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On character tables related to the alternating groups

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Abstract

There is a simple formula for the absolute value of the determinant of the character table of the symmetric group S_n. It equals $a_{\mathcal{P}}$, the product of all parts of all partitions of n (see [4, Corollary 6.5]). In this paper we calculate the absolute values of the determinants of certain submatrices of the character table $\mathcal{X}$ of the alternating group A_n, including that of $\mathcal{X}$ itself (Section 2). We also study explicitly the powers of 2 occurring in these determinants using generating functions (Section 3).

1. Preliminaries

We fix a positive integer n. We will use the same notation as in [2], which we recall here.

If $\mu = (\mu_1, \mu_2, \ldots)$ is a partition of n we write $\mu \in \mathcal{P}$ and then z_μ denotes the order of the centralizer of an element of (conjugacy) type μ in S_n. Suppose $\mu = (1^{m_1(\mu)}, 2^{m_2(\mu)}, \ldots)$, is written in exponential notation. Then we may factor $z_\mu = a_\mu b_\mu$, where

$$a_\mu = \prod_{i\geq 1} i^{m_i(\mu)}, \quad b_\mu = \prod_{i\geq 1} m_i(\mu)!$$

Whenever $\mathcal{Q} \subseteq \mathcal{P}$ we define

$$a_{\mathcal{Q}} = \prod_{\mu\in\mathcal{Q}} a_\mu, \quad b_{\mathcal{Q}} = \prod_{\mu\in\mathcal{Q}} b_\mu \,.$$

We consider the alternating group A_n. We let $\mathcal{P}^+$ denote the even partitions in $\mathcal{P}$, $\mathcal{O}$ the partitions into odd parts, and $\mathcal{D}$ the partitions into distinct parts.

The conjugacy classes in A_n are of two types. The classes labelled by partitions $\mu \in \mathcal{P}^+ \setminus (\mathcal{O} \cap \mathcal{D})$ are the non-split classes, which contain all

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S_n-permutations of this type; we denote a representative by σ_μ and note that the corresponding centralizer is then of order $z'_\mu = z_\mu/2$. For the partitions $\mu \in \mathcal{D} \cap \mathcal{O}$, the corresponding S_n-class splits into two conjugacy classes in A_n, for which we denote representatives by σ_μ^+ and σ_μ^-; their centralizers are of order $z'_\mu = z_\mu$.

We briefly recall some information on the irreducible A_n-characters (see [5, sect. 2.5]).

Let μ be a partition of n. For $\mu \neq \tilde{\mu}$, i.e., μ non-symmetric, $[\mu] \downarrow_{A_n} = [\tilde{\mu}] \downarrow_{A_n}$ is irreducible. Let $\{\mu\} = \{\tilde{\mu}\}$ denote this irreducible character of A_n.
For $\mu = \tilde{\mu}$, i.e., μ symmetric, $[\mu] \downarrow_{A_n} = \{\mu\}_+ + \{\mu\}_-$ is a sum of two distinct irreducible A_n-characters (which are conjugate in S_n).
This gives all the irreducible complex characters of A_n, i.e.,

$$\mathrm{Irr}(A_n) = \{\{\mu\}_\pm \mid \mu \vdash n, \mu = \tilde{\mu}\} \cup \{\{\mu\} \mid \mu \vdash n, \mu \neq \tilde{\mu}\} \,.$$

The characters $\{\mu\}_\pm$, for symmetric μ, usually have non-rational values on the corresponding "critical" classes of cycle type $h(\mu) = (h_1, \ldots, h_l)$, where $h_1, \ldots, h_l$ are the principal hook lengths in μ; note that $h(\mu) \in \mathcal{D} \cap \mathcal{O}$, so the corresponding S_n-class splits. Then we have $[\mu](\sigma_{h(\mu)}) = (-1)^{\frac{n-l}{2}} =: \varepsilon_\mu$ and

$$\{\mu\}_+(\sigma^\pm_{h(\mu)}) = \frac{1}{2}\left(\varepsilon_\mu \pm \sqrt{\varepsilon_\mu \prod_{i=1}^{l} h_i}\right)$$

$$\{\mu\}_-(\sigma^\pm_{h(\mu)}) = \frac{1}{2}\left(\varepsilon_\mu \mp \sqrt{\varepsilon_\mu \prod_{i=1}^{l} h_i}\right)$$

All other irreducible A_n-characters have the same value on these two classes.

For later use, we want to recall the Jacobi minor theorem (see [3, p. 21]).
Let $A = (a_{ij})$ be an $n \times n$ matrix. Let M_v be a v-rowed minor of the determinant $\det A$, corresponding to the rows $i_1, \ldots, i_v$ and the columns $k_1, \ldots, k_v$. Then we take the $(n-v)$-rowed complementary minor for A by deleting all the rows and columns chosen for M_v before, and define the signed complementary minor $M^{(v)}$ to M_v by multiplying this complementary minor by the sign ± 1, depending on $\sum_{j=1}^{v} i_j + \sum_{j=1}^{v} k_j$ being even or odd, respectively. (Note that for principal minors the sign is always $+$.)

Let $A' = (A_{ij})$ be the $n \times n$-matrix of cofactors A_{ij} for A, i.e., the adjoint matrix to A. Let M_v and M'_v be corresponding v-rowed minors of A and A', respectively, then

$$M'_v = (\det A)^{v-1} M^{(v)} \,.$$

2. Determinants of submatrices of the character table of A_n

We observe that by the Murnaghan-Nakayama formula we have for any symmetric partition μ and any $\nu \in \mathcal{D} \cap \mathcal{O}$:

$$\{\mu\}_\pm(\sigma_\nu^\pm) = 0 \text{ for all } \nu > h(\mu) \text{ (in lexicographic order)}$$

Hence, if we order the k (say) partitions in $\mathcal{D} \cap \mathcal{O}$ in decreasing lexicographic order, and the k symmetric partitions according to their principal hook lengths, then the corresponding $2k \times 2k$ part of the character table of A_n is almost an upper triangular matrix, except that we have 2×2 blocks along the diagonal. We call this matrix $\mathcal{X}_s$.

Knowing the entries of these diagonal blocks explicitly, we can easily compute their determinant and hence the (absolute value of the) determinant of this submatrix of the character table. A 2×2 block corresponding to the characters $\{\mu\}_\pm$ on the classes $\sigma^\pm_{h(\mu)}$ gives a contribution of absolute value

$$|\varepsilon_\mu \sqrt{\varepsilon_\mu \prod_i h_i}| = \sqrt{\prod_i h_i} = \sqrt{a_{h(\mu)}}\ ,$$

where $h(\mu) = (h_1, h_2, \ldots)$. Hence the absolute value of the determinant of the whole submatrix is given by:

Proposition 2.1.

$$|\det \mathcal{X}_s| = \prod_{\nu\in\mathcal{D}\cap\mathcal{O}} \sqrt{a_\nu} = \sqrt{a_{\mathcal{D}\cap\mathcal{O}}}\ .$$

We can also easily determine the (absolute value of the) determinant for the whole character table $\mathcal{X}$ of A_n. By character orthogonality, we know that $\bar{\mathcal{X}}^t\mathcal{X}$ is a diagonal matrix with the centralizer orders as its diagonal entries. Set $\mathcal{P}^{(+)} = \mathcal{P}^+ \setminus (\mathcal{D} \cap \mathcal{O})$. Hence we have

$$\begin{aligned} |\det \mathcal{X}|^2 &= \left(\prod\nolimits_{\mu\in\mathcal{P}^{(+)}} \tfrac{z_\mu}{2}\right)\left(\prod\nolimits_{\mu\in\mathcal{D}\cap\mathcal{O}} z_\mu^2\right) \\ &= 2^{-|\mathcal{P}^{(+)}|} z_{\mathcal{P}^+} z_{\mathcal{D}\cap\mathcal{O}} = 2^{-|\mathcal{P}^{(+)}|} a_{\mathcal{P}^+} b_{\mathcal{P}^+} a_{\mathcal{D}\cap\mathcal{O}} \end{aligned}$$

Now we have $b_{\mathcal{P}^+} = 2^{e^+} a_{\mathcal{P}^+}$, for some integer $e^+ \in \mathbb{Z}$. (This is not hard to prove by a combinatorial argument, see Lemma 3.3.)

Hence we obtain

Proposition 2.2.

$$|\det \mathcal{X}|^2 \;=\; 2^{e^+ - |\mathcal{P}^{(+)}|} a^2_{\mathcal{P}^+} a_{\mathcal{D}\cap\mathcal{O}}\ .$$

In the next section we will see that $e^+ = e^+(n) \in \mathbb{N}$, and that there is a nice generating function for the numbers $e^+(n)$ (Proposition 3.4). In particular, an explicit formula for $e^+(n)$ is given by

$$e^+(n) = \sum_{i=1}^{[n/2]} \tau(i) p'(n-2i)\ ,$$

where $\tau(i)$ is the number of divisors of i, and $p'(j) = |\mathcal{D}(j) \cap \mathcal{O}(j)|$.

We are interested in determining the determinant of the integral part of the character table of A_n corresponding to the non-symmetric partitions and the non-split conjugacy classes; let us call this matrix $\mathcal{X}_u$ (with some ordering of rows and columns chosen). (Note that this is also a submatrix of the character table of S_n.) This part of the character table of A_n is complementary to the submatrix we have considered above, and we want to compute its determinant by employing Jacobi's theorem.

Theorem 2.3. *The determinant of the matrix $\mathcal{X}_u$ has absolute value*

$$|\det \mathcal{X}_u| = 2^{(e^+ - |\mathcal{P}^{(+)}|)/2} a_{\mathcal{P}^{(+)}} .$$

Proof. We assume that the rows of the character table $\mathcal{X}$ of A_n are labelled such that the rows corresponding to the symmetric partitions come first, and that the columns are labelled such that the $v = |\mathcal{P}^{(+)}|$ partitions in $\mathcal{P}^{(+)}$ come first. Let Δ be the diagonal matrix with the centralizer orders z'_μ as its diagonal entries, and let $\Delta^{(+)}$ be the diagonal submatrix corresponding to the partitions $\mu \in \mathcal{P}^{(+)}$.

As we have $\bar{\mathcal{X}}^t \cdot \mathcal{X} = \Delta$, we know that the adjoint matrix to $\mathcal{X}$ is

$$\mathcal{X}' = (\det \mathcal{X}) \Delta^{-1} \bar{\mathcal{X}}^t .$$

We now want to apply Jacobi's minor theorem as it is stated in Section 1. We take the v-rowed minor M_v corresponding to the upper left square part in $\mathcal{X}$, i.e., $M_v = \det \mathcal{X}_u$. The corresponding minor of $\mathcal{X}'$ is then the determinant of

$$(\det \mathcal{X})(\Delta^{(+)})^{-1} \mathcal{X}_u$$

(remember that $\mathcal{X}_u$ is integral). The signed complementary minor to M_v in $\mathcal{X}$ is then just $\det \mathcal{X}_s$. By Jacobi's theorem we know that

$$(\det \mathcal{X})^v \Big(\prod_{\mu \in \mathcal{P}^{(+)}} z'_\mu \Big)^{-1} \det \mathcal{X}_u = (\det \mathcal{X})^{v-1} \det \mathcal{X}_s .$$

Hence

$$\det \mathcal{X}_u = (\det \mathcal{X})^{-1} 2^{-v} a_{\mathcal{P}^{(+)}} b_{\mathcal{P}^{(+)}} \det \mathcal{X}_s ,$$

and thus

$$\begin{aligned} |\det \mathcal{X}_u| &= 2^{-(e^+ - v)/2} (a_{\mathcal{P}^+} \sqrt{a_{\mathcal{D} \cap \mathcal{O}}})^{-1} 2^{-v} a_{\mathcal{P}^{(+)}} b_{\mathcal{P}^+} \sqrt{a_{\mathcal{D} \cap \mathcal{O}}} \\ &= 2^{(e^+ - v)/2} a_{\mathcal{P}^{(+)}} \end{aligned}$$

where we have used the relation $b_{\mathcal{P}^+} = 2^{e^+} a_{\mathcal{P}^+}$. $\square$

3. Powers of 2

We compute the generating functions for the powers of 2 occurring in the determinants of the previous section.

Let $P(q), P^+(q), P^-(q)$ be the generating function for the number of partitions (resp. even/odd partitions) of n. The following is well-known:

Lemma 3.1. *$P^+(q) - P^-(q) = \Delta(q)$, where*

$$\Delta(q) = \prod_{k\geq 0}(1+q^{2k+1}) \quad (= \frac{P(q)P(q^4)}{P(q^2)^2})$$

is the generating function for the number of partitions of n into distinct odd parts.

Indeed, using that in $P(q) = \prod_{k\geq 1}\frac{1}{1-q^k}$ the factor $\frac{1}{1-q^k}$ accounts for the parts equal to k we see that

$$P^+(q) - P^-(q) = \prod_{k\geq 1}\frac{1}{1+(-q)^k}\ .$$

Substituting $q \to -q$ in the Euler identity $\prod_{k\geq 1}(1+q^k) = \prod_{k\geq 0}\frac{1}{1-q^{2k+1}}$ and inverting we get

$$\prod_{k\geq 1}\frac{1}{1+(-q)^k} = \prod_{k\geq 0}(1+q^{2k+1})\ ,$$

proving the Lemma. □

We assume in the following always that $\delta = +$ or $-$ is a sign.

Corollary 3.2. *We have*

$$P^\delta(q) = \frac{P(q)+\delta\Delta(q)}{2}\ .$$

We let $\mathcal{P}^\delta(n)$ be the set of partitions of n with sign δ. Then define

$$a^\delta(n) = a_{\mathcal{P}^\delta(n)} = \prod_{\mu\in\mathcal{P}^\delta(n)} a_\mu, \quad b^\delta(n) = b_{\mathcal{P}^\delta(n)} = \prod_{\mu\in\mathcal{P}^\delta(n)} b_\mu\ .$$

We factor each $i \in \mathbb{N}$ as a product $i = i_2 i'$, where i_2 is a power of 2 and i' is odd and consider two involutory bijections ι, ι' on the set

$$\mathcal{T}(n) = \{(\mu, d, k) | \mu \in \mathcal{P}(n), m_d(\mu) \geq k\}\ .$$

Here

$$\iota : (\mu, d, k) \mapsto (\hat{\mu}, k, d)$$

where $\hat{\mu}$ is obtained from μ by replacing k parts equal to d by d parts equal to k and leaving all other parts unchanged and

$$\iota' : (\mu, d, k) \mapsto (\tilde{\mu}, d_2 k', k_2 d')$$

where $\tilde{\mu}$ is obtained from μ by replacing k parts equal to d by k_2d' parts equal to d_2k' and leaving all other parts unchanged. Let

$$\mathcal{T}^{\delta}_{d,k}(n) = \{\mu \in \mathcal{P}^{\delta}(n) | m_d(\mu) \geq k\} .$$

Then

$$|\mathcal{T}^{\delta}_{d,k}(n)| = p^{(-1)^{(d-1)k}\delta}(n - dk) \tag{1}$$

Indeed removing k parts equal to d from a partition μ with sign δ gives you a partition with sign $(-1)^{(d-1)k}\delta$ and of cardinality $|\mu| - dk$.

Note that this means that the partitions $\mu, \hat{\mu}$ in the definition of ι have different signs if and only if $(d-1)k$ and $d(k-1)$ have different parities, ie. if and only if d and k have different parities. Moreover the partitions $\mu, \tilde{\mu}$ in the definition of ι' have the same sign.

Thus ι induces a bijection between $\mathcal{T}^{\delta}_{d,k}(n)$ and $\mathcal{T}^{\delta}_{k,d}(n)$ if d, k have the same parity and between $\mathcal{T}^{\delta}_{d,k}(n)$ and $\mathcal{T}^{-\delta}_{k,d}(n)$ if d, k have different parities. Moreover the bijection ι' shows that

$$a^{\delta}(n)' = b^{\delta}(n)', \tag{2}$$

and hence

Lemma 3.3. *$b^{\delta}(n)/a^{\delta}(n) = 2^{e^{\delta}(n)}$ for some integer $e^{\delta}(n)$.*

The power of 2 in $a^{\delta}(n)$ is

$$x^{\delta}(n) = \prod_{\substack{d,k \\ d \text{ even}}} d_2^{|\mathcal{T}^{\delta}_{d,k}(n)|}$$

and the power of 2 in $b^{\delta}(n)$ is

$$y^{\delta}(n) = \prod_{\substack{d,k \\ k \text{ even}}} k_2^{|\mathcal{T}^{\delta}_{d,k}(n)|}$$

Let $x^{\delta}_o(n), x^{\delta}_e(n)$ be the product of the factors in $x^{\delta}(n)$, where k is odd/even and correspondingly $y^{\delta}_o(n), y^{\delta}_e(n)$ be the product of the factors in $y^{\delta}(n)$, where d is odd/even. Using the map ι we see that

$$x^{\delta}_e(n) = y^{\delta}_e(n), \quad x^{\delta}_o(n) = y^{-\delta}_o(n)$$

Thus the power of 2 in $b^{\delta}(n)/a^{\delta}(n)$ is $x^{-\delta}_o(n)/x^{\delta}_o(n)$.
Suppose that $x^{\delta}_o(n) = 2^{f^{\delta}_o(n)}$ and $x^{\delta}_e(n) = 2^{f^{\delta}_e(n)}$. Then $e^{\delta}(n) = f^{-\delta}_o(n) - f^{\delta}_o(n)$.
We have (since $\nu_2(d) = 0$, when d is odd)

$$f^{\delta}_o(n) = \sum_{\substack{d,k \\ k \text{ odd}}} \nu_2(d)|\mathcal{T}^{\delta}_{d,k}(n)| = \sum_{\substack{d,k \\ k \text{ odd}}} \nu_2(d) p^{-\delta}(n - dk) .$$

Let $\tau_o(n)$ the number of odd divisors of n. Note that $\tau_o(n)\nu_2(n)$ equals the number $\tau_e(n)$ of *even* divisors of n. We then get (substituting $dk = t$ in the above sum and noting that then $\nu_2(d) = \nu_2(t)$)

$$f_o^\delta(n) = \sum_{t=1}^{n} \tau_o(t)\nu_2(t)p^{-\delta}(n-t) = \sum_{t=1}^{n} \tau_e(t)p^{-\delta}(n-t)\,.$$

Let $T(q) = \sum_{t\geq 1} \frac{q^t}{1-q^t}$ be the generating function for $\tau(n)$. Then $T(q^2)$ is the generating function for the number $\tau_e(n)$ of even divisors of n. If $F_o^\delta(q)$ is the generating function for $f_o^\delta(n)$ we obtain

$$F_o^\delta(q) = T(q^2)P^{-\delta}(q). \tag{3}$$

Using Lemmas 3.1 and 3.3 we deduce

Proposition 3.4. *The generating function for $e^\delta(n)$ is*

$$E^\delta(n) = F_o^{-\delta}(q) - F_o^\delta(q) = \delta T(q^2)\Delta(q)\,.$$

Remark 3.5. This Proposition was also proved in [6] in a different way. Our approach was partially inspired by an unpublished note of John Graham.

Note that the proposition shows that $e^+ = e^+(n)$ is always a *positive* integer.

Let us also consider $F_e^\delta(q)$. We have

$$f_e^\delta(n) = \sum_{\{d,k|k \text{ even}\}} \nu_2(d)|\mathcal{T}_{d,k}^\delta(n)| = \sum_{\{d,k|k \text{ even}\}} \nu_2(d)p^\delta(n-dk)\,.$$

We substitute $dk = 2t$ in the above and obtain

$$f_e^\delta(n) = \sum_{t\geq 1} \tau^*(t)p^\delta(n-2t)\,,$$

where $\tau^*(t) = \sum_{d|t} \nu_2(d)$. We have

$$\tau^*(t) = \binom{\nu_2(t)+1}{2} \prod_{p \text{ odd}} (\nu_p(t)+1)\,.$$

Thus if $T^*(q)$ is the generating function for $\tau^*(t)$ then

$$F_e^\delta(q) = T^*(q^2)P^\delta(q)\,.$$

It is easily seen that

$$T^*(q) = \sum_{j\geq 1} T(q^{2^j})\,.$$

Proposition 3.6. *The exponent of 2 in $a^\delta(n)$ has the generating function*

$$F_e^\delta(q) + F_o^\delta(q) = T^*(q^2)P^\delta(q) + T(q^2)P^{-\delta}(q)\,.$$

In Theorem 2.3 we have seen that $|\det \mathcal{X}_u| = 2^{(e-|\mathcal{P}^{(+)}|)/2} a_{\mathcal{P}(+)}$.
By Proposition 3.4, $e = e^+(n)$ has generating function $E^+(q) = \Delta(q)T(q^2)$. Moreover $|\mathcal{P}^{(+)}(n)|$ has generating function $P^+(q) - \Delta(q) = P^-(q)$ (Lemma 3.1). Clearly, $a_{P(+)}(n)$ is divided by the same power of 2 as $a^+(n)$, as the removed partitions have only odd parts. The generating function for the corresponding exponent is given by Proposition 3.6. Hence the exponent of 2 in $\det \mathcal{X}_u$ has the generating function

$$G(q) = \frac{1}{2}\big(T(q^2)\Delta(q) - P^-(q)\big) + T^*(q^2)P^+(q) + T(q^2)P^-(q),$$

and this then yields

Theorem 3.7. *The exponent of 2 in* $\det \mathcal{X}_u$ *has the generating function*

$$G(q) = \frac{1}{2}\big(T(q^2)P(q) - P^-(q)\big) + T^*(q^2)P^+(q)\,.$$

According to MAPLE the first values of the coefficients of $G(q)$ are the following for $n = 2, \ldots, 14$: 0 0 2 2 4 6 15 19 30 43 70 94 138.

Let us finally remark that the Propositions 3.4 and 3.6 also allow to compute the generating function for the exponent of 2 in $|\det(\mathcal{X})|$, using Proposition 2.2.

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