t+h-2)}{2m}}$$
$$\cdot\sum_{k=1}^{n}\frac{(-1)^{n-k}(\lfloor m\rfloor-n+2k)}{\prod_{h=1}^{n}\sin\frac{\pi(\lfloor m\rfloor-n+k+h)}{2m}\prod_{h=1}^{k-1}\sin\frac{\pi h}{2m}\prod_{h=1}^{n-k}\sin\frac{\pi h}{2m}}$$
$$+\frac{\prod_{h=1}^{n}\sin\frac{\pi(\lfloor m\rfloor-n+2h-1)}{2m}}{\prod_{h=1}^{n-1}\sin\frac{\pi h}{2m}}\prod_{1\le h<t\le n}\frac{\sin\frac{\pi(\lfloor m\rfloor-n+h+t)}{2m}\cdot\sin\frac{\pi(\lfloor m\rfloor-n+t+h-1)}{2m}}{\sin^2\frac{\pi(t+h-2)}{2m}}$$

$$\left.\cdot\sum_{k=1}^{n}\frac{(-1)^{n-k}(\lfloor m\rfloor-n+2k-1)}{\prod_{h=1}^{n}\sin\frac{\pi(\lfloor m\rfloor-n+k+h-1)}{2m}\prod_{h=1}^{k-1}\sin\frac{\pi h}{2m}\prod_{h=1}^{n-k}\sin\frac{\pi h}{2m}}\right)\tag{7.5}$$

if m is a half-integer.

Proof. Here, in view of Theorem 29, we have to carry out the sum of (7.1) over all possible λ, i.e., over all $\lambda_1>\lambda_2>\cdots>\lambda_{n-1}>|\lambda_n|$ with $\lambda_1+\lambda_2<2m$, where $|\lambda|\equiv k+|\eta| \bmod 2$. Leaving away factors which are independent of λ, the problem is to compute the sum

$$\sideset{}{'}\sum_{\substack{\lambda_1>\lambda_2>\cdots>\lambda_{n-1}>|\lambda_n|\\ \lambda_1+\lambda_2<2m}}\prod_{1\le h<t\le n}\left(\sin\frac{\pi(\lambda_h-\lambda_t)}{2m}\cdot\sin\frac{\pi(\lambda_h+\lambda_t)}{2m}\right),\tag{7.6}$$

where the sum $\sum'$ is either restricted to those λ for which $|\lambda|$ is even, or to those for which $|\lambda|$ is odd, depending on the parity of $k+|\eta|$.

Let first m be an integer. As in the proof of Theorem 26, instead of a sum $\sum'$ where we have to deal with the unwieldy constraints $\lambda_1+\lambda_2<2m$ and $\lambda_{n-1}>|\lambda_n|$, we would rather prefer sums where the only constraint is of the form $\lambda_1>\lambda_2>\cdots>\lambda_n$. In order to get rid of the constraint $\lambda_1+\lambda_2<2m$, we apply again the argument of replacement of λ_1 by $2m-\lambda_1$ that we already used in the proofs of Theorems 26 and 28. Similarly, in order to get rid of the constraint $\lambda_{n-1}>|\lambda_n|$, we observe that the summand in (7.6) remains invariant under the replacement of λ_n by $-\lambda_n$, and that $(\lambda_1,\lambda_2,\dots,\lambda_n)$ satisfies the conditions on the summation indices if and only if $(\lambda_1,\lambda_2,\dots,-\lambda_n)$ does. Hence, we may rewrite the sum in (7.6) as

$$\begin{aligned}\sideset{}{'}\sum_{m\ge\lambda_1>\cdots>\lambda_n\ge0}\prod_{1\le h<t\le n}&\left(\sin\frac{\pi(\lambda_h-\lambda_t)}{2m}\cdot\sin\frac{\pi(\lambda_h+\lambda_t)}{2m}\right)\\&+\sideset{}{'}\sum_{m-1\ge\lambda_1>\cdots>\lambda_n\ge0}\prod_{1\le h<t\le n}\left(\sin\frac{\pi(\lambda_h-\lambda_t)}{2m}\cdot\sin\frac{\pi(\lambda_h+\lambda_t)}{2m}\right)\\&+\sideset{}{'}\sum_{m\ge\lambda_1>\cdots>\lambda_n>0}\prod_{1\le h<t\le n}\left(\sin\frac{\pi(\lambda_h-\lambda_t)}{2m}\cdot\sin\frac{\pi(\lambda_h+\lambda_t)}{2m}\right)\\&+\sideset{}{'}\sum_{m-1\ge\lambda_1>\cdots>\lambda_n>0}\prod_{1\le h<t\le n}\left(\sin\frac{\pi(\lambda_h-\lambda_t)}{2m}\cdot\sin\frac{\pi(\lambda_h+\lambda_t)}{2m}\right),\end{aligned}\tag{7.7}$$

where, again, the sums $\sum'$ are either restricted to those λ for which $|\lambda|$ is even, or to those for which $|\lambda|$ is odd, depending on the parity of $k+|\eta|$.

The task of evaluating the sums in (7.7) will be accomplished if we are able to evaluate the (complete) sums

$$W_1(c)=\sum_{c\ge\lambda_1>\cdots>\lambda_n\ge0}\prod_{1\le h<t\le n}\left(\sin\frac{\pi(\lambda_h-\lambda_t)}{2m}\cdot\sin\frac{\pi(\lambda_h+\lambda_t)}{2m}\right)$$

and

$$W_2(c)=\sum_{c\ge\lambda_1>\cdots>\lambda_n>0}\prod_{1\le h<t\le n}\left(\sin\frac{\pi(\lambda_h-\lambda_t)}{2m}\cdot\sin\frac{\pi(\lambda_h+\lambda_t)}{2m}\right),$$

and their "signed variants"

$$W_3(c)=\sum_{c\ge\lambda_1>\dots>\lambda_n\ge0}(-1)^{|\lambda|}\prod_{1\le h<t\le n}\left(\sin\frac{\pi(\lambda_h-\lambda_t)}{2m}\cdot\sin\frac{\pi(\lambda_h+\lambda_t)}{2m}\right)$$

and

$$W_4(c)=\sum_{c\ge\lambda_1>\dots>\lambda_n>0}(-1)^{|\lambda|}\prod_{1\le h<t\le n}\left(\sin\frac{\pi(\lambda_h-\lambda_t)}{2m}\cdot\sin\frac{\pi(\lambda_h+\lambda_t)}{2m}\right)$$

for $c=m$ and $c=m-1$. The expression (7.7) is then equal to

$$\frac{1}{2}\Big(W_1(m)+W_1(m-1)+W_2(m)+W_2(m-1)+W_3(m)+W_3(m-1)+W_4(m)+W_4(m-1)\Big)$$

if the sum $\sum'$ is restricted to the λ's for which $|\lambda|$ is even, and it is equal to

$$\frac{1}{2}\Big(W_1(m)+W_1(m-1)+W_2(m)+W_2(m-1)-W_3(m)-W_3(m-1)-W_4(m)-W_4(m-1)\Big)$$

if the sum $\sum'$ is restricted to the λ's for which $|\lambda|$ is odd.

We first show how to evaluate $W_1(c)$ and $W_3(c)$. Using (7.3), we rewrite them uniformly as

$$\sum_{c\ge\lambda_1>\dots>\lambda_n\ge0}\varepsilon^{|\lambda|}2^{-n^2+2n-1}\det_{1\le h,t\le n}\big(\cos(\pi(h-1)\lambda_t/m)\big)$$
$$=\sum_{c\ge\lambda_1>\dots>\lambda_n\ge0}\frac{1}{2^{n^2-n+1}}\det_{1\le h,t\le n}\Big((\varepsilon e^{\pi i(h-1)/m})^{\lambda_t}+(\varepsilon e^{\pi i(h-1)/m})^{-\lambda_t}\Big),$$

with $\varepsilon=1$ or $\varepsilon=-1$ depending on whether we want to express $W_1(c)$ or $W_3(c)$. Replacing λ_t by λ_t+n-t, $t=1,2,\dots,n$, we obtain the expression

$$\sum_{c-n+1\ge\lambda_1\ge\dots\ge\lambda_n\ge0}\frac{1}{2^{n^2-n+1}}\det_{1\le h,t\le n}\Big((\varepsilon e^{\pi i(h-1)/m})^{\lambda_t+n-t}+(\varepsilon e^{\pi i(h-1)/m})^{-(\lambda_t+n-t)}\Big).\tag{7.8}$$

The determinant in the summand appears as a part in the definition of an *even orthogonal character* given in (C.49). Specifically, if we write q for $e^{\pi i/m}$, the sum in (7.8) equals

$$\frac{1}{2^{n^2-n+1}}\det_{1\le h,t\le n}\Big((\varepsilon q^{h-1})^{n-t}+(\varepsilon q^{h-1})^{-(n-t)}\Big)$$
$$\times\sum_{c-n+1\ge\lambda_1\ge\dots\ge\lambda_n\ge0}so^{even}_{\lambda}(\varepsilon q^{n-1},\varepsilon q^{n-2},\dots,\varepsilon q,\varepsilon).\tag{7.9}$$

(The reader should note that the second determinant in (C.49) vanishes if one of the variables x_t, $t=1,2,\dots,n$, is 1 or -1, which is the case in our situation.) The sum in (7.9) can be simplified by means of another character identity from [27]: given non-negative integers or half-integers a,b with $a\ge b$, the identity [27, (3.15)] implies

$$so^{even}_{(a^n)}(x_1,x_2,\dots,x_n)\cdot\sum_{p=-b}^{b}so^{even}_{(b^{n-1},p)}(x_1,x_2,\dots,x_n)$$

$$= \sum_{a+b\ge\nu_1\ge\nu_2\ge\cdots\ge\nu_n\ge a-b} so^{even}_\nu(x_1,x_2,\dots,x_n), \quad (7.10)$$

with the understanding that the sum on the left-hand side ranges over half-integral p if b is a half-integer, and that the sum on the right-hand side ranges over integers ν_t if $a+b$ is an integer, and over half-integers ν_t if $a+b$ is a half-integer. The notation (a^n) is short for the vector consisting of n components all of which equal to a, while the notation (b^{n-1},p) means the vector in which the first $n-1$ components are b, followed by a component p. If we use (7.10), then (7.9) becomes

$$\begin{aligned}
&\frac{1}{2^{n^2-n+1}}\det_{1\le h,t\le n}\left((\varepsilon q^{h-1})^{n-t}+(\varepsilon q^{h-1})^{-(n-t)}\right) so^{even}_{\left(((c-n+1)/2)^n\right)}(\varepsilon q^{n-1},\varepsilon q^{n-2},\dots,\varepsilon q,\varepsilon)\\
&\qquad\times\sum_{p=-(c-n+1)/2}^{(c-n+1)/2} so^{even}_{\left(((c-n+1)/2)^{n-1},p\right)}(\varepsilon q^{n-1},\varepsilon q^{n-2},\dots,\varepsilon q,\varepsilon)\\
&=\varepsilon^{(c-n+1)n}\frac{1}{2^{n^2-n+1}}\det_{1\le h,t\le n}\left((\varepsilon q^{h-1})^{n-t}+(\varepsilon q^{h-1})^{-(n-t)}\right)\\
&\qquad\qquad\times so^{even}_{\left(((c-n+1)/2)^n\right)}(q^{n-1},q^{n-2},\dots,q,1)\\
&\qquad\times\sum_{p=-(c-n+1)/2}^{(c-n+1)/2}\varepsilon^{(c-n+1-2p)/2}\, so^{even}_{\left(((c-n+1)/2)^{n-1},p\right)}(q^{n-1},q^{n-2},\dots,q,1).
\end{aligned}$$

Clearly, the first determinant can be evaluated by means of (B.3). The even orthogonal character of shape $\left(((c-n+1)/2)^n\right)$ can be evaluated by using the definition (C.49) and (B.3). (It should be observed that the second determinant in the numerator on the right-hand side of (C.49) vanishes if $x_n=\pm1$.) Finally, the sum over p of even orthogonal characters is evaluated in Lemma C7 if $\varepsilon=1$, respectively in Lemma C8 if $\varepsilon=-1$.

In order to evaluate $W_2(c)$ and $W_4(c)$, we proceed in a completely analogous fashion. In particular, the two sums $W_2(c)$ and $W_4(c)$ are equal to

$$\sum_{c-n+1\ge\lambda_1\ge\cdots\ge\lambda_n>0}\frac{1}{2^{n^2-n+1}}\det_{1\le h,t\le n}\left((\varepsilon e^{\pi i(h-1)/m})^{\lambda_t+n-t}+(\varepsilon e^{\pi i(h-1)/m})^{-(\lambda_t+n-t)}\right), \quad (7.11)$$

with $\varepsilon=1$ or $\varepsilon=-1$ depending on whether we want to express $W_2(c)$ or $W_4(c)$. By the use of (5.13), this sum becomes

$$\frac{1}{2^{n^2-n+1}}\det_{1\le h,t\le n}\left((\varepsilon q^{h-1})^{n-t}+(\varepsilon q^{h-1})^{-(n-t)}\right)\sum_{c-n+1\ge\lambda_1\ge\cdots\ge\lambda_n\ge1} so^{even}_\lambda(\varepsilon,\varepsilon q,\dots,\varepsilon q^{n-1}). \quad (7.12)$$

The only difference to (7.9) is that in the sum λ_n must be *positive*, instead of just *non-negative*. Hence, the remaining steps are the same, with the small modification that we use (7.10) with $a=(c-n+2)/2$ and $b=(c-n)/2$ to see that (7.12) equals

$$\varepsilon^{(c-n+1)n}\frac{1}{2^{n^2-n+1}}\det_{1\le h,t\le n}\left((\varepsilon q^{h-1})^{n-t}+(\varepsilon q^{h-1})^{-(n-t)}\right)$$

$$\times\, so^{even}_{\left(((c-n+2)/2)^n\right)}(q^{n-1},q^{n-2},\dots,q,1)$$
$$\times \sum_{p=-(c-n)/2}^{(c-n)/2} \varepsilon^{(c-n-2p)/2}\, so^{even}_{\left(((c-n)/2)^{n-1},p\right)}(q^{n-1},q^{n-2},\dots,q,1).$$

Again, the first determinant can be evaluated by means of (B.3), the even orthogonal character of shape $\left(((c-n+2)/2)^n\right)$ can be evaluated by using the definition (C.49) and (B.3), while the sum over p of even orthogonal characters is evaluated in Lemma C7, respectively in Lemma C8.

If everything is combined and simplified, the claimed formula (7.4) is eventually obtained.

In the case that m is a half-integer, an adaption of the above arguments shows that (7.6) equals

$$\sum_{\lfloor m\rfloor\ge\lambda_1>\dots>\lambda_n\ge 0}\ \prod_{1\le h<t\le n}\left(\sin\frac{\pi(\lambda_h-\lambda_t)}{2m}\cdot\sin\frac{\pi(\lambda_h+\lambda_t)}{2m}\right)$$
$$+\sum_{\lfloor m\rfloor\ge\lambda_1>\dots>\lambda_n>0}\ \prod_{1\le h<t\le n}\left(\sin\frac{\pi(\lambda_h-\lambda_t)}{2m}\cdot\sin\frac{\pi(\lambda_h+\lambda_t)}{2m}\right),$$

regardless of the parity of $k+|\eta|$. Hence, it equals $W_1(\lfloor m\rfloor)+W_2(\lfloor m\rfloor)$. On the other hand, we have seen above how to evaluate $W_1(\lfloor m\rfloor)$ and $W_2(\lfloor m\rfloor)$. If this is substituted and the resulting expressions are simplified, we eventually obtain (7.5). □

We conclude the section with the analogous results for the case of walks with diagonal steps.

Theorem 31. *Let m be a positive integer or half-integer. Furthermore, let $\eta=(\eta_1,\eta_2,\dots,\eta_n)$ and $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_n)$ be vectors of integers or of half-integers in the alcove $\mathcal{A}_m^{\tilde D_n}$ of type $\tilde D_n$ (defined in (2.4)). Then, as k tends to infinity such that $k\equiv 2\eta_j+2\lambda_j \bmod 2$, the number of random walks from η to λ with exactly k diagonal steps, which stay in the alcove $\mathcal{A}_m^{\tilde D_n}$, is asymptotically*

$$\frac{4^{n^2}}{2(8m)^n}\left(2^n\prod_{j=0}^{n-1}\cos\frac{j\pi}{2m}\right)^k$$
$$\times\prod_{1\le h<t\le n}\left(\sin\frac{\pi(\eta_h-\eta_t)}{2m}\cdot\sin\frac{\pi(\lambda_h-\lambda_t)}{2m}\sin\frac{\pi(\eta_h+\eta_t)}{2m}\cdot\sin\frac{\pi(\lambda_h+\lambda_t)}{2m}\right).\tag{7.13}$$

Proof. We proceed as in the proof of Theorem 27. We shall again be brief here.

We have to find the asymptotic behaviour of (2.16) (instead of the rather similar (2.14), with which we dealt in the proof of Theorem 27). By applying arguments very similar to those in the proof of Theorem 27, we infer that the dominating terms in the expansion of (2.16) come from the expansion of the third determinant. To be precise, these dominating terms add up to 2^{n-1} times

$$\frac{1}{4m^n}\left(2^n\prod_{j=0}^{n-1}\cos\frac{\pi j}{2m}\right)^k\det_{1\le h,t\le n}\left(\cos\frac{\pi(h-1)\lambda_t}{m}\right)\det_{1\le h,t\le n}\left(\cos\frac{\pi(h-1)\eta_t}{m}\right).\tag{7.14}$$

We have already evaluated these determinants in (7.3). If we use these evaluations in (7.14), and if the result is multiplied by 2^{n-1}, then we obtain exactly (7.13). $\square$

Theorem 32. *Let m be a positive integer or half-integer. Furthermore, let $\eta = (\eta_1, \eta_2, \dots, \eta_n)$ be a vector of integers or of half-integers in the alcove $\mathcal{A}_m^{\tilde{D}_n}$ of type $\tilde{D}_n$ (defined in (2.4)). Then, as k tends to infinity, the number of random walks which start at η and proceed for exactly k diagonal steps, which stay in the alcove $\mathcal{A}_m^{\tilde{D}_n}$, is asymptotically*

$$\begin{aligned}
&\frac{2\cdot 4^{n^2-n}}{(8m)^n}\left(2^n\prod_{j=0}^{n-1}\cos\frac{j\pi}{2m}\right)^k\\
&\quad\times\prod_{1\le h<t\le n}\left(\sin\frac{\pi(\eta_h-\eta_t)}{2m}\cdot\sin\frac{\pi(t-h)}{2m}\sin\frac{\pi(\eta_h+\eta_t)}{2m}\cdot\sin\frac{\pi(t+h-2)}{2m}\right)\\
&\qquad\times\left(\frac{\prod_{h=1}^n\sin\frac{\pi(m-n+2h)}{2m}}{\prod_{h=1}^{n-1}\sin\frac{\pi h}{2m}}\prod_{1\le h<t\le n}\frac{\sin\frac{\pi(m-n+h+t-1)}{2m}\cdot\sin\frac{\pi(m-n+t+h)}{2m}}{\sin^2\frac{\pi(t+h-2)}{2m}}\right.\\
&\qquad\qquad\cdot\sum_{k=1}^n\frac{(-1)^{n-k}(m-n+2k)}{\prod_{h=1}^n\sin\frac{\pi(m-n+k+h)}{2m}\prod_{h=1}^{k-1}\sin\frac{\pi h}{2m}\prod_{h=1}^{n-k}\sin\frac{\pi h}{2m}}\\
&\qquad+\frac{\prod_{h=1}^n\sin\frac{\pi(m-n+2h-1)}{2m}}{\prod_{h=1}^{n-1}\sin\frac{\pi h}{2m}}\prod_{1\le h<t\le n}\frac{\sin\frac{\pi(m-n+h+t-2)}{2m}\cdot\sin\frac{\pi(m-n+t+h-1)}{2m}}{\sin^2\frac{\pi(t+h-2)}{2m}}\\
&\qquad\qquad\cdot\sum_{k=1}^n\frac{(-1)^{n-k}(m-n+2k-2)}{\prod_{h=1}^n\sin\frac{\pi(m-n+k+h-1)}{2m}\prod_{h=1}^{k-1}\sin\frac{\pi h}{2m}\prod_{h=1}^{n-k}\sin\frac{\pi h}{2m}}\\
&\qquad+\frac{\prod_{h=1}^n\sin\frac{\pi(m-n+2h-1)}{2m}}{\prod_{h=1}^{n-1}\sin\frac{\pi h}{2m}}\prod_{1\le h<t\le n}\frac{\sin\frac{\pi(m-n+h+t)}{2m}\cdot\sin\frac{\pi(m-n+t+h-1)}{2m}}{\sin^2\frac{\pi(t+h-2)}{2m}}\\
&\qquad\qquad\cdot\sum_{k=1}^n\frac{(-1)^{n-k}(m-n+2k-1)}{\prod_{h=1}^n\sin\frac{\pi(m-n+k+h-1)}{2m}\prod_{h=1}^{k-1}\sin\frac{\pi h}{2m}\prod_{h=1}^{n-k}\sin\frac{\pi h}{2m}}\\
&\qquad+\frac{\prod_{h=1}^n\sin\frac{\pi(m-n+2h-2)}{2m}}{\prod_{h=1}^{n-1}\sin\frac{\pi h}{2m}}\prod_{1\le h<t\le n}\frac{\sin\frac{\pi(m-n+h+t-1)}{2m}\cdot\sin\frac{\pi(m-n+t+h-2)}{2m}}{\sin^2\frac{\pi(t+h-2)}{2m}}\\
&\qquad\qquad\left.\cdot\sum_{k=1}^n\frac{(-1)^{n-k}(m-n+2k-3)}{\prod_{h=1}^n\sin\frac{\pi(m-n+k+h-2)}{2m}\prod_{h=1}^{k-1}\sin\frac{\pi h}{2m}\prod_{h=1}^{n-k}\sin\frac{\pi h}{2m}}\right)
\end{aligned}\tag{7.15}$$

if m is an integer and $k+2\eta_j$ is even, it is asymptotically

$$\begin{aligned}
&\frac{4^{n^2-n+1}}{(8m)^n}\left(2^n\prod_{j=0}^{n-1}\cos\frac{j\pi}{2m}\right)^k\\
&\quad\times\prod_{1\le h<t\le n}\left(\sin\frac{\pi(\eta_h-\eta_t)}{2m}\cdot\sin\frac{\pi(t-h)}{2m}\sin\frac{\pi(\eta_h+\eta_t)}{2m}\cdot\sin\frac{\pi(t+h-2)}{2m}\right)
\end{aligned}$$

$$\times\left(\frac{\prod_{h=1}^{n}\sin\frac{\pi(\lfloor m\rfloor-n+2h)}{2m}}{\prod_{h=1}^{n-1}\sin\frac{\pi h}{2m}}\prod_{1\le h<t\le n}\frac{\sin\frac{\pi(\lfloor m\rfloor-n+h+t-1)}{2m}\cdot\sin\frac{\pi(\lfloor m\rfloor-n+t+h)}{2m}}{\sin^2\frac{\pi(t+h-2)}{2m}}\right.$$
$$\cdot\sum_{k=1}^{n}\frac{(-1)^{n-k}(\lfloor m\rfloor-n+2k)}{\prod_{h=1}^{n}\sin\frac{\pi(\lfloor m\rfloor-n+k+h)}{2m}\prod_{h=1}^{k-1}\sin\frac{\pi h}{2m}\prod_{h=1}^{n-k}\sin\frac{\pi h}{2m}}$$
$$+\frac{\prod_{h=1}^{n}\sin\frac{\pi(\lfloor m\rfloor-n+2h-1)}{2m}}{\prod_{h=1}^{n-1}\sin\frac{\pi h}{2m}}\prod_{1\le h<t\le n}\frac{\sin\frac{\pi(\lfloor m\rfloor-n+h+t)}{2m}\cdot\sin\frac{\pi(\lfloor m\rfloor-n+t+h-1)}{2m}}{\sin^2\frac{\pi(t+h-2)}{2m}}$$
$$\left.\cdot\sum_{k=1}^{n}\frac{(-1)^{n-k}(\lfloor m\rfloor-n+2k-1)}{\prod_{h=1}^{n}\sin\frac{\pi(\lfloor m\rfloor-n+k+h-1)}{2m}\prod_{h=1}^{k-1}\sin\frac{\pi h}{2m}\prod_{h=1}^{n-k}\sin\frac{\pi h}{2m}}\right)\tag{7.16}$$

if m is a half-integer and $k+2\eta_j$ is even, it is asymptotically

$$\frac{2\cdot 4^{n^2-n+1}}{(8m)^n}\left(2^n\prod_{j=0}^{n-1}\cos\frac{j\pi}{2m}\right)^k$$
$$\times\prod_{1\le h<t\le n}\left(\sin\frac{\pi(\eta_h-\eta_t)}{2m}\cdot\sin\frac{\pi(t-h)}{2m}\sin\frac{\pi(\eta_h+\eta_t)}{2m}\cdot\sin\frac{\pi(t+h-2)}{2m}\right)$$
$$\times\frac{\prod_{h=1}^{n}\sin\frac{\pi(m-n+2h-1)}{2m}}{\prod_{h=1}^{n-1}\sin\frac{\pi h}{2m}}\prod_{1\le h<t\le n}\frac{\sin^2\frac{\pi(m-n+h+t-1)}{2m}}{\sin^2\frac{\pi(t+h-2)}{2m}}$$
$$\times\sum_{k=1}^{n}\frac{(-1)^{n-k}(m-n+2k-1)}{\prod_{h=1}^{n}\sin\frac{\pi(m-n+k+h-1)}{2m}\prod_{h=1}^{k-1}\sin\frac{\pi h}{2m}\prod_{h=1}^{n-k}\sin\frac{\pi h}{2m}}\tag{7.17}$$

if m is an integer and $k+2\eta_j$ is odd, and it is asymptotically

$$\frac{4^{n^2-n+1}}{(8m)^n}\left(2^n\prod_{j=0}^{n-1}\cos\frac{j\pi}{2m}\right)^k$$
$$\times\prod_{1\le h<t\le n}\left(\sin\frac{\pi(\eta_h-\eta_t)}{2m}\cdot\sin\frac{\pi(t-h)}{2m}\sin\frac{\pi(\eta_h+\eta_t)}{2m}\cdot\sin\frac{\pi(t+h-2)}{2m}\right)$$
$$\times\left(\frac{\prod_{h=1}^{n}\sin\frac{\pi(m-n+2h-1/2)}{2m}}{\prod_{h=1}^{n-1}\sin\frac{\pi h}{2m}}\prod_{1\le h<t\le n}\frac{\sin^2\frac{\pi(m-n+h+t-1/2)}{2m}}{\sin^2\frac{\pi(t+h-2)}{2m}}\right.$$
$$\times\sum_{k=1}^{n}\frac{(-1)^{n-k}(m-n+2k-1/2)}{\prod_{h=1}^{n}\sin\frac{\pi(m-n+k+h-1/2)}{2m}\prod_{h=1}^{k-1}\sin\frac{\pi h}{2m}\prod_{h=1}^{n-k}\sin\frac{\pi h}{2m}}$$
$$+\frac{\prod_{h=1}^{n}\sin\frac{\pi(m-n+2h-3/2)}{2m}}{\prod_{h=1}^{n-1}\sin\frac{\pi h}{2m}}\prod_{1\le h<t\le n}\frac{\sin^2\frac{\pi(m-n+h+t-3/2)}{2m}}{\sin^2\frac{\pi(t+h-2)}{2m}}$$
$$\left.\times\sum_{k=1}^{n}\frac{(-1)^{n-k}(m-n+2k-3/2)}{\prod_{h=1}^{n}\sin\frac{\pi(m-n+k+h-3/2)}{2m}\prod_{h=1}^{k-1}\sin\frac{\pi h}{2m}\prod_{h=1}^{n-k}\sin\frac{\pi h}{2m}}\right)\tag{7.18}$$

if m is a half-integer and $k+2\eta_j$ is odd.

Proof. Here, in view of Theorem 31, we have to carry out the sum of (7.13) over all possible λ, i.e., over all $\lambda_1 > \lambda_2 > \cdots > \lambda_{n-1} > |\lambda_n|$ with $\lambda_1 + \lambda_2 < 2m$, where $k \equiv 2\eta_j + 2\lambda_j \mod 2$. Leaving away factors which are independent of λ, the problem is to compute the sum

$$\sum_{\substack{\lambda_1>\lambda_2>\cdots>\lambda_{n-1}>|\lambda_n|\\ \lambda_1+\lambda_2<2m}}{}' \prod_{1\le h<t\le n} \left(\sin\frac{\pi(\lambda_h-\lambda_t)}{2m}\cdot\sin\frac{\pi(\lambda_h+\lambda_t)}{2m}\right), \tag{7.19}$$

where the sum $\sum'$ is restricted to integral, respectively to half-integral $\lambda_1, \lambda_2, \dots, \lambda_n$, depending on whether $k + 2\eta_j$ is even or odd.

As in previous proofs, we have to distinguish between several cases. Let us first restrict the sum (7.19) to integral $\lambda_1, \lambda_2, \dots, \lambda_n$. Then, by repeating the argument from the proof of Theorem 30 of replacement of λ_1 by $2m - \lambda_1$ and of replacement of λ_n by $-\lambda_n$, and using notation from that proof, we obtain that the sum (7.19) equals $W_1(m) + W_1(m-1) + W_2(m) + W_2(m-1)$ if m is an integer, and it equals $2W_1(\lfloor m\rfloor) + 2W_2(\lfloor m\rfloor)$ if m is a half-integer. In the proof of Theorem 30 it was shown how to evaluate the sum $W_1(c)$ and $W_2(c)$ by means of Lemmas C7 and C8. If the results are substituted, we arrive at (7.15) and (7.16).

If we restrict the sum (7.19) to half-integral $\lambda_1, \lambda_2, \dots, \lambda_n$, then we are faced with the problem of evaluating the sum

$$X(c) = \sum_{c\ge\lambda_1>\cdots>\lambda_n>0}{}' \prod_{1\le h<t\le n} \left(\sin\frac{\pi(\lambda_h-\lambda_t)}{2m}\cdot\sin\frac{\pi(\lambda_h+\lambda_t)}{2m}\right)$$

where the sum $\sum'$ is over all *half-integral* $\lambda_1, \lambda_2, \dots, \lambda_n$. More specifically, using the replacement of λ_1 by $2m - \lambda_1$ and the replacement of λ_n by $-\lambda_n$ another time, one sees readily that the sum (7.19) equals $4X(m-\frac12)$ if m is an integer, and it equals $2X(m) + 2X(m-1)$ if m is a half-integer.

In order to evaluate $X(c)$, where c is a half-integer, we use (7.3) to rewrite it as

$$\sum_{c\ge\lambda_1>\cdots>\lambda_n>0}{}' 2^{-n^2+2n-1} \det_{1\le h,t\le n}\big(\cos(\pi(h-1)\lambda_t/m)\big)$$
$$= \sum_{c\ge\lambda_1>\cdots>\lambda_n>0}{}' \frac{1}{2^{n^2-n+1}} \det_{1\le h,t\le n}\left((e^{\pi i(h-1)/m})^{\lambda_t} + (e^{\pi i(h-1)/m})^{-\lambda_t}\right).$$

Replacing λ_t by $\lambda_t + n - t$, $t = 1, 2, \dots, n$, we obtain the expression

$$\sum_{c-n+1\ge\lambda_1\ge\cdots\ge\lambda_n\ge\frac12}{}' \frac{1}{2^{n^2-n+1}} \det_{1\le h,t\le n}\left((e^{\pi i(h-1)/m})^{\lambda_t+n-t} + (e^{\pi i(h-1)/m})^{-(\lambda_t+n-t)}\right). \tag{7.20}$$

Again, this determinant can be expressed in terms of an *even orthogonal character.* (See (C.49) for the definition.) Writing, as before, q for $e^{\pi i/2m}$, the sum in (7.20) equals

$$\frac{1}{2^{n^2-n+1}} \det_{1\le h,t\le n}\left((q^{h-1})^{n-t} + (q^{h-1})^{-(n-t)}\right) \sum_{c-n+1\ge\lambda_1\ge\cdots\ge\lambda_n\ge\frac12}{}' so_\lambda^{even}(q^{n-1}, q^{n-2}, \dots, 1), \tag{7.21}$$

where, still, the sum $\sum'$ is over all half-integral $\lambda_1, \lambda_2, \dots, \lambda_n$. (The reader should note that the second determinant in (C.49) vanishes if one of the variables x_t, $t = 1, 2, \dots, n$, is 1, which is the case in our situation.) The sum in (7.21) can also be simplified by means of (7.10). Use of this formula in (7.21) gives

$$\frac{1}{2^{n^2-n+1}}\det_{1\le h,t\le n}\left((q^{h-1})^{n-t}+(q^{h-1})^{-(n-t)}\right)\\ \times so^{even}_{\left(((2c-2n+3)/4)^n\right)}(q^{n-1},q^{n-2},\dots,1)\\ \times\sum_{p=-(2c-2n+1)/4}^{(2c-2n+1)/4} so^{even}_{\left(((2c-2n+3)/4)^{n-1},p\right)}(q^{n-1},q^{n-2},\dots,1).$$

Again, the first determinant can be evaluated by means of (B.3), the even orthogonal character of shape $\left(((2c-2n+3)/4)^n\right)$ can be evaluated by using the definition (C.49) and (B.3), while the sum over p of even orthogonal characters is evaluated in Lemma C7. Substitution of the result and simplification eventually leads to the expressions (7.17) and (7.18). □

Acknowledgements

I am indebted to the referee for an extremely careful reading of the manuscript, for the many suggestions which improved the presentation, and for pointing out a flaw in the original statement and proof of Lemma A.

Appendix A. A saddle point approximation

The subject of this appendix is the saddle point approximation which is needed in the proof of Theorem 16.

Lemma A. *Let m be a positive integer and d be an integer or half-integer. Let $\omega = e^{2\pi i/m}$, and let $\{r_1, r_2, \dots, r_n\}$ be a subset of $\{0,1,\dots,m-1\}$, where the r_j's are pairwise distinct. Furthermore, let*

$$C = C(r_1,\dots,r_n;m) := \max_\theta \prod_{j=1}^n 2\left|\cos\left(\frac{\pi(\theta+r_j)}{m}\right)\right|, \tag{A.1}$$

and let $\theta_1,\theta_2,\dots,\theta_s$ be the values of θ where the maximum in (A.1) *is attained. Then, as k tends to infinity such that $d+\frac{nk}{2}$ is an integer, the coefficient of $z^{d+\frac{nk}{2}}$ in*

$$\prod_{j=1}^n (1+\omega^{r_j}z)^k \tag{A.2}$$

is asymptotically

$$\frac{1}{\sqrt{2\pi k}}C^k\sum_{\ell=1}^s \varepsilon(r_1,\dots,r_n;m)\frac{\omega^{\frac{k}{2}\sum_{j=1}^n r_j - d\theta_\ell}}{\sqrt{c_0(\theta_\ell)}}, \tag{A.3}$$

where

$$\varepsilon(r_1,\dots,r_n;m) = \operatorname{sgn}\prod_{j=1}^n \cos\left(\frac{\pi(\theta+r_j)}{m}\right)$$

and

$$c_0(\theta)=\sum_{j=1}^{n}\left(2\cos\frac{\pi(\theta+r_j)}{m}\right)^{-2},$$

except if the sum in (A.3) *vanishes, in which case the asymptotic order of the coefficient of* $z^{d+\frac{nk}{2}}$ *in* (A.2) *is strictly less than* C^k.

Remark. The proof of the lemma below shows that, in fact, each θ_ℓ is a solution of the equation

$$\sum_{j=1}^{n}\tan\frac{\pi(\theta+r_j)}{m}=0. \tag{A.4}$$

Proof of Lemma A. We apply the saddle point method (see [9]). (We would in fact like to directly apply a general "law of large powers," such as for example Theorem 6.5 in [9]. However, I was not able to find an applicable theorem in the literature. In particular, Theorem 6.5 from [9] does not apply in our situation since it requires that the coefficients of the series of which the powers are formed are positive.)

We begin by writing the coefficient of $z^{d+\frac{nk}{2}}$ in the product (A.2) as an integral,

$$\frac{1}{2\pi i}\int_{\mathcal{C}}\frac{\prod_{j=1}^{n}(1+\omega^{r_j}z)^k}{z^{d+\frac{nk}{2}}}\frac{dz}{z}, \tag{A.5}$$

where $\mathcal{C}$ is some contour encircling the origin counter-clockwise. The task is to choose a contour $\mathcal{C}$ such that the contribution of the integrand is concentrated very close to a finite number of points on the contour (the saddle points), whereas otherwise the contributions are negligible. The equation for the saddle points is (see [9, Sec. 6.3])

$$\frac{d}{dz}\left(\frac{\prod_{j=1}^{n}(1+\omega^{r_j}z)^k}{z^{d+\frac{nk}{2}}}\right)=0.$$

Equivalently, this is

$$\frac{k}{2}\sum_{j=1}^{n}\frac{\omega^{r_j}z-1}{\omega^{r_j}z+1}-d=0. \tag{A.6}$$

For the saddle point method to work, it is often not necessary to determine the solution(s) of the saddle point equation exactly. It is often sufficient to find a suitable approximation for large k. This is also the case here. Clearly, if k is large, then the coefficient of $k/2$ in (A.6) must be very small. Therefore, as a first approximation, we consider solutions of the equation

$$\sum_{j=1}^{n}\frac{\omega^{r_j}z-1}{\omega^{r_j}z+1}=0. \tag{A.7}$$

This is (after denominators have been cleared) a polynomial equation of degree n. Therefore we must expect (up to) n different solutions to (A.7).

We claim that all n solutions to (A.7) are distinct and moreover have modulus 1. To see this, we substitute $e^{2\pi i\theta/m}$ for z in (A.7). After a little manipulation, we obtain the equivalent equation

$$\sum_{j=1}^{n}\tan\frac{\pi(\theta+r_j)}{m}=0. \tag{A.8}$$

The left-hand side of (A.8) is defined for $0 \le \theta \le m$ except at the values (which have to be taken modulo m)

$$\theta = -r_1 + \frac{m}{2}, -r_2 + \frac{m}{2}, \dots, -r_n + \frac{m}{2},$$

all of which are distinct by assumption. These values are simple poles of the function on the left-hand side of (A.8). Between two successive poles the function is monotone increasing and continuous (here, we regard the interval $[0, m]$ as a circular interval, identifying 0 and m), ranging there from $-\infty$ to $+\infty$. Hence, in each open interval bounded by two successive poles there lies exactly one solution to (A.8). Since the number of poles, and, thus, of such intervals, is n, there are n values of θ in the range $[0, m)$ satisfying equation (A.8). In turn, each solution to (A.8) produces a solution to (A.7) with modulus 1. This establishes that, indeed, all n solutions to (A.7) have modulus 1 and are pairwise distinct.

Let $\theta_1, \theta_2, \dots, \theta_n$ be the above solutions to (A.8), and, for a given θ_ℓ, let $z_\ell = e^{2\pi i\theta_\ell/m}$ be the corresponding solution to (A.7). Obviously, we have $1 + \omega^{r_j} z_\ell \neq 0$ for all ℓ and j.

We shall need a slightly better approximation of the saddle points, which will approximate (A.6) up to second order. Let $y_\ell := z_\ell(1 + \frac{c_\ell}{k})$. Substitution of y_ℓ in (A.6) leads to the equation

$$\begin{aligned}
&\frac{k}{2}\sum_{j=1}^{n}\frac{\omega^{r_j} y_\ell - 1}{\omega^{r_j} y_\ell + 1} - d\\
&\quad = \frac{k}{2}\sum_{j=1}^{n}\frac{\omega^{r_j} z_\ell - 1}{\omega^{r_j} z_\ell + 1}\left(1 + \frac{c_\ell\omega^{r_j} z_\ell}{k(\omega^{r_j} z_\ell - 1)} - \frac{c_\ell\omega^{r_j} z_\ell}{k(\omega^{r_j} z_\ell + 1)} + O\left(k^{-2}\right)\right) - d\\
&\quad = c_\ell\sum_{j=1}^{n}\frac{\omega^{r_j} z_\ell}{(\omega^{r_j} z_\ell + 1)^2} - d + O\left(k^{-1}\right)\\
&\quad = c_\ell\sum_{j=1}^{n}\frac{1}{\left(2\cos\frac{\pi(r_j+\theta_\ell)}{m}\right)^2} - d + O\left(k^{-1}\right)\\
&\quad = 0,
\end{aligned}$$

where we used (A.7) in the third line. Thus, y_ℓ will approximate a solution to (A.6) up to second order, if we choose

$$c_\ell = \frac{d}{\sum_{j=1}^{n}\left(2\cos\dfrac{\pi(r_j + \theta_\ell)}{m}\right)^{-2}}.$$

Returning to the integral (A.5) that we want to approximate, we choose as the contour $\mathcal{C}$ the circle

$$\left\{\left(1 + \tfrac{c_\ell}{k}\right)e^{it} : 0 \le t \le 2\pi\right\}, \tag{A.9}$$

which contains the (approximated) saddle points y_ℓ. Thus, writing $\bar\theta$ for $2\pi\theta_\ell/m$ for short, the portion of the integral (A.5) contributed by a neighbourhood $[\bar\theta - \varepsilon, \bar\theta + \varepsilon]$ of

$\bar\theta$ becomes

$$\frac{1}{2\pi}\int_{-\varepsilon}^{\varepsilon}\frac{\prod_{j=1}^{n}\left(1+\omega^{r_j}(1+\frac{c_\ell}{k})e^{i(\bar\theta+t)}\right)^k}{\left((1+\frac{c_\ell}{k})e^{i(\bar\theta+t)}\right)^{d+\frac{nk}{2}}}dt. \tag{A.10}$$

In order to estimate this integral, we compute a Taylor expansion of

$$\log\left(1+\omega^{r_j}\left(1+\frac{c_\ell}{k}\right)e^{i(\bar\theta+t)}\right),$$

thereby also neglecting terms which are of order $O(k^{-2})$:

$$\begin{aligned}\log&\left(1+\omega^{r_j}\left(1+\frac{c_\ell}{k}\right)e^{i(\bar\theta+t)}\right)\\&=\log(1+e^{i\bar\theta}\omega^{r_j})+\frac{c_\ell}{k}\frac{e^{i\bar\theta}\omega^{r_j}}{\left(1+e^{i\bar\theta}\omega^{r_j}\right)}+it\left(\frac{e^{i\bar\theta}\omega^{r_j}}{1+e^{i\bar\theta}\omega^{r_j}}+\frac{c_\ell}{k}\frac{e^{i\bar\theta}\omega^{r_j}}{\left(1+e^{i\bar\theta}\omega^{r_j}\right)^2}\right)\\&\qquad-\frac{t^2}{2}\left(\frac{e^{i\bar\theta}\omega^{r_j}}{\left(1+e^{i\bar\theta}\omega^{r_j}\right)^2}+\frac{c_\ell}{k}\frac{e^{i\bar\theta}\omega^{r_j}\left(1-e^{i\bar\theta}\omega^{r_j}\right)}{\left(1+e^{i\bar\theta}\omega^{r_j}\right)^3}\right)+O(t^3)+O\left(k^{-2}\right).\end{aligned}$$

This expansion is only valid for

$$\left|\frac{\left(1+\frac{c_\ell}{k}\right)e^{i(\bar\theta+t)}-e^{i\bar\theta}\omega^{r_j}}{1+e^{i\bar\theta}\omega^{r_j}}\right|<1, \tag{A.11}$$

whence, in general, it will only be valid for t in a neighbourhood of 0 and k large enough. More precisely, choose k_0 so that the left-hand side of (A.11) with $t=0$ is less than 1, i.e.,

$$\left|\frac{c_\ell}{k_0(1+e^{i\bar\theta}\omega^{r_j})}\right|<1.$$

Then the above expansion of the logarithm will be valid for $-\varepsilon\le t\le\varepsilon$ and $k\ge k_0$ for some ε which is independent of k.

If this is substituted in (A.10), then, after some simplification based on (A.8) and the definition of c_ℓ, we obtain

$$\begin{aligned}&\frac{1}{2\pi e^{i\bar\theta(d+\frac{nk}{2})}}\prod_{j=1}^{n}(1+e^{i\bar\theta}\omega^{r_j})^k\\&\quad\times\int_{-\varepsilon}^{\varepsilon}\exp\left(-\frac{kt^2}{2}\Big(\sum_{j=1}^{n}\frac{e^{i\bar\theta}\omega^{r_j}}{\left(1+e^{i\bar\theta}\omega^{r_j}\right)^2}+\frac{c_\ell}{k}\sum_{j=1}^{n}\frac{e^{i\bar\theta}\omega^{r_j}\left(1-e^{i\bar\theta}\omega^{r_j}\right)}{\left(1+e^{i\bar\theta}\omega^{r_j}\right)^3}\Big)+O(kt^3)+O\left(k^{-1}\right)\right).\end{aligned} \tag{A.12}$$

Next we estimate the integral in (A.12). One argues as is usual in such a situation: the essential contribution to the integral comes from the range $|t|<(\log k)/\sqrt{k}$, the rest being of order $O\left(k^{-1}\right)$. In this range, the error terms $O(kt^3)+O\left(k^{-1}\right)$ are of the order $O\big((\log^3 k)/\sqrt{k}\big)$. Finally, one extends the integral to the complete range $-\infty<t<\infty$, upon making another error of the order $O\left(k^{-1}\right)$. Leaving these details to the reader, up to an error of $O\big((\log^3 k)/\sqrt{k}\big)$ the integral in (A.12) is asymptotically

$$\int_{-\infty}^{\infty} \exp\left(-S\frac{kt^2}{2}\right) = \sqrt{\frac{2\pi}{kS}}, \tag{A.13}$$

where

$$S = \sum_{j=1}^{n} \frac{e^{i\bar\theta}\omega^{r_j}}{\big(1+e^{i\bar\theta}\omega^{r_j}\big)^2} + \frac{c_\ell}{k}\sum_{j=1}^{n} \frac{e^{i\bar\theta}\omega^{r_j}\big(1-e^{i\bar\theta}\omega^{r_j}\big)}{\big(1+e^{i\bar\theta}\omega^{r_j}\big)^3} = c_0(\theta_\ell) + O\left(\frac{1}{k}\right).$$

with $c_0(\theta_\ell)$ as given in the statement of the theorem. The evaluation (A.13) is valid since $c_0 > 0$, and hence $\mathfrak{Re}\, S > 0$ for k large enough.

Clearly, the second term in the definition of S makes only a negligible contribution to the asymptotic behaviour of (A.13). If the result is substituted in (A.12), we obtain that the portion of the integral (A.5) near the saddle point z_ℓ, given explicitly in (A.10), contributes

$$\frac{\prod_{j=1}^n (1+\omega^{r_j+\theta_\ell})^k}{\omega^{\theta_\ell(d+\frac{nk}{2})}} \frac{1}{\sqrt{2\pi c_0(\theta_\ell)k}},$$

as k tends to infinity, up to an error of $O\big((\log^3 k)/\sqrt{k}\big)$. Summation of these contributions, neglecting those which are asymptotically smaller, yields indeed the expression (A.3) after some manipulation.

It remains to show that the contribution of the remaining portions of the integral (A.5) is negligible. This is a routine matter, the details of which we leave to the reader. We content ourselves to indicate that one may start with the observation that the modulus of the integrand of (A.5) along the contour (A.9) is bounded by

$$\left|\frac{\prod_{j=1}^n \big(1+\omega^{r_j}(1+\frac{c_\ell}{k})e^{it}\big)^k}{\big((1+\frac{c_\ell}{k})e^{it}\big)^{d+\frac{nk}{2}}}\right| < \text{constant}^{\sqrt{k}} \left|\prod_{j=1}^{n}\big(1+\omega^{r_j}e^{it}\big)^k\right|, \tag{A.14}$$

the constant depending on $r_1, r_2, \dots, r_n$ but not on k, as long as t stays away from tiny neighbourhoods of $\pi - \frac{2\pi r_j}{m}$, $j = 1, 2, \dots, n$, determined by the condition that $|1+\omega^{r_j}e^{it}| \le 1/\sqrt{k}$, and that the right-hand side of (A.14) is exponentially smaller than C^k for any t in these remaining portions, while the contributions in these tiny neighbourhoods is even super-exponentially small.. □

Appendix B. Some determinants

In our computations we need frequently the following determinant evaluations. All of them are readily proved by the standard argument that proves Vandermonde-type determinant evaluations.

Lemma B. *Let n by a non-negative integer. Then*

$$\det_{1\le h,t\le n}\left(x_h^{t-1/2} + x_h^{1/2-t}\right) = (x_1x_2\cdots x_n)^{-n+1/2} \prod_{1\le h<t\le n}(x_h-x_t)(1-x_hx_t)\prod_{h=1}^{n}(x_h+1), \tag{B.1}$$

$$\det_{1\le h,t\le n}\left(x_h^{t-1/2}-x_h^{-t+1/2}\right)=(x_1x_2\cdots x_n)^{-n+1/2}\prod_{1\le h<t\le n}(x_h-x_t)(1-x_hx_t)\prod_{h=1}^{n}(x_h-1), \tag{B.2}$$

$$\det_{1\le h,t\le n}(x_h^{t-1}+x_h^{-(t-1)})=2\cdot(x_1\cdots x_n)^{-n+1}\prod_{1\le h<t\le n}(x_h-x_t)(1-x_hx_t), \tag{B.3}$$

$$\det_{1\le h,t\le n}(x_h^{t}-x_h^{-t})=(x_1\cdots x_n)^{-n}\prod_{1\le h<t\le n}(x_h-x_t)(1-x_hx_t)\prod_{h=1}^{n}(x_h^2-1). \tag{B.4}$$

Appendix C. Odd and even orthogonal characters, and Schur functions at special values of the arguments

In this appendix we provide the evaluations of odd orthogonal characters, Schur functions of rectangular shape and nearly rectangular shape, and of certain sums of even orthogonal characters, where the variables are specialized in peculiar ways. The evaluations of the odd orthogonal characters are needed for the proof of Theorem 18 (on which, in turn, hinges also the proof of Theorem 13) and the proof of Theorem 20, the evaluations of the special Schur functions are needed for the proofs of Theorems 22, 26, and 28, while the evaluations of the sums of specialized even orthogonal characters are needed for the proofs of Theorems 30 and 32.

Recall that, given a partition $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_n)$ (i.e., a non-increasing sequence of non-negative integers) or half-partition (which, by definition, is a non-increasing sequence of positive odd integers divided by 2), the *odd orthogonal character* $so_\lambda^{odd}(x_1,x_2,\dots,x_n)$ is defined by (see [13, (24.28)])

$$so_\lambda^{odd}(x_1,x_2,\dots,x_n)=\frac{\det\limits_{1\le h,t\le n}\left(x_h^{\lambda_t+n-t+1/2}-x_h^{-(\lambda_t+n-t+1/2)}\right)}{\det\limits_{1\le h,t\le n}\left(x_h^{n-t+1/2}-x_h^{-(n-t+1/2)}\right)}. \tag{C.1}$$

It is not difficult to see that the denominator in (C.1) does in fact cancel out, so that any odd orthogonal character $so_\lambda^{odd}(x_1,x_2,\dots,x_n)$ is in fact a *Laurent polynomial* in $x_1,x_2,\dots,x_n$ (i.e., a polynomial in $x_1,x_1^{-1},x_2,x_2^{-1},\dots,x_n,x_n^{-1}$), and is thus well-defined for any choice of the variables $x_1,x_2,\dots,x_n$ such that all of them are non-zero.

As earlier in Section 4, we shall use the notation $\left((p/2)^n\right)$ for the vector of n components, all of them equal to $p/2$.

Lemma C1. *Let m and n be positive integers with $m\ge n$, and let $q=e^{\pi i/m}$. Then we have*

$$so^{odd}_{\left(((m-n)/2)^n\right)}(q^{n-1},q^{n-3},\dots,q^{-n+3},q^{-n+1})=\frac{m^{n/2}\prod\limits_{h=1}^{n/2}\cot\frac{(2h-1)\pi}{2m}}{2^{\binom n2}\prod\limits_{1\le h<t\le n}\sin\frac{(t-h)\pi}{m}} \tag{C.2}$$

if n is even, and

$$so^{odd}_{\left(((m-n)/2)^n\right)}(q^{n-1},q^{n-3},\dots,q^{-n+3},q^{-n+1})=\frac{m^{(n+1)/2}\prod_{h=1}^{(n-1)/2}\cot\frac{h\pi}{m}}{2^{\binom n2}\prod_{1\le h<t\le n}\sin\frac{(t-h)\pi}{m}} \tag{C.3}$$

if n is odd.

Proof. In principle, we would like to specialize in the definition (C.1) of the odd orthogonal character. However, we face the difficulty that, because of the peculiar choice of the variables $x_1,x_2,\dots,x_n$ in (C.2) and (C.3), direct substitution gives an indeterminate expression $0/0$. More precisely, the problem is the pairs of reciprocal variables (such as q^{n-1} and q^{-n+1}, q^{n-3} and q^{-n+3}, etc.). In the case that n is odd, we have the additional problem that one of the variables is $q^0=1$.

To overcome this problem, we have recourse to de l'Hôpital's rule. We have to distinguish between two cases, depending on whether n is even or odd.

Let first n be even. In that case we must compute the limit as

$$x_1\to q^{n-1},\ x_2\to q^{n-3},\ \dots,\ x_{n/2}\to q \tag{C.4}$$

of the right-hand side of (C.1) with $x_{n/2+1}=q^{-1},\dots,x_{n-1}=q^{-n+3}$, $x_n=q^{-n+1}$. Upon a simultaneous rearrangement and change of signs of the rows of the two determinants in (C.1), one sees that this is equivalent to computing the limit (C.4) of $(\det A_1)/(\det B_1)$, where

$$A_1=\begin{pmatrix}x_h^{\frac{m+n+1}{2}-t}-x_h^{-(\frac{m+n+1}{2}-t)} & 1\le h\le n/2\\ q^{(2n-2h+1)(\frac{m+n+1}{2}-t)}-q^{-(2n-2h+1)(\frac{m+n+1}{2}-t)} & n/2<h\le n\end{pmatrix}_{1\le h,t\le n}$$

and

$$B_1=\begin{pmatrix}x_h^{n-t+\frac12}-x_h^{-(n-t+\frac12)} & 1\le h\le n/2\\ q^{(2n-2h+1)(n-t+\frac12)}-q^{-(2n-2h+1)(n-t+\frac12)} & n/2<h\le n\end{pmatrix}_{1\le h,t\le n}.$$

Now applying de l'Hôpital's rule, this limit is equal to the limit (C.4) of

$$\frac{\det A_2}{\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}\cdots\frac{\partial}{\partial x_{n/2}}\det B_1}, \tag{C.5}$$

where

$$\begin{aligned}A_2&=\begin{pmatrix}(\frac{m+n+1}{2}-t)(q^{(n-2h+1)(\frac{m+n+1}{2}-t-1)}+q^{-(n-2h+1)(\frac{m+n+1}{2}-t+1)}) & 1\le h\le n/2\\ q^{(2n-2h+1)(\frac{m+n+1}{2}-t)}-q^{-(2n-2h+1)(\frac{m+n+1}{2}-t)} & n/2<h\le n\end{pmatrix}\\ &=\begin{pmatrix}(-1)^{\frac n2-h}i\,q^{-(n-2h+1)}(\frac{m+n+1}{2}-t) & \\ \quad\times(q^{(n-2h+1)(\frac{n+1}{2}-t)}-q^{-(n-2h+1)(\frac{n+1}{2}-t)}) & 1\le h\le n/2\\ (-1)^{n-h}i\,(q^{(2n-2h+1)(\frac{n+1}{2}-t)}+q^{-(2n-2h+1)(\frac{n+1}{2}-t)}) & n/2<h\le n\end{pmatrix}.\end{aligned} \tag{C.6}$$

The limit of the denominator of (C.5) is readily obtained, because the determinant of B_1 can actually be evaluated by means of (B.2). To be precise, with

$$x_{n/2+1}=q^{n-1},\ \dots,\ x_{n-1}=q^3,\ x_n=q, \tag{C.7}$$

we have

$$\det B_1 = (x_1 x_2 \cdots x_n)^{-n+1/2} \prod_{1\le h<t\le n} (x_t - x_h)(1 - x_h x_t) \prod_{h=1}^{n} (x_h - 1). \tag{C.8}$$

Thus, if we differentiate the product on the right-hand side of (C.8) with respect to x_1, say, (using the product rule, of course), and subsequently set $x_1 = q^{n-1}$ (which is exactly what we want to do; see (C.4)), then it is just one term in the derivative which contributes, namely

$$\frac{1}{x_1 - x_{n/2+1}} (x_1 x_2 \cdots x_n)^{-n+1/2} \prod_{1\le h<t\le n} (x_t - x_h)(1 - x_h x_t) \prod_{h=1}^{n} (x_h - 1),$$

in which the factor $(x_1 - x_{n/2+1})$ cancels; all other terms vanish because of the occurrence of the factor $(x_1 - x_{n/2+1})$. (Recall that $x_{n/2+1} = q^{n-1}$; see (C.7).) An analogous argument applies to the other pairs of variables. Hence, the denominator of (C.5) is equal to

$$(-1)^{\binom{n/2}{2}} (q^{n^2/2})^{-n+1/2} \prod_{1\le h<t\le n/2} (q^{n-2t+1} - q^{n-2h+1})^4 (1 - q^{2n-2h-2t+2})^4 \\ \times \prod_{h=1}^{n/2} (q^{n-2h+1} - 1)^2 (1 - q^{2n-4h+2}). \tag{C.9}$$

Next we devote ourselves to the evaluation of the determinant of A_2, with A_2 given by (C.6). For $t = 1, 2, \dots, \frac{n}{2}$ we subtract column $n - t + 1$ from column t. As a result we obtain that $\det A_2$ is equal to $\det A_3$, where A_3 is the $n \times n$ block matrix

$$A_3 = \begin{pmatrix} A_3^{(1)} & * \\ 0 & A_3^{(2)} \end{pmatrix}, \tag{C.10}$$

with $A_3^{(1)}$ the $\frac{n}{2} \times \frac{n}{2}$ matrix

$$A_3^{(1)} = \left((-1)^{\frac{n}{2}-h} i\, q^{-(n-2h+1)} m \big(q^{(n-2h+1)(\frac{n+1}{2}-t)} - q^{-(n-2h+1)(\frac{n+1}{2}-t)} \big) \right)_{1\le h,t\le n/2},$$

and $A_3^{(2)}$ the $\frac{n}{2} \times \frac{n}{2}$ matrix

$$A_3^{(2)} = \left((-1)^{\frac{n}{2}-h} i\, \big(q^{(n-2h+1)(\frac{1}{2}-t)} + q^{-(n-2h+1)(\frac{1}{2}-t)} \big) \right)_{1\le h,t\le n/2}.$$

Clearly, the determinant $\det A_3$ is equal to the product $(\det A_3^{(1)}) \cdot (\det A_3^{(2)})$. Using again (B.2), we have

$$\det A_3^{(1)} = i^{n/2} q^{-n^2/4} m^{n/2} (q^{n^2/4})^{-n/2+1/2} \\ \times \prod_{1\le h<t\le n/2} (q^{n-2h+1} - q^{n-2t+1})(1 - q^{2n-2h-2t+2}) \prod_{h=1}^{n/2} (q^{n-2h+1} - 1), \tag{C.11}$$

whereas by means of (B.1) we have

$$\det A_3^{(2)} = (-1)^{\binom{n/2}{2}} i^{n/2} (q^{n^2/4})^{(1-n)/2}$$

$$\times \prod_{1\le h<t\le n/2}(q^{n-2h+1}-q^{n-2t+1})(1-q^{2n-2h-2t+2})\prod_{h=1}^{n/2}(1+q^{n-2h+1}). \quad \text{(C.12)}$$

If we now combine (C.10), (C.11), (C.12), and (C.9), use the fact that $\det A_2=\det A_3$, and substitute all this in (C.5), then, after some simplification, we obtain

$$so^{odd}_{\left(((m-n)/2)^n\right)}(q^{n-1},q^{n-3},\dots,q^{-n+3},q^{-n+1})$$
$$=\frac{(-1)^{n/2}m^{n/2}\prod_{h=1}^{n/2}(1+q^{n-2h+1})}{\prod_{1\le h<t\le n/2}(q^{h-t}-q^{t-h})^2\prod_{1\le h,t\le n/2}(q^{-n+h+t-1}-q^{n-h-t+1})\prod_{h=1}^{n/2}(1-q^{n-2h+1})},$$

which can be rewritten as (C.2).

Now let n be odd. We proceed in a completely analogous manner. Again, the task is to compute the specialized odd orthogonal character, by means of a limit of its definition (C.1). Here, the difficulty is not only the pairs of reciprocal variables, but also that one of the variables is $q^0=1$.

What we must compute is the limit as

$$x_1\to q^{n-1},\ x_2\to q^{n-3},\ \dots,\ x_{(n-1)/2}\to q^2,\ x_{(n+1)/2}\to 1, \quad \text{(C.13)}$$

of the right-hand side of (C.1) with $x_{(n+3)/2}=q^{-2},\dots,x_{n-1}=q^{-n+3}$, $x_n=q^{-n+1}$. Upon a simultaneous rearrangement and change of signs of the rows of the two determinants in (C.1), one sees that this is equivalent to computing the limit (C.13) of $(\det A_4)/(\det B_4)$, where

$$A_4=\begin{pmatrix} x_h^{\frac{m+n+1}{2}-t}-x_h^{-(\frac{m+n+1}{2}-t)} & 1\le h\le (n+1)/2\\ q^{(2n-2h+2)(\frac{m+n+1}{2}-t)}-q^{-(2n-2h+2)(\frac{m+n+1}{2}-t)} & (n+1)/2<h\le n\end{pmatrix}_{1\le h,t\le n}$$

and

$$B_4=\begin{pmatrix} x_h^{n-t+\frac12}-x_h^{-(n-t+\frac12)} & 1\le h\le (n+1)/2\\ q^{(2n-2h+2)(n-t+\frac12)}-q^{-(2n-2h+2)(n-t+\frac12)} & (n+1)/2<h\le n\end{pmatrix}_{1\le h,t\le n}.$$

Applying de l'Hôpital's rule, this limit is equal to the limit (C.13) of

$$\frac{\det A_5}{\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}\cdots\frac{\partial}{\partial x_{(n+1)/2}}\det B_4}, \quad \text{(C.14)}$$

where

$$A_5=\begin{pmatrix} (\frac{m+n+1}{2}-t)(q^{(n-2h+1)(\frac{m+n+1}{2}-t-1)}+q^{-(n-2h+1)(\frac{m+n+1}{2}-t+1)}) & 1\le h\le (n+1)/2\\ q^{(2n-2h+2)(\frac{m+n+1}{2}-t)}-q^{-(2n-2h+2)(\frac{m+n+1}{2}-t)} & (n+1)/2<h\le n\end{pmatrix}$$
$$=\begin{pmatrix} (-1)^{\frac{n+1}{2}-h}q^{-(n-2h+1)}(\frac{m+n+1}{2}-t) & \\ \quad\times(q^{(n-2h+1)(\frac{n+1}{2}-t)}+q^{-(n-2h+1)(\frac{n+1}{2}-t)}) & 1\le h\le (n+1)/2\\ (-1)^{n+1-h}(q^{(2n-2h+2)(\frac{n+1}{2}-t)}-q^{-(2n-2h+2)(\frac{n+1}{2}-t)}) & (n+1)/2<h\le n\end{pmatrix}. \quad \text{(C.15)}$$

The limit of the denominator of (C.14) is obtained in the same way as before, by using the complete factorization of $\det B_4$ by means of (B.2). (It should be noted that the difference between the denominator of (C.14) and that of (C.5) is the number of differentiations, and the number of variables with respect to which the limit is taken.) After a small calculation, it turns out that the denominator of (C.14) is equal to

$$(-1)^{\binom{(n+1)/2}{2}}(q^{(n^2-1)/2})^{-n+1/2}\prod_{1\le h<t\le (n-1)/2}(q^{n-2t+1}-q^{n-2h+1})^4(1-q^{2n-2h-2t+2})^4 \\ \times\prod_{h=1}^{(n-1)/2}(q^{n-2h+1}-1)^6(1-q^{2n-4h+2}). \quad \text{(C.16)}$$

Next we devote ourselves to the evaluation of the determinant of A_5, with A_5 given by (C.15). For $t=1,2,\dots,\frac{n-1}{2}$ we add column $n-t+1$ to column t. As a result we obtain that $\det A_5$ is equal to $\det A_6$, where A_6 is the $n\times n$ block matrix

$$A_6=\begin{pmatrix} A_6^{(1)} & * \\ 0 & A_6^{(2)}\end{pmatrix}, \quad \text{(C.17)}$$

with $A_6^{(1)}$ the $\frac{n+1}{2}\times\frac{n+1}{2}$ matrix

$$A_6^{(1)}=\Bigg((-1)^{\frac{n+1}{2}-h}q^{-(n-2h+1)}m \\ \times\begin{cases} q^{(n-2h+1)(\frac{n+1}{2}-t)}+q^{-(n-2h+1)(\frac{n+1}{2}-t)} & 1\le t\le\frac{n-1}{2}\\ 1 & t=\frac{n+1}{2}\end{cases}\Bigg)_{1\le h,t\le (n+1)/2},$$

and $A_6^{(2)}$ the $\frac{n-1}{2}\times\frac{n-1}{2}$ matrix

$$A_6^{(2)}=\left((-1)^{\frac{n+1}{2}-h}(q^{(n-2h+1)(-t)}-q^{-(n-2h+1)(-t)})\right)_{1\le h,t\le (n-1)/2}.$$

(It should be noted that the entries in column $(n+1)/2$ of $A_6^{(1)}$ are exactly a half of what would result from substituting $t=(n+1)/2$ in the definition of the entries of the other columns.) Clearly, the determinant $\det A_6$ is equal to the product $(\det A_6^{(1)})\cdot(\det A_6^{(2)})$. Using (B.3), we have

$$\det A_6^{(1)}=(-1)^{(n^2-1)/8}q^{-(n^2-1)/4}m^{(n+1)/2}(q^{(n^2-1)/4})^{-n/2+1/2} \\ \times\prod_{1\le h<t\le (n+1)/2}(q^{n-2h+1}-q^{n-2t+1})(1-q^{2n-2h-2t+2}), \quad \text{(C.18)}$$

whereas by means of (B.4) we have

$$\det A_6^{(2)}=q^{-(n^2-1)/4}(q^{(n^2-1)/4})^{-(n-1)/2} \\ \times\prod_{1\le h<t\le (n-1)/2}(q^{n-2h+1}-q^{n-2t+1})(1-q^{2n-2h-2t+2})\prod_{h=1}^{(n-1)/2}(q^{2n-4h+2}-1). \quad \text{(C.19)}$$

If we now combine (C.17), (C.18), (C.19), and (C.16), use the fact that $\det A_5 = \det A_6$, and substitute all this in (C.14), then, after some simplification, we obtain

$$so^{odd}_{\left(((m-n)/2)^n\right)}(q^{n-1},q^{n-3},\dots,q^{-n+3},q^{-n+1})$$
$$= \frac{m^{(n+1)/2}}{\prod\limits_{1\le h<t\le (n-1)/2}(q^{h-t}-q^{t-h})^2\prod\limits_{1\le h,t\le (n-1)/2}(q^{n-h-t+1}-q^{-n+h+t-1})}$$
$$\times\prod_{h=1}^{(n-1)/2}\frac{(q^{\frac{n+1}{2}-h}+q^{-\frac{n+1}{2}+h})}{(q^{\frac{n+1}{2}-h}-q^{-\frac{n+1}{2}+h})^3},$$

which can be rewritten as (C.3). □

The next lemma provides a similar evaluation of a rectangularly shaped odd orthogonal character, the special values of the arguments at which the character is evaluated being exactly the negative values of those in Lemma C1. This evaluation is needed in the proof of Theorem 18, however only in the case that m is even. For the sake of completeness, we state also the corresponding result for odd m without proof.

Lemma C2. *Let m and n be positive integers with $m\ge n$, and let $q=e^{\pi i/m}$. Then we have*

$$so^{odd}_{\left(((m-n)/2)^n\right)}(-q^{n-1},-q^{n-3},\dots,-q^{-n+3},-q^{-n+1}) = \frac{m^{n/2}\prod\limits_{h=1}^{n/2}\tan\dfrac{(2h-1)\pi}{2m}}{2^{\binom n2}\prod\limits_{1\le h<t\le n}\sin\dfrac{(t-h)\pi}{m}} \tag{C.20}$$

if n is even (regardless of m),

$$so^{odd}_{\left(((m-n)/2)^n\right)}(-q^{n-1},-q^{n-3},\dots,-q^{-n+3},-q^{-n+1}) = \frac{(-1)^{(m-n)/2}m^{(n-1)/2}\prod\limits_{h=1}^{(n-1)/2}\tan\dfrac{h\pi}{m}}{2^{\binom n2}\prod\limits_{1\le h<t\le n}\sin\dfrac{(t-h)\pi}{m}} \tag{C.21}$$

if both n and m are odd, and

$$so^{odd}_{\left(((m-n)/2)^n\right)}(-q^{n-1},-q^{n-3},\dots,-q^{-n+3},-q^{-n+1}) = 0 \tag{C.22}$$

if n is odd and m is even.

Proof. Again, when we would directly specialize in the definition (C.1) of the odd orthogonal character, then we face the difficulty that we obtain an indeterminate expression $0/0$. As we mentioned before the statement of the theorem, we are only going to discuss the case that m is even. The arguments are however completely analogous if m is odd.

Let first n be even. In that case we must compute the limit as

$$x_1\to -q^{n-1},\ x_2\to -q^{n-3},\ \dots,\ x_{n/2}\to -q \tag{C.23}$$

of the right-hand side of (C.1) with $x_{n/2+1} = -q^{-1}, \dots, x_{n-1} = -q^{-n+3}$, $x_n = -q^{-n+1}$. Upon a simultaneous rearrangement and change of signs of the rows of the two determinants in (C.1), and using the equality $-1 = q^m$, one sees that this is equivalent to computing the limit (C.23) of $(\det A_7)/(\det B_7)$, where

$$A_7 = \begin{pmatrix} x_h^{\frac{m+n+1}{2}-t} - x_h^{-(\frac{m+n+1}{2}-t)} & 1 \le h \le n/2 \\ q^{(2n+m-2h+1)(\frac{m+n+1}{2}-t)} - q^{-(2n+m-2h+1)(\frac{m+n+1}{2}-t)} & n/2 < h \le n \end{pmatrix}_{1\le h,t\le n}$$

and

$$B_7 = \begin{pmatrix} x_h^{n-t+\frac{1}{2}} - x_h^{-(n-t+\frac{1}{2})} & 1 \le h \le n/2 \\ q^{(2n+m-2h+1)(n-t+\frac{1}{2})} - q^{-(2n+m-2h+1)(n-t+\frac{1}{2})} & n/2 < h \le n \end{pmatrix}_{1\le h,t\le n}.$$

Now applying de l'Hôpital's rule, this limit is equal to the limit (C.23) of

$$\frac{\det A_8}{\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}\cdots\frac{\partial}{\partial x_{n/2}}\det B_7}, \tag{C.24}$$

where

$$\begin{aligned} A_8 &= \begin{pmatrix} (\frac{m+n+1}{2}-t)(q^{(n+m-2h+1)(\frac{m+n+1}{2}-t-1)} + q^{-(n+m-2h+1)(\frac{m+n+1}{2}-t+1)}) & 1 \le h \le n/2 \\ q^{(2n+m-2h+1)(\frac{m+n+1}{2}-t)} - q^{-(2n+m-2h+1)(\frac{m+n+1}{2}-t)} & n/2 < h \le n \end{pmatrix} \\ &= \begin{pmatrix} (-1)^{\frac{m}{2}-h-t}q^{-(n-2h+1)}(\frac{m+n+1}{2}-t) & \\ \quad\times(q^{(n-2h+1)(\frac{n+1}{2}-t)} + q^{-(n-2h+1)(\frac{n+1}{2}-t)}) & 1 \le h \le n/2 \\ (-1)^{\frac{n+m}{2}-h-t+1}(q^{(2n-2h+1)(\frac{n+1}{2}-t)} - q^{-(2n-2h+1)(\frac{n+1}{2}-t)}) & n/2 < h \le n \end{pmatrix}. \end{aligned} \tag{C.25}$$

The limit of the denominator of (C.24) is again readily obtained, because the determinant of B_7 can actually be evaluated by means of (B.2). The result is that the denominator of (C.24) is equal to

$$(q^{n^2/2})^{-n+1/2}\prod_{1\le h<t\le n/2}(q^{n-2t+1} - q^{n-2h+1})^4(1-q^{2n-2h-2t+2})^4 \\ \times\prod_{h=1}^{n/2}(q^{n-2h+1}+1)^2(1-q^{2n-4h+2}). \tag{C.26}$$

Next we devote ourselves to the evaluation of the determinant of A_8, with A_8 given by (C.25). For $t = 1, 2, \dots, \frac{n}{2}$ we subtract column $n-t+1$ from column t. As a result we obtain that $\det A_8$ is equal to $\det A_9$, where A_9 is the $n \times n$ block matrix

$$A_9 = \begin{pmatrix} A_9^{(1)} & * \\ 0 & A_9^{(2)} \end{pmatrix}, \tag{C.27}$$

with $A_9^{(1)}$ the $\frac{n}{2} \times \frac{n}{2}$ matrix

$$A_9^{(1)} = \left((-1)^{\frac{m}{2}-h-t}q^{-(n-2h+1)}m(q^{(n-2h+1)(\frac{n+1}{2}-t)} + q^{-(n-2h+1)(\frac{n+1}{2}-t)})\right)_{1\le h,t\le n/2},$$

and $A_9^{(2)}$ the $\frac{n}{2} \times \frac{n}{2}$ matrix

$$A_9^{(2)} = \left((-1)^{\frac{m}{2}-h-t+1}(q^{(n-2h+1)(\frac{1}{2}-t)} - q^{-(n-2h+1)(\frac{1}{2}-t)})\right)_{1\le h,t\le n/2}.$$

Clearly, the determinant $\det A_9$ is equal to the product $(\det A_9^{(1)})\cdot(\det A_9^{(2)})$. Using (B.1), we have

$$\det A_9^{(1)} = (-1)^{\binom{n/2+1}{2}} q^{-n^2/4} m^{n/2} (q^{n^2/4})^{-n/2+1/2} \\ \times \prod_{1\le h<t\le n/2} (q^{n-2h+1}-q^{n-2t+1})(1-q^{2n-2h-2t+2}) \prod_{h=1}^{n/2} (q^{n-2h+1}+1), \quad (C.28)$$

whereas by means of (B.2) we have

$$\det A_9^{(2)} = (q^{n^2/4})^{1/2-n/2} \\ \times \prod_{1\le h<t\le n/2} (q^{n-2h+1}-q^{n-2t+1})(1-q^{2n-2h-2t+2}) \prod_{h=1}^{n/2} (q^{n-2h+1}-1). \quad (C.29)$$

If we now combine (C.27), (C.28), (C.29), and (C.26), use the fact that $\det A_8 = \det A_9$, and substitute all this in (C.24), then, after some simplification, we obtain

$$so^{odd}_{\left(((m-n)/2)^n\right)}(q^{n-1},q^{n-3},\dots,q^{-n+3},q^{-n+1}) \\ = \frac{m^{n/2}\prod_{h=1}^{n/2}(1-q^{n-2h+1})}{\prod_{1\le h<t\le n/2}(q^{h-t}-q^{t-h})^2 \prod_{1\le h,t\le n/2}(q^{-n+h+t-1}-q^{n-h-t+1}) \prod_{h=1}^{n/2}(1+q^{n-2h+1})},$$

which can be rewritten as (C.20).

Now let n be odd. We proceed in a completely analogous manner. Again, the task is to compute the specialized odd orthogonal character in (4.15), by means of a limit of its definition (C.1).

What we must compute is the limit as

$$x_1 \to -q^{n-1},\ x_2 \to -q^{n-3},\ \dots,\ x_{(n-1)/2} \to -q^2, \quad (C.30)$$

of the right-hand side of (C.1) with $x_{(n+1)/2} = -1, x_{(n+3)/2} = -q^{-2}, \dots, x_{n-1} = -q^{-n+3}$, $x_n = -q^{-n+1}$. Upon a simultaneous rearrangement and change of signs of the rows of the two determinants in (C.1), one sees that this is equivalent to computing the limit (C.30) of $(\det A_{10})/(\det B_{10})$, where

$$A_{10} = \begin{pmatrix} x_h^{\frac{m+n+1}{2}-t} - x_h^{-(\frac{m+n+1}{2}-t)} & 1\le h\le (n-1)/2 \\ q^{(2n+m-2h)(\frac{m+n+1}{2}-t)} - q^{-(2n+m-2h)(\frac{m+n+1}{2}-t)} & (n-1)/2 < h \le n \end{pmatrix}_{1\le h,t\le n}$$

and

$$B_{10} = \begin{pmatrix} x_h^{n-t+\frac12} - x_h^{-(n-t+\frac12)} & 1\le h\le (n-1)/2 \\ q^{(2n+m-2h)(n-t+\frac12)} - q^{-(2n+m-2h)(n-t+\frac12)} & (n-1)/2 < h \le n \end{pmatrix}_{1\le h,t\le n}.$$

The determinant $\det B_{10}$ can again be evaluated explicitly by means of (B.2), and it turns out to be non-zero. However, the determinant $\det A_{10}$ vanishes because all the entries in its n-th row are 0 (because of $q^m = -1$). Therefore the quotient

$(\det A_{10})/(\det B_{10})$ is zero, whence also its limit (C.30). Thus, the claim (C.22) is established. $\square$

Our next lemma provides special evaluations of so-called *Schur functions*, which are needed in the proof of Theorem 22. Recall that for any partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_N)$ (i.e., a non-increasing sequence of non-negative integers) the Schur function $s_\lambda(x_1, x_2, \dots, x_N)$ is defined by (see [13, p. 403, (A.4)], [33, I, (3.1)], or [32, Prop. 1.4.4])

$$s_\lambda(x_1, x_2, \dots, x_N) = \frac{\det\limits_{1\le h,t\le N}\left(x_h^{\lambda_t+N-t}\right)}{\det\limits_{1\le h,t\le N}\left(x_h^{N-t}\right)}. \tag{C.31}$$

Again, it is not difficult to see that the denominator in (C.31) does in fact cancel out, so that any Schur function $s_\lambda(x_1, x_2, \dots, x_N)$ is in fact a *polynomial* in $x_1, x_2, \dots, x_N$, and is thus well-defined for any choice of the variables $x_1, x_2, \dots, x_N$.

In the lemma below, the notation $\big((m-n-1)^n\big)$ is a short notation for the partition in which the first n parts are $m-n-1$, followed by $n+1$ parts all of which are 0.

Lemma C3. *Let $q = e^{\pi i/m}$. Then we have*

$$s_{\left((m-n-1)^n\right)}(q^n, q^{n-1}, \dots, q, -1, q^{-1}, \dots, q^{-n+1}, q^{-n}) = 2^{-n^2} \frac{\prod\limits_{h=1}^{n/2} \tan^2 \frac{(2h-1)\pi}{2m}}{\prod\limits_{h=1}^{n+1}\prod\limits_{t=1}^{n} \left|\sin \frac{(2t-2h+1)\pi}{2m}\right|} \tag{C.32}$$

if both m and n are even,

$$s_{\left((m-n-1)^n\right)}(q^n, q^{n-1}, \dots, q, -1, q^{-1}, \dots, q^{-n+1}, q^{-n}) = 2^{-n^2} \frac{\prod\limits_{h=1}^{(n+1)/2} \tan^2 \frac{(2h-1)\pi}{2m}}{\prod\limits_{h=1}^{n+1}\prod\limits_{t=1}^{n} \left|\sin \frac{(2t-2h+1)\pi}{2m}\right|} \tag{C.33}$$

if m is even and n is odd,

$$\begin{aligned} s_{\left((m-n-1)^n\right)}&(q^n, q^{n-1}, \dots, q, -1, q^{-1}, \dots, q^{-n+1}, q^{-n}) \\ &= 2^{-n^2} \prod_{h=1}^{n/2} \frac{\sin^2 \frac{(2h-1)\pi}{2m}}{\cos^2 \frac{h\pi}{m}} \frac{1}{\prod\limits_{h=1}^{n+1}\prod\limits_{t=1}^{n} \left|\sin \frac{(2t-2h+1)\pi}{2m}\right|} \end{aligned} \tag{C.34}$$

if m is odd and n is even, and

$$s_{\left((m-n-1)^n\right)}(q^n, q^{n-1}, \dots, q, -1, q^{-1}, \dots, q^{-n+1}, q^{-n}) = 0 \tag{C.35}$$

if both m and n are odd.

Proof. We have to evaluate (C.31) with $N = 2n+1$, $\lambda = (m-n-1, m-n-1, \dots, m-n-1, 0, \dots, 0)$ (where $m-n-1$ is repeated n times), $x_1 = q^n$, $x_2 = q^{n-1}$, ..., $x_n = q$, $x_{n+1} = -1$, $x_{n+2} = q^{-1}$, ..., $x_{2n} = q^{-n+1}$, $x_{2n+1} = q^{-n}$. The denominator of (C.31) can be easily evaluated as it is just a Vandermonde determinant. If we use that $q^m = -1$, under these specializations the numerator becomes

$$\det \begin{pmatrix} D_{11} & D_{12} \\ l_1 & l_2 \\ D_{21} & D_{22} \end{pmatrix}, \tag{C.36}$$

where D_{11} is the $n \times n$ matrix

$$D_{11} = \left((-1)^{n+1-h}(q^{n+1-h})^{n-t}\right)_{1\le h,t\le n},$$

D_{12} is the $n \times (n+1)$ matrix

$$D_{12} = \left((q^{n+1-h})^{n+1-t}\right)_{1\le h\le n,\,1\le t\le n+1},$$

l_1 is the (row) vector

$$l_1 = \left((-1)^{m+n-t}\right)_{1\le t\le n},$$

l_2 is the (row) vector

$$l_2 = \left((-1)^{n+1-t}\right)_{1\le t\le n+1},$$

D_{21} is the $n \times n$ matrix

$$D_{21} = \left((-1)^{h}(q^{-h})^{n-t}\right)_{1\le h,t\le n},$$

and D_{22} is the $n \times (n+1)$ matrix

$$D_{22} = \left((q^{-h})^{n+1-t}\right)_{1\le h\le n,\,1\le t\le n+1}.$$

For the evaluation of the determinant (C.36), we have to distinguish between four cases, depending on the parities of m and n.

If both m and n are even, we subtract column $n+1+t$ from column t, $t = 1, 2, \ldots, n$. The result is that the first n entries in the odd numbered rows become zero. If we rearrange the rows so that these zeroes are moved to the bottom (i.e., row 2 is moved up to first position, row 4 is moved up to second position, etc.), then we obtain that the determinant (C.36) is equal to

$$(-1)^{\binom{n}{2}} \det \begin{pmatrix} E_1 & * \\ 0 & E_2 \end{pmatrix}, \tag{C.37}$$

where E_1 is the $n \times n$ matrix

$$E_1 = \left(2(-1)^{n+1-2h}(q^{n+1-2h})^{n-t}\right)_{1\le h,t\le n},$$

and E_2 is the $(n+1) \times (n+1)$ matrix

$$E_2 = \begin{pmatrix} (q^{n+2-2h})^{n+1-t} & h \ne \frac{n}{2}+1 \\ (-1)^{n+1-t} & h = \frac{n}{2}+1 \end{pmatrix}_{1\le h,t\le n+1}.$$

Clearly, the determinant in (C.37) (and thus the determinant in (C.36) that we want to evaluate) is equal to the product $(\det E_1)\cdot(\det E_2)$. Both of the latter determinants are Vandermonde determinants, and are therefore easily evaluated. If the result is substituted in (C.31), together with the denominator evaluation, the claimed expression (C.32) is obtained after some simplification.

In the other three cases we proceed in a similar manner. There, in the determinant (C.36) we *add* column $n+1+t$ to column t, $t = 1, 2, \ldots, n$. If the parities of m and n are different, then, again, the rows can be rearranged so that a block form is obtained (this rearrangement is different in the two cases), in which both the upper-left and the lower-right blocks are Vandermonde matrices. Finally, if both m and n are odd, then, after addition of the columns as described above, the first n entries in $n+2$ (!) rows become zero (to be precise, these are the odd numbered rows *and* the $(n+1)$-st row). Thus, the determinant of this matrix vanishes. □

Next, we turn to two further special evaluations of Schur functions, which are needed in the proofs of Theorems 26 and 28. These special evaluations are given in Lemmas C5 and C6 below). In the proofs, we make use of the standard basic hypergeometric notation introduced earlier in (6.21), and in particular of the following summation formula.

Lemma C4. *For any non-negative integer N and any indeterminate b, we have*

$$ {}_2\phi_1\left[\begin{matrix} q^{-2N}, b \\ q^{4-2N}/b \end{matrix}; q^2, \frac{q^3}{b}\right] = \frac{(b/q;q)_N\,(q^2;q^2)_N}{q^N\,(b/q^2;q^2)_N\,(q;q)_N}. \tag{C.38}$$

Proof. We write the hypergeometric series on the left-hand side of (C.38) as a sum over k, say. The reader should observe that, because of the upper parameter q^{-2N}, the sum is in fact a finite sum, with k running from 0 up to N. Now we reverse the order of summation in the sum, i.e., we replace k by $N-k$. If we rewrite the resulting sum in basic hypergeometric notation, then we obtain

$$\frac{\left(1-bq^{2N-2}\right)}{q^N\left(1-bq^{-2}\right)}{}_2\phi_1\left[\begin{matrix} q^{-2N}, b/q^2 \\ q^{2-2N}/b \end{matrix}; q^2, \frac{q^3}{b}\right].$$

This series can be summed by means of the summation formula (see ([14, Ex. 1.8])

$$ {}_2\phi_1\left[\begin{matrix} a^2, a^2/b \\ b \end{matrix}; q^2, \frac{bq}{a^2}\right] = \frac{1}{2}\left(\frac{(-b/a;q)_\infty\,(a;q)_\infty\,(q;q^2)_\infty}{(b;q^2)_\infty\,(bq/a^2;q^2)_\infty} + \frac{(b/a;q)_\infty\,(-a;q)_\infty\,(q;q^2)_\infty}{(b;q^2)_\infty\,(bq/a^2;q^2)_\infty}\right),$$

upon letting a tend to q^{-N} and replacing b by q^{2-2N}/b. After some simplification, one arrives at the right-hand side of (C.38). □

Similar to earlier notational conventions, in the lemma below, the notation $(c^{n-p},(c-1)^p)$ is a short notation for the partition in which the first $n-p$ parts are c, the next p parts are $c-1$, followed by $n+1$ parts all of which are 0.

Lemma C5. *Let n and c be positive integers, let p be a non-negative integer with $0\le p\le n$, and let q be an indeterminate. Then we have*

$$\begin{aligned} &s_{\left(c^{n-p},(c-1)^p\right)}\left(q^{2n-1},q^{2n-3},\dots,q^3,q,1,q^{-1},q^{-3},\dots,q^{-2n+3},q^{-2n+1}\right)\\ &\quad=\prod_{h=1}^{2n}\frac{\left(q^{\frac{c+h}{2}}-q^{-\frac{c+h}{2}}\right)}{\left(q^{\frac{h}{2}}-q^{-\frac{h}{2}}\right)}\prod_{h=1}^{n}\prod_{t=1}^{n}\frac{\left(q^{c+n+t-h}-q^{-c-n-t+h}\right)}{\left(q^{n+t-h}-q^{-n-t+h}\right)}\prod_{h=1}^{n}\frac{\left(q^h-q^{-h}\right)^2}{\left(q^{c+p+h}-q^{-c-p-h}\right)}\\ &\qquad\times\frac{1}{\prod_{h=1}^{p}\left(q^h-q^{-h}\right)\prod_{h=1}^{n-p}\left(q^h-q^{-h}\right)}\frac{(q^{\frac{c}{2}}-q^{-\frac{c}{2}})(q^{\frac{c}{2}+p}+q^{-\frac{c}{2}-p})}{(q^{c+p}-q^{-c-p})}. \end{aligned}\tag{C.39}$$

Proof. Using the symmetry of the Schur function, we have to evaluate (C.31) with $N=2n+1$, $\lambda=(c,\dots,c,c-1,\dots,c-1,0,\dots,0)$ (where c is repeated $n-p$ times and $c-1$ is repeated p times), $x_1=q^{2n-1}$, $x_2=q^{2n-3}$, …,$x_{n-1}=q^3$, $x_n=q$, $x_{n+1}=q^{-1}$, $x_{n+2}=q^{-3}$, …, $x_{2n-1}=q^{-2n+3}$, $x_{2n}=q^{-2n+1}$, $x_{2n+1}=1$. The denominator of (C.31) can be easily evaluated as it is just a Vandermonde determinant. On the other hand, under these specializations the numerator becomes

$$\det\begin{pmatrix} F_1 & F_2 & F_3\\ l_3 & l_4 & l_5\end{pmatrix},$$

where F_1 is the $(2n)\times(n-p)$ matrix

$$F_1=\left((q^{2n+1-2h})^{c+2n+1-t}\right)_{1\le h\le 2n,\,1\le t\le n-p},$$

F_2 is the $(2n)\times p$ matrix

$$F_2=\left((q^{2n+1-2h})^{c+n+p-t}\right)_{1\le h\le 2n,\,1\le t\le p},$$

F_3 is the $(2n)\times(n+1)$ matrix

$$F_3=\left((q^{2n+1-2h})^{n+1-t}\right)_{1\le h\le 2n,\,1\le t\le n+1},$$

l_3 is the (row) vector of length $n-p$ consisting entirely of 1's, l_4 is the (row) vector of length p consisting entirely of 1's, and l_5 is the (row) vector of length $n+1$ consisting entirely of 1's.

We consider first the case where p is strictly between 0 and n, that is, where $0<p<n$. We subtract the $(t+1)$-st column from the t-th column, $t=1,2,\dots,2n$. Clearly, this makes the last row become $(0,0,\dots,0,1)$. So, if we subsequently expand the determinant with respect to the last row and factor $q^{2n+1-2h}-1$ out of the h-th row, $h=1,2,\dots,2n$, we obtain the expression

$$\left(\prod_{h=1}^{2n}(q^{2n+1-2h}-1)\right)\det\begin{pmatrix}G_1 & G_2 & G_3 & G_4 & G_5\end{pmatrix}, \tag{C.40}$$

where G_1 is the $(2n)\times(n-p-1)$ matrix

$$G_1=\left((q^{2n+1-2h})^{c+2n-t}\right)_{1\le h\le 2n,\,1\le t\le n-p-1},$$

G_2 is the $(2n)\times 1$ matrix (i.e., column of length $2n$)

$$G_2=\left((q^{2n+1-2h})^{c+n+p-1}\left(1+q^{2n+1-2h}\right)\right)_{1\le h\le 2n},$$

G_3 is the $(2n)\times(p-1)$ matrix

$$G_3=\left((q^{2n+1-2h})^{c+n+p-t-1}\right)_{1\le h\le 2n,\,1\le t\le p-1},$$

G_4 is the $(2n)\times 1$ matrix (i.e., column of length $2n$)

$$G_4=\left((q^{2n+1-2h})^{n}\sum_{k=0}^{c-1}q^{(2n+1-2h)k}\right)_{1\le h\le 2n},$$

and G_5 is the $(2n)\times n$ matrix

$$G_5=\left((q^{2n+1-2h})^{n-t}\right)_{1\le h\le 2n,\,1\le t\le n}.$$

We now use linearity in the $(n-p)$-th and n-th columns to convert (C.40) into

$$\left(\prod_{h=1}^{2n}(q^{2n+1-2h}-1)\right)\left(\sum_{k=0}^{c-1}\det\begin{pmatrix}G_1 & G_2^{(1)} & G_3 & G_4^{(k)} & G_5\end{pmatrix}\right.\\ \left.+\sum_{k=0}^{c-1}\det\begin{pmatrix}G_1 & G_2^{(2)} & G_3 & G_4^{(k)} & G_5\end{pmatrix}\right), \tag{C.41}$$

where G_1, G_3 and G_5 are as above, where $G_2^{(1)}$ is the $(2n)\times 1$ matrix

$$G_2^{(1)}=\left((q^{2n+1-2h})^{c+n+p}\right)_{1\le h\le 2n},$$

where $G_2^{(2)}$ is the $(2n)\times 1$ matrix

$$G_2^{(2)}=\left((q^{2n+1-2h})^{c+n+p-1}\right)_{1\le h\le 2n},$$

and where $G_4^{(k)}$ is the $(2n)\times 1$ matrix

$$G_4^{(k)}=\left((q^{2n+1-2h})^{n+k}\right)_{1\le h\le 2n}.$$

The determinants in (C.41) are determinants of the form

$$\det\left(X_t^{2n+1-2h}\right)_{1\le h,t\le 2n}=\left(\prod_{t=1}^{2n}x_t^{2n-1}\right)\det\left((X_t^{-2})^{h-1}\right)_{1\le h,t\le 2n}. \tag{C.42}$$

Since the last determinant is a Vandermonde determinant, we can evaluate the determinants in (C.41). Thus, under our specializations, the numerator in (C.31) becomes

$$\left(\prod_{h=1}^{2n}(q^{2n+1-2h}-1)\right)\left(\prod_{1\le h<t\le n}(q^{t-h}-q^{-t+h})^2\right)\left(\prod_{h=1}^{n}\prod_{t=1}^{n}(q^{c+n+t-h}-q^{-c-n-t+h})\right)\\ \times\left(\sum_{k=0}^{c-1}\frac{\prod_{h=1}^{n}(q^{c+n-h-k}-q^{-c-n+h+k})\prod_{h=1}^{n}(q^{h+k}-q^{-h-k})}{(q^{c+p-k-1}-q^{-c-p+k+1})\prod_{h=1}^{p-1}(q^h-q^{-h})\prod_{h=1}^{n-p}(q^h-q^{-h})\prod_{h=1}^{n}(q^{c+p+h-1}-q^{-c-p-h+1})}\right.\\ \left.+\sum_{k=0}^{c-1}\frac{\prod_{h=1}^{n}(q^{c+n-h-k}-q^{-c-n+h+k})\prod_{h=1}^{n}(q^{h+k}-q^{-h-k})}{(q^{c+p-k}-q^{-c-p+k})\prod_{h=1}^{p}(q^h-q^{-h})\prod_{h=1}^{n-p-1}(q^h-q^{-h})\prod_{h=1}^{n}(q^{c+p+h}-q^{-c-p-h})}\right). \tag{C.43}$$

The task now is to simplify the two sums. In the first sum in (C.43) we replace k by $k-1$. After this replacement, we put the two sums together, that is, for any fixed k we add the k-th summands of the two sums. Thus, the expression (C.43) simplifies to

$$\frac{\left(\prod_{h=1}^{2n}(q^{2n+1-2h}-1)\right)\left(\prod_{1\le h<t\le n}(q^{t-h}-q^{-t+h})^2\right)\left(\prod_{h=1}^{n}\prod_{t=1}^{n}(q^{c+n+t-h}-q^{-c-n-t+h})\right)}{\left(\prod_{h=1}^{p}(q^h-q^{-h})\right)\left(\prod_{h=1}^{n-p}(q^h-q^{-h})\right)\left(\prod_{h=0}^{n}(q^{c+p+h}-q^{-c-p-h})\right)}$$
$$\times\left(\sum_{k=0}^{c}\left(\prod_{h=1}^{n-1}(q^{c+n-h-k}-q^{-c-n+h+k})(q^{h+k}-q^{-h-k})\right)(q^n-q^{-n})\right.$$
$$\left.\cdot\Big(q^{-k}(q^{-c-n-p}+q^{c+n-p}-q^{c+p-n}-q^{c+n+p})-q^k(q^{-c-n-p}+q^{-c+n-p}-q^{-c+p-n}-q^{c+n+p})\Big)\right). \tag{C.44}$$

In what follows, we concentrate on the sum in (C.44). We split the sum in two sums according to the additive decomposition of the last factor of the summand and rewrite the two sums in basic hypergeometric notation,

$$\begin{aligned}(-1)^n q^{cn-c-n}(q^{-2n-2c+2};q^2)_{n-1}\,(q^2;q^2)_n\\ \times\Bigg((q^{-c-n-p}+q^{c+n-p}-q^{c+p-n}-q^{c+n+p})\;{}_2\phi_1\!\left[\begin{matrix}q^{-2c},q^{2n}\\ q^{2-2n-2c}\end{matrix};q^2,q^{1-2n}\right]\\ -(q^{-c-n-p}+q^{-c+n-p}-q^{-c+p-n}-q^{c+n+p})\;{}_2\phi_1\!\left[\begin{matrix}q^{-2c},q^{2n}\\ q^{2-2n-2c}\end{matrix};q^2,q^{3-2n}\right]\Bigg).\end{aligned} \tag{C.45}$$

We write the second ${}_2\phi_1$-series as a sum over k, say, reverse the order of summation, that is, we replace k by $c-k$, and then we write the resulting sum again in basic hypergeometric notation. We obtain a ${}_2\phi_1$-series which turns out to be identical with the first ${}_2\phi_1$-series in (C.45). If we put everything together and simplify, then (C.45) becomes

$$\begin{aligned}(-1)^n q^{cn-2c-2n-p}(q^{-2n-2c+2};q^2)_{n-1}\,(q^2;q^2)_n(1-q^c)(1-q^{c+2n})(1+q^{c+2p})\\ \times\,{}_2\phi_1\!\left[\begin{matrix}q^{-2c},q^{2n}\\ q^{2-2n-2c}\end{matrix};q^2,q^{1-2n}\right].\end{aligned} \tag{C.46}$$

Next we apply the contiguous relation

$${}_2\phi_1\!\left[\begin{matrix}a,b\\ c\end{matrix};q,z\right]={}_2\phi_1\!\left[\begin{matrix}a,bq\\ c\end{matrix};q,z\right]-bz\frac{(1-a)}{(1-c)}\,{}_2\phi_1\!\left[\begin{matrix}aq,bq\\ cq\end{matrix};q,z\right]$$

to the ${}_2\phi_1$-series in (C.46). This transforms the expression (C.46) into

$$\begin{aligned}(-1)^n q^{cn-2c-2n-p}(q^{-2n-2c+2};q^2)_{n-1}\,(q^2;q^2)_n(1-q^c)(1-q^{c+2n})(1+q^{c+2p})\\ \times\left({}_2\phi_1\!\left[\begin{matrix}q^{-2c},q^{2n+2}\\ q^{2-2n-2c}\end{matrix};q^2,q^{1-2n}\right]-q\frac{1-q^{-2c}}{1-q^{2-2n-2c}}\,{}_2\phi_1\!\left[\begin{matrix}q^{-2c+2},q^{2n+2}\\ q^{4-2n-2c}\end{matrix};q^2,q^{1-2n}\right]\right).\end{aligned} \tag{C.47}$$

Both ${}_2\phi_1$-series in the last line can be evaluated by means of Lemma C4. If we substitute the result in (C.47), put this back in (C.44), and divide the result by the numerator in (C.31) subject to our specializations, which we evaluated by means of the Vandermonde determinant evaluation, we arrive at the right-hand side of (C.39) after some simplification.

If $p = 0$, then we can proceed in a completely analogous manner. In fact, the computations are somewhat simpler in this case, so that we leave the details to the reader. That the formula works also for $p = n$ can then be checked by verifying that (C.39) for $p = n$ agrees with (C.39) for $p = 0$ and c replaced by $c - 1$.

This completes the proof of the lemma. □

Lemma C6. *Let n and c be positive integers, and let q be an indeterminate. Then we have*

$$s_{(c^n)}(q^{2n-1}, q^{2n-3}, \dots, q^3, q, -1, q^{-1}, q^{-3}, \dots, q^{-2n+3}, q^{-2n+1})$$
$$= \prod_{h=1}^{2n} \frac{\left(q^{\frac{c+h}{2}} - (-1)^{c+h} q^{-\frac{c+h}{2}}\right)}{\left(q^{\frac{h}{2}} - (-1)^h q^{-\frac{h}{2}}\right)} \prod_{h=1}^{n-1} \prod_{t=1}^{n} \frac{\left(q^{c+n+t-h} - q^{-c-n-t+h}\right)}{\left(q^{n+t-h} - q^{-n-t+h}\right)}. \quad \text{(C.48)}$$

Proof. We replace q by $-q$ in the $p = 0$ case of Lemma C5. Then, the left-hand side of (C.39) agrees exactly with the left-hand side of (C.48). The products on the right-hand sides are not in the same form, but they are equivalent because the extra terms in Lemma C5 cancel out if $p = 0$.

□

The final two lemmas concern sums of *even orthogonal characters*, which are needed in the proofs of Theorems 30 and 32. Given a non-increasing sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ of integers or half-integers with $\lambda_{n-1} \geq |\lambda_n|$, the even orthogonal character $so^{even}_\lambda(x_1, x_2, \dots, x_n)$ is defined by (see [13, (24.40)])

$$so^{even}_\lambda(x_1, x_2, \dots, x_n) = \frac{\det_{1\le h,t\le n}\left(x_h^{\lambda_t+n-t} + x_h^{-(\lambda_t+n-t)}\right) + \det_{1\le h,t\le n}\left(x_h^{\lambda_t+n-t} - x_h^{-(\lambda_t+n-t)}\right)}{\det_{1\le h,t\le n}\left(x_h^{n-t} + x_h^{-(n-t)}\right)}. \quad \text{(C.49)}$$

Lemma C7. *Let n be a positive integer and c be a non-negative half-integer. Then we have*

$$\sum_{p=-c}^{c} so^{even}_{\left((c^{n-1},p)\right)}(q^{n-1}, q^{n-2}, \dots, 1)$$
$$= \prod_{1\le h<t\le n} \frac{q^{(2c+t+h-1)/2} - q^{-(2c+t+h-1)/2}}{q^{(t+h-2)/2} - q^{-(t+h-2)/2}} \frac{\prod_{h=1}^{n}\left(q^{c+h-1/2} - q^{-(c+h-1/2)}\right)}{\prod_{h=1}^{n-1}\left(q^{h/2} - q^{-h/2}\right)}$$
$$\times \sum_{k=1}^{n} \frac{(-1)^{k-1}(2c+2k-1)}{\prod_{h=1}^{n}\left(q^{(2c+k+h-1)/2} - q^{-(2c+k+h-1)/2}\right) \prod_{h=1}^{k-1}\left(q^{h/2} - q^{-h/2}\right) \prod_{h=1}^{n-k}\left(q^{h/2} - q^{-h/2}\right)}. \quad \text{(C.50)}$$

Proof. For convenience, let us write $\widetilde{so}_\lambda(x_1,x_2,\dots,x_n)$ for just "one half" in the definition (C.49) of even orthogonal characters, that is,

$$\widetilde{so}_\lambda(x_1,x_2,\dots,x_n)=\frac{\det\limits_{1\le h,t\le n}\left(x_h^{\lambda_t+n-t}+x_h^{-(\lambda_t+n-t)}\right)}{\det\limits_{1\le h,t\le n}\left(x_h^{n-t}+x_h^{-(n-t)}\right)}. \tag{C.51}$$

It should be observed that we have

$$\widetilde{so}_\lambda(x_1,\dots,x_{n-1},1)=so_\lambda^{even}(x_1,\dots,x_{n-1},1), \tag{C.52}$$

since the second determinant in (C.49) vanishes if $x_n=1$. Now, by (C.51) and the determinant evaluation (B.3), we have

$$\sum_{p=-c}^{c}\widetilde{so}_{\left((c^{n-1},p)\right)}(x_1,x_2,\dots,x_n)$$
$$=\frac{1}{2}\left(\prod_{1\le h<t\le n}\frac{1}{(x_h^{\frac12}x_t^{-\frac12}-x_h^{-\frac12}x_t^{\frac12})(x_h^{\frac12}x_t^{\frac12}-x_h^{-\frac12}x_t^{-\frac12})}\right)$$
$$\times\sum_{p=-c}^{c}\det_{1\le h,t\le n}\begin{pmatrix}x_h^{c+n-t}+x_h^{-(c+n-t)} & 1\le t<n\\ x_h^{p}+x_h^{-p} & t=n\end{pmatrix}$$
$$=\frac{1}{2}\left(\prod_{1\le h<t\le n}\frac{1}{(x_h^{\frac12}x_t^{-\frac12}-x_h^{-\frac12}x_t^{\frac12})(x_h^{\frac12}x_t^{\frac12}-x_h^{-\frac12}x_t^{-\frac12})}\right)$$
$$\times\det_{1\le h,t\le n}\begin{pmatrix}x_h^{c+n-t}+x_h^{-(c+n-t)} & 1\le t<n\\ 2\dfrac{x_h^{c+\frac12}-x_h^{-(c+\frac12)}}{x_h^{\frac12}-x_h^{-\frac12}} & t=n\end{pmatrix}$$
$$=\left(\prod_{1\le h<t\le n}\frac{1}{(x_h^{\frac12}x_t^{-\frac12}-x_h^{-\frac12}x_t^{\frac12})(x_h^{\frac12}x_t^{\frac12}-x_h^{-\frac12}x_t^{-\frac12})}\right)\left(\prod_{h=1}^{n}\frac{1}{x_h^{\frac12}-x_h^{-\frac12}}\right)$$
$$\times\det_{1\le h,t\le n}\begin{pmatrix}x_h^{c+n-t+\frac12}-x_h^{-(c+n-t+\frac12)}+x_h^{c+n-t-\frac12}-x_h^{-(c+n-t-\frac12)} & 1\le t<n\\ x_h^{c+\frac12}-x_h^{-(c+\frac12)} & t=n\end{pmatrix}$$
$$=\left(\prod_{1\le h<t\le n}\frac{1}{(x_h^{\frac12}x_t^{-\frac12}-x_h^{-\frac12}x_t^{\frac12})(x_h^{\frac12}x_t^{\frac12}-x_h^{-\frac12}x_t^{-\frac12})}\right)\left(\prod_{h=1}^{n}\frac{1}{x_h^{\frac12}-x_h^{-\frac12}}\right)$$
$$\times\det_{1\le h,t\le n}\left(x_h^{c+n-t+\frac12}-x_h^{-(c+n-t+\frac12)}\right),$$

where the last line arises by obvious elementary column operations from the next-to-last line. In this identity, we want to put $x_1=q^{n-1}$, $x_2=q^{n-2}$, …, $x_n=1$. However, we cannot directly put $x_n=1$, because this would lead to an expression $0/0$. Therefore instead, we have to take the limit $x_n\to1$. Thereby, using (C.52), we get

$$\sum_{p=-c}^{c}so^{even}_{\left((c^{n-1},p)\right)}(q^{n-1},q^{n-2},\dots,1)$$

$$= \left(\prod_{1\le h<t\le n} \frac{1}{(q^{(t-h)/2} - q^{-(t-h)/2})(q^{(h+t-2)/2} - q^{-(h+t-2)/2}} \right) \left(\prod_{h=1}^{n-1} \frac{1}{q^{h/2} - q^{-h/2}} \right) \\ \times \det_{1\le h,t\le n} \begin{pmatrix} q^{(n-h)(c+n-t+\frac12)} - q^{-(n-h)(c+n-t+\frac12)} & 1\le h<n \\ 2c+2n-2t+1 & h=n \end{pmatrix}. \tag{C.53}$$

Now we expand the determinant along the last row. Then each of the appearing minors can be evaluated by means of (B.4). If we substitute the result in (C.53), we obtain

$$\sum_{p=-c}^{c} so^{even}_{\left((c^{n-1},p)\right)}(q^{n-1}, q^{n-2}, \dots, 1) \\ = \left(\prod_{1\le h<t\le n} \frac{1}{(q^{(t-h)/2} - q^{-(t-h)/2})(q^{(h+t-2)/2} - q^{-(h+t-2)/2})} \right) \left(\prod_{h=1}^{n-1} \frac{1}{q^{h/2} - q^{-h/2}} \right) \\ \times \sum_{k=1}^{n} (-1)^{n+k}(2c+2n-2k+1) \prod_{\substack{t=1\\ t\ne k}}^{n} (q^{c+n-t+\frac12} - q^{-(c+n-t+\frac12)}) \\ \cdot \prod_{\substack{1\le h<t\le n\\ h,t\ne k}} (q^{(t-h)/2} - q^{-(t-h)/2})(q^{(2c+2n-h-t+1)/2} - q^{-(2c+2n-h-t+1)/2}).$$

After replacing k by $n+1-k$ and performing some simplification, we arrive at the claimed expression. ☐

Lemma C8. *Let n be a positive integer and c be a non-negative half-integer. Then we have*

$$\sum_{p=-c}^{c} (-1)^{c-p}\, so^{even}_{\left((c^{n-1},p)\right)}(q^{n-1}, q^{n-2}, \dots, 1) \\ = \prod_{1\le h<t\le n} \frac{q^{(2c+t+h-1)/2} - q^{-(2c+t+h-1)/2}}{q^{(t+h-2)/2} - q^{-(t+h-2)/2}} \prod_{h=1}^{n-1} \frac{1}{\left(q^{h/2} + q^{-h/2}\right)}. \tag{C.54}$$

Proof. According to the definition (C.49) of even orthogonal characters and the determinant evaluation (B.3), we have

$$\sum_{p=-c}^{c} (-1)^{c-p}\, \widetilde{so}_{\left((c^{n-1},p)\right)}(x_1, x_2, \dots, x_n) \\ = \frac12 \left(\prod_{1\le h<t\le n} \frac{1}{(x_h^{\frac12} x_t^{-\frac12} - x_h^{-\frac12} x_t^{\frac12})(x_h^{\frac12} x_t^{\frac12} - x_h^{-\frac12} x_t^{-\frac12})} \right) \\ \times \sum_{p=-c}^{c} (-1)^{c-p} \det_{1\le h,t\le n} \begin{pmatrix} x_h^{c+n-t} + x_h^{-(c+n-t)} & 1\le t<n \\ x_h^{p} + x_h^{-p} & t=n \end{pmatrix} \\ = \frac12 \left(\prod_{1\le h<t\le n} \frac{1}{(x_h^{\frac12} x_t^{-\frac12} - x_h^{-\frac12} x_t^{\frac12})(x_h^{\frac12} x_t^{\frac12} - x_h^{-\frac12} x_t^{-\frac12})} \right)$$

$$\times \det_{1\le h,t\le n}\begin{pmatrix} x_h^{c+n-t}+x_h^{-(c+n-t)} & 1\le t<n\\ 2\dfrac{x_h^{c+\frac12}+x_h^{-(c+\frac12)}}{x_h^{\frac12}+x_h^{-\frac12}} & t=n\end{pmatrix}$$

$$=\left(\prod_{1\le h<t\le n}\frac{1}{(x_h^{\frac12}x_t^{-\frac12}-x_h^{-\frac12}x_t^{\frac12})(x_h^{\frac12}x_t^{\frac12}-x_h^{-\frac12}x_t^{-\frac12})}\right)\left(\prod_{h=1}^{n}\frac{1}{x_h^{\frac12}+x_h^{-\frac12}}\right)$$

$$\times \det_{1\le h,t\le n}\begin{pmatrix} x_h^{c+n-t+\frac12}+x_h^{-(c+n-t+\frac12)}+x_h^{c+n-t-\frac12}+x_h^{-(c+n-t-\frac12)} & 1\le t<n\\ x_h^{c+\frac12}+x_h^{-(c+\frac12)} & t=n\end{pmatrix}$$

$$=\left(\prod_{1\le h<t\le n}\frac{1}{(x_h^{\frac12}x_t^{-\frac12}-x_h^{-\frac12}x_t^{\frac12})(x_h^{\frac12}x_t^{\frac12}-x_h^{-\frac12}x_t^{-\frac12})}\right)\left(\prod_{h=1}^{n}\frac{1}{x_h^{\frac12}+x_h^{-\frac12}}\right)$$

$$\times \det_{1\le h,t\le n}\left(x_h^{c+n-t+\frac12}+x_h^{-(c+n-t+\frac12)}\right),$$

where $\widetilde{so}_\lambda(x_1,x_2,\dots,x_n)$ is defined in (C.51), and where the last line arises by obvious elementary column operations from the next-to-last line. Now we put $x_1=q^{n-1}$, $x_2=q^{n-2}$, …, $x_n=1$ in this identity, in which case, due to (C.52), the "incomplete" even orthogonal character $\widetilde{so}_{\left((c^{n-1},p)\right)}(x_1,x_2,\dots,x_n)$ becomes the specialized even orthogonal character $so^{even}_{\left((c^{n-1},p)\right)}(q^{n-1},q^{n-2},\dots,1)$, and we use (B.3) to evaluate the determinant in the last line. After some simplification we arrive at (C.54). □

References

[1] D. André, *Solution directe du problème résolu par M. Bertrand*, Comptes Rendus Acad. Sciences Paris Sér. I **105** (1887), 436–437.

[2] T. H. Baker and P. J. Forrester, *Random walks and random fixed point free involutions*, J. Phys. A: Math. Gen. **34** (2001), L381–L390.

[3] C. Banderier and P. Flajolet, *Basic analytic combinatorics of directed lattice paths*, Theoret. Comput. Sci. **281** (2002), 37–80.

[4] P. Biane, *Minuscule weights and random walks on lattices*, Quant. Probab. Related Topics **7** (1992), 51–65.

[5] P. Biane, *Estimation asymptotique des multiplicités dans les puissances tensorielles d'un q-module*, Comptes Rendus Acad. Sci. Paris, Sér. I **316** (1993), 849–852.

[6] L. Comtet, *Advanced Combinatorics*, D. Reidel, Dordrecht, Holland, 1974.

[7] M. Filaseta, *A new method for solving a class of ballot problems*, J. Combin. Theory Ser. A **39** (1985), 102–111.

[8] M. E. Fisher, *Walks, walls, wetting and melting*, J. Stat. Phys. **34** (1984), 667-729.

[9] P. Flajolet and R. Sedgewick, *The average analysis of algorithms: Saddle point asymptotics*, INRIA Res. Rep. #2376, Chapter 6 in the book project "Analytic Combinatorics," available at `http://algo.inria.fr/flajolet/Publications/books.html`.

[10] P. J. Forrester, *Exact solution of the lock step model of vicious walkers*, J. Phys. A: Math. Gen. **23** (1990), 1259–1273.

[11] M. Fulmek, *Nonintersecting lattice paths on the cylinder*, Séminaire Lotharingien Combin. **52** (2004), Article #B52b, 16 pages.

[12] M. Fulmek and C. Krattenthaler, *Lattice path proofs for determinant formulas for symplectic and orthogonal characters*, J. Combin. Theory Ser. A **77** (1997), 3–50.

[13] W. Fulton and J. Harris, *Representation Theory*, Springer–Verlag, New York, 1991.

[14] G. Gasper and M. Rahman, *Basic hypergeometric series*, 2nd edition, Encyclopedia of Mathematics And Its Applications 96, Cambridge University Press, Cambridge, 2004.

[15] I. M. Gessel and C. Krattenthaler, *Cylindric partitions*, Trans. Amer. Math. Soc. **349** (1997), 429–479.

[16] I. M. Gessel and D. Zeilberger, *Random walk in a Weyl chamber*, Proc. Amer. Math. Soc. **115** (1992), 27–31.

[17] D. J. Grabiner, *Walks and representation theory*, Ph.D. Thesis, Harvard University, 1995.

[18] D. J. Grabiner, *Brownian motion in a Weyl chamber, non-colliding particles, and random matrices*, Ann. Inst. H. Poincaré Probab. Statist. **35** (1999), 177–204.

[19] D. J. Grabiner, *Random walk in an alcove of an affine Weyl group, and non-colliding random walks on an interval*, J. Combin. Theory Ser. A **97** (2002), 285–306.

[20] D. J. Grabiner, *Asymptotics for the distributions of subtableaux in Young and up-down tableaux*, Electron. J. Combin. **11**(2) (2006), Article #R29, 22 pp.

[21] D. J. Grabiner and P. Magyar, *Random walks in Weyl chambers and the decomposition of tensor powers*, J. Alg. Combin. **2** (1993), 239–260.

[22] A. J. Guttmann, A. L. Owczarek and X. G. Viennot, *Vicious walkers and Young tableaux I: without walls*. J. Phys. A: Math. Gen. **31** (1998), 8123–8135.

[23] B. R. Handa and S. G. Mohanty, *Enumeration of higher-dimensional paths under restrictions*, Discrete Math. **26** (1979), 119–128.

[24] J. E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge University Press, Cambridge, 1990.

[25] S. Karlin, *Coincident probabilities and applications in combinatorics*, J. Appl. Probab. **25A** (1988), 185–200.

[26] C. Krattenthaler, *Oscillating tableaux and nonintersecting lattice paths*, J. Statist. Plann. Inference **54** (1996), 75–85.

[27] C. Krattenthaler, *Identities for classical group characters of nearly rectangular shape*, J. Algebra **209** (1998), 1–64.

[28] C. Krattenthaler, A. J. Guttmann and X. G. Viennot, *Vicious walkers, friendly walkers and Young tableaux II: with a wall*, J. Phys. A: Math. Gen. **33** (2000), 8835–8866.

[29] C. Krattenthaler, A. J. Guttmann and X. G. Viennot, *Vicious walkers, friendly walkers and Young tableaux III: between two walls*, J. Statist. Phys. **110** (2003), 1069–1086.

[30] G. Kreweras and H. Niederhausen, *Solution of an enumerative problem connected with lattice paths*, Europ. J. Combin. **2** (1981), 55–60.

[31] G. Kuperberg, *Random words, quantum statistics, central limits, random matrices*, Methods Appl. Anal. **9** (2002), 101–119.

[32] A. Lascoux, *Symmetric functions and combinatorial operators on polynomials*, CBMS Regional Conference Series in Mathematics, vol. 99, Amer. Math. Soc., Providence, RI, 2003.

[33] I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, second edition, Oxford University Press, New York/London, 1995.

[34] S. G. Mohanty, *Lattice path counting and applications*, Academic Press, New York, 1979.

[35] H. Niederhausen, *Linear recurrences under side conditions*, Europ. J. Combin. **1** (1980), 353–368.

[36] H. Niederhausen, *The ballot problem with three candidates*, Europ. J. Combin. **4** (1983), 161–167.

[37] H. Niederhausen, *The enumeration of restricted random walks by Sheffer polynomials with applications to statistics*, J. Statist. Plann. Inference **14** (1986), 95–114.

[38] R. A. Proctor, *Reflection and algorithm proofs of some more Lie group dual pair identities*, J. Combin. Theory Ser. Ser. A **62** (1993), 107–127.

[39] A, Regev, *Asymptotic values for degrees associated with strips of Young diagrams*, Adv. in Math. **41** (1981), 115–136.

[40] M. Rubey, *Nonintersecting lattice paths in combinatorics, commutative algebra and statistical mechanics*, Ph.D. thesis, Universität Wien, 2002; available at `http://www.mat.univie.ac.at/~rubey/`.

[41] M. Rubey, *Asymptotic values of vicious walkers with arbitrary end points*, in preparation.

[42] M. Sato, *Generating functions for the number of lattice paths between two parallel lines with a rational incline*, Math. Japonica **34** (1989), 123–137.

[43] M. Sato, *Generating functions for the number of lattice paths restricted by two parallel hyperplanes*, J. Statist. Plann. Inference **34** (1993), 251–258.
[44] M. Sato and T. Sado, *Lattice paths restricted by two parallel hyperplanes*, Bull. Inf. Cybernetics **21** (1985), 97–105.
[45] D. Zeilberger, *André's reflection proof generalized to the many candidate ballot problem*, Discrete Math. **44** (1983), 325–326.

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