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PROJECTIVE REPRESENTATIONS OF GENERALIZED SYMMETRIC GROUPS

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1. **Introduction**

The representation theory of generalized symmetric groups has been of interest over a long period dating back to the classical work of W. Specht [28],[29] and M.Osima [22] — an exposition of this work and other references may be found in [12]. Furthermore, the projective representations of these groups have been considered by a number of authors, much of the this work was not published or was published in journals not readily accessible in the western world. The first comprehensive work on the projective representations of the generalized symmetric groups was due to E. W. Read [24] which was followed later by an improvement in the work of M. Saeed-ul-Islam, see, for example, [26]. Of equal interest has been the representation theory of the hyperoctahedral groups, which are a special case of the generalized symmetric groups. The projective representations of these groups was considered by M. Munir in his thesis [20] which elaborated on the earlier work of E. W. Read and M. Saeed-ul-Islam and also by J. Stembridge [31] who gave an independent development which was more complete and satisfactory in many respects. This approach later influenced that used by H. I. Jones in his thesis [13] where the use of Clifford algebras was emphasized.

More recently, the generalized symmetric groups have become far more predominant in the context of complex reflection groups and the corresponding cyclotomic Hecke algebras where they and their subgroups form the infinite family $G(m, p, n)$, see for example [3],[4] and [5]. In view of this interest, it was thought worthwhile to present this work which is based on the earlier work of H. I. Jones which has not been published. As this article is also meant to be partially expository, a great deal of the background material is also presented.

There are eight non-equivalent 2-cocycles for the generalized symmetric group $G(m, 1, n)$, which will be denoted by B_n^m in this paper. Thus, in addition to the ordinary irreducible representations, there are seven other classes of projective representations to be considered. However, the position is not too complicated in that all of the non-equivalent irreducible projective representations can be expressed in terms of certain 'building blocks'. These are the ordinary and spin representations of the symmetric group S_n, that is, the generalized symmetric group $G(1, 1, n)$, which are well known and date back to the early work of F.G. Frobenius and A. Young (see [12]) and I. Schur [30] respectively. Also, required are basic spin representations P, Q and R of B_n^m for certain 2-cocycles. All of these can be constructed in a uniform way using Clifford algebras and the basic spin representations of the orthogonal groups. Thus, we will present all of the required information for constructing these building blocks.

The paper is organised as follows. In Section 2 we present all of the background information and notation required later, there are short subsections on partitions, the projective representations of groups, the method of J. R. Stembridge on Clifford theory (A. H. Clifford) for $\mathbb{Z}_2^2$-quotients [31] and Clifford algebras (W. K. Clifford) and their representations. Section 3 contains all of the information required about the generalized symmetric groups B_n^m; a presentation, classes of conjugate elements and its linear characters are given. In Section 4, the main aim is to construct the three classes of basic spin representations P, Q and R of B_n^m mentioned above and some additional information required later — these are mainly based on the authors earlier work, [17], [18], [19]. For the sake of completeness we also include a brief description of the elegant construction of the irreducible spin representations of the symmetric groups given by M. L. Nazarov [21]. The final section then contains the construction of the irreducible projective representations for the eight 2-cocycles. In this section, we follow J. R. Stembridge's work in the special case B_n^2. Our results are not as complete as his and an indication of proof only is given in some cases. A detailed description, including the construction of the irreducible representations for three closely connected subgroups will appear later.

2. **Background and Notation**

2.1. **Partitions.** The notation follows [14]. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be a partition of n, then $l(\lambda) = k$ is the **length** of λ and $|\lambda| = n$ is the *weight* of λ. The **conjugate** of λ is denoted by λ'. A partition λ is called an **even(odd) partition** if the number of even parts in λ is even(odd). A partition is sometimes written as $\lambda = (1^{a_1}2^{a_2}\ldots n^{a_n})$, $0 \leq a_i \leq n$ indicating that a_i parts of λ are equal to i, $1 \leq i \leq n$, $|\lambda| = \sum_{i=1}^n ia_i$ and $l(\lambda) = \sum_{i=1}^n a_i$.

Let $P(n)$ denote the set of all partitions of n, then $DP(n) = \{\lambda \in P(n) \mid \lambda_1 > \lambda_2 > \cdots > \lambda_k > 0\}$ is the set of all partitions of n into distinct parts, $DP^+(n) = \{\lambda \in DP(n) \mid |\lambda| - l(\lambda) \text{ is even}\}$, $DP^-(n) = \{\lambda \in DP(n) \mid |\lambda| - l(\lambda) \text{ is odd}\}$, $OP(n) = \{\lambda = (1^{\alpha_1}3^{\alpha_3}\ldots)\}$ is the set of all partitions of n into odd parts, $EP(n) = \{\lambda = (2^{\alpha_2}4^{\alpha_4}\ldots)\}$ is the set of all partitions of n into even parts and $SCP(n) = \{\lambda \in P(n) \mid \lambda = \lambda'\}$ is the set of self-conjugate partitions of n.

An m-**partition** of n is a partition comprising of m partitions $(\lambda_{(1)}; \lambda_{(2)}; \ldots; \lambda_{(m)})$ such that $\lambda_{(i)} \in P(n_i)$, $1 \leq i \leq m$ and $\sum_{i=1}^m n_i = n$. The partition $\lambda_{(i)}$ is written as $(\lambda_{i1}, \lambda_{i2}, \ldots, \lambda_{ik_i})$, where $k_i = l(\lambda_{(i)})$ for $1 \leq i \leq m$. The **conjugate** of $(\lambda_{(1)}; \lambda_{(2)}; \ldots; \lambda_{(m)})$ is the m-partition $(\lambda'_{(1)}; \lambda'_{(2)}; \ldots; \lambda'_{(m)})$. An m-partition is said to be **even(odd)** if the total number of even parts of $(\lambda_{(1)}; \lambda_{(2)}; \ldots; \lambda_{(m)})$ is even(odd). An m-partition is sometimes written in the form

$$((1^{\alpha_{11}}2^{\alpha_{12}}\ldots); (1^{\alpha_{21}}2^{\alpha_{22}}\ldots); \ldots; (1^{\alpha_{m1}}2^{\alpha_{m2}}\ldots));$$

$l(\lambda_{(1)}; \lambda_{(2)}; \ldots; \lambda_{(m)}) = l(\lambda_{(1)}) + l(\lambda_{(2)}) + \cdots + l(\lambda_{(m)})$ is the **length of** $(\lambda_{(1)}; \lambda_{(2)}; \ldots; \lambda_{(m)})$ and $|(\lambda_{(1)}; \lambda_{(2)}; \ldots; \lambda_{(m)})| = |(\lambda_{(1)})| + |(\lambda_{(2)}| + \cdots + |(\lambda_{(m)})|$ is the **weight of** $(\lambda_{(1)}; \lambda_{(2)}; \ldots; \lambda_{(m)})$. We note that $l(\lambda_{(1)}; \lambda_{(2)}; \ldots; \lambda_{(m)}) = \Sigma_{i=1}^m \Sigma_{j=1}^n a_{ij}$.

2.2. **Projective representations.** We present some basic background material on the projective representations of groups which is required later.

Let G be a group with identity 1 of order $|G|$, $\mathbb{C}$ the field of complex numbers, $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ and $GL(n, \mathbb{C})$ the group of invertible $n \times n$ matrices over $\mathbb{C}$.

A **projective representation of degree n** of G is a map $P : G \to GL(n, \mathbb{C})$ such that for $g, h \in G$

$$P(g)P(h) = \alpha(g, h)P(gh)$$

and $P(1) = I_n$, where I_n is the identity $n \times n$ matrix and $\alpha(g,h) \in \mathbb{C}^{\times}$. Since multiplication in G and $GL(n, \mathbb{C})$ is associative, it follows that

$$\alpha(g,h)\alpha(gh,k) = \alpha(g,hk)\alpha(h,k) \tag{2.1}$$

for all $g, h, k \in G$. A map $\alpha : G \times G \to \mathbb{C}^{\times}$ which satisfies (2.1) is called a **2-cocycle**(**factor set**) of G in $\mathbb{C}$ and we shall say that P is a projective representation with 2-cocycle α.

Projective representations P and Q of degree n with 2-cocycles α and β respectively are said to be **projectively equivalent** if there exists a map $\mu : G \to \mathbb{C}^{\times}$ and a matrix $S \in GL(n, \mathbb{C})$ such that

$$Q(g) = \mu(g)S^{-1}P(g)S$$

for all $g \in G$. If P and Q are projectively equivalent, it follows that

$$\beta(g,h) = \frac{\mu(g)\mu(h)}{\mu(gh)}\alpha(g,h) \tag{2.2}$$

for all $g, h \in G$. The corresponding 2-cocycles β and α are then said to be **equivalent**.

Let $H^2(G, \mathbb{C}^{\times})$ denote the set of equivalence classes of 2-cocycles; then $H^2(G, \mathbb{C}^{\times})$ is an abelian group which is called the **Schur multiplier** of G. The Schur multiplier gives a measure of the number of different classes of projectively inequivalent representations which a group G possesses. If G is a finite group, then $H^2(G, \mathbb{C}^{\times})$ is a finite abelian group.

All projective representations of G may be obtained from ordinary representations of a larger group; thus the problem of determining all the projective representations of a group G is essentially reduced to that of determining ordinary representations of a larger finite group.

A **central extension** (H, ϕ) of a group G is a group H together with a homomorphism $\phi : H \to G$ such that $ker\phi \subset Z(H)$, where $Z(H)$ is the centre of H, that is,

$$1 \to ker\phi \to H \xrightarrow{\phi} G \to \{1\}$$

is exact. Let $A = ker\phi$, and let $\{\gamma(g) \mid g \in G\}$ be a set of coset representatives of A in H which are in $1-1$ correspondence with the elements of G; thus

$$H = \bigcup_{g \in G} A\gamma(g).$$

Then, for all $g, h \in G$, let $a(g,h)$ be the unique element in A such that

$$\gamma(g)\gamma(h) = a(g,h)\gamma(gh).$$

The associative law in H and G implies that

$$a(g,h)a(gh,k) = a(g,hk)a(h,k) \tag{2.3}$$

for all $g, h, k \in G$. Now, let γ be a linear character of the abelian group A and put

$$\alpha(g,h) = \gamma(a(g,h))$$

for all $g, h \in G$, then (2.3) implies that α is a 2-cocycle of G.

Now, let T be an ordinary irreducible representation of H of degree n put $P(g) = T(\gamma(g))$ for all $g \in G$, then P is a projective representation of G with 2-cocycle α. A projective representation P of G arising from an irreducible ordinary representation T of H in this way is said to be **linearized** by the ordinary representation T.

If G is a finite group, then there exists a central extension H of G with kernel $H^2(G, \mathbb{C}^\times)$ which linearizes every projective representation of G. Such a group H is called a **representation group** of G; this implies that every finite group has at least one representation group. Thus, the problem of determining all the irreducible projective representations of G for all possible 2-cocycles is reduced to determining all the ordinary irreducible representations of a representation group H.

In practice, we shall see that it will be sufficient to determine a complete set of irreducible projective representations of a group G for a **fixed 2-cocycle** α whose values are roots of unity. In that case, we can calculate in terms of a subgroup of the representation group of G which will be called a α-**covering group** of G.

Let α be a 2-cocycle such that $\{\alpha\}$ has order n and let ω be a primitive n-th root of unity, then $\alpha(g,h) = \omega^{\eta(g,h)}$ for some $0 \le \eta(g,h) < n$. Suppose that $\{\nu(g) \mid g \in G\}$ is a set of distinct symbols in one-one correspondence with the elements of G. Let $G(\alpha) = \{(\alpha^j, \nu(g)) \mid 0 \le j < n, g \in G\}$, then it is easily verified that $G(\alpha)$ is a group with composition defined by

$$(\alpha^j, \nu(g))(\alpha^k, \nu(h)) = (\alpha^{j+k+\eta(g,h)}, \nu(gh))$$

for all $g, h \in G, 0 \le j, k < n$.

If now P is a projective representation of G of degree n with 2-cocycle α, then define $T : G(\alpha) \to GL(n, \mathbb{C})$ by

$$T(\alpha^j, \nu(g)) = \omega^j P(g),$$

then T is an ordinary representation of $G(\alpha)$. That is, P has been lifted to an ordinary representation of $G(\alpha)$. Such a group $G(\alpha)$ is called an α-**covering group** of the group G.

In the case of the generalized symmetric group, the 2-cocycles are of order two, thus we shall then refer to the $G(\alpha)$ as **double covers**. As we are basically working with ordinary representations of the $G(\alpha)$, we can apply all the usual results from representation theory. However, we shall be interested in the non-ordinary projective representations, namely the ones in which the central element $-1 \in G(\alpha)$ is represented faithfully, we refer to these as **spin representations** of G with 2-cocycle α.

If $\mathcal{C}$ denotes a class of conjugate elements in G, let $\mathcal{C}(\alpha) \in G(\alpha)$ denote the inverse image in $G(\alpha)$. If any $g \in \mathcal{C}(\alpha)$ is conjugate to $-g$, then $\mathcal{C}(\alpha)$ is a class of conjugate elements in $G(\alpha)$, otherwise $\mathcal{C}(\alpha)$ splits into two classes. The spin character will only be non-zero on the splitting classes; thus it will be necessary to determine the splitting classes for each 2-cocycle.

2.3. **Clifford theory for $\mathbb{Z}_2^2$-quotients.** Let G be a group with a subgroup H of index 2 and let η be a linear character of G defined by

$$\eta(g) \quad = \quad \begin{cases} 1 & \text{if } g \in H \\ -1 & \text{if } g \notin H. \end{cases}$$

If T is an irreducible representation of G with character χ, then $\eta \otimes T$ is also an irreducible representation of G, if these representations are equivalent, then we say that T is **self-associate**, but if not, they are said to be η-**associate**, and are denoted by T_+ and T_-, their characters are denoted by χ_+ and χ_-; clearly $\chi_-(g) = \eta(g)\chi_+(g)$ for all $g \in G$. If T is self-associate, then the unique(up to sign) matrix S such that

$$T(g)S = \eta(g)ST(g)$$

for all $g \in G$, is called the η-**associator** of T. If T is self-associate, then $T|_H$ decomposes into two inequivalent irreducible representations of H of equal degree, say T_1 and T_2 with characters χ_1 and χ_2 respectively, then the **difference character** $\Delta^\eta\chi$, is defined by

$$\Delta^\eta\chi(g) = trST(g) = \chi_1(g) - \chi_2(g)$$

for all $g \in H$. Knowledge of the difference character then gives the corresponding characters of H,

$$\frac{1}{2}(\chi \pm \Delta^\eta\chi).$$

All the above results are classic [12] and date back to A. H. Clifford. Recently, J. R. Stembridge [31] has extended this detailed analysis to the case where $G/H \cong \mathbb{Z}_2 \times \mathbb{Z}_2$; we briefly recall his results. Let $L = \{1, \eta, \sigma, \eta\sigma\}$ be the four corresponding linear characters of G. If T is an irreducible representation of G, then $\nu \otimes T$ for all $\nu \in L$ is also an irreducible representation of G. As before, the question is whether these are equivalent or not. Let $L_T = \{\nu \in L | \nu \otimes T \ \sim T\}$. Then, the following proposition gives the behavior of T on restriction to H.

Proposition 2.1. *Let T be an irreducible representation of degree d of G.*
(i) If $L_T = \{1\}$, then T_H is an irreducible representation of degree d of H.
(ii) If $L_T = \{1, \nu\}$, where $\nu \in L$, $\nu \neq 1$, then T_H is the direct sum of two inequivalent irreducible representation of degree $d/2$ of H.
(iii) If $L_T = L$, and R, S are the η, σ-associators of T respectively, then
(a) if $RS = SR$, then T_H is the direct sum of four inequivalent irreducible representation of degree $d/4$ of H,
(b) if $RS = -SR$, then T_H is the direct sum of two copies of one irreducible representation of degree $d/2$ of H.

As in the above, knowledge of the difference characters enables one to write out the irreducible representations of H, the only additional case which needs to be considered is (iii)(b); in that case, the four irreducible characters are

$$\frac{1}{4}(\chi \pm \Delta^\eta\chi \pm \Delta^\sigma\chi \pm \Delta^{\eta\sigma}\chi),$$

where an even number of the $-$ signs occur.

2.4. **Clifford algebras and their representations.** Let $C(n)$ be the Clifford algebra generated by $1, e_1, \ldots, e_n$ subject to the relations

$$e_j^2 = 1, \ \ e_je_k = -e_ke_j, \ \ 1 \leq j, k \leq n, \ \ j \neq k.$$

If $Pin(n)$ is defined to be the set of invertible elements s of $C(n)$ such that $(s\alpha(s^t))^2 = 1$, where α is the natural $\mathbb{Z}_2$-grading on $C(n)$ and ^{t} is the transpose, then we have the short exact sequence

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow Pin(n) \xrightarrow{\rho_n} O(n) \longrightarrow 1 \quad , \tag{2.4}$$

where ρ_n is defined by $\rho_n(s)e_j = \alpha(s)e_js^{-1}$, for all $s \in Pin(n)$, $1 \leq j \leq n$.
In fact, the Schur multiplier of $O(n)$ is given by

$$H^2(O_n, \mathbb{C}^*) = \mathbb{Z}_2. \tag{2.5}$$

Furthermore, if $Spin(n) = \rho_n^{-1}(SO(n))$, then we also have the classical double covering of the special orthogonal (rotation) group $SO(n)$

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow Spin(n) \xrightarrow{\rho_n} SO(n) \longrightarrow 1 \quad , \tag{2.6}$$

Clearly, $Spin(n)$ is of index 2 in $Pin(n)$; let η denote the corresponding linear character of $Pin(n)$.

We now construct the so-called **basic spin representation** of Clifford algebras. Let

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ I = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ J = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \ K = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

then

$$I^2 = K^2 = E, \ \ J^2 = E, \ \ JI = -IJ = iK,$$

$$KI = -IK = iJ, \ \ KJ = -JK = I.$$

Then, if $n = 2\mu$ is even, we define an isomorphism $P_n : C_n \to \mathbb{C}(2^\mu)$ by

$$\begin{cases} P_n(e_{2j-1}) & = & M_{2j-1} := K^{\otimes(j-1)} \otimes I \otimes E^{\otimes(\mu-j)} \\ P_n(e_{2j}) & = & M_{2j} := K^{\otimes(j-1)} \otimes J \otimes E^{\otimes(\mu-j)} \end{cases} \tag{2.7}$$

for $1 \leq j \leq \mu$ and if $n = 2\mu + 1$ is odd, we define an isomorphism $P_{n,+} : C_n \to \mathbb{C}(2^\mu)$ by

$$\begin{cases} P_{n,+}(e_j) & = & P_n(e_j) \\ P_{n,+}(e_{2\mu+1}) & = & M_n = K^{\otimes\mu} \end{cases} \tag{2.8}$$

for $1 \leq j \leq 2\mu$. Furthermore, for $1 \leq j \leq n$, put

$$P_{n,-}(e_j) = -P_{n,+}(e_j)$$

Then we note that

$$M_j^2 = I, \ \ M_j M_k = -M_k M_j \ \text{ for } \ 1 \leq j, k \leq n. \tag{2.9}$$

Then, if n is even, P_n is the unique irreducible complex representation of degree $2^{n/2}$ of C_n and if n is odd, $P_{n,+}$ and $P_{n,-}$ are the two inequivalent irreducible complex representations of degree $2^{n/2}$ of C_n which are clearly η-associate representations. From now on, we denote these by $P, P_\pm$. We shall refer to these as the **basic spin representations** of the Clifford algebra. It is easily checked that an $\eta-$associator of P is $K^{\otimes\mu}$. In [18], it was proved that the basic spin representation of a Clifford algebra $C(n)$ is irreducible when restricted to the orthogonal group, or to be more precise, to its double cover $Pin(n)$. This restricted representation is called the **basic spin representation** of the orthogonal group.

We now define a twisted outer product of spin representations. Let m and n be positive integers such that $m + n = l$. We show how to construct irreducible spin representations of $Pin(m, n)$ by taking a product of an irreducible spin representation of $Pin(m)$ with an irreducible spin representation of $Pin(n)$.

Let P_1 and P_2 be irreducible spin representations of $Pin(m)$ and $Pin(n)$ respectively of degrees d_1 and d_2 respectively. Then the **twisted product** $P_1 \hat{\otimes} P_2$ is a spin representation of the twisted product $Pin(m, n) \cong Pin(m) \hat{\otimes} Pin(n)$ (see [18]) defined as follows; there are 3 cases to be considered.

Case 1: If P_1 and P_2 are η-associate spin representations of $Pin(m)$ and $Pin(n)$ respectively, then put

$$\begin{aligned}(P_1\hat{\otimes}P_2)(\tau,\sigma) &= E\otimes P_1(\tau)\otimes P_2(\sigma) \text{ if } \tau\in Spin(m),\sigma\in Spin(n),\\ (P_1\hat{\otimes}P_2)(\tau,1) &= I\otimes P_1(\tau)\otimes I_{d_2} \text{ if } \tau\in Pin(m)\setminus Spin(m),\\ (P_1\hat{\otimes}P_2)(1,\sigma) &= J\otimes I_{d_2}\otimes P_2(\sigma) \text{ if } \sigma\in Pin(n)\setminus Spin(n);\end{aligned}$$

the relation $IJ=-JI$ ensures that $P_1\hat{\otimes}P_2$ is a spin representation of $Pin(m)\hat{\otimes}Pin(n)$ of degree $2d_1d_2$. Furthermore, $P_1\hat{\otimes}P_2$ is self-associate, since $tr(I)=tr(J)=0$ and so $P_1\hat{\otimes}P_2$ and $\eta\otimes(P_1\hat{\otimes}P_2)$ have equal characters.
Case 2: If P_1 is a self-associate spin representation of $Pin(m)$ with η-associator S_1 and P_2 is an η-associate spin representation of $Pin(n)$, then

$$S_1P_1(\sigma)=\begin{cases} P_1(\sigma)S_1 & \text{if } \sigma\in Spin(m)\\ -P_1(\sigma)S_1 & \text{if } \sigma\in Pin(m)\setminus Spin(m).\end{cases}$$

Now, define

$$\begin{aligned}(P_1\hat{\otimes}P_2)_\pm(\tau,\sigma) &= P_1(\tau)\otimes P_{2\pm}(\sigma) \text{ if } \tau\in Spin(m),\sigma\in Spin(n),\\ (P_1\hat{\otimes}P_2)_\pm(\tau,1) &= P_1(\tau)\otimes I_{d_2} \text{ if } \tau\in Pin(m)\setminus Spin(m),\\ (P_1\hat{\otimes}P_2)_\pm(1,\sigma) &= S_1\otimes P_{2\pm}(\sigma) \text{ if } \sigma\in Pin(n)\setminus Spin(n).\end{aligned}$$

Then $(P_1\hat{\otimes}P_2)_\pm$ are η-associate irreducible spin representations of $Pin(m)\hat{\otimes}Pin(n)$ of degree d_1d_2.
Case 3: If P_1 and P_2 are both self-associate representations, then define $(P_1\hat{\otimes}P_2)_\pm$ as in Case 2, but replacing $P_{2\pm}$ by P_2, then $(P_1\hat{\otimes}P_2)_+$ and $(P_1\hat{\otimes}P_2)_-$ are equivalent irreducible spin representations of $Pin(m)\hat{\otimes}Pin(n)$, thus $(P_1\hat{\otimes}P_2)_+$ is a self-associate spin representation of degree d_1d_2 in this case.

If we let χ_{P_1},χ_{P_2} and $\chi_{P_1\hat{\otimes}P_2}$ denote the characters of P_1,P_2 and $(P_1\hat{\otimes}P_2)$ respectively, and $\Delta_{P_1},\Delta_{P_2}$ and $\Delta_{P_1\hat{\otimes}P_2}$ denote the difference characters if P_1,P_2 or $P_1\hat{\otimes}P_2$ are self-associate, then as a consequence of the above we have the following proposition.

Proposition 2.2. *If P_1 and P_2 are spin representations of $Pin(m)$ and $Pin(n)$ respectively and*

(i) if P_1 and P_2 are η-associate representations then

$$\chi_{P_1\hat{\otimes}P_2}(\tau,\sigma)=\begin{cases} 2\chi_{P_1}(\tau)\chi_{P_2}(\sigma) & \text{if } \tau\in Spin(m),\sigma\in Spin(n)\\ 0 & \text{otherwise.}\end{cases}$$

(ii) if one of P_1 or P_2 is self-associate, then

$$\chi_{P_1\hat{\otimes}P_2}(\tau,\sigma)=\begin{cases} \chi_{P_1}(\tau)\chi_{P_2}(\sigma) & \text{if } \tau\in Spin(m),\sigma\in Spin(n)\\ \Delta_{P_1}(\tau)\chi_{P_2}(\sigma) & \text{if } \tau\in Pin(m)\setminus Spin(m),\sigma\in Pin(n)\setminus Spin(n)\\ 0 & \text{otherwise.}\end{cases}$$

The above can be generalized, that is, we can define the twisted product $P_1\hat{\otimes}\cdots\hat{\otimes}P_t$, where $\hat{\otimes}$ is an associative 'multiplication'.

Let $m_1,\ldots,m_t$ be positive integers such that $m_1+\cdots+m_t=l$ and for $1\le j\le t$, let P_j be an irreducible spin representation of $Pin(m_j)$ of degree d_j. For simplicity, we assume that $P_j,\ 1\le j\le r\le t$, are self-associate representations and that the remaining

$s=t-r$ representations P_j are η-associate representations. Let $\pm S_j,\ 1\le j\le r$, be the η-associators of the representations P_j, then

$$P_j(\sigma_j)=\begin{cases} S_jP_j(\sigma_j) & \text{if } \sigma_j\in Spin(m_j)\\ -S_jP_j(\sigma_j) & \text{if } \sigma_j\notin Spin(m_j).\end{cases} \tag{2.10}$$

Let σ_j also denote the element $1\otimes\cdots\otimes 1\otimes\sigma_j\otimes 1\otimes\cdots\otimes 1$ in $Pin(m_1)\hat{\otimes}\cdots\hat{\otimes}Pin(m_t)$, with σ_j in the j-th position, where $\sigma_j\in Pin(m_j),\ 1\le j\le t$. If $\sigma_j\in Spin(m_j)$, $1\le j\le t$, put

$$P(\sigma_j)=I_{2^{\lfloor s/2\rfloor}}\otimes I_{d_1}\otimes\cdots\otimes I_{d_{j-1}}\otimes P_j(\sigma_j)\otimes I_{d_{j+1}}\otimes\cdots\otimes I_{d_t} \tag{2.11}$$

and if $\sigma_j\notin Spin(m_j)$, put

$$P(\sigma_j)=\begin{cases} I_{2^{\lfloor s/2\rfloor}}\otimes S_1\otimes\cdots\otimes S_{j-1}\otimes P_j(\sigma_j)\otimes I_{d_{j+1}}\otimes\cdots\otimes I_{d_t} \ \text{ if } 1\le j\le r,\\ M_{j-r}\otimes S_1\otimes\cdots\otimes S_r\otimes I_{d_{r+1}}\otimes\cdots\otimes I_{d_{j-1}}P_j(\sigma_j)\otimes I_{d_{j+1}}\otimes\cdots\otimes I_{d_t}\\ \ \text{if } r+1\le j\le r+s=t.\end{cases} \tag{2.12}$$

The relations (2.3) ensure that P is a spin representation of

$$Pin(m_1,\ldots,m_t)\cong Pin(m_1)\hat{\otimes}\cdots\hat{\otimes}Pin(m_t).$$

The degree of P is $2^{\lfloor s/2\rfloor}d_1\cdots d_t$.

The character of this representation was also calculated in [18] to give the following proposition.

Proposition 2.3. *Let ζ be the character of P and $\zeta_j,\ 1\le j\le t$, be the characters of P_j.*

(i) If $\sigma_j\in Spin(m_j),\ 1\le j\le t$, then

$$\zeta(\sigma_1\cdots\sigma_t)=2^{\lfloor s/2\rfloor}\zeta_1(\sigma_1)\cdots\zeta_t(\sigma_t).$$

(ii) If s is odd and Δ_j is the difference character of the self-associate representations $P_j,\ 1\le j\le r$, and if $\sigma_j\in Spin(m_j),\ 1\le j\le r, \sigma_j\notin Spin(m_j),\ r+1\le j\le t$, then

$$\zeta(\sigma_1\cdots\sigma_t)=\pm(2i)^{[s/2]}\Delta_1(\sigma_1)\cdots\Delta_r(\sigma_r)\zeta_{r+1}(\sigma_{r+1})\cdots\zeta_t(\sigma_t).$$

(iii) In all other cases

$$\zeta(\sigma_1\cdots\sigma_t)=0$$

The above proposition can be applied in particular to the special case where the P_i are the basic spin representations of $Pin(m_i)$. Then, the assumption that the first r of the representations are self-associate is equivalent to assuming that the m_i are even for $1\le i\le r$ and that the m_i are odd for $r+1\le i\le t$. The degree of the representation P will therefore be $2^{\lfloor s/2\rfloor}2^{m_1/2}\cdots 2^{(m_{r+1}-1)/2}\cdots 2^{(m_t-1)/2}=2^{\lfloor s/2\rfloor}2^{(l-s)/2}=2^{\lfloor l/2\rfloor}$. Furthermore, the explicit formulae of Proposition 2.3 could be used to give more explicit values for the characters in terms of the eigenvalues of the elements $\sigma_1,\ldots,\sigma_t$. This will not be done at this point, it is postponed for consideration later when these results are applied to certain reflection groups.

3. The Generalized Symmetric Group $\mathbb{Z}_m^n\rtimes S_n$

3.1. **Presentation.** If $\mathbb{Z}_m$ is the cyclic group of order m and S_n is the symmetric group of order $n!$, the *generalized symmetric group* is the *wreath product* $\mathbb{Z}_m\wr S_n$ or the semi-direct product $\mathbb{Z}_m^n\rtimes S_n$. This group is of order $m^nn!$; in the sequel, it is denoted by B_n^m (when $m=1$, we have the symmetric group S_n or the Weyl group of type A_{n-1} and when $m=2$, we have the hyperoctahedral group or the Weyl group of type B_n).

If S_n is considered as a permutation group acting on the set $\{1, 2, \ldots, n\}$, then S_n is generated by s_i, $1 \le i \le n-1$ with relations

$$s_i^2 = 1, (s_i s_{i+1})^3 = 1,\ 1 \le i \le n-2, (s_i s_j)^2 = 1, |i-j| \ge 2,\ 1 \le i, j \le n-1,$$

where s_i is the transposition $(i, i+1), 1 \le i \le n-1$ The group B_n^m can be considered as the group generated by s_i, $1 \le i \le n-1$, w_j, $1 \le j \le n$ with relations

$$s_i^2 = 1, w_j^m = 1; (s_i s_{i+1})^3 = 1,\ 1 \le i \le n-2,\ s_i w_i = w_{i+1} s_i,\ s_i w_j = w_j s_i, j \ne i, i+1$$

$$(s_i s_j)^2 = 1, |i-j| \ge 2,\ 1 \le i, j \le n-1,\ w_i w_j = w_j w_i, i \ne j,\ 2 \le i \le n-1\ .$$

Comparing this with the presentation of S_n , we see the natural embedding of S_n in B_n^m; also w_i may be regarded as the mapping which takes i onto ζi, with $\{1, 2, \ldots, i-1, i+1, \ldots, n\}$ fixed, where ζ is a primitive m-th root of unity. It can be verified that $w_j = s_{j-1} s_{j-2} \cdots s_1 w_1 s_1 \cdots s_{j-2} s_{j-1}$ for $1 \le j \le n$. That is, B_n^m is the permutation group acting on the set $\{1, 2, \ldots, n\}$, but also with the 'sign' changes w_i which are written as $w_i = \begin{pmatrix} i \\ \zeta i \end{pmatrix}$.

3.2. **Classes of conjugate elements.** The classes of conjugate elements of S_n are parameterized by the partitions $(1^{n_1} 2^{n_2} \ldots n^{n_n})$ of n, where $n_i \ge 0$, $1 \le i \le n$.

The classes of B_n^m are defined similarly in terms of m-partitions (see, for example, [12]). The elements of B_n^m permute the set $\{1, 2, \ldots, n\}$ and multiply each of the elements of this set by a power of ζ. Thus the elements of B_n^m are of the form

$$x = \begin{pmatrix} 1 & 2 & \ldots & n \\ \zeta^{k_1} b_1 & \zeta^{k_2} b_2 & \ldots & \zeta^{k_n} b_n \end{pmatrix},$$

where $\{b_1, b_2, \ldots, b_n\}$ is a permutation of the set $\{1, 2, \ldots, n\}$ and $1 \le k_i \le m$, $1 \le i \le n$. Any element of B_n^m can be uniquely expressed as a product of disjoint cycles $x = \prod_{i=1}^t \theta_i$. where

$$\theta_i = \begin{pmatrix} b_{i_1} & b_{i_2} & \ldots & b_{i_{l_i}} \\ \zeta^{k_{i_1}} b_{i_2} & \zeta^{k_{i_2}} b_{i_3} & \ldots & \zeta^{k_{i_{l_i}}} b_{i_1} \end{pmatrix},$$

where $\sum_{i=1}^t l_i = n$; put $f(\theta_i) = \sum_{j=1}^{l_i} k_{i_j}$.

Then the classes of conjugate elements of B_n^m correspond to the m-partitions of n

$$(1^{a_{11}} 2^{a_{12}} \ldots n^{a_{1n}}; 1^{a_{21}} 2^{a_{22}} \ldots n^{a_{2n}}; \ldots; 1^{a_{m1}} 2^{a_{m2}} \ldots n^{a_{mn}}),$$

where $\sum_{i=1}^n a_{ij} = n_j$ $1 \le j \le m$, where a_{pq} denotes the number of cycles θ_i in the above decomposition of σ of length q such that $f(\theta_i) \equiv p-1 \pmod m$. The order of this class is

$$\frac{m^n n!}{\prod_{p,q} a_{pq}! (qm)^{a_{pq}}} \tag{3.1}$$

We have, by definition, the short exact sequence

$$1 \longrightarrow \mathbb{Z}_m^n \longrightarrow B_n^m \xrightarrow{\upsilon_n} S_n \longrightarrow 1 \quad , \tag{3.2}$$

where υ_n is defined by $\upsilon_n(s_i) = s_i$, $\upsilon_n(w_i) = 1$ for all $1 \le i \le n$, where $\mathbb{Z}_m^n = \mathbb{Z}_m \otimes \ldots \otimes \mathbb{Z}_m$, ($n$ copies), where the i-th copy of $\mathbb{Z}_m$ should be regarded as the cyclic group generated by w_i. In the case where m is even, there is a corresponding short exact sequence

$$1 \longrightarrow \mathbb{Z}_{m/2}^n \longrightarrow B_n^m \xrightarrow{\tau_n} B_n^2 \longrightarrow 1 \quad , \tag{3.3}$$

where τ_n is defined by $\tau_n(s_i) = s_i$, $\tau_n(w_i) = w_i$ for all $1 \le j \le n$, where now the i-th copy of $\mathbb{Z}_{m/2}$ should be regarded as the cyclic group generated by w_i^2.

Under the homomorphism υ_n the class

$$(1^{a_{11}}2^{a_{12}}\dots n^{a_{1n}};1^{a_{21}}2^{a_{22}}\dots n^{a_{2n}};\dots;1^{a_{m1}}2^{a_{m2}}\dots n^{a_{mn}}),$$

of B_n^m fuses to the class $(1^{\sum_{i=1}^m a_{i1}}2^{\sum_{i=1}^m a_{i2}}\dots n^{\sum_{i=1}^m a_{in}})$ of S_n and under the homomorphism τ_n this class fuses to the class

$$(1^{\sum_{\substack{i=1\\ i\ odd}}^m a_{i1}}2^{\sum_{\substack{i=1\\ i\ odd}}^m a_{i2}}\dots n^{\sum_{\substack{i=1\\ i\ odd}}^m a_{in}};1^{\sum_{\substack{i=1\\ i\ even}}^m a_{i1}}2^{\sum_{\substack{i=1\\ i\ even}}^m a_{i2}}\dots n^{\sum_{\substack{i=1\\ i\ even}}^m a_{in}})$$

of B_n^2.

These two isomorphisms will allow us to use known results about the spin representations of the symmetric group S_n and the hyperoctahedral group B_n^2 to determine the spin representations of B_n^m.

The group B_n^m has a total of $2m$ linear characters defined by

$$\left\{\begin{array}{ll} \sigma_k(s_i)=1, 1\le i\le n-1 & \sigma_k(w_j)=\zeta^k, 1\le j\le n\\ \eta(s_i)=-1, 1\le i\le n-1 & \eta(w_j)=1, 1\le j\le n\\ \epsilon_k(s_i)=-1, 1\le i\le n-1 & \epsilon_k(w_j)=\zeta^k, 1\le j\le n, \end{array}\right. \tag{3.4}$$

where $1 \le k \le m-1$, together with the identity character. In the special case $k = m/2$, we write ϵ for $\epsilon_{m/2}$ and σ for $\sigma_{m/2}$. The values of these characters for an element in the class $(\lambda_{(1)};\lambda_{(2)};\dots;\lambda_{(m)})$ are as follows

$$\begin{aligned} \eta(\lambda_{(1)};\lambda_{(2)};\dots;\lambda_{(m)}) &= (-1)^{\sum_{i=1}^m l(\lambda_{(i)})},\\ \sigma(\lambda_{(1)};\lambda_{(2)};\dots;\lambda_{(m)}) &= (-1)^{n-\sum_{i=2,i\ even}^m l(\lambda_{(i)})},\\ \epsilon(\lambda_{(1)};\lambda_{(2)};\dots;\lambda_{(m)}) &= (-1)^{n-\sum_{i=1,i\ odd}^{m-1} l(\lambda_{(i)})}. \end{aligned}$$

Then, we prove the following lemma which describes the kernels of some of the characters. The descriptions are given in terms of the classes of conjugate elements of B_n^m.

Lemma 3.1. *(i) ker* $\eta = \{x \in (\lambda_{(1)};\lambda_{(2)};\dots;\lambda_{(m)}) \mid (\lambda_{(1)};\lambda_{(2)};\dots;\lambda_{(m)})$ *is even*$\}$,

(ii) ker $\sigma = \{x \in (\lambda_{(1)};\lambda_{(2)};\dots;\lambda_{(m)}) \mid \sum_{i=2\ i\ even}^m \sum_{j=1}^n a_{ij}$ *is even* $\}$,

(iii) ker $\epsilon = \eta\sigma = \{x \in (\lambda_{(1)};\lambda_{(2)};\dots;\lambda_{(m)}) \mid (\lambda_{(1)};\lambda_{(2)};\dots;\lambda_{(m)})$ *is even and* $\sum_{i=2\ i\ even}^m \sum_{j=1}^n a_{ij}$ *is even , or* $(\lambda_{(1)};\lambda_{(2)};\dots;\lambda_{(m)})$ *is odd and* $\sum_{i=2\ i\ even}^m \sum_{j=1}^n a_{ij}$ *is odd* $\}$.

Proof. (i) Since $\eta(w_j) = 1$ for $1 \le j \le n$ and $\eta(s_j) = -1$ for $1 \le j \le n$, then for $x \in ker\ \eta$, the total number of the generators s_i in any expression for x must be even, that is, the number of even cycles in this expression must be even, thus $[\lambda_{(1)};\lambda_{(2)};\dots;\lambda_{(m)}]$ is even.

(ii) Since $\sigma(w_j) = -1$ for $1 \le j \le n$ and $\sigma(s_j) = 1$ for $1 \le j \le n$, then for $x \in ker\ \sigma$, the total number of the generators w_i in any expression for x must be even. In an expression $x = \prod_{i=1}^t \theta_i$ of x as a product of cycles, the cycles θ_i for which $f(\theta_i$ is even (odd) give rise to an even (odd) number of w_j. Thus, for $\sigma(x) = 1$, we require an even number of cycles θ_i with $f(\theta_i)$ odd. This can only occur if $\sum_{i=2\ i\ even}^m \sum_{j=1}^n a_{ij}$ is even.

(iii) Since $\epsilon(w_j) = -1$ for $1 \le j \le n$ and $\epsilon(s_j) = -1$ for $1 \le j \le n$, then for $x \in ker\ \sigma$, the total number of the generators w_i and s_i in any expression for x must be even. Then, for similar reasons to those in the proof of (i) and (ii), there are two possible cases. Thus, either both $(\lambda_{(1)};\lambda_{(2)};\ldots;\lambda_{(m)})$ and $\sum_{i=2\ i\ even}^{m}\sum_{j=1}^{n}a_{ij}$ are even or both are odd which results in the required conclusion.

If we now let $M = ker\ \eta \bigcap ker\ \sigma \bigcap ker\ \epsilon$, then $M = \{x \in (\lambda_{(1)};\lambda_{(2)};\ldots;\lambda_{(m)}) \mid (\lambda_{(1)};\lambda_{(2)};\ldots;\lambda_{(m)})$ is even and $\sum_{i=2\ i\ even}^{m}\sum_{j=1}^{n}a_{ij}$ is even $\}$. Then, the following lemma can be proved.

Lemma 3.2. *If m is even, the following is a short exact sequence*

$$1 \longrightarrow M \longrightarrow B_n^m \longrightarrow \mathbb{Z}_2\times\mathbb{Z}_2 \longrightarrow 1\ .$$

Proof. Define $\phi : B_n^m \longrightarrow \mathbb{Z}_2\times\mathbb{Z}_2$ by

$$\phi(x) = (1,-1)^{k_1}(-1,1)^{k_2},$$

where k_1 and k_2 are the number of the s_i and w_i respectively in any expression for x in terms of the generators of B_n^m. Then ϕ is well-defined. Clearly, the map ϕ is surjective and it only remains to determine $ker\phi$.

For $x \in ker\phi$, then it is necessary for both k_1 and k_2 to be even. It now suffices to check against the calculation of all the kernels in Lemma 3.1 to verify that $ker\phi$ is indeed the subgroup M.

4. A Covering Group $\tilde{B}_n^m$ of B_n^m and its Basic Spin Representations

The Schur multiplier of B_n^m was obtained in [8]

$$H^2(B_n^m,\mathbb{C}^*) = \begin{cases} \mathbb{Z}_2=\{\gamma\} & \text{if } m \text{ is odd, } n\geq 4,\\ \mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2=\{(\gamma,\lambda,\mu)\} & \text{if } m \text{ is even, } n\geq 4,\\ \mathbb{Z}_2\times\mathbb{Z}_2=\{(\lambda,\mu)\} & \text{if } m \text{ is even, } n=3,\\ \mathbb{Z}_2=\{\mu\} & \text{if } m \text{ is even, } n=2,\\ \{1\} & \text{otherwise,}\end{cases} \tag{4.1}$$

where $\gamma=\lambda=\mu=\pm1$.

This means that if m is even B_n^m has eight 2-cocycles $\{(\gamma,\lambda,\mu)|\gamma^2=\lambda^2=\mu^2=1\}$ and two 2-cocycles if m is odd, $\{(\gamma)|\gamma^2=1\}$. A corresponding representation group is denoted by $\tilde{B}_n^m$ which has a presentation

$$\begin{aligned}\tilde{B}_n^m \;=\; & < t_i,\ 1\leq i\leq n-1,\ u_j,\ 1\leq j\leq n \ \mid\ t_i^2=1, u_j^m=1\\ & (t_it_{i+1})^3=1,\ 1\leq i\leq n-2,\ t_iu_i=u_{i+1}t_i,\ t_iu_j=\lambda u_jt_i, j\neq i,i+1\\ & (t_it_j)^2=\gamma 1, |i-j|\geq 2,\ 1\leq i,j\leq n-1,\\ & u_iu_j=\mu u_ju_i, i\neq j,\ 2\leq i\leq n-1\rangle,\end{aligned} \tag{4.2}$$

where

$$\gamma^2=\lambda^{(2,m)}=\mu^{(2,m)}=1$$

and γ,λ,μ commute with each other and with the t_i,u_j.

For simplicity, from now on, we will fix a 2-cocycle $[\gamma,\lambda,\mu]\in(\gamma,\lambda,\mu)$, with $\gamma^2=\lambda^{(2,m)}=\mu^{(2,m)}=1$ and with the convention that $\lambda=\mu=1$ if m is odd; $\gamma=1$ if m is even and $n=3$; $\gamma=\lambda=1$ if m is even and $n=2$; and $\gamma=\lambda=\mu=1$ if $n=1$. Thus, the 2-cocycles will be denoted by $[\pm1,\pm1,\pm1]$; we note that only the 2-cocycles $[\pm1,1,1]$ appear in the case m odd (and in particular for the group S_n).

The splitting classes for spin representations of B_n^m for all 2-cocycles were first given by Read [23] (who in [24] was the first to determine all the irreducible spin representations of B_n^m for all 2-cocycles). Later, Stembridge [31] did the same for the hyperoctahedral groups, the special case $m = 2$. He showed that the splitting classes are given as in Table 1. This table is broken into four columns according to the four possible values of η and σ. The entry indicates the splitting classes of B_n corresponding to the 2-cocycle. For example, for the 2-cocycle $[1, -1, -1]$, the splitting classes (λ, μ) of B_n for which $\eta = -1, \sigma = -1$ are of the form $(DOP; DEP)$, that is, λ has distinct odd parts and μ has distinct even parts.

2-cocycle	$\eta = 1, \sigma = 1$	$\eta = -1, \sigma = 1$	$\eta = 1, \sigma = -1$	$\eta = -1, \sigma = -1$
$[1, -1, 1]$	$(P; P)$	$(EP; \emptyset)$	$(DOP; DOP)$	$(\emptyset; EP)$
$[-1, 1, 1]$	$(OP; OP)$	$(DP; DP)$	$(OP; OP)$	$(DP; DP)$
$[-1, -1, 1]$	$(OP; OP)$	$(DEP; \emptyset)$	$(DP; DP)$	$(\emptyset; DEP)$
$[1, 1, -1]$	$(OP; \emptyset)$	$\emptyset$	$(\emptyset; DP)$	$(\emptyset; DP)$
$[1, -1, -1]$	$(OP; \emptyset)$	$(\emptyset; DP)$	$(\emptyset; OP)$	$(DOP; DEP)$
$[-1, 1, -1]$	$(OP; EP)$	$\emptyset$	$(\emptyset; DOP)$	$(\emptyset; P)$
$[-1, -1, -1]$	$(OP; EP)$	$(\emptyset; P)$	$(\emptyset; P)$	$(OP; EP)$

TABLE 1. Splitting classes for B_n^2

We now obtain splitting classes for the group B_n^m for all the 2-cocycles. Indeed, the table in the case m even can be obtained directly from Table 1 using the homomorphism $B_n^m \xrightarrow{\tau_n} B_n^2$ given in (3.3). Alternatively, these results can be proved directly without invoking those obtained by Stembridge. Reinterpreting the results of Read [23] in our notation, shows that our results are consistent with those obtained very much earlier by him. We again note that only the second row of Table 2 is relevant in the case m odd.

2-cocycle	$\eta = 1, \sigma = 1$	$\eta = -1, \sigma = 1$	$\eta = 1, \sigma = -1$	$\eta = -1, \sigma = -1$
$[1, -1, 1]$	$(P; \ldots; P)$	$(EP; \emptyset; \ldots; EP; \emptyset)$	$(DOP; \ldots; DOP)$	$(\emptyset; EP; \ldots; \emptyset; EP)$
$[-1, 1, 1]$	$(OP; \ldots; OP)$	$(DP; \ldots; DP)$	$(OP; \ldots; OP)$	$(DP; \ldots; DP)$
$[-1, -1, 1]$	$(OP; \ldots; OP)$	$(DEP; \emptyset; \ldots; DEP; \emptyset)$	$(DP; \ldots; DP)$	$(\emptyset; DEP; \ldots; \emptyset; DEP)$
$[1, 1, -1]$	$(OP; \emptyset; \ldots; OP; \emptyset)$	$\emptyset$	$(\emptyset; DP; \ldots; \emptyset; DP)$	$(\emptyset; DP; \ldots; \emptyset; DP)$
$[1, -1, -1]$	$(OP; \emptyset; \ldots; OP; \emptyset)$	$(\emptyset; DP; \ldots; \emptyset; DP)$	$(\emptyset; OP; \ldots; \emptyset; OP)$	$(DOP; DEP; \ldots; DOP; DEP)$
$[-1, 1, -1]$	$(OP; EP; \ldots; OP; EP)$	$\emptyset$	$(\emptyset; DOP; \ldots; \emptyset; DOP)$	$(\emptyset; P; \ldots; \emptyset; P)$
$[-1, -1, -1]$	$(OP; EP; \ldots; OP; EP)$	$(\emptyset; P; \ldots; \emptyset; P)$	$(\emptyset; P; \ldots; \emptyset; P)$	$(OP; EP; \ldots; OP; EP)$

TABLE 2. Splitting classes for B_n^m

For example, for the 2-cocycle $[-1, 1, 1]$, the splitting classes of B_n^m (or of $\tilde{B}_n^m$) in the notation of this paper are classes of the m-partition form $(OP, OP, \ldots, OP)$ and $(DP, DP, \ldots, DP)$.

4.1. **Basic spin representations of generalized symmetric groups.** Let $W(\Phi)$ be the irreducible finite reflection group of rank l with root system Φ and simple system $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ and let $\tau_j = \tau_{\alpha_j}$ be the reflection corresponding to $\alpha_j \in \Pi$. Then the group $W(\Phi)$ is generated by the simple reflections τ_j, $1 \le j \le l$ subject to the relations

$$\tau_j^2 = 1, 1 \le j \le l, \ \ (\tau_j \tau_k)^{m_{jk}} = 1, 1 \le j, k \le l, j \ne k,$$

where m_{jk} are positive integers such that $m_{kj} = m_{jk}$.

If the group $W(\Phi)$ of rank l is embedded in the orthogonal group $O(n)$; say $\phi : W(\Phi) \hookrightarrow O(n)$ is an embedding of $W(\Phi)$ into an orthogonal group $O(n)$, for some n, then let

$M_\phi(\Phi) = \rho_n^{-1}(W(\Phi))$. Then we have the following (see [2] and [18] for the details including notation)

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & Pin(n) & \xrightarrow{\rho_n} & O(n) & \longrightarrow & 1 \\ & & & & & & \uparrow \phi & & \\ 1 & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & M_\phi(\Phi) & \xrightarrow{\rho_n} & W(\Phi) & \longrightarrow & 1. \end{array} \tag{4.3}$$

It is clear that the lower sequence in (4.3) is also an exact sequence, but to show that $M_\phi(\Phi)$ is a covering group of $W(\Phi)$, it is necessary to show that $M_\phi(\Phi)$ is a stem extension of $W(\Phi)$, that is, to verify that

$$\mathbb{Z}_2 \subset Z(M_\phi(\Phi)) \cap (M_\phi(\Phi))'.$$

This will ensure that the basic spin representation of $O(n)$ will still be a non-trivial spin representation, that is, not projectively equivalent to an ordinary representation, on restriction to the subgroup $W(\Phi)$. Furthermore, it was shown in [18], that if $n = l$ the basic spin representations $P, P_\pm$ remain irreducible on restriction to the finite irreducible reflection groups $W(\Phi)$, where $rank(\Phi) = l$.

This is now used to construct a number of basic spin representations of B_n^m for certain 2-cocycles. We first consider the natural embedding $\eta : W(\Phi) \hookrightarrow O(l)$, where rank$\Phi = l$. In this case, put

$$M(\Phi) = \phi_l^{-1}(W(\Phi)).$$

Then, a presentation of $M(\Phi)$ is obtained. We have that

$$\rho_l(\alpha_j) = \tau_j,\ \ 1 \le j \le l.$$

and if we let $r_j = \alpha_j / \| \alpha_j \|,\ \ 1 \le j \le l$, then we also have

$$\rho_l(r_j) = \tau_j = \tau_{r_j},\ \ 1 \le j \le l.$$

If, in addition, $z \in Pin(l)$, is such that $\rho_l(z) = I_l$, then $z \in \mathbb{Z}_2$, that is, $z^2 = 1$. Then, we have that the group $M(\Phi)$ is generated by $r_j, 1 \le j \le l,\ \ z$ subject to the relations

$$(r_j r_k)^{m_{jk}} = z^{m_{jk}-1},\ \ 1 \le j,k \le l,\ \ z^2 = 1, z r_j = r_j z,\ \ 1 \le j \le l.$$

We apply these results in particular to the reflection groups of type A_{n-1} (the symmetric group S_n), B_n (the hyperoctahedral group B_n^2) and $I_2(2)$ (the dihedral group of order 4).
Type A_{n-1}. In order to apply the above, we use an embedding of the root system A_{n-1} in $\mathbb{R}^{n-1}$ where the simple system is given by

$$\{\alpha_j = \sqrt{j-1} e_{j-1} - \sqrt{j+1} e_j,\ 1 \le j \le n-1\},$$

(rather than the usual one) then

$$P(s_j) = \frac{1}{\sqrt{2j}} (\sqrt{j-1} M_{j-1} - \sqrt{j+1} M_j),\ 1 \le j \le n-1$$

is the irreducible basic spin representation of S_n if n is odd and $P_\pm$ are the two associate basic spin representation of $W(A_{n-1})$ if n is even. In the above, the generators τ_j have been replaced by the corresponding ones in this setting. In fact, we obtain the presentation

$$\begin{aligned} \tilde{A}_{n-1} \ = \ & \langle t_i,\ 1 \le i \le n-1, z \mid t_i^2 = 1,\ z^2 = 1,\ (t_i t_{i+1})^3 = 1,\ 1 \le i \le n-2, \\ & (t_i t_j)^2 = z,\ t_i z = z t_i,\ |i-j| \ge 2,\ 1 \le i,j \le n-1 \rangle \end{aligned}$$

and thus, this representation, as was shown in [18] is the irreducible basic spin representation of S_n for the 2-cocycle [-1,1,1]. Furthermore, the value of its character was determined as given in the following proposition.

Proposition 4.1. *Let $\psi, (\psi_\pm)$ be the character of the basic spin representation $P, (P_\pm)$.*
(i) If $x \in (\rho), \rho \in OP(n)$, then

$$\psi(x) = \ 2^{\lfloor \frac{1}{2}(l(\rho)-1) \rfloor}.$$

(ii) If $x \in (n)$, then

$$\psi_\pm(x) = \ \pm i^{\frac{1}{2}(n-2)}\sqrt{n/2} \quad \textit{if } n \textit{ is even.}$$

(iii)

$$\psi(x) = 0 \quad \textit{otherwise.}$$

These representations can in turn be lifted to give an irreducible basic spin representation of B_n^m again denoted by P which corresponds to the 2-cocycle $[-1, 1, 1]$ using the homomorphism υ_n defined in (3.2). This results in the following proposition.

Proposition 4.2. *Let $\psi, (\psi_\pm)$ be the character of the basic spin representation $P, (P_\pm)$.*
(i) If $x \in (\lambda_{(1)}; \lambda_{(2)}; \ldots; \lambda_{(m)})$, $\lambda_{(i)} \in (OP(|\lambda_{(i)}|)$, $1 \le i \le m$, then

$$\psi(x) = \ 2^{\lfloor \frac{1}{2}(l(\lambda_{(1)};\lambda_{(2)};\ldots;\lambda_{(m)})-1) \rfloor}.$$

(ii) If $x \in (\emptyset; \ldots; \emptyset; n; \emptyset; \ldots; \emptyset)$, then

$$\psi_\pm(x) = \ \pm i^{\frac{1}{2}(n-2)}\sqrt{n/2} \quad \textit{if } n \textit{ is even,}$$

where n can be in any one of m possible positions.
(iii)

$$\psi(x) = \ 0 \quad \textit{otherwise.}$$

It was I. Schur [30] who first showed that the irreducible representations for this 2-cocycle correspond to partitions $\lambda \in DP(n)$. These were constructed in a remarkable way by M. L. Nazarov [21] which is a generalization of the above construction which corresponds to the partition (n). We briefly recall his results.

Let $\lambda \in DP(n)$, the **shifted diagram** for λ is

$$D_\lambda = \{(i, j) \in \mathbb{Z}^2 \mid 1 \le i \le l(\lambda);\ i \le j \le \lambda_i + i - 1\}.$$

This is represented graphically where a point $(i, j) \in \mathbb{Z}^2$ is represented by the unit square in the plane $\mathbb{R}^2$ with centre (i, j), the coordinates i and j increasing from top to bottom and from left to right respectively. A **shifted tableau** of shape λ is a bijection $\Delta : D_\lambda \to \{1, 2, \ldots, n\}$; a bijection is represented as a filling of the squares of D_λ with the numbers $1, 2, \ldots, n$, each of these numbers being used once only. A shifted tableau Δ is **standard** if the numbers increase down its columns and across its rows. Now, let $\mathcal{S}_\lambda$ denote the set of all standard shifted tableaux of shape λ. Let $\Delta \in \mathcal{S}_\lambda$ and let $k \in \{1, 2, \ldots, n\}$ be fixed. Let k and $k+1$ have the coordinates (i, j) and (i', j') in Δ. Put $a = j-i+1,\ b = j'-i'+1$. Consider Δ and $s_k\Delta$, then $s_k\Delta \in \mathcal{S}_\lambda$ or $s_k\Delta \notin \mathcal{S}_\lambda$. Assume $a < b$, otherwise work with $s_k\Delta$, even if $s_k\Delta \notin \mathcal{S}_\lambda$. Put

$$f(a, b) = \frac{\sqrt{2b(b-1)}}{(a-b)(a+b-1)},$$

$x = (-1)^{b+k} f(a,b)$, $y = (-1)^{a+k} f(b,a)$, $z = \frac{\sqrt{1-x^2-y^2}}{2}$ and $u = \sqrt{(1-x^2)}$. Let

$$A = \begin{pmatrix} x & z \\ z & y \end{pmatrix},\ B = \begin{pmatrix} -y & z \\ z & -x \end{pmatrix} \text{ and } C = \begin{pmatrix} x & u \\ u & -x \end{pmatrix}.$$

Let h be the number of rows in Δ occupied by $1, 2, \ldots, k+1$.Then, if $\Delta, \Delta' \in \mathcal{S}_\lambda$, put

$$X_\pm^{\langle\lambda\rangle}(s_k) = \begin{cases} A \otimes M_{k-h+1} + B \otimes M_{k-h} & \text{if } 1 < a < b \\ C \otimes M_{k-h+1} & \text{if } 1 = a < b \end{cases}$$

and if $\Delta' \notin \mathcal{S}_\lambda$, replace the matrices A, B and C by the element which appears in the $(2,2)$-position. This is repeated for all the tableaux $\Delta \in \mathcal{S}_\lambda$ and we obtain the following proposition.

Proposition 4.3. *The $X^{\langle\lambda\rangle}$, $\lambda \in DP^+(n)$, $X_\pm^{\langle\lambda\rangle}$, $\lambda \in DP^-(n)$ form a complete set of irreducible spin representations of S_n of degree $2^{\lfloor\frac{n-l(\lambda)}{2}\rfloor} g_\lambda$, where g_λ is the number of shifted standard tableaux of shape λ.*

We note that the η-associator of $X^{\langle\lambda\rangle}$ is $id \otimes K^{\otimes\mu}$, where $\mu = \lfloor n/2\rfloor$.

Type B_n. In order to apply the above, we use an embedding of the root system B_n in $\mathbb{R}^n$ where the simple system is given by

$$\{\alpha_j = e_{j-1} - e_j,\ 1 \le j \le n-1,\ \alpha_n = e_n\},$$

then

$$Q(s_j) = \frac{1}{\sqrt{2}}(M_{j-1} - M_j),\ 1 \le j \le n-1,\ Q(w_1) = M_n$$

is the irreducible basic spin representation of $W(B_n)$ if n is even and $Q_\pm$ are the two associate basic spin representation of $W(B_n)$ if n is odd. Here, we have replaced the notation P $(P_\pm)$ by Q $(Q_\pm)$ for obvious reasons.

In this case, we obtain the presentation

$$\begin{aligned} \tilde{B}_n \;=\; & \langle t_i,\ 1 \le i \le n-1,\ u_j,\ 1 \le j \le n \mid t_i^2 = 1, u_j^2 = 1,\ (t_i t_{i+1})^3 = 1, \\ & 1 \le i \le n-2,\ t_i u_i = u_{i+1} t_i,\ t_i u_j = \lambda u_j t_i, j \neq i, i+1,\ (t_i t_j)^2 = \gamma, \\ & |i-j| \ge 2,\ 1 \le i, j \le n-1,\ u_i u_j = \mu u_j u_i, i \neq j,\ 2 \le i \le n-1\rangle, \end{aligned}$$

where $\gamma^2 = \lambda^{(2,m)} = \mu^{(2,m)} = 1$ and γ, λ, μ commute with each other and with the t_i, u_j (note that $u_{n-i} = t_{n-i} u_{n-i+1} t_{n-i}$, $1 \le i \le n-1$) for the covering group of B_n. From this we deduce that this representation is the irreducible basic spin representation for the 2-cocycle $[-1,-1,-1]$. Furthermore, the value of its character can be determined [18] as given in the following proposition.

Proposition 4.4. *Let $\chi, (\chi_\pm)$ be the character of the basic spin representation $Q, (Q_\pm)$.*

(i) If $x \in (\rho; \varrho)$, $(\rho; \varrho) \in (OP(|\rho|); EP(|\varrho|))$, then

$$\chi(x) = \begin{cases} 2^{\frac{1}{2}(l(\rho;\varrho))} & \text{if } n \text{ is even} \\ 2^{\frac{1}{2}(l(\rho;\varrho)-1)} & \text{if } n \text{ is odd} \end{cases}$$

(ii) If $x \in (\emptyset; \varrho)$, $\varrho \in P(n)$, then

$$\chi_\pm(x) = \ \pm i^{\frac{1}{2}(n-1)} 2^{\frac{1}{2}(l(\varrho)-1)} \quad \text{if } n \text{ is odd,}$$

(iii)

$$\chi(x) = \ 0 \quad \text{otherwise.}$$

In the same way as above, these representations and characters are now lifted to B_n^m using the homomorphism τ_n defined in (3.3) to give the following proposition.

Proposition 4.5. *Let $\chi, (\chi_\pm)$ be the character of the basic spin representation $Q, (Q_\pm)$.*
(i) If $x \in (\rho_1; \varrho_1; \ldots; \rho_{m/2}; \varrho_{m/2})$, $\rho_i \in OP$; $\varrho_i \in EP$, $1 \leq i \leq m/2$, then

$$\chi(x) = \begin{cases} 2^{\frac{1}{2}(\sum_{i=1}^{m/2} l(\rho_i;\varrho_i))} & \text{if } n \text{ is even} \\ 2^{\frac{1}{2}(\sum_{i=1}^{m/2} l(\rho_i;\varrho_i)-1)} & \text{if } n \text{ is odd} \end{cases}$$

(ii) If $x \in (\emptyset, \varrho_1, \ldots, \emptyset, \varrho_{m/2})$, $\varrho_i \in P$, $1 \leq i \leq m/2$, then

$$\chi_\pm(x) = \pm i^{\frac{1}{2}(n-1)} 2^{\frac{1}{2}(\sum_{i=1}^{m/2} l(\varrho_i)-1)} \quad \text{if } n \text{ is odd,}$$

(iii)

$$\chi(x) = 0 \quad \text{otherwise.}$$

In [19] it was shown how to determine an irreducible basic spin representation of B_n^m for the 2-cocycle $[1, -1, 1]$. In fact, we use the embedding $B_n^m \hookrightarrow O(2)$ given by

$$(\sigma \oplus \eta)(s_i) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ 1 \leq i \leq l-1, \ (\sigma \oplus \eta)(w_l) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Now, if we use the exact sequence

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & Pin(2) & \xrightarrow{\rho_2} & O(2) & \longrightarrow & 1 \\ & & & & & & \uparrow{\scriptstyle \sigma\oplus\eta} & & \\ & & & & & & B_n^m/M & & \end{array}$$

we ultimately, by putting

$$R(s_i) = M_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \ 1 \leq i \leq n-1, \ R(w_l) = M_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

obtain an irreducible spin representation R of degree 2 of B_n^m corresponding to the 2-cocycle $[1, -1, 1]$. The $\sigma-, \eta-$ and $\epsilon-$associators of this representation are J, I and K respectively.

The character of this representation is given by the following proposition.

Proposition 4.6. *If ξ denotes the character of R, then*

$$\xi(x) = \begin{cases} 2 & \text{if } x \in M \\ 0 & \text{otherwise.} \end{cases}$$

We have now constructed irreducible basic spin representations P, Q and R of B_n^m corresponding to the 2-cocycles $[-1, 1, 1], [-1, -1, -1]$ and $[1, -1, 1]$ (respectively) of degrees $2^{\lfloor \frac{1}{2}(n-1) \rfloor}$, $2^{\frac{1}{2}n}$ or $2^{\frac{1}{2}(n-1)}$ according as n is even or odd and 2 (respectively). These are now shown to be absolutely fundamental in the construction of the irreducible projective representations for the remaining 2-cocycles. But before proceeding to show how this is done, following Stembridge [31], we apply Proposition 2.1 to obtain a general result which proves to be extremely helpful in many of these cases.

The 2-cocycle of R is $\alpha = [1, -1, 1]$, let P be any projective representation of B_n^m with 2-cocycle β, then $R \otimes P$ is a projective representation of B_n^m with 2-cocycle $\alpha\beta$. Then we have the following proposition.

Proposition 4.7. *Let P be an irreducible projective representation of degree d of B_n^m with 2-cocycle β.*

(i) If $L_P = \{1\}$, then $R \otimes P$ is an irreducible representation of degree $2d$ of B_n^m with 2-cocycle $\alpha\beta$.

(ii) If $L_P = \{1, \nu\}$, where $\nu \in L$, then $R \otimes P$ is the direct sum of two inequivalent irreducible projective representation of degree d of B_n^m with 2-cocycle $\alpha\beta$.

(iii) If $L_P = L$, and U, V are the η, σ-associators of P respectively, then

(a) if $UV = -VU$, then $R \otimes P$ is the direct sum of four inequivalent irreducible projective representation of degree $d/2$ of B_n^m with 2-cocycle $\alpha\beta$.

(b) if $UV = VU$, then $R \otimes P$ is the direct sum of two equivalent irreducible projective representation of degree d of B_n^m with 2-cocycle $\alpha\beta$.

5. **Irreducible spin representations of generalized symmetric groups.**

5.1. **The 2-cocycle [1,1,1] — ordinary representations.** We first review the construction of the irreducible ordinary representations of the generalized symmetric groups, these are the ones corresponding to the 2-cocycle $[1, 1, 1]$, see [12], but also for a treatment which is more in line with our requirements, see the work of M. Saeed-ul Islam [27]. As this work is not easily available a review of his presentation is given below. Furthermore, H. Can [6] has given a description of the construction of the corresponding Specht modules and also in [7], he gives a description of these in the context of complex reflection groups. For recent work on the calculation of the characters from a combinatorial point of view, see [1].

Let $X^{[\lambda]}$ denote the irreducible representation of S_n corresponding to the partition λ of n, let $\chi^{[\lambda]}$ denote the corresponding irreducible character. The irreducible representations of B_n^m are indexed by m-partitions $(\lambda_{(1)}; \lambda_{(2)}; \ldots; \lambda_{(m)})$ of n, the corresponding representations and characters will be denoted by $X^{[\lambda_{(1)};\lambda_{(2)};\ldots;\lambda_{(m)}]}$ and $\chi^{[\lambda_{(1)};\lambda_{(2)};\ldots;\lambda_{(m)}]}$ respectively.

If we let $p_0 = 0$ and $p_i = \sum_{j=1}^{i} k_j,\ 1 \leq i \leq m$, then $B_{k_i}^m$ is the generalized symmetric group acting on the set of k_i elements $P_i = \{p_{i-1} + 1, \ldots, p_i\},\ 1 \leq i \leq m$, where $\sum_{i=1}^{m} k_i = n$. Then let $B_{(k_1,\ldots,k_m)}^m = B_{k_1}^m \times \cdots \times B_{k_m}^m$ be the corresponding generalized Young subgroup. Recall that we have defined earlier the linear characters $\sigma_k,\ 1 \leq k \leq m-1$ by

$$\sigma_k(s_i) = 1,\ 1 \leq i \leq n-1;\ \sigma_k(w_j) = \zeta^k, 1 \leq j \leq n.$$

The representation $X^{[\lambda;\emptyset;\ldots;\emptyset]}$ is obtained by lifting $X^{[\lambda]}$ from S_n to B_n^m, we define $X^{[\emptyset;\ldots;\emptyset;\lambda;\emptyset;\ldots;\emptyset]}$, where λ is in the $(k+1)$-th position, $1 \leq k \leq m-1$ to be $\sigma_k \otimes X^{[\lambda;\emptyset;\ldots;\emptyset]}$. If $|\lambda_i| = k_i$ $1 \leq i \leq m$, define

$$X^{[\lambda_{(1)};\lambda_{(2)};\ldots;\lambda_{(m)}]} = \left(X^{[\lambda_{(1)};\emptyset;\ldots;\emptyset]} \otimes X^{[\emptyset;\lambda_{(2)};\ldots;\emptyset]} \otimes \cdots \otimes X^{[\emptyset;\emptyset;\ldots;\lambda_{(m)}]}\right) \uparrow B_n^m,$$

inducing from $B_{(k_1,\ldots,k_m)}^m$ to B_n^m. If we let

$$\chi_{(k_1,\ldots,k_m)} = 1 \otimes \sigma_1 \otimes \cdots \otimes \sigma_{m-1},$$

then we have

$$X^{[\lambda_{(1)};\lambda_{(2)};\ldots;\lambda_{(m)}]} = \left(\chi_{(k_1,\ldots,k_m)} \otimes X^{[\lambda_{(1)};\emptyset;\ldots;\emptyset]} \otimes X^{[\lambda_{(2)};\emptyset;\ldots;\emptyset]} \otimes \cdots \otimes X^{[\lambda_{(m)};\emptyset;\ldots;\emptyset]}\right) \uparrow B_n^m.$$

We can now prove the following lemma and theorem.

Lemma 5.1. *$\chi_{(k_1,\ldots,k_m)}$ is a linear character of $B^m_{(k_1,\ldots,k_m)}$ such that $\chi^g_{(k_1,\ldots,k_m)}(x) \neq \chi_{(\tilde{k}_1,\ldots,\tilde{k}_m)}(x)$ for some $x \in \ker \upsilon_n$ and for all $g \in B^m_n$ unless $(k_1,\ldots,k_m) = (\tilde{k}_1,\ldots,\tilde{k}_m)$ in which case this holds for all $g \in B^m_n \backslash B^m_{(k_1,\ldots,k_m)}$.*

Proof. If $(\tilde{k}_1,\ldots,\tilde{k}_m) \neq (k_1,\ldots,k_m)$, assume without loss of generality that $\tilde{k}_i > k_i$ for some i, that is, $P_i \subset \tilde{P}_i$. If $g \in B^m_n$ is such that $\upsilon_n(g)P_i = P_i$, if $j \in \tilde{P}_i \backslash P_i$, then $\upsilon_n(j) \in P_l,\ l \neq i$ and we put $x = \binom{1}{1}\cdots\binom{j}{\zeta j}\cdots\binom{n}{n}$.

If $\upsilon_n(g)P_i \neq P_i$, then there exists $j \in P_i \subset \tilde{P}_i$ such that $\upsilon_n(j) \in P_l,\ 1 \leq l \leq m,\ l \neq i$ and for this j, we define x as above.

In each case, $\chi_{(\tilde{k}_1,\ldots,\tilde{k}_m)}(x) = \zeta^i$, but

$$\chi^g_{(k_1,\ldots,k_m)}(x) = \chi_{(k_1,\ldots,k_m)}(\upsilon_n(g)x\upsilon_n(g)^{-1}) = \zeta^l,\ l \neq i.$$

If $(\tilde{k}_1,\ldots,\tilde{k}_m) = (k_1,\ldots,k_m)$ and $g \in B^m_n \backslash B^m_{(k_1,\ldots,k_m)}$, then there exists at least one index $i,\ 1 \leq i \leq m$ and an integer $j \in P_j$ such that $\upsilon_n(j) \in P_l,\ 1 \leq l \leq m,\ l \neq i$. Once again we define $x \in \ker \upsilon_n$ as above for this particular j. Clearly,

$$\chi^g_{(k_1,\ldots,k_m)}(x) = \zeta^l \neq \zeta^i = \chi_{(\tilde{k}_1,\ldots,\tilde{k}_m)}(x),$$

which completes the proof of the lemma.

Theorem 5.2. *A complete set of inequivalent irreducible (ordinary) representations of B^m_n is given by the $X^{[\lambda_{(1)};\lambda_{(2)};\ldots;\lambda_{(m)}]}$, where $(\lambda_{(1)};\lambda_{(2)};\ldots;\lambda_{(m)})$ is any m-partition of n.*

Proof. Let $(k_1,\ldots,k_m)$ and $(\tilde{k}_1,\ldots,\tilde{k}_m)$ be two arbitrary m-tuples of n and let $X^{[\lambda_{(1)},\lambda_{(2)},\ldots,\lambda_{(m)}]} = X^{[\lambda_{(1)};\emptyset;\ldots;\emptyset]} \otimes X^{[\emptyset;\lambda_{(2)};\ldots;\emptyset]} \otimes \cdots \otimes X^{[\emptyset;\emptyset;\ldots;\lambda_{(m)}]}$ and $X^{[\tilde{\lambda}_{(1)},\tilde{\lambda}_{(2)},\ldots,\tilde{\lambda}_{(m)}]} = X^{[\tilde{\lambda_{(1)}};\emptyset;\ldots;\emptyset]} \otimes X^{[\emptyset;\tilde{\lambda_{(2)}};\ldots;\emptyset]} \otimes \cdots \otimes X^{[\emptyset;\emptyset;\ldots;\tilde{\lambda_{(m)}}]}$ be two corresponding representations of $B^m_{(k_1,\ldots,k_m)}$ and $B^m_{(\tilde{k}_1,\ldots,\tilde{k}_m)}$ with characters $\chi^{[\lambda_{(1)},\lambda_{(2)},\ldots,\lambda_{(m)}]}$ and $\chi^{[\tilde{\lambda}_{(1)},\tilde{\lambda}_{(2)},\ldots,\tilde{\lambda}_{(m)}]}$ respectively as defined above. We will prove that

$$(\chi^{[\lambda_{(1)};\lambda_{(2)};\ldots;\lambda_{(m)}]}, \chi^{[\tilde{\lambda}_{(1)};\tilde{\lambda}_{(2)};\ldots;\tilde{\lambda}_{(m)}]})_{B^m_n} = 0$$

unless $k_i = \tilde{k}_i,\ 1 \leq i \leq m$, in which case it is equal to 1.

By Frobenius' reciprocity theorem and Mackey's subgroup theorem, the above inner product is equal to

$$\begin{aligned}
& (\chi^{[\lambda_{(1)},\lambda_{(2)},\ldots,\lambda_{(m)}]}, (\chi^{[\tilde{\lambda}_{(1)};\tilde{\lambda}_{(2)};\ldots;\tilde{\lambda}_{(m)}]}) \downarrow B^m_{(k_1,\ldots,k_m)})_{B^m_{(k_1,\ldots,k_m)}} \\
= & \sum_x (\chi^{[\lambda_{(1)},\lambda_{(2)},\ldots,\lambda_{(m)}]}, (\chi^{[\tilde{\lambda}_{(1)},\tilde{\lambda}_{(2)},\ldots,\tilde{\lambda}_{(m)}]} \downarrow H_x) \uparrow B^m_{(k_1,\ldots,k_m)})_{B^m_{(k_1,\ldots,k_m)}} \\
= & \sum_x (\chi^{[\lambda_{(1)},\lambda_{(2)},\ldots,\lambda_{(m)}]} \downarrow H_x, (\chi^{[\tilde{\lambda}_{(1)},\tilde{\lambda}_{(2)},\ldots,\tilde{\lambda}_{(m)}]})^x \downarrow H_x)_{H_x},
\end{aligned}$$

where $H_x = B^m_{(k_1,\ldots,k_m)} \bigcap x^{-1} B^m_{(\tilde{k}_1,\ldots,\tilde{k}_m)} x$, $(\chi^{[\tilde{\lambda}_{(1)},\tilde{\lambda}_{(2)},\ldots,\tilde{\lambda}_{(m)}]})^x(x^{-1}gx) = (\chi^{[\tilde{\lambda}_{(1)},\tilde{\lambda}_{(2)},\ldots,\tilde{\lambda}_{(m)}]})(g)$ for all $g \in B^m_{(\tilde{k}_1,\ldots,\tilde{k}_m)}$ and x ranges over the double coset representatives of the generalized Young subgroups $B^m_{(k_1,\ldots,k_m)}$ and $B^m_{(\tilde{k}_1,\ldots,\tilde{k}_m)}$ in B^m_n.

We now show that each term in the above summation is zero except in the case noted above. If for some x,

$$\sum_x \chi^{[\lambda_{(1)},\lambda_{(2)},\ldots,\lambda_{(m)}]} \downarrow H_x = \sum_x (\chi^{[\tilde{\lambda}_{(1)},\tilde{\lambda}_{(2)},\ldots,\tilde{\lambda}_{(m)}]})^x \downarrow H_x$$

have an irreducible component in common, then so do

$$\sum_x \chi^{[\lambda_{(1)},\lambda_{(2)},\ldots,\lambda_{(m)}]} \downarrow \ker \upsilon_n = \sum_x (\chi^{[\tilde{\lambda}_{(1)},\tilde{\lambda}_{(2)},\ldots,\tilde{\lambda}_{(m)}]})^x \downarrow \ker \upsilon_n,$$

since $\ker \upsilon_n \subset H_x$. Then, using the alternative form for $X^{[\lambda_{(1)},\lambda_{(2)},\ldots,\lambda_{(m)}]}$ given above, we see that these respectively coincide with $\chi_{(k_1,\ldots,k_m)} \downarrow \ker \upsilon_n$ and $\chi^x_{(\tilde{k}_1,\ldots,\tilde{k}_m)} \downarrow \ker \upsilon_n$. Since both of these are irreducible, they are equal on $\ker \upsilon_n$ and so, by Lemma 5.1, we have $(k_1,\ldots,k_m) = (\tilde{k}_1,\ldots,\tilde{k}_m)$ and $x \in B^m_{(k_1,\ldots,k_m)}$. Now, using this information, an elementary inner product calculation shows that

$$(\chi^{[\lambda_{(1)},\lambda_{(2)},\ldots,\lambda_{(m)}]}, \chi^{[\tilde{\lambda}_{(1)},\tilde{\lambda}_{(2)},\ldots,\tilde{\lambda}_{(m)}]})_{B^m_{(k_1,\ldots,k_m)}} = (\chi^{\lambda_1}\cdots\chi^{\lambda_m}, \chi^{\tilde{\lambda}_1}\cdots\chi^{\tilde{\lambda}_m})_{S_{(k_1,\ldots,k_m)}},$$

which is non-zero only if these two characters are equal and we have the desired result.

5.2. **The 2-cocycle [-1,1,1].** The approach in this section follows closely that of the previous section and thus the proof is only outlined, but now the irreducible spin representations of S_n are used in place of the ordinary representations.

As constructed in Proposition 4.3, if $\lambda \in DP(n)^+$, $X^{\langle\lambda\rangle}$ are the irreducible spin representation of S_n and if $\lambda \in DP(n)^-$, $X^{\langle\lambda\rangle}_\pm$ are the two η-associate irreducible spin representations. The corresponding spin characters are denoted by $\chi^{\langle\lambda\rangle}$, $\chi^{\langle\lambda\rangle}_\pm$.

We show that the irreducible representations of B^m_n for the 2-cocycle [-1,1,1] are indexed by m-partitions $(\lambda_{(1)};\lambda_{(2)};\ldots;\lambda_{(m)})$ of n, where $\lambda_{(i)} \in DP(|\lambda_{(i)}|)$, $1 \le i \le m$, the corresponding representations and characters will be denoted by $X^{\langle\lambda_{(1)};\lambda_{(2)};\ldots;\lambda_{(m)}\rangle}$ ($X^{\langle\lambda_{(1)};\lambda_{(2)};\ldots;\lambda_{(m)}\rangle}_\pm$)and $\chi^{\langle\lambda_{(1)};\lambda_{(2)};\ldots;\lambda_{(m)}\rangle}$ ($\chi^{\langle\lambda_{(1)};\lambda_{(2)};\ldots;\lambda_{(m)}\rangle}_\pm$)(if they are η-associate) respectively.

The representation $X^{\langle\lambda;\emptyset;\ldots;\emptyset\rangle}$ is obtained by lifting $X^{\langle\lambda\rangle}$ from S_n to B^m_n, we define $X^{\langle\emptyset;\ldots;\emptyset;\lambda;\emptyset;\ldots;\emptyset\rangle}$, where λ is in the $(k+1)$-th position, $1 \le k \le m-1$ to be $\sigma_k \otimes X^{\langle\lambda;\emptyset;\ldots;\emptyset\rangle}$. If $|\lambda_i| = k_i$ $1 \le i \le m$, define

$$X^{\langle\lambda_{(1)};\lambda_{(2)};\ldots;\lambda_{(m)}\rangle} = (X^{\langle\lambda_{(1)};\emptyset;\ldots;\emptyset\rangle}\hat{\otimes}X^{\langle\emptyset;\lambda_{(2)};\ldots;\emptyset\rangle}\hat{\otimes}\cdots\hat{\otimes}X^{\langle\emptyset;\emptyset;\ldots;\lambda_{(m)}\rangle}) \uparrow B^m_n,$$

inducing from $B^m_{(k_1,\ldots,k_m)}$ to B^m_n, where $\hat{\otimes}$ is the twisted tensor product [18],[9]. Then we have

$$X^{\langle\lambda_{(1)};\lambda_{(2)};\ldots;\lambda_{(m)}\rangle} = (\chi_{(k_1,\ldots,k_m)} \otimes X^{\langle\lambda_{(1)};\emptyset;\ldots;\emptyset\rangle}\hat{\otimes}X^{\langle\lambda_{(2)};\emptyset;\ldots;\emptyset\rangle}\hat{\otimes}\cdots\hat{\otimes}X^{\langle\lambda_{(m)};\emptyset;\ldots;\emptyset\rangle}) \uparrow B^m_n.$$

There are similar statements for the η-associate representations and characters.

The representation $X^{\langle\lambda_{(1)};\lambda_{(2)};\ldots;\lambda_{(m)}\rangle}$ can be written down explicitly using formulae (2.11) and (2.12) by simply replacing the representation P_j by the representation $X^{\langle\emptyset;\ldots;\emptyset;\lambda_j;\emptyset;\ldots;\emptyset\rangle}$. Furthermore, using Proposition 2.3 we get a far more explicit formula for the character $\chi^{\langle\lambda_{(1)};\lambda_{(2)};\ldots;\lambda_{(m)}\rangle}$, in fact this formula is obtained by a slight modification of the one in Proposition 2.3 and will not thus be repeated.

Theorem 5.3. *A complete set of inequivalent irreducible spin representations of B^m_n for the 2-cocycle $[-1,1,1]$ is given by the*

$$X^{\langle\lambda_{(1)};\lambda_{(2)};\ldots;\lambda_{(m)}\rangle} \text{ if } n - l(\lambda_{(1)};\lambda_{(2)};\ldots;\lambda_{(m)}) \text{ is even}$$

and

$$X^{\langle\lambda_{(1)};\lambda_{(2)};\ldots;\lambda_{(m)}\rangle}_\pm \text{ if } n - l(\lambda_{(1)};\lambda_{(2)};\ldots;\lambda_{(m)}) \text{ is odd,}$$

where $(\lambda_{(1)};\lambda_{(2)};\ldots;\lambda_{(m)})$ is an m-partition of n, with $\lambda_{(i)} \in DP(|\lambda_{(i)}|)$, $1 \le i \le m$.

The details of the proof will not be given in that it now follows very closely the structure of the proof of Theorem 5.2 in the previous section. The only major difference will be in the character calculations.

From now on in this paper we assume that m is even as the remaining 2-cocycles only exist in this case.

5.3. **The 2-cocycle [-1,-1,-1].** Let Q, $Q_\pm$ be the basic spin representation of B_n^m for the 2-cocycle [-1,-1,-1] constructed in Section 4.1. Then, for the m-partition of n $(\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset)$, put

$$QX^{[\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset]} = Q\otimes X^{[\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset]}\ (n\ even)$$

and

$$Q_\pm X^{[\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset]} = Q_\pm\otimes X^{[\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset]}\ (n\ odd).$$

Then, we prove the following theorem.

Theorem 5.4. *A complete set of inequivalent irreducible spin representations of B_n^m for the 2-cocycle $[-1,-1,-1]$ is given by the $QX^{[\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset]}$ $(n$ even$)$ and $Q_\pm X^{[\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset]}$ $(n$ odd$)$, where $[\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset]$ is an m-partition of n.*

Proof. We shall give a proof in the case n even only, the odd case is dealt with similarly.

If we consider the representation $X^{[\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset]}$, for any element of cycle type $(P(|\lambda_{(1)}|),0,P(|\lambda_{(3)}|),0,\ldots,P(|\lambda_{(m-1)}|),0)$ in B_n^m, by dividing each partition into its odd and even parts, we obtain classes of B_n^m of type $(OP(|\rho_1|),EP(|\varrho_1|),\ldots,OP(|\rho_{m/2}|),EP(|\varrho_{m/2}|))$, where $|\rho_i|+|\varrho_i| = |\lambda_{(2i-1)}|$, $1\le i(odd)\le m-1$. If ζ is the character of Q, then by Proposition 4.4, we have that $\zeta(\rho_1,\varrho_1,\ldots,\rho_{m/2},\varrho_{m/2})$ are nonzero on the classes of type $(OP(|\rho_1|),EP(|\varrho_1|),\ldots,OP(|\rho_{m/2}|),EP(|\varrho_{m/2}|))$, thus it follows that the characters $(\zeta\chi^{[\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset]})$ are linearly independent.

Conversely, from Table 2 in Section 4 we see that the splitting classes of B_n^m for the 2-cocycle $[-1,-1,-1]$ are of the form $(OP,EP,\ldots,OP,EP)$ and $(P,\emptyset,P,\emptyset,\ldots,P,\emptyset)$; it can be shown that the latter only occurs for n odd. Thus, for the case n even, the above characters span the space of spin characters. It only remains to show that these characters are irreducible.

For, the case n even, using Proposition 4.4, we have that

$$\begin{aligned}
&||\zeta\chi^{[\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset]}||^2\\
&= \sum_{\substack{\rho_i\in OP,\varrho_i\in EP\\ 1\le i\le m/2}} \frac{1}{z_{\rho_1,\varrho_1,\ldots,\rho_{m/2},\varrho_{m/2}}}|\zeta\chi^{[\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset]}(\rho_1,\varrho_1,\ldots,\rho_{m/2},\varrho_{m/2})|^2\\
&= \sum_{\substack{\rho_i\in OP,\varrho_i\in EP\\ 1\le i\le m/2}} \frac{1}{z_{\rho_1,\varrho_1,\ldots,\rho_{m/2},\varrho_{m/2}}}2^{l(\rho,\varrho))}|\chi^{\lambda_{(1)}}(\rho_1\cup\varrho_1)\cdots\chi^{\lambda_{m-1}}(\rho_{m/2}\cup\varrho_{m/2})|^2\\
&= \sum_{\substack{\rho_i\in OP,\varrho_i\in EP\\ 1\le i\le m/2}} \frac{1}{z_{\rho_1,\varrho_1},\ldots,z_{\rho_{m/2},\varrho_{m/2}}}\prod_{i=1}^{m/2}2^{l(\rho_i,\varrho_i))}|\prod_{i(odd)}^{m/2}\chi^{\lambda_{(2i-1)}}(\rho_i\cup\varrho_i)|^2 = 1
\end{aligned}$$

using the corresponding result, Theorem 9.2, in [31] and where $z_{\rho_1,\varrho_1,\ldots,\rho_{m/2},\varrho_{m/2}}$ is the order of the centralizer of that class in B_n^m.

5.4. **The 2-cocycle [1,-1,-1].** In this case, a similar process to that used in the previous section is applied to the representations

$$\begin{aligned} QX^{\langle\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset\rangle} &= Q\otimes X^{\langle\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset\rangle},\\ QX_\pm^{\langle\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset\rangle} &= Q\otimes X_\pm^{\langle\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset\rangle},\\ Q_\pm X^{\langle\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset\rangle} &= Q_\pm\otimes X^{\langle\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset\rangle}\end{aligned}$$

or

$$Q_\pm X_\pm^{\langle\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset\rangle} = Q_\pm\otimes X_\pm^{\langle\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset\rangle},$$

where $(\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset)$ is an m-partition of n, with $\lambda_{(i)}\in DP(|\lambda_{(i)}|)$, $1\le i\le m-1$ as the case may be. These representations have 2-cocycle $[1,-1,-1]$.

We prove the following theorem.

Theorem 5.5. *A complete set of inequivalent irreducible spin representations of B_n^m for the 2-cocycle $[1,-1,-1]$ is given by the*

(i) if n is even

$$QX^{\langle\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset\rangle} \text{ if } n-l(\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset) \text{ is even}$$

and

$$QX_\pm^{\langle\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset\rangle} \text{ if } n-l(\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset) \text{ is odd,}$$

and

(ii) if n is odd

$$Q_\pm X^{\langle\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset\rangle} \text{ if } n-l(\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset) \text{ is even}$$

and

$$Q_\pm X_\pm^{\langle\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset\rangle} \text{ if } n-l(\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset) \text{ is odd,}$$

where $(\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset)$ is an m-partition of n, with $\lambda_{(i)}\in DP(|\lambda_{(i)}|)$, $1\le i\le m-1$.

The proof follows along the same lines as the one given in the previous section, we note that the splitting classes in this case are

$$(OP;\emptyset;\ldots;OP;\emptyset),\ (DOP;DEP;\ldots;DOP;DEP),$$
$$(\emptyset;OP;\ldots;\emptyset;OP) \text{ and } (\emptyset;DP;\ldots;\emptyset;DP),$$

the latter two again only occur in the case n odd. In the even case, we use the well known one-one correspondence between the sets $OP(n)$ and $DP(n)$ (which is used in the case of Schur's theory for irreducible spin representations osf S_n) and the clear one-one correspondence between the sets $DP(n)$ and $DOP(k), DEP(n-k)$ (separate the odd and even parts), for all values of k.

5.5. **The 2-cocycle [1,-1,1].** The following lemma is required.

Lemma 5.6. *If $X^{[\lambda_{(1)};\lambda_{(2)};\dots;\lambda_{(m)}]}$ is an ordinary representation of B_n^m with character $\chi^{[\lambda_{(1)};\lambda_{(2)};\dots;\lambda_{(m)}]}$ corresponding to the m-partition $(\lambda_{(1)};\lambda_{(2)};\dots;\lambda_{(m)})$ of n, then*

$$\begin{array}{rcl} (i)\ \eta\chi^{[\lambda_{(1)};\lambda_{(2)};\dots;\lambda_{(m)}]} & = & \chi^{[\lambda'_{(1)};\lambda'_{(2)};\dots;\lambda'_{(m)}]}, \\ (ii)\ \sigma\chi^{[\lambda_{(1)};\lambda_{(2)};\dots;\lambda_{(m)}]} & = & \chi^{[\lambda_{(\frac{m}{2}+1)};\dots;\lambda_{(m)};\lambda_{(1)};\dots;\lambda_{(\frac{m}{2})}]}, \\ (iii)\ \varepsilon\chi^{[\lambda_{(1)};\lambda_{(2)};\dots;\lambda_{(m)}]} & = & \chi^{[\lambda'_{(\frac{m}{2}+1)};\dots;\lambda'_{(m)};\lambda'_{(1)};\dots;\lambda'_{(\frac{m}{2})}]}, \end{array}$$

where $\eta,\sigma,\varepsilon=\zeta\sigma$ are linear characters of B_n^m.

Proof Using the well-known fact [12] that $\eta\chi^{[\lambda]}=\chi^{[\lambda']}$, where $\chi^{[\lambda']}$ is the character of S_n $\eta-$associate to $\chi^{[\lambda]}$ and noting that

$$\chi^{[\lambda_{(1)};\lambda_{(2)};\dots;\lambda_{(m)}]}=\chi^{[\lambda_{(1)};\emptyset;\dots;\emptyset]}\otimes(\sigma_1\otimes X^{[\lambda_{(2)};\emptyset;\dots;\emptyset]})\otimes\cdots\otimes(\sigma_{m-1}\otimes X^{[\lambda_{(m)};\emptyset;\dots;\emptyset]})$$

then (i) follows. Furthermore, since by definition

$$\sigma\chi^{[\lambda;\emptyset;\dots;\emptyset]}=\sigma_{m/2}\otimes\chi^{[\lambda;\emptyset;\dots;\emptyset]}=\chi^{[\emptyset;\dots;\emptyset;\lambda;\emptyset;\dots;\emptyset]},$$

(ii) also follows. (iii) is now a direct consequence of (i) and (ii).

Recall that the subgroup M of B_n^m is defined by $M=ker\ \eta\bigcap ker\ \sigma\bigcap ker\ \varepsilon$, thus we have the following corollary.

Corollary 5.7.

$$\begin{aligned}\chi_M^{[\lambda_{(1)};\lambda_{(2)};\dots;\lambda_{(m)}]}=\chi_M^{[\lambda'_{(1)};\lambda'_{(2)};\dots;\lambda'_{(m)}]}&=\chi_M^{[\lambda_{(\frac{m}{2}+1)};\dots;\lambda_{(m)};\lambda_{(1)};\dots;\lambda_{(\frac{m}{2})}]}\\ &=\chi_M^{[\lambda'_{(\frac{m}{2}+1)};\dots;\lambda'_{(m)};\lambda'_{(1)};\dots;\lambda'_{(\frac{m}{2})}]}.\end{aligned}$$

If we now define $\xi^{[\lambda_{(1)};\lambda_{(2)};\dots;\lambda_{(m)}]}=\xi\otimes\chi^{[\lambda_{(1)};\lambda_{(2)};\dots;\lambda_{(m)}]}$, where ξ is the character of the irreducible basic spin representation R of B_n^m for the 2-cocycle $[1,-1,1]$, then it follows from Proposition 4.5 that

$$\begin{aligned}\xi^{[\lambda_{(1)};\lambda_{(2)};\dots;\lambda_{(m)}]}=\xi^{[\lambda'_{(1)};\lambda'_{(2)};\dots;\lambda'_{(m)}]}&=\xi^{[\lambda_{(\frac{m}{2}+1)};\dots;\lambda_{(m)};\lambda_{(1)};\dots;\lambda_{(\frac{m}{2})}]}\\ &=\xi^{[\lambda'_{(\frac{m}{2}+1)};\dots;\lambda'_{(m)};\lambda'_{(1)};\dots;\lambda'_{(\frac{m}{2})}]}.\end{aligned}$$

If we now put $RX^{[\lambda_{(1)};\lambda_{(2)};\dots;\lambda_{(m)}]}=R\otimes X^{[\lambda_{(1)};\lambda_{(2)};\dots;\lambda_{(m)}]}$ for each m-partition $(\lambda_{(1)};\lambda_{(2)};\dots;\lambda_{(m)})$ of n, then the $RX^{[\lambda_{(1)};\lambda_{(2)};\dots;\lambda_{(m)}]}$ are spin representations of B_n^m with 2-cocycle $[1,-1,1]$. We use Proposition 4.7 to show that the irreducible spin representations of B_n^m for this 2-cocycle appear as constituents of these.

Theorem 5.8. *The representation $RX^{[\lambda_{(1)};\lambda_{(2)};\dots;\lambda_{(m)}]}$ is*

(i) is a sum of two equivalent irreducible representations if $\lambda_{(i)}=\lambda_{(\frac{m}{2}+i)}\in SCP$, $1\le i\le\frac{m}{2}$ and $n\equiv 0 \pmod 4$,

(ii) is a sum of four inequivalent irreducible spin representations of equal degrees if $\lambda_{(i)}=\lambda_{(\frac{m}{2}+i)}\in SCP$, $1\le i\le\frac{m}{2}$ and $n\equiv 2 \pmod 4$,

(iii) is a sum of two inequivalent representations of equal degrees if $\lambda_{(i)}\in SCP$, $1\le i\le m$ or $\lambda_{(i)}=\lambda_{(\frac{m}{2}+i)}$ or $\lambda_{(i)}=\lambda'_{(\frac{m}{2}+i)}$, $1\le i\le\frac{m}{2}$ but not $\lambda_{(i)}=\lambda_{(\frac{m}{2}+i)}\in SCP$, $1\le i\le\frac{m}{2}$.

In all other cases, the four representations
$RX^{[\lambda_{(1)};\lambda_{(2)};\dots;\lambda_{(m)}]}$, $RX^{[\lambda'_{(1)};\lambda'_{(2)};\dots;\lambda'_{(m)}]}$, $RX^{[\lambda_{(\frac{m}{2}+1)};\dots;\lambda_{(m)};\lambda_{(1)};\dots;\lambda_{(\frac{m}{2})}]}$, $RX^{[\lambda'_{(\frac{m}{2}+1)};\dots;\lambda'_{(m)};\lambda'_{(1)};\dots;\lambda'_{(\frac{m}{2})}]}$
are equivalent irreducible spin representations of B_n^m.

5.6. **The 2-cocycle [-1,-1,1].** The procedure in this case follows that of the previous section, but now we consider the representations
$RX^{\langle\lambda_{(1)};\lambda_{(2)};\ldots;\lambda_{(m)}\rangle} = R\otimes X^{\langle\lambda_{(1)};\lambda_{(2)};\ldots;\lambda_{(m)}\rangle}$ $(RX_{\pm}^{\langle\lambda_{(1)};\lambda_{(2)};\ldots;\lambda_{(m)}\rangle} = R\otimes X_{\pm}^{\langle\lambda_{(1)};\lambda_{(2)};\ldots;\lambda_{(m)}\rangle})$
if $n - l(\lambda_{(1)};\lambda_{(2)};\ldots;\lambda_{(m)})$ is odd (even), where $(\lambda_{(1)};\lambda_{(2)};\ldots;\lambda_{(m)})$ is an m-partition of n, with $\lambda_{(i)}\in DP(|\lambda_{(i)}|)$, $1\le i\le m$. Then the $RX^{\langle\lambda_{(1)};\lambda_{(2)};\ldots;\lambda_{(m)}\rangle}$, $RX_{\pm}^{\langle\lambda_{(1)};\lambda_{(2)};\ldots;\lambda_{(m)}\rangle}$ are spin representations of B_n^m with 2-cocycle $[-1,-1,1]$.

In this case the following lemma is required.

Lemma 5.9. *If $\chi^{\langle\lambda_{(1)};\lambda_{(2)};\ldots;\lambda_{(m)}\rangle}$, $\chi_{\pm}^{\langle\lambda_{(1)};\lambda_{(2)};\ldots;\lambda_{(m)}\rangle}$ are the spin character of B_n^m then*

$$\begin{aligned}
(i)\ \eta\chi_{\pm}^{\langle\lambda_{(1)};\lambda_{(2)};\ldots;\lambda_{(m)}\rangle} &= \chi_{\mp}^{\langle\lambda'_{(1)};\lambda'_{(2)};\ldots;\lambda'_{(m)}\rangle},\\
\eta\chi^{\langle\lambda_{(1)};\lambda_{(2)};\ldots;\lambda_{(m)}\rangle} &= \chi^{\langle\lambda'_{(1)};\lambda'_{(2)};\ldots;\lambda'_{(m)}\rangle},\\
(ii)\ \sigma\chi^{\langle\lambda_{(1)};\lambda_{(2)};\ldots;\lambda_{(m)}\rangle} &= \chi^{\langle\lambda_{(\frac{m}{2}+1)};\ldots;\lambda_{(m)};\lambda_{(1)};\ldots;\lambda_{(\frac{m}{2})}\rangle},\\
\sigma\chi_{\pm}^{\langle\lambda_{(1)};\lambda_{(2)};\ldots;\lambda_{(m)}\rangle} &= \chi_{\pm}^{\langle\lambda_{(\frac{m}{2}+1)};\ldots;\lambda_{(m)};\lambda_{(1)};\ldots;\lambda_{(\frac{m}{2})}\rangle},\\
(iii)\ \varepsilon\chi_{\pm}^{\langle\lambda_{(1)};\lambda_{(2)};\ldots;\lambda_{(m)}\rangle} &= \chi_{\mp}^{\langle\lambda'_{(\frac{m}{2}+1)};\ldots;\lambda'_{(m)};\lambda'_{(1)};\ldots;\lambda'_{(\frac{m}{2})}\rangle},\\
\varepsilon\chi^{\langle\lambda_{(1)};\lambda_{(2)};\ldots;\lambda_{(m)}\rangle} &= \chi^{\langle\lambda'_{(\frac{m}{2}+1)};\ldots;\lambda'_{(m)};\lambda'_{(1)};\ldots;\lambda'_{(\frac{m}{2})}\rangle},
\end{aligned}$$

where $\zeta,\sigma,\varepsilon=\zeta\sigma$ are linear characters of B_n^m.

Proof From [30] we have that $\eta\chi_{\pm}^{\langle\lambda\rangle} = \chi_{\mp}^{\langle\lambda\rangle}$, and noting that

$$\chi^{\langle\lambda_{(1)};\lambda_{(2)};\ldots;\lambda_{(m)}\rangle} = \chi^{\langle\lambda_{(1)};\emptyset;\ldots;\emptyset\rangle}\otimes(\sigma_1\otimes X^{\langle\lambda_{(2)};\emptyset;\ldots;\emptyset\rangle})\otimes\cdots\otimes(\sigma_{m-1}\otimes X^{\langle\lambda_{(m)};\emptyset;\ldots;\emptyset\rangle})$$

then (i) follows. Furthermore, (ii) follows using the same argument as in Lemma 5.6 (ii) and (iii) is now a direct consequence of (i) and (ii).

Corollary 5.10.

$$\begin{aligned}
\chi_{\pm M}^{\langle\lambda_{(1)};\lambda_{(2)};\ldots;\lambda_{(m)}\rangle} &= \chi_{\mp M}^{\langle\lambda'_{(1)};\lambda'_{(2)};\ldots;\lambda'_{(m)}\rangle} = \chi_{\pm M}^{\langle\lambda_{(\frac{m}{2}+1)};\ldots;\lambda_{(m)};\lambda_{(1)};\ldots;\lambda_{(\frac{m}{2})}\rangle}\\
&= \chi_{\mp M}^{\langle\lambda'_{(\frac{m}{2}+1)};\ldots;\lambda'_{(m)};\lambda'_{(1)};\ldots;\lambda'_{(\frac{m}{2})}\rangle}.
\end{aligned}$$

If we now define $\xi^{\langle\lambda_{(1)};\lambda_{(2)};\ldots;\lambda_{(m)}\rangle} = \xi\otimes\chi^{\langle\lambda_{(1)};\lambda_{(2)};\ldots;\lambda_{(m)}\rangle}$, where ξ is the character of the irreducible basic spin representation R of B_n^m for the 2-cocycle $[1,-1,1]$, then it follows from Proposition 4.6 that

$$\begin{aligned}
\xi_{\pm}^{\langle\lambda_{(1)};\lambda_{(2)};\ldots;\lambda_{(m)}\rangle} &= \xi_{\mp}^{\langle\lambda'_{(1)};\lambda'_{(2)};\ldots;\lambda'_{(m)}\rangle} = \xi_{\pm}^{\langle\lambda_{(\frac{m}{2}+1)};\ldots;\lambda_{(m)};\lambda_{(1)};\ldots;\lambda_{(\frac{m}{2})}\rangle}\\
&= \xi_{\mp}^{\langle\lambda'_{(\frac{m}{2}+1)};\ldots;\lambda'_{(m)};\lambda'_{(1)};\ldots;\lambda'_{(\frac{m}{2})}\rangle}.
\end{aligned}$$

Theorem 5.11. *The representation $RX^{\langle\lambda_{(1)};\lambda_{(2)};\ldots;\lambda_{(m)}\rangle}$ is*

(i) is a sum of two irreducible equivalent representations if $\lambda_{(i)} = \lambda_{(\frac{m}{2}+i)}$, $1\le i\le\frac{m}{2}$ and $l(\lambda_{(1)};\lambda_{(2)};\ldots;\lambda_{(m)})$ is even ,

(ii) is a sum of four inequivalent irreducible representations of equal degrees if $\lambda_{(i)} = \lambda_{(\frac{m}{2}+i)}$, $1\le i\le\frac{m}{2}$ and $l(\lambda_{(1)};\lambda_{(2)};\ldots;\lambda_{(m)})$ is odd,

(iii) is a sum of two inequivalent representations of equal degrees if

$$n - l(\lambda_{(1)};\lambda_{(2)};\ldots;\lambda_{(m)})$$

is even, but $\lambda_{(i)}\neq\lambda_{(\frac{m}{2}+i)}$, for some $1\le i\le\frac{m}{2}$.

If $n-l(\lambda_{(1)};\lambda_{(2)};\ldots;\lambda_{(m)})$ is odd, the four representations

$$RX_{\pm}^{\langle\lambda_{(1)};\lambda_{(2)};\ldots;\lambda_{(m)}\rangle}, RX_{\mp}^{\langle\lambda_{(1)};\lambda_{(2)};\ldots;\lambda_{(m)}\rangle},$$
$$RX_{\pm}^{\langle\lambda_{(\frac{m}{2}+1)};\ldots;\lambda_{(m)};\lambda_{(1)};\ldots;\lambda_{(\frac{m}{2})}\rangle}, RX_{\mp}^{\langle\lambda_{(\frac{m}{2}+1)};\ldots;\lambda_{(m)};\lambda_{(1)};\ldots;\lambda_{(\frac{m}{2})}\rangle}$$

are equivalent irreducible spin representations of B_n^m.

5.7. **The 2-cocycle [-1,1,-1].** In this case, for the m-partition of n $(\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset)$, if n is even, put

$$RQX^{[\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset]} = R\otimes QX^{[\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset]}$$

and if n is odd, put

$$RQ_{\pm}X^{[\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset]} = R\otimes Q_{\pm}X^{[\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset]}.$$

Then, these representations have factor set $[-1,1,-1]$ and we prove the following theorem.

Theorem 5.12. *The representation $RQX^{[\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset]}$ is*
(i) is a sum of two equivalent irreducible representations if $\lambda_{(i)}\in SCP$, $1\le i(odd)\le m$ and n is even
(ii) is a sum of two inequivalent representations of equal degrees if $\lambda_{(i)}\in SCP$, $1\le i(odd)\le m$ and n is odd or $\lambda_{(i)}\notin SCP$, for some $1\le i(odd)\le m$ and n is even.
If $\lambda_{(i)}\notin SCP, 1\le i(odd)\le m$ and n is odd, the representations $RQ_{\pm}X^{[\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset]}$, $RQ_{\pm}X^{[\lambda'_{(1)};\emptyset;\lambda'_{(3)};\emptyset;\ldots;\lambda'_{(m-1)};\emptyset]}$, $RQ_{\pm}X^{[\lambda_{(\frac{m}{2})};\emptyset;\ldots;\lambda_{(m-1)};\emptyset;\ldots;\lambda_{(1)};\emptyset;\ldots;\lambda_{(\frac{m}{2}-1)};\emptyset]}$, $RQ_{\pm}X^{[\lambda'_{(\frac{m}{2})};\emptyset;\ldots;\lambda'_{(m-1)};\emptyset;\ldots;\lambda'_{(1)};\emptyset;\ldots;\lambda'_{(\frac{m}{2}-1)};\emptyset]}$ are equivalent irreducible spin representations of B_n^m.

For the proof of this theorem, the lemma below will be needed in a similar way to the preceding sections; we denote the character of $QX^{[\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset]}$ by $\zeta\chi^{[\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset]}$.

Lemma 5.13. *If n is even, then*

(i) $\eta(\zeta\chi^{[\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset]})=\zeta\chi^{[\lambda'_{(1)};\emptyset;\lambda'_{(3)};\emptyset;\ldots;\lambda'_{(m-1)};\emptyset]}$,
(ii) $\sigma(\zeta\chi^{[\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset]})=\zeta\chi^{[\lambda'_{(1)};\emptyset;\lambda'_{(3)};\emptyset;\ldots;\lambda'_{(m-1)};\emptyset]}$,
(iii) $\varepsilon(\zeta\chi^{[\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset]})=\zeta\chi^{[\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset]}$.

If n is odd, then

(i) $\eta(\zeta_{\pm}\chi^{[\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset]})=\zeta_{\mp}\chi^{[\lambda'_{(1)};\emptyset;\lambda'_{(3)};\emptyset;\ldots;\lambda'_{(m-1)};\emptyset]}$,

$$(ii)\ \sigma(\zeta_{\pm}\chi^{[\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset]})=\begin{cases}\zeta_{\pm}\chi^{[\lambda_{(\frac{m}{2}+1)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset;\lambda_1;\emptyset;\ldots;\lambda_{(\frac{m}{2}-1)};\emptyset]} & \textit{if } m\equiv 0 \pmod 4\\ \zeta_{\pm}\chi^{[\emptyset;\lambda_{(\frac{m}{2}+2)};\ldots;\lambda_{(m-1)};\emptyset;\lambda_1;\emptyset;\ldots;\emptyset;\lambda_{(\frac{m}{2})}]} & \textit{if } m\equiv 2 \pmod 4\end{cases}$$

$$(iii)\ \varepsilon(\zeta_{\pm}\chi^{[\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset]})=\begin{cases}\zeta_{\pm}\chi^{[\lambda'_{(\frac{m}{2}+1)};\emptyset;\ldots;\lambda'_{(m-1)};\emptyset;\lambda'_1;\emptyset;\ldots;\lambda'_{(\frac{m}{2}-1)};\emptyset]} & \textit{if } m\equiv 0 \pmod 4\\ \zeta_{\pm}\chi^{[\emptyset;\lambda'_{(\frac{m}{2}+2)};\ldots;\lambda'_{(m-1)};\emptyset;\lambda'_1;\emptyset;\ldots;\emptyset;\lambda'_{(\frac{m}{2})}]} & \textit{if } m\equiv 2 \pmod 4\end{cases}$$

where $\zeta,\sigma,\varepsilon=\zeta\sigma$ are linear characters of B_n^m.

From this lemma, we obtain, as before in Section 5.5, the following corollary.

Corollary 5.14. *If n is even, then*

$$\zeta\chi_M^{[\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset]} = \zeta\chi_M^{[\lambda'_{(1)};\emptyset;\lambda'_{(3)};\emptyset;\ldots;\lambda'_{(m-1)};\emptyset]}.$$

If n is odd, then

$$\zeta\chi_M^{[\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset]} = \zeta\chi_M^{[\lambda'_{(1)};\emptyset;\lambda'_{(3)};\emptyset;\ldots;\lambda'_{(m-1)};\emptyset]}$$

$$= \begin{cases} \zeta_\pm \chi^{[\lambda_{(\frac{m}{2}+1)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset;\lambda_1;\emptyset;\ldots;\lambda_{(\frac{m}{2}-1)};\emptyset]} & \text{if } m \equiv 0 \pmod 4 \\ \zeta_\pm \chi^{[\emptyset;\lambda_{(\frac{m}{2}+2)};\ldots;\lambda_{(m-1)};\emptyset;\lambda_1;\emptyset;\ldots;\emptyset;\lambda_{(\frac{m}{2})}]} & \text{if } m \equiv 2 \pmod 4 \end{cases}$$

$$= \begin{cases} \zeta_\pm \chi^{[\lambda'_{(\frac{m}{2}+1)};\emptyset;\ldots;\lambda'_{(m-1)};\emptyset;\lambda'_1;\emptyset;\ldots;\lambda'_{(\frac{m}{2}-1)};\emptyset]} & \text{if } m \equiv 0 \pmod 4 \\ \zeta_\pm \chi^{[\emptyset;\lambda'_{(\frac{m}{2}+2)};\ldots;\lambda'_{(m-1)};\emptyset;\lambda'_1;\emptyset;\ldots;\emptyset;\lambda'_{(\frac{m}{2})}]} & \text{if } m \equiv 2 \pmod 4 \end{cases}.$$

In turn, this leads to the corresponding results for the characters $\xi\zeta\chi_M^{[\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset]}$ and in turn to the proof of Theorem 5.12.

5.8. **The 2-cocycle [1,1,-1].** In this case, let $(\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset)$ be the m-partition of n with $\lambda_{(i)} \in DP(|\lambda_{(i)}|)$, $1 \le i \le m$.
If n is even and $n - l(\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset)$ is even, put

$$RQX^{\langle\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset\rangle} = R \otimes QX^{\langle\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset\rangle}$$

and if $n - l(\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset)$ is odd, put

$$RQX_\pm^{\langle\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset\rangle} = R \otimes QX_\pm^{\langle\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset\rangle};$$

and if n is odd and $n - l(\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset)$ is even, put

$$RQ_\pm X^{\langle\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset\rangle} = R \otimes Q_\pm X^{\langle\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset\rangle}$$

and if $n - l(\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset)$ is odd, put

$$RQ_\pm X_\pm^{\langle\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset\rangle} = R \otimes Q_\pm X_\pm^{\langle\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset\rangle}.$$

These representations have 2-cocycle $[1,1,-1]$.

Theorem 5.15. *The representation* $RQX^{\langle\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset\rangle}$ *is*

(i) is a sum of two equivalent irreducible representations if $n-l(\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset)$ *is even and* n *is even ,*

(ii) is a sum of two inequivalent representations of equal degrees if $n-l(\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset)$ *is even and* n *is odd or if* $n-l(\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset)$ *is odd and* n *is even.*

If $n - l(\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset)$ *is odd and* n *is odd, the representations* $RQ_\pm X_\pm^{\langle\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset\rangle}$ *and* $RQ_\pm X_\pm^{\langle\lambda_{(\frac{m}{2})};\emptyset;\ldots;\lambda_{(m-1)};\emptyset;\ldots;\lambda_{(1)};\emptyset;\ldots;\lambda_{(\frac{m}{2}-1)};\emptyset\rangle}$ *are equivalent irreducible spin representations of* B_n^m.

In this case, we merely state the corresponding lemma to Lemma 5.13.

Lemma 5.16. *If* n *is even, then*

(i) $\eta(\zeta\chi_\pm^{\langle\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset\rangle}) = \zeta\chi_\mp^{\langle\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset]}$,

(ii) $\sigma(\zeta\chi_\pm^{\langle\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset\rangle}) = \zeta\chi_\mp^{\langle\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset\rangle}$,

(iii) $\varepsilon(\zeta\chi_\pm^{\langle\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset\rangle}) = \zeta\chi_\mp^{\langle\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset\rangle}$.

If n *is odd, then*

(i) $\eta(\zeta_\pm\chi^{\langle\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset\rangle}) = \zeta_\mp\chi^{\langle\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset\rangle}$,

$$(ii)\ \sigma(\zeta_\pm\chi^{\langle\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset\rangle}) = \begin{cases} \zeta_\mp\chi^{\langle\lambda_{(\frac{m}{2}+1)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset;\lambda_1;\emptyset;\ldots;\lambda_{(\frac{m}{2}-1)};\emptyset\rangle} & \text{if } m \equiv 0 \pmod 4 \\ \zeta_\mp\chi^{\langle\emptyset;\lambda_{(\frac{m}{2}+2)};\ldots;\lambda_{(m-1)};\emptyset;\lambda_1;\emptyset;\ldots;\emptyset;\lambda_{(\frac{m}{2})}\rangle} & \text{if } m \equiv 2 \pmod 4 \end{cases}$$

$$(iii)\ \varepsilon(\zeta_\pm\chi^{\langle\lambda_{(1)};\emptyset;\lambda_{(3)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset\rangle}) = \begin{cases} \zeta_\mp\chi^{\langle\lambda_{(\frac{m}{2}+1)};\emptyset;\ldots;\lambda_{(m-1)};\emptyset;\lambda_1;\emptyset;\ldots;\lambda_{(\frac{m}{2}-1)};\emptyset\rangle} & \text{if } m \equiv 0 \pmod 4 \\ \zeta_\mp\chi^{\langle\emptyset;\lambda_{(\frac{m}{2}+2)};\ldots;\lambda'_{(m-1)};\emptyset;\lambda_1;\emptyset;\ldots;\emptyset;\lambda_{(\frac{m}{2})}\rangle} & \text{if } m \equiv 2 \pmod 4 \end{cases}$$

where $\zeta, \sigma, \varepsilon = \zeta\sigma$ *are linear characters of* B_n^m.

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