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LITTELMANN'S REFINED DEMAZURE CHARACTER FORMULA REVISITED

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ABSTRACT. We give a purely combinatorial derivation of Littelmann's refined Demazure character formula.

1. INTRODUCTION

The Demazure character formula is a generalization of Weyl's character formula. It was first stated by Demazure in [D], who showed that it would follow from a certain string property. However, it turned out that this property did not hold in the original setting. The first correct proofs of the formula were therefore given by Andersen, [A] and Ramanan-Ramanathan [RR], using methods closely related to Frobenius splitting.

This work is concerned with the crystal basis approach to the Demazure character formula. In that setting the string property indeed *does* hold as demonstrated by Kashiwara in [K1]. We briefly review the deduction of the character formula from it.

We then go on to show that the string property can be obtained using only combinatorial properties of the crystals: the Kashiwara operators $\tilde{e}_i$, $\tilde{f}_i$, together with the $*$-operation. This is different from the previous deductions of the formula, which use either a representation theoretical interpretation of the formula or appeal to Littelmann's path models. Our deduction should be contrasted with the remarks following Proposition **6.3.10** in Joseph's book, [J1].

I would like to thank the referee for many useful suggestions.

2. THE REFINED DEMAZURE CHARACTER FORMULA

2.1. Let us briefly recall the notion of crystal as introduced by Kashiwara. We refer to [K1,K2,J] for all unexplained notation. Let $C := (c_{i,j})_{i,j\in I}$ be a generalized Cartan matrix. Crystals are certain combinatorial objects associated to C. They consist of a set B with maps $\tilde{e}_i, \tilde{f}_i : B \to B \cup \{0\}$ and maps $\epsilon_i, \varphi_i : B \to \mathbb{Z} \cup \{-\infty\}$, $wt_i : B \to P$, $\forall i \in I$, that satisfy certain conditions.

There is a crystal $B(\lambda)$ associated to the Weyl module $V(\lambda)$ of the quantized universal algebra $U_q(\mathfrak{g})$. The limit crystal is called $B(\infty)$.

Given two crystals B_1 and B_2 one can make $B_1 \times B_2$ into a crystal, which is called the tensor product $B_1 \otimes B_2$. For example we have that

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \epsilon_i(b_2), \\ b_1 \otimes \tilde{f}_i b_2 & \text{otherwise.} \end{cases}$$

There is also a sum construction. (But notice that not all crystals arise from the representation theoretical crystals using such constructions).

We shall mainly view crystals as combinatorial objects in the above sense, but shall also appeal to Kashiwara's $*$-operation on $B(\infty)$ (see [K1]). We first of all need the following property: for all $i \in I$ there is an injective morphism of crystals $\Psi_i : B(\infty) \to B(\infty) \otimes B_i$ where B_i is the crystal defined in example 1.2.6. of [K1]. It satisfies the following conditions

$$\Psi_i : u_\infty \mapsto u_\infty \otimes b_i, \tag{2.1.1}$$

$$\Psi_i(\tilde{f}_i^* b) = b' \otimes \tilde{f}_i b'' \text{ where } \Psi_i(b) = b' \otimes b'', \tag{2.1.2}$$

$$\tilde{f}_i \Psi_i(b) = \Psi_i(\tilde{f}_i b) \text{ and } \tilde{e}_i \Psi_i(b) = \Psi_i(\tilde{e}_i b), \tag{2.1.3}$$

where u_∞ is the unique element of B_∞ of weight 0, and where $B(\infty) \otimes B_i$ has the above structure of a tensor product. Joseph has given a purely combinatorial proof of the existence of Ψ_i, [J2].

Now, for a reduced expression $s_{i_n} s_{i_{n-1}} \dots s_{i_1}$ of the Weyl group element w, we define $B_w(\infty) \subset B(\infty)$ and $B_w(\lambda) \subset B(\lambda)$ in the following recursive way

$$B_w(\infty) := \bigcup_k \tilde{f}_{i_n}^k B_{s_{i_n} w}(\infty), \qquad B_1(\infty) := \{u_\infty\},$$

$$B_w(\lambda) := \bigcup_k \tilde{f}_{i_n}^k B_{s_{i_n} w}(\lambda), \qquad B_1(\lambda) := \{u_\lambda\}.$$

A priori, these definitions might depend on the choice of reduced expression $s_{i_n} s_{i_{n-1}} \dots s_{i_1}$ of w. We shall later show that in fact $B_w(\infty)$ and $B_w(\lambda)$ are independent of this choice.

2.2. Let $\mathcal{D}_i$ be the additive operator on $\mathbb{Z}[B(\lambda)]$ given by

$$\mathcal{D}_i b = \begin{cases} \sum_{0 \le k \le wt_i(b)} \tilde{f}_i^k b & \text{if } \; wt_i(b) \ge 0, \\ -\sum_{1 \le k \le -wt_i(b)-1} \tilde{e}_i^k b & \text{if } \; wt_i(b) < 0. \end{cases}$$

Then the refined Demazure character formula, [K1,L1], is the following equality in $\mathbb{Z}[B(\lambda)]$

$$\sum_{b \in B_w(\lambda)} b = \mathcal{D}_{i_n}\mathcal{D}_{i_{n-1}} \dots \mathcal{D}_{i_1} u_\lambda. \tag{2.2.1}$$

The $\mathcal{D}_i$'s induce the usual Demazure operators on the group ring of the weight lattice $\mathbb{Z}[P]$ under the weight map $w : \mathbb{Z}[B(\lambda)] \to \mathbb{Z}[P]$. Thus, if W is finite and we take $w = w_0$ the longest element of the Weyl group, (2.2.1) generalizes the original Demazure expression of the Weyl character, see e.g. [A].

2.3. In the rest of this section we shall review Kashiwara's proof of (2.2.1). The idea is to reduce to the verification of the following three properties of $B_w(\lambda)$:

(1) $B_w^*(\infty) = B_{w^{-1}}(\infty)$,
(2) $\tilde{e}_i B_w(\infty) \subset B_w(\infty) \cup \{0\} \quad \forall i \in I$,
(3) $\tilde{f}_j b \in B_w(\infty) \Rightarrow \tilde{f}_j^k b \in B_w(\infty) \quad \forall b \in B_w(\infty), \forall k \in \mathbb{N}, \forall j \in I$.

Let us denote any subset of $B(\infty)$ or of $B(\lambda)$ an i-string if it is of the form

$$S = \{\tilde{f}_i^k b \mid k \geq 0, \text{ where } b \in B(\lambda) \text{ satisfies } \tilde{e}_i b = 0\}. \tag{2.3.1}$$

We call b the highest weight vector of S. The key "Demazure string property" of these i-strings is then the following: for any i-string $S \subset B(\infty)$ we have that

$$B_w(\infty) \cap S \text{ is either } S \text{ or } \{b\} \text{ or the empty set }. \tag{2.3.2}$$

This is seen by combining (2) and (3).

2.4. The string property is also valid for $B(\lambda)$: to see this one defines for $\lambda \in P$ the crystal on one element $T_\lambda := \{t_\lambda\}$ as follows:

$$wt_i(t_\lambda) = \langle \lambda, \alpha_i \rangle \quad \varepsilon_i(t_\lambda) = -\infty, \quad \varphi_i(t_\lambda) = -\infty,$$
$$\tilde{e}_i(t_\lambda) = 0 = \tilde{f}_i(t_\lambda).$$

Let $\lambda \in P^+$. Then $u_\lambda \mapsto u_\infty \otimes t_\lambda$ defines an embedding of crystals $\iota_\lambda : B(\lambda) \mapsto B(\infty) \otimes T_\lambda$ that commutes with the $\tilde{e}_i$'s.

Now, $B_w(\lambda)$ is the inverse image of $B_w(\infty) \otimes T_\lambda$ under ι_λ. Furthermore, the inverse image under ι_λ of an i-string for $B(\infty)$ is an i-string for $B(\lambda)$. Thus (2.3.2) implies the string property for $B(\lambda)$.

2.5. For completeness we now include Kashiwara's proof of the following lemma.

Lemma 2.1. *The refined Demazure formula (2.2.1) follows from the string property for $B(\lambda)$.*

Proof. If $\tilde{e}_i b = 0$ for $b \in B(\lambda)$ then clearly $\mathcal{D}_i b$ is an i-string having b as its highest weight vector. Moreover, an easy calculation shows that $\mathcal{D}_i S = S$ for S any i-string. Now Theorem 2 of [K2] says that

$$B(\lambda) = \bigcup_{k_i \geq 0, j_i \in I, m \geq 0} \tilde{f}_{j_m}^{k_m} \tilde{f}_{j_{m-1}}^{k_{m-1}} \dots \tilde{f}_{j_1}^{k_1} u_\lambda. \tag{2.5.1}$$

Hence, $B(\lambda)$ is the disjoint union of i-strings for any $i \in I$, since i-strings are either disjoint or coincide.

We now prove (2.2.1) by induction on $l(w)$. We thus assume the formula for $s_{i_n} w = s_{i_{n-1}} s_{i_{n-2}} \dots s_{i_1}$ and need to check the equality

$$\sum_{b \in B_w(\lambda)} b = \mathcal{D}_{i_n}\Big(\sum_{b \in B_{s_{i_n} w}(\lambda)} b \Big). \tag{2.5.2}$$

As $\mathcal{D}_i$ leaves any i-string invariant it is enough to verify the following equality

$$\sum_{b \in B_w(\lambda) \cap S} b = \mathcal{D}_{i_n}\Big(\sum_{b \in B_{s_{i_n} w}(\lambda) \cap S} b \Big). \tag{2.5.3}$$

Now (2.3.2) severely restricts the shape of these intersections, and even further restrictions are imposed by the condition

$$B_w(\lambda) \cap S = \bigcup_k \tilde{f}_{i_n}^k (B_{s_{i_n} w}(\lambda) \cap S),$$

which is a consequence of the definitions. All together, we are left with only three possibilities, namely

(1) $B_w(\lambda) \cap S = B_{s_{i_n} w}(\lambda) \cap S = \emptyset$,
(2) $B_w(\lambda) \cap S = B_{s_{i_n} w}(\lambda) \cap S = S$,
(3) $B_w(\lambda) \cap S = S$ and $B_{s_{i_n} w}(\lambda) \cap S = \{b\}$ where $\tilde{e}_i b = 0$.

In all three cases it is straightforward to check that (2.5.3) holds true. □

We have thus reduced ourselves to the verification of (1), (2) and (3) of 2.3. Kashiwara proves (1) and (2) by realizing the $B_w(\lambda)$'s as crystals of the Demazure modules whereas the proof of the string property (3) relies on the combinatorial properties of the operators $\tilde{e}_i^*$ and $\tilde{f}_i^*$ together with (1) and (2).

Here we shall demonstrate that (1) and (2) can be obtained in the same combinatorial spirit that is employed for (3), that is without relying on an interpretation of $B_w(\lambda)$'s as crystals for any modules. Now it is known that Littelmanns's Path model is equivalent to the crystal combinatorics, see eg. [J1] and references therein, and that (2) and (3) (which suffice to obtain the string property (2.3.2)) can be obtained in that setting, [L2]. Still, Joseph remarks on page 181 in [J1] that it seems extremely difficult to establish (2) purely combinatorially.

3. Properties of $B_w(\infty)$

3.1. Recall the injective morphism $\Psi_i : B(\infty) \to B(\infty) \otimes B_i$ from the previous section. Using its properties (2.1.1), (2.1.2) and (2.1.3) one can obtain information about the commutation of $\tilde{f}_i$ and $\tilde{e}_i$; this is illustrated by the following lemma.

Lemma 3.1. *For any $i, j \in I$ and $b \in B(\infty)$ we have*

$$\bigcup_{k,n} \tilde{f}_i^n \tilde{f}_j^{*k} b = \bigcup_{k,n} \tilde{f}_j^{*k} \tilde{f}_i^n b.$$

Proof. If $i \neq j$ then by Corollary 2.2.2 of [K1] $\tilde{f}_i$ and $\tilde{f}_j^*$ commute and there is nothing to prove. So we assume $i = j$. Write

$$\Psi_i(b) = b_0 \otimes \tilde{f}_i^m b_i,$$

and let $\varphi := \varphi_i(b_0)$ and $\varepsilon := m$. Now, Ψ_i is an embedding so to show the equality of the lemma it is enough to see that both sides have the same image under Ψ_i. So we replace b by $b_0 \otimes \tilde{f}_i^m b_i$ and keep in mind that the action of $\tilde{f}_j^{*k}$ is on the right factor while $\tilde{f}_i$ acts as on a tensor product.

Let now $\Psi_i(b) = b_0 \otimes \tilde{f}_i^m b_i$ be represented as a point in the crystal graph associated to $B(\infty) \otimes B_i$. The crystal graph is a representation of the action of $\tilde{f}_i$ on $B(\infty) \otimes B_i$, so there is an arrow between two points in the graph if $\tilde{f}_i$ carries the corresponding crystal elements to each other.

If $\varphi \leq m$ the action of $\tilde{f}_i$ is on the second factor and there is a horizontal arrow leaving $b_0 \otimes \tilde{f}_i^m b_i$ and if $\varphi > m$ there is a vertical arrow leaving $b_0 \otimes \tilde{f}_i^m b_i$

One typically gets a picture as the following one.

$$
\begin{array}{c|ccccccccccccccc}
 & B_i & & & & & & & & & & & & & & \\
\hline
B(\infty) & & & & & & & & & & & & & & & \\
 & * & & * & & * & & * & \rightarrow & * & \rightarrow & * & \rightarrow & * & \rightarrow & * \\
 & \downarrow & & \downarrow & & \downarrow & & & & & & & & & & \\
 & * & & * & & * & \rightarrow & * & \rightarrow & * & \rightarrow & * & \rightarrow & * & \rightarrow & * \\
 & \downarrow & & \downarrow & & & & & & & & & & & & \\
 & * & & * & \rightarrow & * & \rightarrow & * & \rightarrow & * & \rightarrow & * & \rightarrow & * & \rightarrow & * \\
 & \downarrow & & & & & & & & & & & & & & \\
 & * & \rightarrow & * & \rightarrow & * & \rightarrow & * & \rightarrow & * & \rightarrow & * & \rightarrow & * & \rightarrow & *
\end{array}
$$

The subset of $B(\lambda)$,

$$\bigcup_k \tilde{f}_i^k (b_0 \otimes \tilde{f}_i^m b_i),$$

is represented by the points of the graph that can be hit by a sequence of arrows starting in $b_0 \otimes \tilde{f}_i^m b_i$.

On the other hand the action of $\tilde{f}_i^*$ is always on the second factor of the tensor product, so $\tilde{f}_i^*$ always takes a point in the graph to its right neighbour. Using this information one can now calculate the two sides of the lemma; in both cases one gets the infinite rectangle whose upper left corner is $\Psi_i(b) = b_0 \otimes \tilde{f}_i^m b_i$ and whose lower left corner is the point below $b_0 \otimes \tilde{f}_i^m b_i$ in which the arrows change direction. The lemma is proved. □

3.2. We can use the above to show the following result.

Theorem 3.2. $B_w(\infty) = \bigcup_{k_1,\dots k_n} \tilde{f}_{i_1}^{*k_1} \dots \tilde{f}_{i_n}^{*k_n} u_\infty$.

Proof. By definition $\tilde{f}_i^{*k} u_\infty = \tilde{f}_i^k u_\infty$ for all k and all i. So we get that

$$B_w(\infty) = \bigcup_{k_1,\dots k_n} \tilde{f}_{i_n}^{k_n} \dots \tilde{f}_{i_2}^{k_2} \tilde{f}_{i_1}^{*k_1} u_\infty.$$

Using Lemma 2.1 we can move $\tilde{f}_{i_1}^{*k_1}$ to the front position. We then proceed with $\tilde{f}_{i_2}^{k_2}$ etc. The theorem is proved. □

3.3. We can now deduce the property (1) of $B_w(\infty)$:

Corollary 3.3. $B_w^*(\infty) = B_{w^{-1}}(\infty)$.

Proof. Let $b \in B_w(\infty)$, i.e. $b = \tilde{f}_{i_n}^{k_n} \dots \tilde{f}_{i_1}^{k_1} u_\infty$ for some $k_1, \dots k_n$. The definition of $\tilde{f}_i^*$ then gives that

$$b^* = \tilde{f}_{i_n}^{*k_n} \tilde{f}_{i_{n-1}}^{*k_{n-1}} \dots \tilde{f}_{i_1}^{*k_1} u_\infty$$

But from Theorem 3.2 we see that $b^* \in B_{w^{-1}}(\infty)$ and the corollary is proved. □

3.4. We shall now consider the property (2). To that end we prove the following lemma

Lemma 3.4. *For all $i, j \in I$ and for all $b \in B(\infty)$ we have that*

$$\tilde{e}_i \bigcup_k \tilde{f}_j^{*k} b \subset \bigcup_k \tilde{f}_j^{*k} \tilde{e}_i b \cup \bigcup_k \tilde{f}_j^{*k} b \cup \{0\}$$

Proof. Again only the case $i = j$ is nontrivial; otherwise $\tilde{e}_i$ and $\tilde{f}_j^*$ commute. We apply the morphism Ψ_i to both sides of the lemma and can then check the inclusion in the crystal graph:

$$\begin{array}{c|ccccccccccccccc}
 & B_i & & & & & & & & & & & & & & \\
\hline
B(\infty) & & & & & & & & & & & & & & & \\
 & * & & * & & * & & * & \to & * & \to & * & \to & * & \to & * \\
 & \downarrow & & \downarrow & & \downarrow & & & & & & & & & & \\
 & * & & * & & * & \to & * & \to & * & \to & * & \to & * & \to & * \\
 & \downarrow & & \downarrow & & & & & & & & & & & & \\
 & * & & * & \to & * & \to & * & \to & * & \to & * & \to & * & \to & * \\
 & \downarrow & & & & & & & & & & & & & & \\
 & * & \to & * & \to & * & \to & * & \to & * & \to & * & \to & * & \to & *
\end{array}$$

The graph is infinite to the right. We understand that $\tilde{e}_i b = 0$ if there is no arrow ending at the point corresponding to b. Again, $\tilde{f}_i^*$ acts by shifting a b to the right while $\tilde{f}_i$ follows the arrows (and hence $\tilde{e}_i$ follows the arrows in negative direction).

Let us start out by verifying that there are no points missing in the above picture. So we must check that if the arrow leaving b is vertical and there is no arrow ending at b then neither should there be any arrow ending at b's right neighbour.

Let thus b be as indicated and write $\Psi_i(b) = b_0 \otimes \tilde{f}_i^m b_i$. Then $\varphi(b_0) > \varepsilon(\tilde{f}_i^m b_i) = m$ because the arrow leaving b is vertical. Now $\tilde{e}_i(b) = 0$ implies that $\tilde{e}_i(b_0) = 0$ because Ψ_i commutes with $\tilde{e}_i$ and no element of B_i is mapped to 0 under $\tilde{e}_i$. Since $\varphi(b_0) \geq \varepsilon(\tilde{f}_i^{m+1} b_i)$ we indeed get that

$$\tilde{e}_i(\Psi_i(\tilde{f}_i^* b)) = \tilde{e}_i(b_0 \otimes \tilde{f}_i^{m+1} b_i) = \tilde{e}_i b_0 \otimes \tilde{f}_i^{m+1} b_i = 0$$

We now split the verification of the lemma into several cases. Firstly we consider the case of a b with $\tilde{e}_i(b) = 0$. Then the left hand side of the lemma consists of those points in the row of b from which a horizontal arrow is leaving. But this is contained in the right hand side of the lemma.

Then we consider the case of a vertical arrow entering and a vertical arrow leaving b. In that case the left hand side of the lemma consists of all the points that are positioned to the right of b (including b itself) together with the points in the row above b that have an arrow leading into one of the first points. In addition, the right hand side consists of the first points together with their upper neighbours. Thus the inclusion also holds in this case.

We then consider the case of a vertical arrow entering and a horizontal arrow leaving b. Then the left hand side of the lemma consists of the points positioned to the right of b together with b itself and its immediate predecessor. This is contained in the right hand side of the lemma (only the $k = 0$ part of the first union is needed).

Finally we consider the case of horizontal arrows entering as well as leaving b. In that case the left hand side consists of all points to the right of b together with b's immediate predecessor, which is included in the right hand side (only the first union is needed). □

3.5. We can now show the property (2) of $B_w(\infty)$:

Theorem 3.5. *For $i \in I$ we have $\tilde{e}_i B_w(\infty) \subset B_w(\infty) \cup \{0\}$*

Proof. We argue by induction on $l(w)$ and thus assume the theorem for $l(w) - 1$. By Theorem 3.2 $B_w(\infty)$ satisfies the equality

$$B_w(\infty) = \bigcup_{k_1} \tilde{f}_{i_1}^{*k_i} B_{ws_{i_1}}(\infty)$$

By induction hypothesis $\tilde{e}_i B_{ws_{i_1}}(\infty) \subset B_{ws_{i_1}}(\infty) \cup \{0\}$. Combining this with lemma 3.4 we obtain the induction step. The theorem is proved. □

4. The Braid Relations

In this section we verify that the crystal Demazure operators $\mathcal{D}_i$ satisfy the braid relations on dominant weights. From this it follows that $B_w(\lambda)$ is independent of the choice of reduced expression for w. Note that Kashiwara has observed that the $\mathcal{D}_i$ do *not* satisfy the braid relations in general.

4.1. Since W is a Weyl group, it is enough to check the braid relations for W of type $\mathrm{A}_2, \mathrm{B}_2$ or G_2. Indeed, for

$$w = w_1 s_{i_k} s_{i_{k-1}} s_{i_k} w_2 = w_1 s_{i_{k-1}} s_{i_k} s_{i_{k-1}} w_2$$

a braid relation of type A_2 it is enough to check the case $w_1 = 1$. By the refined sum formula (2.2.1) applied to w_2 one should then show that

$$\mathcal{D}_{i_k}\mathcal{D}_{i_{k-1}}\mathcal{D}_{i_k}\left(\sum_{b\in B_{w_2}(\lambda)} b\right) = \mathcal{D}_{i_{k-1}}\mathcal{D}_{i_k}\mathcal{D}_{i_{k-1}}\left(\sum_{b\in B_{w_2}(\lambda)} b\right). \tag{4.1.1}$$

Using (2.2.1) once more, the left hand side of this is the sum over all elements of $B_{s_k s_{k-1} s_k w_2}(\lambda)$ while the right hand side is the sum over the elements of $B_{s_{k-1} s_k s_{k-1} w_2}(\lambda)$. We write $w_2 = s_{i_l} s_{i_{l-1}} \dots s_{i_1}$ and get then by repeated use of Lemma 3.1 like in Theorem 3.2 that

$$B_{s_k s_{k-1} s_k w_2}(\lambda) = \bigcup_{k_1,\dots k_l} \tilde{f}_{i_1}^{*k_1} \dots \tilde{f}_{i_{l-1}}^{*k_{l-1}} \tilde{f}_{i_l}^{*k_l} B_{s_k s_{k-1} s_k}(\lambda).$$

Similarly, the right hand side of (4.1.1) is the sum over

$$\bigcup_{k_1,\dots k_l} \tilde{f}_{i_1}^{*k_1} \dots \tilde{f}_{i_{l-1}}^{*k_{l-1}} \tilde{f}_{i_l}^{*k_l} B_{s_k s_{k-1} s_k}(\lambda).$$

The A_2-case of the braid relations then implies (4.1.1). Similarly, one reduces the other braid relations to rank 2 cases.

4.2. To check the $\mathrm{A}_2, \mathrm{B}_2$ or G_2 cases, we appeal to the representation theoretical interpretation of $B(\lambda)$ as basis at $q = 0$ of the irreducible highest weight module $V(\lambda)$ for the quantum group $U_q(\mathfrak{g})$.

Let us consider the A_2-case and write $\lambda = (\lambda_1, \lambda_2)$ in terms of the fundamental weights (ω_1, ω_2). Then $\tilde{f}_1^{\lambda_2} u_\lambda$ is nonzero, since it is the lowest element of the 2-string with highest element u_λ. But by weight considerations $\tilde{f}_1^{\lambda_2} u_\lambda$ must be mapped to 0 under $\tilde{e}_1$ and therefore it is the highest element of the 1-string, whose lowest element is $\tilde{f}_2^{\lambda_1+\lambda_2} \tilde{f}_1^{\lambda_1} u_\lambda$ and especially nonzero. Continuing, we find that

$$\tilde{f}_1^{\lambda_2} \tilde{f}_2^{\lambda_1+\lambda_2} \tilde{f}_1^{\lambda_1} u_\lambda \in B_{s_1 s_2 s_1}(\lambda) \subset B(\lambda)$$

is nonzero. The lowest weight vector space of $V(\lambda)$ is one dimensional and so this element is the unique lowest element if $B(\lambda)$.

Now, by (2) of (2.3), $B_{s_1s_2s_1}(\lambda)$ is invariant under all the $\tilde{e}_i$ operators. Since it moreover contains the lowest element, it must be equal to all of $B(\lambda)$. The same conclusion holds for $B_{s_2s_1s_2}(\lambda)$ and then $B_{s_2s_1s_2}(\lambda) = B_{s_1s_2s_1}(\lambda)$ as promised.

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