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Feynman Diagrams in Algebraic Combinatorics

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Abstract

We show, in great detail, how the perturbative tools of quantum field theory allow one to rigorously obtain: a "categorified" Faà di Bruno type formula for multiple composition, an explicit formula for reversion and a proof of Lagrange-Good inversion, all in the setting of multivariable power series. We took great pains to offer a self-contained presentation that, we hope, will provide any mathematician who wishes, an easy access to the wonderland of quantum field theory.

Key words : Quantum field theory, Combinatorial species.

I Introduction

In our last articles [1, 2], we showed the connection between quantum field theory (QFT) and two research fields in pure mathematics. The first field is the formal inverse approach to the Jacobian conjecture which is a problem of commutative algebra; and the second is the Lagrange-Good multivariable inversion which belongs to enumerative combinatorics. Although the arguments we succintly presented in these articles are mathematically rigorous, it would be difficult to appreciate this fact without a sufficient mastery of perturbation theory in QFT. There already exist a few carefully written introductions to QFT aimed at a mathematical audience ([27] is especially recommended for a general introduction, as well as [16] for the particulars of Fermionic theories, and since we are at the age of multimedia one can also *watch* [11]). Nevertheless, we feel that these references would benefit most the mathematical analyst, rather than the algebraist or combinatorialist. It is partly to fill this need, and also to develop an adequate mathematical theory encompassing our above mentioned papers, that this article was conceived.

Before we proceed, let us first explain what we mean by QFT and why we feel that it is important for algebraic combinatorics. As any theory in physics, QFT has both "grammar" and "meaning". By "grammar" we refer to the mathematical structure of the theory, independently of any physical interpretation. The latter corresponds to what we called the "meaning". To give an analogy, the Navier-Stokes equation as an example of nonlinear PDE, susceptible of a purely mathematical investigation, belongs to the "grammar" of hydrodynamical theory; while its physical interpretation, as describing the evolution of a real fluid like air in the atmosphere, belongs to its "meaning". Of course this distinction is an idealization, but it will help to avoid misunderstandings in what follows. Before we concentrate exclusively on the "grammar" of QFT let us quickly though imperfectly define the "meaning" of QFT as the description of the interaction via radiation fields of constituents of matter, considered as point-like objects, in a way that is compatible with the principles of quantum mechanics and special relativity. As such, it belongs to the rather specialized field of high energy physics and it would be hard to justify its interest for mathematicians at large. However, much more is at stake concerning the "grammar" of QFT, which we expect in the future to pervade most fields of mathematics as it has those of physics (see [29] for some prospective). Indeed the "grammar" of QFT is fundamentally a *generalization of calculus* and certainly the most exciting one since Newton and Leibniz.

This generalization proceeds in two main directions: functional or infinite-dimensional integration, and symbolic integration. Roughly speaking, in the first direction one is interested in defining "natural" measures on spaces of functions ϕ from a *base* manifold $\mathcal{B}$ to a *target* manifold $\mathcal{T}$. The calculus one learns in the first years of university corresponds to the situation where $\mathcal{B}$ is finite; the integrals involved are the familiar ones in finitely many dimensions. A one dimensional manifold $\mathcal{B}$ corresponds for instance to the Wiener measure and to stochastic processes related to Brownian motion that are extensively studied in probability theory. One truly starts doing QFT when the dimension of $\mathcal{B}$ is at least two. In fact, the most interesting and challenging situations occur in dimensions 2, 3 and 4, a pattern which surprisingly is also familiar in topology. The great difficulty of *constructive field theory*, which is the branch of mathematical physics that adresses the problem of giving a rigorous construction of these measures, and which has been honored by the choice of its most outstanding problem among the prize problems of the Clay Foundation [22], comes from the requirement of "naturality". The latter grosso modo means that the density of such measures, with respect to the (ill-defined) Lebesgue measure (in case $\mathcal{T} = \mathbb{R}$ for instance) has to be defined only in terms of the *local* geometry of $\mathcal{B}$, $\mathcal{T}$ and the maps $\phi : \mathcal{B} \to \mathcal{T}$ that one is summing over. For more on this we refer the reader to [20, 27, 12]. We indulged in this digression because we are writing for combinatorialists who might perhaps be agreeably surprised to learn that the most successful methods to tackle this problem of mathematical analysis are combinatorial! Were it not already taken, a suitable denomination for constructive field theory would be "combinatorial analysis".

We now come to the second direction of generalization we mentioned, that of *symbolic integration*. It will be the focus of this work which we hope will deserve a place under the banner of [13]. A first and very fecund example of symbolic integral calculus stemming from QFT is that of integration with respect to anticommuting variables (Fermions). It was introduced by F. A. Berezin, the rightful heir to Grassmann and the elder Cartan. One can define this operation, without speaking of "integration" at all, as that of taking the "top form" coefficient in an element of the exterior algebra of a finite-dimensional vector space. Howerver one would then lose the *suggestive power* of the integral notation among whose benefits has been the discovery of the celebrated Berezin change of variable formula which underlies supersymmetry. A "bijective proof" of this identity seems to us an urgent matter, and an interesting question for the combinatorial community. We will comment on this in Section IV. Besides, what physicists have discovered over the last half-century are substantial fragments of a *dictionary* between "integrals" susceptible of a formal calculus (Feynman path-integrals), and generating series in terms of discrete combinatorial structures (Feynman diagrams). The bridge between the two is given by Wick's theorem which we now state in its *complex Bosonic* version.

Theorem 1 *Let $A \in \mathcal{M}_n(\mathbb{C})$ be a matrix such that $Re\ A \stackrel{\text{def}}{=} \frac{1}{2}(A+A^*)$ is positive definite.*
1) For any J and K, two vectors in $\mathbb{C}^n$, one has

$$\int_{\mathbb{C}^n} d\overline{\phi}d\phi\ e^{-\phi^* A\phi + J^*\phi + \phi^* K} = \frac{e^{J^* A^{-1} K}}{\det(A)}, \tag{1}$$

where the $*$ means Hermitian conjugation, $\phi \in \mathbb{C}^n$ with components $\phi_1, \ldots, \phi_n$ is integrated with respect to the measure

$$d\overline{\phi}d\phi \stackrel{\text{def}}{=} \prod_{i=1}^{n} \frac{d(Re\ \phi_i)d(Im\ \phi_i)}{\pi} \quad . \tag{2}$$

2) Let $i_1, \ldots, i_p$ and $j_1, \ldots, j_q$ be two collections of indices in $\{1, \ldots, n\}$, then

$$\frac{\int_{\mathbb{C}^n} d\overline{\phi}d\phi\ e^{-\phi^* A\phi} \phi_{i_1} \cdots \phi_{i_p} \overline{\phi}_{j_1} \cdots \overline{\phi}_{j_q}}{\int_{\mathbb{C}^n} d\overline{\phi}d\phi\ e^{-\phi^* A\phi}} = \begin{cases} 0 \text{ if } p \neq q \\ \mathrm{per}\left((A^{-1}_{i_\alpha j_\beta})_{1 \le \alpha, \beta \le p} \right) \text{ if } p = q \end{cases}, \tag{3}$$

where $\mathrm{per}(M)$ denotes the permanent of the matrix M, and $\overline{\phi}_j$ is simply the complex conjugate of the component ϕ_j of ϕ.
3) For any polynomial in the ϕ_i's and the $\overline{\phi}_j$'s considered as $2n$ independent variables, the effect of integrating with respect to the Gaussian probability measure

$$\frac{d\overline{\phi}d\phi\ e^{-\phi^* A\phi}}{\int_{\mathbb{C}^n} d\overline{\phi}d\phi\ e^{-\phi^* A\phi}}$$

on $\mathbb{C}^n$ is the same as that of applying the "differential operator"

$$\exp\left(\sum_{i,j=1}^{n} \left(\frac{\partial}{\partial\phi_i} (A^{-1})_{ij} \frac{\partial}{\partial\overline{\phi}_j} \right) \right)$$

followed by evaluation at $\phi = \overline{\phi} = 0$.

The proofs of *2)* and *3)* are easy consequences of *1)* which is an exercise in ordinary calculus. This theorem translates the evaluation of Gaussian integrals into a combinatorial game, which should be evident from the expansion of the permanent of *2)* in terms of permutations or by keeping track of which derivative acts on which factor in the formulation *3)*. In what follows, we reverse the thrust and use the theorem as a *definition* of "integrals" that can then be rigorously constructed in some rings of formal power series. We are certainly not the first, and hopefully not the last, to follow this line of thought to make mathematical use of the "folklore" of perturbative QFT. For instance, knot theorists have cashed in on this idea [6, 7], and as we put this article into type, we learned of [17], where the principal motivation comes from the study of the cohomology of moduli spaces of curves [18], and where an effort similar to ours is made. As for the present work, the most convenient mathematical framework we found is the theory of combinatorial species initiated by Joyal in his seminal work [23] (see also [9]). It provides us with the most reasonable compromise between categorical "abstract nonsense" and the almost childlike simplicity of Feynman diagrammatic notation that has to be preserved at all costs.

Now let us outline the plan of this paper. Part II is concerned with the formal calculus aspect of the dictionary and addresses successively: multiple composition of multivariable power series in II.1, reversion in II.2, and Lagrange-Good inversion with a, perhaps new, generalization of it in II.3. The presentation in this part is deliberately heuristic. Part III introduces a rigorous mathematical framework for the three above mentioned topics which are treated in the same sequence in III.1, III.2 and III.3 respectively. We will end this article in section IV with a few comments indicating some directions for further work.

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II Symbolic Integration

Warning : This section is heuristic.

Let R be a commutative ring with unit containing the field of rationals $\mathbb{Q}$. Our concern in this section is to introduce a notion of integral calculus for formal power series over R. In the sequel a "function" $F : R^n \to R$ means a formal power series $F \in R[[X_1, \ldots, X_n]]$. Likewise a "function" $F : R^n \to R^n$ means a system $F = (F_i)_{1\le i\le n}$ of n power series in $R[[X_1, \ldots, X_n]]$. If $u = (u_1, \ldots, u_n)$ is a vector of n indeterminates, we introduce the corresponding differential symbols $du_1, \ldots, du_n$ and write $du \stackrel{\text{def}}{=} du_1 \cdots du_n$ for their product. Given two such vectors

$u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ we let $uv \stackrel{\text{def}}{=} u_1v_1 + \cdots + u_nv_n$. We now introduce an integration symbol $\int$ in such a way that given a function $F : R^n \to R$, $\int duF(u)$ is an element of R we would like to think of as the integral of F. We also introduce an n-dimensional "delta function" $\delta(u) = \delta(u_1) \cdots \delta(u_n)$, and postulate the following rules of computation.

Rule 1 : For any $F : R^n \to R$,

$$\int duF(u)\delta(u) = F(0) \quad . \tag{4}$$

Rule 2 :

$$\int du\ e^{-uv} = \delta(v) \quad . \tag{5}$$

Rule 3 : All the rules of ordinary calculus are allowed: integration by parts, Fubini, change of variables etc. However, there is no absolute value for the Jacobian factor in a change of variables.

We will now use this apparently nonsensical calculational scheme for three different applications.

II.1 Composition

Let F, G be functions from R^n to R^n, with no constant term

Claim 1 :

$$(F \circ G)_i(X) = \int d\overline{s}dsd\overline{t}dtd\overline{u}du\ s_i e^{-\overline{s}s-\overline{t}t-\overline{u}u+\overline{s}F(t)+\overline{t}G(u)+\overline{u}X} \quad . \tag{6}$$

Here, $F = (F_i)_{1\leq i\leq n}$, $F_i(X) \in R[[X_1, \ldots, X_n]]$, and likewise for G. $F \circ G$ is the result of substituting $G_i(X)$ for X_i in $F(X)$, and $(F \circ G)_i(X)$ is the i-th component of the composition of F with G. Also $\overline{s} = (\overline{s}_1, \ldots, \overline{s}_n)$, $s = (s_1, \ldots, s_n)$, $\overline{t} = (\overline{t}_1, \ldots, \overline{t}_n)$, $t = (t_1, \ldots, t_n)$, $\overline{u} = (\overline{u}_1, \ldots, \overline{u}_n)$ and $u = (u_1, \ldots, u_n)$ are six vectors of indeterminates. An expression like $\overline{s}s$ means $\sum_{i=1}^n \overline{s}_i s_i$, and $\overline{s}F(t) \stackrel{\text{def}}{=} \sum_{i=1}^n \overline{s}_i F_i(t_1, \ldots, t_n)$.

The previous claim is obtained from the following symbolic calculation. We first integrate over $\overline{u}$ in (6) according to Rule 2 :

$$\int d\overline{u}e^{-\overline{u}u+\overline{u}X} = \delta(u - X), \tag{7}$$

therefore

$$\int d\overline{s}dsd\overline{t}dtd\overline{u}du\ s_i e^{-\overline{s}s-\overline{t}t-\overline{u}u+\overline{s}F(t)+\overline{t}G(u)+\overline{u}X}$$

$$= \int d\overline{s}dsd\overline{t}dtdu \; s_i e^{-\overline{s}s-\overline{t}t+\overline{s}F(t)+\overline{t}G(u)}\delta(u-X) \tag{8}$$

$$= \int d\overline{s}dsd\overline{t}dtdv \; s_i e^{-\overline{s}s-\overline{t}t+\overline{s}F(t)+\overline{t}G(v+X)}\delta(v) \; , \tag{9}$$

where we used the translation $v = u - X$, which gives by Rule 1

$$\int d\overline{s}dsd\overline{t}dt \; s_i e^{-\overline{s}s-\overline{t}t+\overline{s}F(t)+\overline{t}G(X)} \quad .$$

Now this becomes after integration with respect to $\overline{t}$

$$\int d\overline{s}dsdt \; s_i e^{-\overline{s}s+\overline{s}F(t)}\delta(t-G(X))$$

$$= \int d\overline{s}dsdw \; s_i e^{-\overline{s}s+\overline{s}F(w+G(X))}\delta(w), \tag{10}$$

where $w = t - G(X)$, and finally we get

$$\int d\overline{s}ds \; s_i e^{-\overline{s}s+\overline{s}F(G(X))} = \int ds \; s_i \delta(s - F(G(X))) \tag{11}$$

$$= F_i(G(X)) \quad . \tag{12}$$

Note that Rule 3 was used in the calculation by applying "Fubini's theorem" to perform the integrations in the chosen order, and by using the change of variable formula in the simple case of a translation. It is easy to generalize this calculation to a multiple composition. That is suppose $F^{(1)}, \ldots, F^{(p)}$ are p functions from R^n to R^n (with no constant term) so that $F^{(j)} = (F_i^{(j)})_{1\le i\le n}$, with each $F_i^{(j)}$ in $R[[X_1, \ldots, X_n]]$. We can then write a similar formula for their p-fold composition.

Claim 2 :

$$(F^{(1)} \circ \cdots \circ F^{(p)})_i(X) = \int \prod_{k=1}^{p+1} (d\overline{u}^{(k)} du^{(k)})$$

$$u_i^{(1)} \exp\left(-\sum_{k=1}^{p+1} \overline{u}^{(k)} u^{(k)} + \sum_{k=1}^{p} \overline{u}^{(k)} F^{(k)}(u^{(k+1)}) + \overline{u}^{(p+1)} X \right), \tag{13}$$

where, for each k, $1 \le k \le p+1$, $\overline{u}^{(k)} = (\overline{u}_1^{(k)}, \ldots, \overline{u}_n^{(k)})$ and $u^{(k)} = (u_1^{(k)}, \ldots, u_n^{(k)})$ are vectors of indeterminates. The above integral is formally over $2n(p+1)$ variables.

II.2 Reversion

Let F be a function from R^n to R^n with no constant term and with invertible linear component. We would like a formula for the compositional inverse $F^{-1} : R^n \to R^n$.

For this purpose we let Ω be any function $R^n \to R$ (not necessarily without constant term) and we consider, with similar notation as in the previous section, the integral

$$\int d\overline{u}du\ \Omega(u) e^{-\overline{u}F(u)+\overline{u}Y} \quad . \tag{14}$$

We use Rule 3 to perform the change of variable $v = F(u) - Y$ or $u = F^{-1}(v+Y)$. If $F = (F_i(X))_{1\le i\le n}$ with $X = (X_1, \ldots, X_n)$ and $F_i(X) \in R[[X_1, \ldots, X_n]]$, for $1 \le i \le n$, we use the notation

$$\partial F(Z) \stackrel{\text{def}}{=} \left(\frac{\partial F_i}{\partial X_j}(Z)\right)_{1\le i,j\le n} \tag{15}$$

for the Jacobian matrix of F "at the point" $Z = (Z_1, \ldots, Z_n)$. It is an element of $\mathcal{M}_n(R[[Z_1, \ldots, Z_n]])$. The change of variable formula asserts that in the above integral we can replace the dummy integration variable u by $F^{-1}(v+Y)$ and du by $\det[\partial(F^{-1})(v+Y)]dv$. Therefore

$$\begin{aligned}\int d\overline{u}du\ \Omega(u) e^{-\overline{u}F(u)+\overline{u}Y} &= \int d\overline{u}dv\ \det[\partial(F^{-1})(v+Y)] \\ &\qquad \Omega(F^{-1}(v+Y))e^{-\overline{u}v} \qquad (16) \\ &= \int dv\ \det[\partial(F^{-1})(v+Y)] \\ &\qquad \Omega(F^{-1}(v+Y))\delta(v) \qquad (17)\end{aligned}$$

by Rule 2. Finally Rule 1 gives

$$\int d\overline{u}du\ \Omega(u) e^{-\overline{u}F(u)+\overline{u}Y} = \Omega(F^{-1}(Y)) \det[\partial(F^{-1})(Y)] \quad . \tag{18}$$

Applying this last formula successively to $\Omega(u) \stackrel{\text{def}}{=} 1$ and $\Omega(u) \stackrel{\text{def}}{=} u_i$, and noting the cancellation of the determinantal factor, we obtain

Claim 3 :

$$F^{-1}(Y)_i = \frac{\int d\overline{u}du\ u_i e^{-\overline{u}F(u)+\overline{u}Y}}{\int d\overline{u}du\ e^{-\overline{u}F(u)+\overline{u}Y}} \ , \tag{19}$$

which is a formula for the i-th component of the compositional inverse F^{-1} applied to $Y = (Y_1, \ldots, Y_n)$. It is a generalization, due to V. Rivasseau and the author [1], of a formula that first appeared in the context of KAM theory [19]. Note the compelling probabilistic interpretation as the average of u_i with respect to the probability measure proportional to $d\overline{u}du\ e^{-\overline{u}F(u)+\overline{u}Y}$.

II.3 Lagrange-Good inversion

Let $G = (G_i)_{1\le i\le n}$ be a given system of formal power series in n indeterminates, and let $F = (F_i)_{1\le i\le n}$ be the system of formal power series in $R[[X_1, \ldots, X_n]]$, without constant term, implicitly defined by the equations

$$F_i = X_i G_i(F) \quad \text{for} \quad 1 \le i \le n \quad . \tag{20}$$

The implicit form of the multivariable Lagrange-Good inversion formula says that for any $\Omega : R^n \to R$ the coefficient of $\frac{X^M}{M!}$ in

$$\Omega(F) \frac{1}{\det(\delta_{ij} - X_i \partial_j G_i(F))}$$

is equal to the coefficient of $\frac{u^M}{M!}$ in $\Omega(u)G(u)^M$. Here we used the multiindex notation $M \stackrel{\text{def}}{=} (M_1, \ldots, M_n) \in \mathbb{N}^n$, $u^M \stackrel{\text{def}}{=} u_1^{M_1} \cdots u_n^{M_n}$, $M! \stackrel{\text{def}}{=} M_1! \cdots M_n!$ and $G(u)^M \stackrel{\text{def}}{=} G_1(u)^{M_1} \cdots G_n(u)^{M_n}$. If $G_i = G_i(u)$, is originally given in terms of the u variables, $\partial_j G_i(F)$ denotes the substitution of u by $F = F(X)$ in $\frac{\partial G_i}{\partial u_j}(u)$. Therefore

$$(\delta_{ij} - X_i \partial_j G_i(F))_{1\le i,j\le n} \in \mathcal{M}_n(R[[X_1, \ldots, X_n]]) \quad . \tag{21}$$

The Lagrange-Good formula can be compactly written as

$$\Omega(F) \frac{1}{\det(\delta_{ij} - X_i \partial_j G_i(F))} = \sum_{M\in\mathbb{N}^n} \frac{X^M}{M!} \left(\frac{\partial}{\partial u}\right)^M \Omega(u)G(u)^M \Bigg|_{u=0} \quad . \tag{22}$$

We will derive this identity using our symbolic calculus. We consider this time the integral

$$\int d\overline{u}du\ \Omega(u)e^{-\overline{u}u+\overline{u}XG(u)} \quad .$$

Here $\overline{u}XG(u) \stackrel{\text{def}}{=} \sum_{i=1}^n \overline{u}_i X_i G_i(u)$. In order to be able to apply Rule 2 to integrate over $\overline{u}$, we need first to perform the change of variables $v = H(u)$ where $H_i(u) = u_i - X_i G_i(u)$. H is considered as a function of u from R^n to R^n. X plays the role of a parameter. Note that $F(X) = H^{-1}(0)$. Now $v = H(u)$ implies $u = H^{-1}(v)$ and $du = \det[\partial(H^{-1})(v)]dv$, therefore by Rule 3

$$\int d\overline{u}du\ \Omega(u)e^{-\overline{u}u+\overline{u}XG(u)} = \int d\overline{u}du\ \Omega(u)e^{-\overline{u}H(u)} \tag{23}$$

$$= \int d\overline{u}dv\ \Omega(H^{-1}(v)) \det[\partial(H^{-1})(v)]e^{-\overline{u}v} \ . \tag{24}$$

Now by Rules 2 and 1 this becomes

$$\int dv\ \Omega(H^{-1}(v)) \det[\partial(H^{-1})(v)]\delta(v) = \Omega(H^{-1}(0)) \det[\partial(H^{-1})(0)] \tag{25}$$

$$= \Omega(H^{-1}(0)) \frac{1}{\det[\partial H(H^{-1}(0))]} \tag{26}$$

$$= \Omega(F) \frac{1}{\det(I - X\partial G(F))} \quad ; \tag{27}$$

where I is the $n \times n$ identity matrix and $[X\partial G(F)]_{ij} \stackrel{\text{def}}{=} X_i \partial_j G_i(F)$. Therefore

$$\int d\overline{u}du\ \Omega(u) e^{-\overline{u}u + \overline{u}XG(u)} = \Omega(F) \frac{1}{\det(\delta_{ij} - X_i \partial_j G_i(F))} \quad . \tag{28}$$

There is a second way to do the computation, namely to expand the exponential. In multiindex notation it reads

$$\int d\overline{u}du\ \Omega(u) e^{-\overline{u}u + \overline{u}XG(u)} = \sum_{M \in \mathbb{N}^n} \frac{X^M}{M!} \int d\overline{u}du\ \overline{u}^M e^{-\overline{u}u} \Omega(u) G(u)^M \quad . \tag{29}$$

Now note that $\overline{u}^M e^{-\overline{u}u} = (-\frac{\partial}{\partial u})^M e^{-\overline{u}u}$ therefore one can integrate by parts

$$\int d\overline{u}du\ \overline{u}^M e^{-\overline{u}u} \Omega(u) G(u)^M = \int d\overline{u}du\ e^{-\overline{u}u} \left(\frac{\partial}{\partial u}\right)^M \left[\Omega(u) G(u)^M\right] \tag{30}$$

$$= \int du\ \delta(u) \left(\frac{\partial}{\partial u}\right)^M \left[\Omega(u) G(u)^M\right] \tag{31}$$

$$= \left(\frac{\partial}{\partial u}\right)^M \left[\Omega(u) G(u)^M\right] \Big|_{u=0} \quad , \tag{32}$$

and finally

$$\int d\overline{u}du\ \Omega(u) e^{-\overline{u}u + \overline{u}XG(u)} = \sum_{M \in \mathbb{N}^n} \frac{X^M}{M!} \left(\frac{\partial}{\partial u}\right)^M \left[\Omega(u) G(u)^M\right] \Big|_{u=0} \quad . \tag{33}$$

Now (22) follows from (28) and (33). Note also that by applying (28) successively to $\Omega(u) \stackrel{\text{def}}{=} 1$ and $\Omega(u) \stackrel{\text{def}}{=} u_i$, and computing the ratio we obtain a formula for $F_i(X)$ which is given by the following.

Claim 4 :

$$F_i(X) = \frac{\int d\overline{u}du\ u_i e^{-\overline{u}u + \overline{u}XG(u)}}{\int d\overline{u}du\ e^{-\overline{u}u + \overline{u}XG(u)}} \quad . \tag{34}$$

Note the similarity with Claim 3. In fact we derived Claim 4 from Claim 3 in [2]. Note also that one of the advantages of our symbolic calculus is that it can suggest generalizations and variations on the Lagrange-Good formula. For instance, given $G = (G_i)_{1 \le i \le n}$, with $G_i = G_i(u)$, $u = (u_1, \ldots, u_n)$, and given n^2 indeterminates

$(X_{ij})_{1\le i,j\le n}$, one can show that there is a unique solution $F = (F_i)_{1\le i\le n}$, with $F_i \in R[[(X_{ij})_{1\le i,j\le n}]]$, without constant term, to the equations

$$F_i = \sum_{j=1}^{n} X_{ij} G_j(F) \ \text{ for } \ 1 \le i \le n \ . \tag{35}$$

The solution is given by the same expression as in Claim 4, except that

$$\overline{u} X G(u) \stackrel{\text{def}}{=} \sum_{i,j=1}^{n} \overline{u}_i X_{ij} G_j(u_1,\ldots,u_n) \ . \tag{36}$$

One also has a Lagrange-Good type identity which seems to be new, and is our first encounter here with the inadequacy of the multiindex notation. With $\Omega : R^n \to R$, it reads

$$\Omega(F)\frac{1}{\det(I - X\partial G(F))} = \sum_{k\ge 0} \sum_{\substack{i_1,\ldots,i_k,\\ j_1,\ldots,j_k=1}}^{n} \frac{X_{i_1 j_1}\cdots X_{i_k j_k}}{k!}$$

$$\int d\overline{u}du\ \Omega(u) e^{-\overline{u}u} \overline{u}_{i_1}\cdots\overline{u}_{i_k} G_{j_1}(u)\cdots G_{j_k}(u) \tag{37}$$

$$= \sum_{k\ge 0} \sum_{\substack{i_1,\ldots,i_k,\\ j_1,\ldots,j_k=1}}^{n} \frac{X_{i_1 j_1}\cdots X_{i_k j_k}}{k!} \frac{\partial}{\partial u_{i_1}}\cdots\frac{\partial}{\partial u_{i_k}} \left[\Omega(u) G_{j_1}(u)\cdots G_{j_k}(u)\right]\Big|_{u=0} \ . \tag{38}$$

III Feynman diagrams

In contrast to the previous heuristic but conceptually important section, we will now do some mathematics. Throughout the remainder of this article $[n]$ will denote the set of the first n nonnegative integers; and $\#(E)$ will denote the cardinality of a finite set E. As we will constantly use the notion of summable families in power series rings, the reader who needs it should consult [10] for a refresher. Let R be a commutative ring with unit containing $\mathbb{Q}$. A single power series $F \in R[[X_1,\ldots,X_n]]$ is usually specified using multiindex notation as

$$F = \sum_{\alpha\in\mathbb{N}^n} \frac{1}{\alpha!} u_\alpha X^\alpha, \tag{39}$$

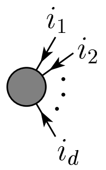
where $\alpha = (\alpha_1,\ldots,\alpha_n)$ is a multiindex in $\mathbb{N}^n$, $\alpha! = \alpha_1!\cdots\alpha_n!$, $X^\alpha = X_1^{\alpha_1}\cdots X_n^{\alpha_n}$, and $(u_\alpha)_{\alpha\in\mathbb{N}^n}$ is the family of coefficients defining F. Note the normalization by $\frac{1}{\alpha!}$ which might seem like an insignificant matter of convention but will be a leitmotiv in the following exposition, namely to normalize by the cardinality of the group of *ambiguity*. We will make this precise later. Multiindex notation is more than highly impractical for our purposes, it is in fact an example of a "bad decategorification" in the sense of [4], the authors of which were also influenced by Joyal's theory of species. It is rather more natural to use a *tensorial* notation for our power series F as

$$F = \sum_{d \geq 0} \frac{1}{d!} \sum_{i_1,\ldots,i_d=1}^{n} F^{[d]}_{i_1 \ldots i_d} X_{i_1} \ldots X_{i_d} \quad . \tag{40}$$

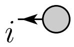
Note again the $\frac{1}{d!}$ as our ambiguity group here is the symmetric group $\mathfrak{S}_d$. If $(i_1, \ldots, i_d) \in [n]^d$ let $\mu(i_1, \ldots, i_d) \in \mathbb{N}^n$ be the associated multiplicity multiindex that is $\mu(i_1, \ldots, i_d) = (\mu_1, \ldots, \mu_n)$, where, for each i, $1 \leq i \leq n$, μ_i is the number of indices $r \in [d]$ such that $i_r = i$. The translation between (39) and (40) is of course

$$F^{[d]}_{i_1 \ldots i_d} = u_{\mu(i_1,\ldots,i_d)} \quad , \tag{41}$$

for any $d \geq 0$ and $(i_1, \ldots, i_d) \in [n]^d$. $F^{[d]}_{i_1 \ldots i_d}$ can be thought of as a tensor element (the multidimensional analog of a matrix entry) of a symmetric d-covariant tensor (i.e., an element of the R-module $Sym^d((R^n)^*)$). Feynman diagrams are first of all, the most efficient notation for tensors and tensor contraction (see the Appendix of [25] and also [14]) and the most natural step after Einstein's convention of summing over repeated indices. We will for instance write

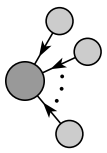


$$\stackrel{\text{def}}{=} F^{[d]}_{i_1 \ldots i_d} \quad , \tag{42}$$



$$\stackrel{\text{def}}{=} X_i \quad , \tag{43}$$

and



$$\stackrel{\text{def}}{=} \sum_{i_1,\ldots,i_d=1}^{n} F^{[d]}_{i_1 \ldots i_d} X_{i_1} \ldots X_{i_d} \quad . \tag{44}$$

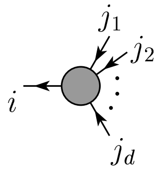
Our presentation will now bifurcate in order to specifically address the three sections of Part II.

III.1 Composition

Let $F = (F_i)_{1 \leq i \leq n}$ with $F_i \in R[[X_1, \ldots, X_n]]$ be a system of n formal power series without constant term, given for any i, $1 \leq i \leq n$, under the tensorial form

$$F = \sum_{d \geq 1} \frac{1}{d!} \sum_{j_1,\ldots,j_d=1}^{n} F^{[d]}_{i,j_1 \ldots j_d} X_{j_1} \cdots X_{j_d} \quad . \tag{45}$$

We will write



$$\stackrel{\text{def}}{=} F^{[d]}_{i,j_1\ldots j_d} = \frac{\partial^d}{\partial X_{j_1}\cdots\partial X_{j_d}} F_i(X)\Big|_{X=0} \quad . \tag{46}$$

$F^{[d]}_{i,j_1\ldots j_d}$ can be thought of as a tensor element of a 1-contravariant and d-covariant tensor, which is symmetric in the d covariant indices i.e., an element of the R-module $R^n \otimes Sym^d((R^n)^*)$. Suppose we have two systems of n formal power series without constant term, $F = (F_i)_{1\le i\le n}$ and $G = (G_i)_{1\le i\le n}$. Our purpose in this section is to give a precise mathematical meaning to formula (6). For this we need to introduce some definitions and notations. Let $\overline{s} = (\overline{s}_1,\ldots,\overline{s}_n)$, $s = (s_1,\ldots,s_n)$, $\overline{t} = (\overline{t}_1,\ldots,\overline{t}_n)$, $t = (t_1,\ldots,t_n)$, $\overline{u} = (\overline{u}_1,\ldots,\overline{u}_n)$ and $u = (u_1,\ldots,u_n)$ be six vectors of indeterminates. Denote by K the ring of formal power series $R[[\overline{s}, s, \overline{t}, t, \overline{u}, u]]$ defined over R using these $6n$ indeterminates. An element $U \in K$ can be written in multiindex notation as

$$U = \sum_{\alpha_1,\ldots,\alpha_6\in\mathbb{N}^n} u_{\alpha_1,\ldots,\alpha_6} \frac{\overline{s}^{\alpha_1} s^{\alpha_2} \overline{t}^{\alpha_3} t^{\alpha_4} \overline{u}^{\alpha_5} u^{\alpha_6}}{\alpha_1!\cdots\alpha_6!} \quad , \tag{47}$$

but also in tensorial notation as

$$U = \sum_{k_1,\ldots,k_6\in\mathbb{N}} \sum_{\tau_1,\ldots,\tau_6} \frac{U^{[k_1,\ldots,k_6]}_{\tau_1,\ldots,\tau_6}}{k_1!\cdots k_6!} \overline{s}_{\tau_1} s_{\tau_2} \overline{t}_{\tau_3} t_{\tau_4} \overline{u}_{\tau_5} u_{\tau_6} \quad , \tag{48}$$

where the sum on τ_1 is over all maps $[k_1] \to [n]$ and likewise for $\tau_2,\ldots,\tau_6$. $\overline{s}_{\tau_1} \stackrel{\text{def}}{=} \overline{s}_{\tau_1(1)}\cdots\overline{s}_{\tau_1(k_1)}$ and likewise for s_{τ_2}, $\overline{t}_{\tau_3}$, t_{τ_4}, $\overline{u}_{\tau_5}$ and u_{τ_6}. Given A, B and C three matrices in $GL_n(R)$ we define

$$\begin{aligned} I_{A,B,C}(\tau_1,\ldots,\tau_6) \stackrel{\text{def}}{=} \sum_{\sigma,\mu,\nu} &\left(\prod_{1\le k\le k_1} [A^{-1}]_{\tau_2(\sigma(k))\tau_1(k)}\right) \\ \times \left(\prod_{1\le k\le k_3} [B^{-1}]_{\tau_4(\mu(k))\tau_3(k)}\right) &\left(\prod_{1\le k\le k_5} [C^{-1}]_{\tau_6(\nu(k))\tau_5(k)}\right) \quad , \end{aligned} \tag{49}$$

where the sum is over all bijective maps $\sigma : [k_1] \to [k_2]$, $\mu : [k_3] \to [k_4]$ and $\nu : [k_5] \to [k_6]$. Of course, the result is zero unless $k_1 = k_2$, $k_3 = k_4$ and $k_5 = k_6$.

Lemma 1 *Let $\rho_1,\ldots,\rho_6$ be in $\mathfrak{S}_{k_1},\ldots,\mathfrak{S}_{k_6}$ respectively, then*

$$I_{A,B,C}(\tau_1\circ\rho_1,\ldots,\tau_6\circ\rho_6) = I_{A,B,C}(\tau_1,\ldots,\tau_6) \quad . \tag{50}$$

Proof : Trivial. ∎

Definition 1 *Let $\alpha_1, \ldots, \alpha_6 \in \mathbb{N}^n$, we define the* formal Gaussian integral *of the monomial $\overline{s}^{\alpha_1} s^{\alpha_2} \overline{t}^{\alpha_3} t^{\alpha_4} \overline{u}^{\alpha_5} u^{\alpha_6}$, with covariances A^{-1}, B^{-1}, C^{-1}, as*

$$(\det\ A)^{-1}(\det\ B)^{-1}(\det\ C)^{-1} I_{A,B,C}(\tau_1, \ldots, \tau_6) \ ,$$

which belongs to R, and where for each i, $1 \le i \le 6$, τ_i is any map from $[k_i]$ to $[n]$ with $k_i = |\alpha_i|$ and such that the associated multiplicity multiindex $\mu(\tau_i) = \alpha_i$. By the previous lemma, this element of R is independent of the choice of $\tau_1, \ldots, \tau_6$. We denote *this expression by*

$$\int d\overline{s} ds d\overline{t} dt d\overline{u} du \ e^{-\overline{s}As - \overline{t}Bt - \overline{u}Cu} \ \overline{s}^{\alpha_1} s^{\alpha_2} \overline{t}^{\alpha_3} t^{\alpha_4} \overline{u}^{\alpha_5} u^{\alpha_6} \quad .$$

Definition 2 *If U is as before a power series in K, we define the* formal Gaussian integral *of U with covariances A^{-1}, B^{-1} and C^{-1} as*

$$\int d\overline{s} ds d\overline{t} dt d\overline{u} du \ e^{-\overline{s}As - \overline{t}Bt - \overline{u}Cu} \ U \stackrel{\text{def}}{=} \sum_{\alpha_1, \ldots, \alpha_6 \in \mathbb{N}^n} \frac{u_{\alpha_1, \ldots, \alpha_6}}{\alpha_1! \cdots \alpha_6!}$$
$$\int d\overline{s} ds d\overline{t} dt d\overline{u} du \ e^{-\overline{s}As - \overline{t}Bt - \overline{u}Cu} \ \overline{s}^{\alpha_1} s^{\alpha_2} \overline{t}^{\alpha_3} t^{\alpha_4} \overline{u}^{\alpha_5} u^{\alpha_6} \quad , \tag{51}$$

if the right hand side is summable. R being equipped with the discrete topology, this simply means that finitely many terms are nonzero.

We now restrict to the case where $A = B = C = I$ the $n \times n$ identity matrix, and can state

Theorem 2 *For any $d \ge 1$ and $i, j_1, \ldots, j_d \in [n]$,*

$$(F \circ G)^{[d]}_{i,j_1 \cdots j_d} = \int d\overline{s} ds d\overline{t} dt d\overline{u} du \ e^{-\overline{s}s - \overline{t}t - \overline{u}u} \ U \quad , \tag{52}$$

where

$$U \stackrel{\text{def}}{=} s_i \overline{u}_{j_1} \cdots \overline{u}_{j_d} \exp\left(\overline{s}F(t) + \overline{t}G(u)\right) \quad . \tag{53}$$

Remark : Again we used our tensorial notation for the coefficients of $F \circ G(X)$, that is

$$(F \circ G)^{[d]}_{i,j_1 \cdots j_d} \stackrel{\text{def}}{=} \frac{\partial^d}{\partial X_{j_1} \cdots \partial X_{j_d}} F_i(G(X)) \bigg|_{X=0} \quad . \tag{54}$$

Remark : As $\overline{s}F(t)$ and $\overline{t}G(u)$ have no constant term, U is a well-defined element of K.

Remark : If one formally expands (6) with respect to X one obtains

$$\sum_{d \ge 1} \sum_{j_1, \ldots, j_d = 1}^{n} \frac{X_{j_1} \cdots X_{j_d}}{d!} (F \circ G)^{[d]}_{i,j_1 \cdots j_d} = \sum_{d \ge 1} \sum_{j_1, \ldots, j_d = 1}^{n} \frac{X_{j_1} \cdots X_{j_d}}{d!}$$
$$\int d\overline{s} ds d\overline{t} dt d\overline{u} du \ e^{-\overline{s}s - \overline{t}t - \overline{u}u} s_i \overline{u}_{j_1} \cdots \overline{u}_{j_d} e^{\overline{s}F(t) + \overline{t}G(u)} \quad . \tag{55}$$

Therefore Theorem 2 is a rigorous restatement of Claim 1.

In order to prove the theorem, we need to define the notions of pre-Feynman and Feynman diagram structures, which find their natural habitat in the Joyal theory of combinatorial species. The proof of Theorem 2 will accordingly be postponed till the end of this section. We suppose that $d \geq 1$. The index i considered as a map from $I \stackrel{\text{def}}{=} [1]$ to $[n]$ and the collection of indices $j_1, \ldots, j_d$ considered as a map from $J \stackrel{\text{def}}{=} [d]$ to $[n]$ are *fixed* in the following. We call these two maps *index assignments*. Now let E be any finite set.

Definition 3 *A pre-Feynman diagram sructure on E is an ordered collection*

$$\mathcal{E} = (E_{\overline{s}}, E_s, E_{\overline{t}}, E_t, E_{\overline{u}}, E_u, E_{int}, E_{ext}, \pi_F, \pi_G, \rho_s, \rho_{\overline{u}}) \tag{56}$$

made of the following data.

- $E_{\overline{s}}, E_s, E_{\overline{t}}, E_t, E_{\overline{u}}, E_u, E_{int}, E_{ext}$ *are subsets of E.*
- π_F, π_G *are (unordered) sets of subsets of E.*
- ρ_s *is a map from I to $E_{ext} \cap E_s$.*
- $\rho_{\overline{u}}$ *is a map from J to $E_{ext} \cap E_{\overline{u}}$.*

We furthermore ask that the previous data satisfy the following constraints.

- $E_{\overline{s}}, E_s, E_{\overline{t}}, E_t, E_{\overline{u}}, E_u$ *are disjoint and their union is E.*
- E_{int}, E_{ext} *are disjoint and their union is E.*
- $(E_{\overline{s}} \cup E_{\overline{t}} \cup E_t \cup E_u) \cap E_{ext} = \emptyset$
- $\rho_s : I \to E_{ext} \cap E_s$ *and* $\rho_{\overline{u}} : J \to E_{ext} \cap E_{\overline{u}}$ *are bijective.*
- $\pi_F \cap \pi_G = \emptyset$ *and* $\pi_F \cup \pi_G$ *forms a partition of E_{int}.*
- *Any block $B \in \pi_F$, also called an F-vertex, is the union of $B \cap E_{\overline{s}}$ and $B \cap E_t$ which must respectively be a singleton and a nonempty set.*
- *Any block $B \in \pi_G$, also called a G-vertex, is the union of $B \cap E_{\overline{t}}$ and $B \cap E_u$ which must respectively be a singleton and a nonempty set.*

We denote the set of pre-Feynman diagram structures on a finite set E by $PreFey(E)$ which is obviously finite too. What we have just done is defining a covariant endofunctor for the groupoid category of finite sets with morphisms given by bijective maps. Indeed, if $\sigma : E \to E'$ is such a morphism, its transform by this functor $PreFey(\sigma) : PreFey(E) \to PreFey(E')$ is the map which to a pre-Feynman diagram structure

$$\mathcal{E} = (E_{\overline{s}}, E_s, E_{\overline{t}}, E_t, E_{\overline{u}}, E_u, E_{int}, E_{ext}, \pi_F, \pi_G, \rho_s, \rho_{\overline{u}}) \tag{57}$$

on E associates the analogous structure

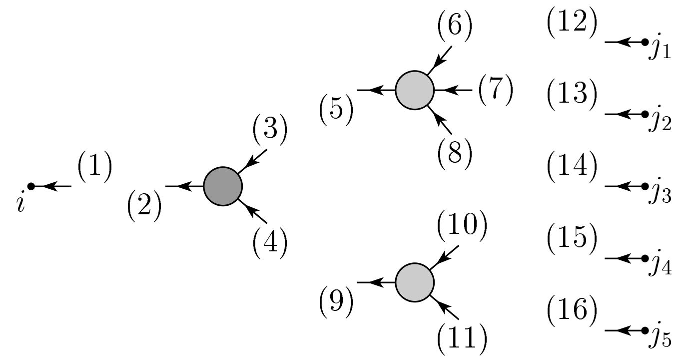
$$\mathcal{E}' = (E'_{\overline{s}}, E'_s, E'_{\overline{t}}, E'_t, E'_{\overline{u}}, E'_u, E'_{int}, E'_{ext}, \pi'_F, \pi'_G, \rho'_s, \rho'_{\overline{u}}) \tag{58}$$

on E' given in the obvious manner by $E'_{\overline{s}} = \sigma(E_{\overline{s}})$, $E'_s = \sigma(E_s)$, $E'_{\overline{t}} = \sigma(E_{\overline{t}})$, $E'_t = \sigma(E_t)$, $E'_{\overline{u}} = \sigma(E_{\overline{u}})$, $E'_u = \sigma(E_u)$, $E'_{int} = \sigma(E_{int})$, $E'_{ext} = \sigma(E_{ext})$, $\pi'_F = \{\sigma(B)|B \in \pi_F\}$, $\pi'_G = \{\sigma(B)|B \in \pi_G\}$, $\rho'_s = \sigma \circ \rho_s$ and finally $\rho'_{\overline{u}} = \sigma \circ \rho_{\overline{u}}$. We have thus constructed an example of combinatorial species in the sense of Joyal, denoted by $PreFey$. $PreFey(\sigma)$ is the transport of structure along the bijection σ between the finite sets E and E'.

As the previous definition might be hard to digest if served dry, let us pause to explain the rationale and give an example. The "job" of a pre-Feynman diagram structure $\mathcal{E}$ on a set E is to encode an algebraic formula. We have so to speak defined a "programming language" with its syntactic rules (the constraints in Definition 3); a "program" in this language (a pre-Feynman diagram structure) serves to compute an element of the ring K. For example take $d = 5$ and $E = [16]$, with $E_{\overline{s}} = \{2\}$, $E_s = \{1\}$, $E_{\overline{t}} = \{5, 9\}$, $E_t = \{3, 4\}$, $E_{\overline{u}} = \{12, 13, 14, 15, 16\}$, $E_u = \{6, 7, 8, 10, 11\}$, $E_{int} = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$, $E_{ext} = \{1, 12, 13, 14, 15, 16\}$, $\pi_F = \{\{2, 3, 4\}\}$, $\pi_G = \{\{5, 6, 7, 8\}\{9, 10, 11\}\}$, $\rho_s : [1] \to E_s \cap E_{ext} = \{1\}$ (two a priori unrelated copies of the set $\{1\}$) is given by $\rho_s(1) = 1$, and finally let $\rho_{\overline{u}} : [d] \to E_{\overline{u}} \cap E_{ext} = \{12, 13, 14, 15, 16\}$ be given by $\rho_{\overline{u}}(1) = 12$, $\rho_{\overline{u}}(2) = 13$, $\rho_{\overline{u}}(3) = 14$, $\rho_{\overline{u}}(4) = 15$ and $\rho_{\overline{u}}(5) = 16$. The element of K computed by this data or "program" is

$$
\begin{aligned}
s_i \overline{u}_{j_1} \overline{u}_{j_2} \overline{u}_{j_3} \overline{u}_{j_4} \overline{u}_{j_5} \times & \left(\sum_{\alpha_2, \alpha_3, \alpha_4 = 1}^{n} \overline{s}_{\alpha_2} F^{[2]}_{\alpha_2, \alpha_3 \alpha_4} t_{\alpha_3} t_{\alpha_4} \right) \\
\times & \left(\sum_{\alpha_5, \alpha_6, \alpha_7, \alpha_8 = 1}^{n} \overline{t}_{\alpha_5} G^{[3]}_{\alpha_5, \alpha_6 \alpha_7 \alpha_8} u_{\alpha_6} u_{\alpha_7} u_{\alpha_8} \right) \\
\times & \left(\sum_{\alpha_9, \alpha_{10}, \alpha_{11} = 1}^{n} \overline{t}_{\alpha_9} G^{[2]}_{\alpha_9, \alpha_{10} \alpha_{11}} u_{\alpha_{10}} u_{\alpha_{11}} \right) . \qquad (59)
\end{aligned}
$$

A quick look at this expression will convince the reader that a much better way to represent it is by the following picture.



Remember that we have fixed in our discussion d, i and $j_1, \ldots, j_5$. This piece of information that is necessary to write and evaluate the expression (59) is not included in the data carried by our pre-Feynman diagram structure. The set E, in our example serves as an *abstract* set of labels for the indeterminates of type $\overline{s}$, s, $\overline{t}$, t, $\overline{u}$ and u that appear in (59) and are represented as oriented half-lines in the picture where we have indicated the labelling between parentheses. These indeterminates are called "fields" in the QFT terminology. The expression (59) is called the *amplitude* of the previous pre-Feynman diagram structure given the additional external structure i, $j_1, \ldots, j_5 \in [n]$. We now can introduce the notion of a Feynman diagram structure.

Definition 4 *A Feynman diagram structure on a finite set E is a quadruple $\mathcal{F} = (\mathcal{E}, \mathcal{C}_s, \mathcal{C}_t, \mathcal{C}_u)$ made of a pre-Feynman diagram structure*

$$\mathcal{E} = (E_{\overline{s}}, E_s, E_{\overline{t}}, E_t, E_{\overline{u}}, E_u, E_{int}, E_{ext}, \pi_F, \pi_G, \rho_s, \rho_{\overline{u}}) \quad , \tag{60}$$

and three bijective *maps $\mathcal{C}_s : E_{\overline{s}} \to E_s$, $\mathcal{C}_t : E_{\overline{t}} \to E_t$ and $\mathcal{C}_u : E_{\overline{u}} \to E_u$.*

The maps $\mathcal{C}_s$, $\mathcal{C}_t$, $\mathcal{C}_u$ are called contraction schemes *for the s-fields, t-fields and u-fields respectively.*

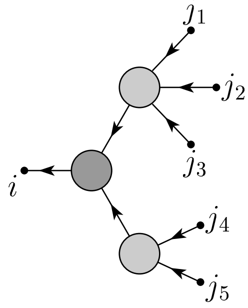
The set of Feynman diagram structures on E is denoted by $Fey(E)$. We are again defining a functor which is a combinatorial species. Indeed, if $\sigma : E \to E'$ is a bijective map, we define the transformed morphism $Fey(\sigma) : Fey(E) \to Fey(E')$ in the obvious manner by letting $Fey(\sigma)(\mathcal{E}, \mathcal{C}_s, \mathcal{C}_t, \mathcal{C}_u) = (\mathcal{E}', \mathcal{C}'_s, \mathcal{C}'_t, \mathcal{C}'_u)$ with $\mathcal{E}' = PreFey(\mathcal{E})$, $\mathcal{C}'_s = \sigma \circ \mathcal{C}_s \circ (\sigma^{-1})|_{E'_{\overline{s}}}$, $\mathcal{C}'_t = \sigma \circ \mathcal{C}_t \circ (\sigma^{-1})|_{E'_{\overline{t}}}$ and $\mathcal{C}'_u = \sigma \circ \mathcal{C}_u \circ (\sigma^{-1})|_{E'_{\overline{u}}}$.

Again the idea is to encode, thanks to such a structure, an algebraic expression whose value lies this time in the ground ring R instead of the formal power series ring K. For instance, if we take the previous example of pre-Feynman diagram structure and add to it the maps $\mathcal{C}_s$, $\mathcal{C}_t$, $\mathcal{C}_u$ given by $\mathcal{C}_s(2) = 1$, $\mathcal{C}_t(5) = 3$, $\mathcal{C}_t(9) = 4$, $\mathcal{C}_u(12) = 6$, $\mathcal{C}_u(13) = 7$, $\mathcal{C}_u(14) = 8$, $\mathcal{C}_u(15) = 10$ and $\mathcal{C}_u(16) = 11$; the resulting algebraic expression, also called the amplitude of this Feynman diagram structure, is

$$\sum_{\alpha_2,\ldots,\alpha_{11}=1}^{n} \delta_{i\alpha_2} F^{[2]}_{\alpha_2,\alpha_3\alpha_4} \delta_{\alpha_3\alpha_5} G^{[3]}_{\alpha_5,\alpha_6\alpha_7\alpha_8}$$

$$\delta_{\alpha_6 j_1} \delta_{\alpha_7 j_2} \delta_{\alpha_8 j_3} \delta_{\alpha_4\alpha_9} G^{[2]}_{\alpha_9,\alpha_{10}\alpha_{11}} \delta_{\alpha_{10} j_4} \delta_{\alpha_{11} j_5} \ ,$$

which belongs to R, and where δ_{ij} is simply Kronecker's symbol whose presence is due to our choice $A = B = C = I$ defining the covariance matrices. Again it does not take long to realize that a much better notation for this messy formula is



.

Note that this expression becomes, when varying the indices i, $j_1, \ldots, j_5$ in $[n]$, the collection of entries of a tensor in $R \otimes (R^*)^{\otimes 5}$ that is built using tensor contraction from three elementary tensors corresponding to some homogenous components of the series F and G. We have thus combinatorially translated a very natural "conceptual" construction of multilinear algebra. Note that this composite tensor has no reason to be symmetric in $j_1, \ldots, j_5$.

Having provided the definitions of pre-Feynman and Feynman diagram structures, and an example illustrating their meaning, we will now, for the sake of mathematical precision, give the formal definition of amplitudes.

Definition 5 *Let as before $\mathcal{E}$ be a pre-Feynman diagram structure on a finite set E, and suppose we are given two assignment maps $\tau_s : I \to [n]$ and $\tau_{\overline{u}} : J \to [n]$ with $I = [1]$, $J = [d]$. We call an* index attribution *any map $\alpha : E \to [n]$ such that $\alpha|_{E_{ext} \cap E_s} = \tau_s \circ \rho_s^{-1}$ and $\alpha|_{E_{ext} \cap E_{\overline{u}}} = \tau_{\overline{u}} \circ \rho_{\overline{u}}^{-1}$. Given such an index attribution map α and a block $B \in \pi_F$, if $B \cap E_{\overline{s}} = \{x\}$ and $B \cap E_t = \{y_1, \ldots, y_p\}$ with $p \geq 1$ we write*

$$F(B, \alpha) \stackrel{\text{def}}{=} F^{[p]}_{\alpha(x), \alpha(y_1) \ldots \alpha(y_p)} , \tag{61}$$

which does not depend on the chosen order of the elements in $B \cap E_t$. Likewise, if $B \in \pi_G$ is such that $B \cap E_{\overline{t}} = \{x\}$ and $B \cap E_u = \{y_1, \ldots, y_p\}$ with $p \geq 1$ we denote

$$G(B, \alpha) \stackrel{\text{def}}{=} G^{[p]}_{\alpha(x), \alpha(y_1) \ldots \alpha(y_p)} . \tag{62}$$

We now define the amplitude *of the pre-Feynman diagram structure $\mathcal{E}$ on E with respect to the assignment maps τ_s and $\tau_{\overline{u}}$ as*

$$\mathcal{A}_{PreFey}(E, \mathcal{E}, \tau_s, \tau_{\overline{u}}) \stackrel{\text{def}}{=} \sum_{\alpha} \left(\prod_{\overline{x} \in E_{\overline{s}}} \overline{s}_{\alpha(\overline{x})} \right) \left(\prod_{x \in E_s} s_{\alpha(x)} \right)$$

$$\left(\prod_{\overline{y}\in E_{\overline{t}}} \overline{t}_{\alpha(\overline{y})}\right)\left(\prod_{y\in E_t} t_{\alpha(y)}\right)\left(\prod_{\overline{z}\in E_{\overline{u}}} \overline{u}_{\alpha(\overline{z})}\right)\left(\prod_{z\in E_u} u_{\alpha(z)}\right)$$
$$\left(\prod_{B\in\pi_F} F(B,\alpha)\right)\left(\prod_{B\in\pi_G} G(B,\alpha)\right) \quad , \tag{63}$$

where the sum is over all index attribution maps α. $\mathcal{A}_{PreFey}(E,\mathcal{E},\tau_s,\tau_{\overline{u}})$ belongs to the ring $K = R[[\overline{s},s,\overline{t},t,\overline{u},u]]$.

Definition 6 *With the same notation as before, to a Feynman diagram structure $\mathcal{F}$ on E and two assignment maps τ_s and $\tau_{\overline{u}}$ we associate the corresponding* amplitude

$$\mathcal{A}_{Fey}(E,\mathcal{F},\tau_s,\tau_{\overline{u}}) \stackrel{\text{def}}{=} \sum_{\alpha}\left(\prod_{\overline{x}\in E_{\overline{s}}} \delta_{\alpha(\mathcal{C}_s(\overline{x}))\alpha(\overline{x})}\right)\left(\prod_{\overline{y}\in E_{\overline{t}}} \delta_{\alpha(\mathcal{C}_t(\overline{y}))\alpha(\overline{y})}\right)$$
$$\times\left(\prod_{\overline{z}\in E_{\overline{u}}} \delta_{\alpha(\mathcal{C}_u(\overline{z}))\alpha(\overline{z})}\right)\left(\prod_{B\in\pi_F} F(B,\alpha)\right)\left(\prod_{B\in\pi_G} G(B,\alpha)\right) \quad , \tag{64}$$

where again the sum is over all index attribution maps α compatible with the external structure provided by τ_s and $\tau_{\overline{u}}$, and δ_{ij} is the Kronecker symbol. Note that this time $\mathcal{A}_{Fey}(E,\mathcal{F},\tau_s,\tau_{\overline{u}})$ belongs to R.

The following important propositions are obvious from the previous definitions, and state the relabelling invariance of the amplitudes.

Proposition 1 *If E, E' are two finite sets equipped with pre-Feynman diagram structures $\mathcal{E}$ and $\mathcal{E}'$ respectively, such that there exists a bijection $\sigma : E \to E'$ that sends $\mathcal{E}$ on $\mathcal{E}'$ by $PreFey(\sigma)$, then*

$$\mathcal{A}_{PreFey}(E,\mathcal{E},\tau_s,\tau_{\overline{u}}) = \mathcal{A}_{PreFey}(E',\mathcal{E}',\tau_s,\tau_{\overline{u}}) \quad . \tag{65}$$

Proposition 2 *If E, E' are two finite sets equipped with Feynman diagram structures $\mathcal{F}$ and $\mathcal{F}'$ respectively, such that there exists a bijection $\sigma : E \to E'$ that sends $\mathcal{F}$ on $\mathcal{F}'$ by $Fey(\sigma)$, then*

$$\mathcal{A}_{Fey}(E,\mathcal{F},\tau_s,\tau_{\overline{u}}) = \mathcal{A}_{Fey}(E',\mathcal{F}',\tau_s,\tau_{\overline{u}}) \quad . \tag{66}$$

An important notion is that of *automorphism group* (the group of *ambiguity* we mentioned earlier) of pre-Feynman and Feynman diagram structures.

Definition 7 *The automorphism group of a pre-Feynman diagram structure $\mathcal{E}$ on E is the group $Aut(E,\mathcal{E})$ of all bijective maps $\sigma : E \to E'$ such that $PreFey(\sigma)$ leaves $\mathcal{E}$ unchanged.*

Proposition 3

$$
\begin{aligned}
&\#(Aut(E,\mathcal{E})) = \\
&\qquad \prod_{p\geq 1}(m_{F,p}!(p!)^{m_{F,p}}) \times \prod_{q\geq 1}(m_{G,q}!(q!)^{m_{G,q}}) \quad , \qquad (67)
\end{aligned}
$$

where for each integer $p \geq 1$, $m_{F,p}$ is the number of blocks $B \in \pi_F$ such that $\#(B \cap E_t) = p$ and for each integer $q \geq 1$, $m_{G,q}$ is the number of blocks $B \in \pi_G$ such that $\#(B \cap E_u) = q$.

Proof : A map $\sigma : E \to E$ that preserves the structure $\mathcal{E}$ is necessarily the identity on $E_{ext} = (E_{ext} \cap E_s) \cup (E_{ext} \cap E_{\overline{u}})$, since the injective maps ρ_s and $\rho_{\overline{u}}$ satisfy $\rho_s \circ \sigma = \rho_s$ on I and $\rho_{\overline{u}} \circ \sigma = \rho_{\overline{u}}$ on J. Besides σ must permute the blocks of π_F that contain the same number of elements from E_t, which accounts for the $m_{F,p}!$ factors. Likewise σ must permute the blocks of π_G that contain the same number of elements from E_u, which accounts for the $m_{G,q}!$ factors. Finally σ permutes the elements within each block of π_F and π_G, which gives the $p!$ and $q!$ factors. ■

Proposition 4 *With the notations of Theorem 2, let i, $j_1, \ldots, j_d$ be elements of $[n]$ that define assignment maps $\tau_s : I = [1] \to [n]$ by $\tau_s(1) = i$ and $\tau_{\overline{u}} : J = [d] \to [n]$ by $\tau_{\overline{u}}(\nu) = j_\nu$, for $1 \leq \nu \leq d$. The element*

$$
U = s_i \overline{u}_{j_1} \cdots \overline{u}_{j_d} \exp\left(\overline{s}F(t) + \overline{t}G(u)\right)
$$

of the ring K can be rewritten as

$$
U = \sum_{[E,\mathcal{E}]} \frac{\mathcal{A}_{preFey}(E,\mathcal{E},\tau_s,\tau_{\overline{u}})}{\#Aut(E,\mathcal{E})} \quad , \qquad (68)
$$

where the sum is over the isomorphism classes of pairs $(E,\mathcal{E})$ made of a finite set E and a pre-Feynman diagram structure $\mathcal{E}$ on E. The sets $I = [1]$, $J = [d]$ and the assignment maps $\tau_s : I \to [n]$ and $\tau_{\overline{u}} : J \to [n]$ are fixed throughout. Two pairs $(E,\mathcal{E})$ and $(E',\mathcal{E}')$ are said isomorphic if there exists a bijection $\sigma : E \to E'$ such that $PreFey(\sigma)$ sends $\mathcal{E}$ to $\mathcal{E}'$. In the sum (68) $(E,\mathcal{E})$ denotes any representative of the class $[E,\mathcal{E}]$.

Proof : If $m \geq 1$, we use the shorthand notation

$$
\overline{s}Ft^m \stackrel{\text{def}}{=} \sum_{\alpha,\beta_1,\ldots,\beta_m=1}^{n} \overline{s}_\alpha F^{[m]}_{\alpha,\beta_1\ldots\beta_m} t_{\beta_1} \ldots t_{\beta_m} \quad , \qquad (69)
$$

and

$$
\overline{t}Gu^m \stackrel{\text{def}}{=} \sum_{\alpha,\beta_1,\ldots,\beta_m=1}^{n} \overline{t}_\alpha G^{[m]}_{\alpha,\beta_1\ldots\beta_m} u_{\beta_1} \cdots u_{\beta_m} \quad . \qquad (70)
$$

Therefore in the formal power series ring K we have

$$U = s_i \overline{u}_{j_1} \cdots \overline{u}_{j_d} \exp\left(\sum_{p \geq 1} \frac{\overline{s} F t^p}{p!} + \sum_{q \geq 1} \frac{\overline{t} G u^q}{q!} \right) \tag{71}$$

$$= s_i \overline{u}_{j_1} \cdots \overline{u}_{j_d} \sum_{m_F, m_G \geq 0} \frac{1}{m_F! m_G!} \left(\sum_{p \geq 1} \frac{\overline{s} F t^p}{p!} \right)^{m_F} \left(\sum_{q \geq 1} \frac{\overline{t} G u^q}{q!} \right)^{m_G} , \tag{72}$$

which by the multinomial theorem becomes

$$U = \sum_{m_F, m_G \geq 0} \sum_{(m_{F,p})_{p \geq 1}} \sum_{(m_{G,q})_{q \geq 1}} s_i \overline{u}_{j_1} \cdots \overline{u}_{j_d}$$
$$\times \prod_{p \geq 1} \left(\frac{1}{m_{F,p}!} \left(\frac{\overline{s} F t^p}{p!} \right)^{m_{F,p}} \right) \times \prod_{q \geq 1} \left(\frac{1}{m_{G,q}!} \left(\frac{\overline{t} G u^q}{q!} \right)^{m_{G,q}} \right) , \tag{73}$$

where the sum is over all families $(m_{F,p})_{p \geq 1}$ and $(m_{G,q})_{q \geq 1}$ of nonnegative integers, necessarily of finite support, such that $\sum_{p \geq 1} m_{F,p} = m_F$ and $\sum_{q \geq 1} m_{G,q} = m_G$. One can then "remove the parentheses" in the above packet summation in K, since for each monomial in the variables $\overline{s}$, s, $\overline{t}$, t, $\overline{u}$ and u, only finitely many m_F's and m_G's contribute. Then

$$U = \sum_{(m_{F,p})_{p \geq 1}, (m_{G,q})_{q \geq 1}} s_i \overline{u}_{j_1} \cdots \overline{u}_{j_d}$$
$$\times \prod_{p \geq 1} \left(\frac{1}{m_{F,p}!} \left(\frac{\overline{s} F t^p}{p!} \right)^{m_{F,p}} \right) \times \prod_{q \geq 1} \left(\frac{1}{m_{G,q}!} \left(\frac{\overline{t} G u^q}{q!} \right)^{m_{G,q}} \right) , \tag{74}$$

where the sum is over all pairs $((m_{F,p})_{p \geq 1}, (m_{G,q})_{q \geq 1})$ of finitely supported families of nonnegative integers. Now notice that such pairs are in bijective correspondence with equivalence classes $[E, \mathcal{E}]$ of finite sets equipped with pre-Feynman diagram structures. The previous proposition does the rest. ∎

We now have, as an immediate consequence, the following.

Proposition 5 *Keeping the same notation, we have in the ring R*

$$\int d\overline{s} ds d\overline{t} dt d\overline{u} du \ e^{-\overline{s}s - \overline{t}t - \overline{u}u} \ U =$$
$$\sum_{[E,\mathcal{E}]} \sum_{(\mathcal{C}_s, \mathcal{C}_t, \mathcal{C}_u)} \frac{\mathcal{A}_{Fey}(E, (\mathcal{E}, \mathcal{C}_s, \mathcal{C}_t, \mathcal{C}_u), \tau_s, \tau_{\overline{u}})}{\# Aut(E, \mathcal{E})} , \tag{75}$$

where again one sums first over equivalence classes of pre-Feynman diagram structures, $(E, \mathcal{E})$ being an arbitrary representative of such a class. The $\mathcal{C}_s$, $\mathcal{C}_t$, $\mathcal{C}_u$ are *summed over contractions of the s, t and u fields respectively within the chosen representative* $(E, \mathcal{E})$. $\mathcal{F} = (\mathcal{E}, \mathcal{C}_s, \mathcal{C}_t, \mathcal{C}_u)$ *is then a Feynman diagram structure with underlying pre-Feynman structure* $\mathcal{E}$. *Also the sum on the right hand side is of finite support.*

Proof : From Definitions 1 and 2 it is clear that term by term, i.e., for each class $[E, \mathcal{E}]$ we have

$$\int d\overline{s}dsd\overline{t}dtd\overline{u}du\ e^{-\overline{s}s-\overline{t}t-\overline{u}u}\ \mathcal{A}_{PreFey}(E, \mathcal{E}, \tau_s, \tau_{\overline{u}}) = \sum_{(\mathcal{C}_s, \mathcal{C}_t, \mathcal{C}_u)} \mathcal{A}_{Fey}(E, (\mathcal{E}, \mathcal{C}_s, \mathcal{C}_t, \mathcal{C}_u), \tau_s, \tau_{\overline{u}}) \quad . \tag{76}$$

Proposition 4 being proven, all one has to check is that the sum over $[E, \mathcal{E}]$ is well-defined in R, i.e., has finite support. However this is an easy consequence of the tree structure of our Feynman diagrams, where the root is the unique element of E_s, the first generation of vertices corresponds to the blocks π_F, the second generation corresponds to those of π_G and the third generation, that is the set of leaves of the tree, is $E_{\overline{u}}$. An easy counting argument using the definition of our pre-Feynman and Feynman diagram structures shows that in order for a triple of bijections $(\mathcal{C}_s, \mathcal{C}_t, \mathcal{C}_u)$ to exist for a given pair $(E, \mathcal{E})$, one must have $\#(\pi_F) = 1$ and $\#(\pi_G) \le d$, because G has no constant term. This shows that finitely many classes $[E, \mathcal{E}]$ contribute. ∎

We now introduce as we did for pre-Feynman diagram structures, the following definition

Definition 8 *The automorphism group of a Feynman diagram structure* $\mathcal{F}$ *on* E *is the group* $Aut(E, \mathcal{F})$ *of all bijective maps* $\sigma : E \to E'$ *such that* $Fey(\sigma)$ *leaves* $\mathcal{F}$ *unchanged.*

We now have the following result

Theorem 3 *With the same notation as before*

$$\int d\overline{s}dsd\overline{t}dtd\overline{u}du\ e^{-\overline{s}s-\overline{t}t-\overline{u}u}\ U = \sum_{[E,\mathcal{F}]} \frac{\mathcal{A}_{Fey}(E, \mathcal{F}, \tau_s, \tau_{\overline{u}})}{\#Aut(E, \mathcal{F})} \quad , \tag{77}$$

where the sum is over equivalence classes of pairs $(E, \mathcal{F})$ *made of a finite set* E *and a Feynman diagram structure* $\mathcal{F}$ *on* E. *Again the sets* $I = [1]$, $J = [d]$ *and the index assignment maps* $\tau_s : I \to [n]$ *and* $\tau_{\overline{u}} : J \to [n]$ *are fixed. Two pairs* $(E, \mathcal{F})$ *and* $(E', \mathcal{F}')$ *are said equivalent if there exists a bijection* $\sigma : E \to E'$ *such that* $Fey(\sigma)(\mathcal{F}) = \mathcal{F}'$. *In the sum on the right hand side of (77),* $(E, \mathcal{F})$ *denotes any representative of the class* $[E, \mathcal{F}]$.

Proof : Starting from the previous proposition, one notices that any equivalence class of Feynman diagrams occurs in Equation (75), where one takes the equivalence class of $(E,(\mathcal{E},\mathcal{C}_s,\mathcal{C}_t,\mathcal{C}_u))$. One simply has to count how many times a given Feynman diagram class appears that way. If $(E,(\mathcal{E},\mathcal{C}_s,\mathcal{C}_t,\mathcal{C}_u))$ is equivalent to $(E',(\mathcal{E}',\mathcal{C}'_s,\mathcal{C}'_t,\mathcal{C}'_u))$, that means that there is a bijection $\sigma : E \rightarrow E'$ such that $PreFey(\sigma)(\mathcal{E}) = \mathcal{E}'$, $\mathcal{C}'_s = \sigma \circ \mathcal{C}_s \circ (\sigma^{-1})|_{E_{\overline{s}}}$, $\mathcal{C}'_t = \sigma \circ \mathcal{C}_t \circ (\sigma^{-1})|_{E_{\overline{t}}}$ and $\mathcal{C}'_u = \sigma \circ \mathcal{C}_u \circ (\sigma^{-1})|_{E_{\overline{u}}}$. But then $(E,\mathcal{E})$ is equivalent to $(E',\mathcal{E}')$ as pre-Feynman diagram structures, which forces $E = E'$ and $\mathcal{E} = \mathcal{E}'$, since in equation (75) one takes only one representative in each class of pre-Feynman diagram structures. Therefore the number of occurences in (75) of the class of $(E,\mathcal{F})$, with $\mathcal{F} = (\mathcal{E},\mathcal{C}_s,\mathcal{C}_t,\mathcal{C}_u)$, is equal to the number of contractions $(\mathcal{C}'_s,\mathcal{C}'_t,\mathcal{C}'_u)$ such that $(E,(\mathcal{E},\mathcal{C}'_s,\mathcal{C}'_t,\mathcal{C}'_u))$ is equivalent to $(E,(\mathcal{E},\mathcal{C}_s,\mathcal{C}_t,\mathcal{C}_u))$. In other words, one is counting the cardinality of the orbit of $(\mathcal{C}_s,\mathcal{C}_t,\mathcal{C}_u)$ under the left-action of $Aut(E,\mathcal{E})$ given by

$$\sigma.(\mathcal{C}_s,\mathcal{C}_t,\mathcal{C}_u) = \left(\sigma \circ \mathcal{C}_s \circ (\sigma^{-1})|_{E_{\overline{s}}},\ \sigma \circ \mathcal{C}_t \circ (\sigma^{-1})|_{E_{\overline{t}}},\ \sigma \circ \mathcal{C}_u \circ (\sigma^{-1})|_{E_{\overline{u}}}\right) \ . \tag{78}$$

It is equal to

$$\frac{\#Aut(E,\mathcal{E})}{\#Aut(E,\mathcal{F})} \ ;$$

since $Aut(E,\mathcal{F})$ is the isotropy subgroup of $(\mathcal{C}_s,\mathcal{C}_t,\mathcal{C}_u)$. Now Equation (77) follows immediately. ∎

Remark : Equation (77) shows that a formal Gaussian integral is analogous to a Hurwitz or exponential generating series (see [23]). Indeed it is easy to check that

$$\sum_{[E,\mathcal{F}]} \frac{\mathcal{A}_{Fey}(E,\mathcal{F},\tau_s,\tau_{\overline{u}})}{\#Aut(E,\mathcal{F})} = \sum_{k\geq 0} \sum_{\mathcal{F}\in Fey([k])} \frac{\mathcal{A}_{Fey}([k],\mathcal{F},\tau_s,\tau_{\overline{u}})}{k!} \ . \tag{79}$$

The classical Hurwitz series for the species of Feynman diagrams corresponds to replacing $\mathcal{A}_{Fey}(E,\mathcal{F},\tau_s,\tau_{\overline{u}})$ by the coarser "constant over the orbits" function $V^{\#(E)}$ for some indeterminate V.

We can now finally proceed to

Proof of Theorem 2 : Let $d \geq 1$ be fixed for the moment, and consider the polynomial in $R[X_1,\ldots,X_n]$

$$\begin{aligned} I_d = \sum_{\tau_{\overline{u}}} X_{\tau_{\overline{u}}(1)} \cdots X_{\tau_{\overline{u}}(d)} \int d\overline{s}ds d\overline{t}dt d\overline{u}du \ e^{-\overline{s}s-\overline{t}t-\overline{u}u} \\ s_i \overline{u}_{\tau_{\overline{u}}(1)} \cdots \overline{u}_{\tau_{\overline{u}}(d)} \exp\left(\overline{s}F(t) + \overline{t}G(u)\right) \ , \end{aligned} \tag{80}$$

where the sum is over all maps $\tau_{\overline{u}} : J = [d] \rightarrow [n]$. We have using Proposition 5, all sums being finite here,

$$I_d = \sum_{\tau_{\overline{u}}} X_{\tau_{\overline{u}}(1)} \cdots X_{\tau_{\overline{u}}(d)} \sum_{[E,\mathcal{E}]} \sum_{(\mathcal{C}_s,\mathcal{C}_t,\mathcal{C}_u)} \frac{\mathcal{A}_{Fey}(E,(\mathcal{E},\mathcal{C}_s,\mathcal{C}_t,\mathcal{C}_u),\tau_s,\tau_{\overline{u}})}{\#Aut(E,\mathcal{E})} \ . \tag{81}$$

Now notice that from the symmetry properties of tensor elements of F and G, the Definition 6 of amplitudes and the tree-like description of the relevent Feynman diagrams given in the proof of Proposition 5, it is easy to see that

$$\sum_{\tau_{\overline{u}}} \mathcal{A}_{Fey}\left(E, (\mathcal{E}, \mathcal{C}_s, \mathcal{C}_t, \mathcal{C}_u), \tau_s, \tau_{\overline{u}}\right) X_{\tau_{\overline{u}}(1)} \cdots X_{\tau_{\overline{u}}(d)} \tag{82}$$

only depends on $[E, \mathcal{E}]$, that is on $(m_{G,q})_{q\geq 1}$, the notation being the same as in Proposition 3. We used the fact that $m_{F,p}$ vanishes for all $p \geq 1$ except for $p = m$ where $m \stackrel{\text{def}}{=} \sum_{q\geq 1} m_{G,q}$. Besides one has $\sum_{q\geq 1} q.m_{G,q} = d$. We denote the sum (82) by $\Omega((m_{G,q})_{q\geq 1})$. By Proposition 3

$$\#Aut(E, \mathcal{E}) = m! \times \prod_{q\geq 1} \left(m_{G,q}!(q!)^{m_{G,q}}\right) \quad . \tag{83}$$

Now the number of triples $(\mathcal{C}_s, \mathcal{C}_t, \mathcal{C}_u)$ of contraction schemes is $1! \times m! \times d!$, therefore

$$\begin{aligned} I_d &= \sum_{\substack{(m_{G,q})_{q\geq 1} \\ \sum_{q\geq 1} q m_{G,q} = d}} \frac{1! \times m! \times d! \times \Omega((m_{G,q})_{q\geq 1})}{m! \times \prod_{q\geq 1} \left(m_{G,q}!(q!)^{m_{G,q}}\right)} && (84) \\ &= \sum_{\substack{(m_{G,q})_{q\geq 1} \\ \sum_{q\geq 1} q m_{G,q} = d}} \frac{d!\ \Omega((m_{G,q})_{q\geq 1})}{\prod_{q\geq 1} \left(m_{G,q}!(q!)^{m_{G,q}}\right)} \quad . && (85) \end{aligned}$$

Now for a given $m \geq 1$ and $\omega = (\omega_1, \ldots, \omega_m)$ with $\omega_i \geq 1$ for all i, $1 \leq i \leq m$ and $\omega_1 + \cdots + \omega_m = d$, we let $\mu(\omega) = (m_{G,q})_{q\geq 1}$ where $m_{G,q}$ counts the number of indices i, $1 \leq i \leq m$ with $\omega_i = q$. It is easy to see that

$$\Omega((m_{G,q})_{q\geq 1}) = \sum_{\alpha_1,\ldots,\alpha_m=1}^{n} F^{[m]}_{i,\alpha_1\ldots\alpha_m} \left(G_{\alpha_1} X^{\omega_1}\right) \cdots \left(G_{\alpha_m} X^{\omega_m}\right) \quad , \tag{86}$$

where we used the shorthand notation

$$G_i X^{\nu} \stackrel{\text{def}}{=} \sum_{j_1,\ldots,j_\nu=1}^{n} F^{[\nu]}_{i,j_1\ldots j_\nu} X_{j_1} \cdots X_{j_\nu} \quad . \tag{87}$$

Since the number of ω's for which $\mu(\omega)$ is equal to a given $(m_{G,q})_{q\geq 1}$ is by the multinomial theorem $\frac{m!}{\prod_{q\geq 1} m_{G,q}!}$, one has

$$I_d = \sum_{\substack{(m_{G,q})_{q\geq 1} \\ \sum_{q\geq 1} q m_{G,q} = d}} \frac{\prod_{q\geq 1} m_{G,q}!}{m!} \times \sum_{\substack{\omega \\ \mu(\omega) = (m_{G,q})_{q\geq 1}}} \frac{d!\ \Omega(\mu(\omega))}{\prod_{q\geq 1} \left(m_{G,q}!(q!)^{m_{G,q}}\right)} \quad , \tag{88}$$

or

$$I_d = d! \sum_{m\geq 1} \sum_{\substack{\omega \\ \omega_1+\cdots+\omega_m = d}} \frac{1}{m!} \times \sum_{\alpha_1,\ldots,\alpha_m=1}^{n} F^{[m]}_{i,\alpha_1\ldots\alpha_m} \left(G_{\alpha_1} X^{\omega_1}\right) \cdots \left(G_{\alpha_m} X^{\omega_m}\right) \quad , \tag{89}$$

and finally summing over $d \geq 1$, we have

$$\sum_{d\geq 1} \frac{I_d}{d!} = F_i(G(X)) \ \ ; \text{ since } \ G_i(X) = \sum_{\nu\geq 1} \frac{1}{\nu!} G_i X^\nu \ \ . \tag{90}$$

The only thing that remains to be checked to prove Theorem 2 is that the right hand side of equation (52) is symmetric with respect to the indices $j_1, \ldots, j_d$, which is obvious from Lemma 1 and Definitions 1 and 2. ■

III.2 Reversion

As in the beginning of Section II.2 we let $F = (F_i)_{1\leq i\leq n}$ be a system of n formal power series without constant term in $R[[X_1, \ldots, X_n]]$, given by

$$F_i(X) = \sum_{d\geq 1} \frac{1}{d!} \sum_{j_1,\ldots,j_d=1}^{n} F^{[d]}_{i,j_1\ldots j_d} X_{j_1} \cdots X_{j_d} \ \ . \tag{91}$$

We will in fact separate the *linear* part

$$L_i(X) \stackrel{\text{def}}{=} \sum_{j=1}^{n} F^{[1]}_{i,j} X_j \tag{92}$$

from the *nonlinear* part

$$H_i(X) \stackrel{\text{def}}{=} -\sum_{d\geq 2} \frac{1}{d!} \sum_{j_1,\ldots,j_d=1}^{n} F^{[d]}_{i,j_1\ldots j_d} X_{j_1} \cdots X_{j_d} \ \ , \tag{93}$$

so that $F_i(X) = L_i(X) - H_i(X)$. The linear part $L(u)$, which becomes quadratic after contraction with $\overline{u}$ in order to form $\overline{u}L(u) = \sum_{i=1}^{n} \overline{u}_i L_i(u)$ is called the *free* or *Gaussian* part in the physics literature. The remaining terms in the exponential in (14), that is $\overline{u}H(u) + \overline{u}Y$, form the *interaction* part.

Let $A \in \mathcal{M}_n(R)$ be the matrix with entries $A_{ij} \stackrel{\text{def}}{=} F^{[1]}_{i,j}$ for $1 \leq i, j \leq n$. We will suppose in this section that $A \in GL_n(R)$. This is a necessary and sufficient condition for $F = (F_i)_{1\leq i\leq n}$ to be invertible for composition of multivariable power series. Our aim here is to give a precise meaning and rigorous justification for Claim 3, giving a formula for the compositional inverse F^{-1}. We introduce the vectors $\overline{u} = (\overline{u}_1, \ldots, \overline{u}_n)$, $u = (u_1, \ldots, u_n)$ and $Y = (Y_1, \ldots, Y_n)$ of indeterminates, and will work only with the formal power series rings $R[[Y]]$ and $R[[\overline{u}, u, Y]]$.

Let k_1, k_2 be in $\mathbb{N}$, and let $\tau_1 : [k_1] \to [n]$ and $\tau_2 : [k_2] \to [n]$ be two given maps. We again define

$$I_A(\tau_1, \tau_2) \stackrel{\text{def}}{=} \sum_{\sigma} \prod_{1\leq k\leq k_1} [A^{-1}]_{\tau_2(\sigma(k))\tau_1(k)} \ \ , \tag{94}$$

where the sum is over all bijective maps $\sigma : [k_1] \to [k_2]$.

Definition 9 *Let $\alpha_1, \alpha_2 \in \mathbb{N}^n$, we define the* formal Gaussian integral *of the monomial $\overline{u}^{\alpha_1} u^{\alpha_2}$, with covariance A^{-1} as the element in R given by*

$$\int d\overline{u}du\ e^{-\overline{u}Au}\ \overline{u}^{\alpha_1} u^{\alpha_2} \stackrel{\text{def}}{=} (\det\ A)^{-1} I_A(\tau_1, \tau_2)\quad , \tag{95}$$

where each τ_i, for $i = 1, 2$, is any map from $[k_i]$ to $[n]$ with $k_i = |\alpha_i|$ and such that the associated multiplicity multiindex $\mu(\tau_i)$ is equal to α_i. Again this definition is independent of the choice of τ_1 and τ_2.

Definition 10 *If*

$$U = \sum_{\alpha_1,\alpha_2,\alpha_3 \in \mathbb{N}^n} \frac{u_{\alpha_1,\alpha_2,\alpha_3}}{\alpha_1!\alpha_2!\alpha_3!}\ \overline{u}^{\alpha_1} u^{\alpha_2} Y^{\alpha_3} \tag{96}$$

is an element of $R[[\overline{u}, u, Y]]$, we define the formal Gaussian integral *of U as the element in $R[[Y]]$ given by*

$$\int d\overline{u}du\ e^{-\overline{u}Au}\ U \stackrel{\text{def}}{=} \sum_{\alpha_1,\alpha_2,\alpha_3 \in \mathbb{N}^n} \frac{u_{\alpha_1,\alpha_2,\alpha_3} Y^{\alpha_3}}{\alpha_1!\alpha_2!\alpha_3!} \int d\overline{u}du\ e^{-\overline{u}Au}\ \overline{u}^{\alpha_1} u^{\alpha_2}\quad , \tag{97}$$

provided that the right hand side is summable in $R[[Y]]$ (i.e., for any $\alpha_3 \in \mathbb{N}^n$ there are only finitely many (α_1, α_2)'s giving a nonzero contribution).

Now the numerator in Claim 3 can be interpreted as the application of this definition in the case where

$$U = u_i \exp\left(\overline{u}H(u) + \overline{u}Y\right)\quad ; \tag{98}$$

while the denominator corresponds to the case

$$U = \exp\left(\overline{u}H(u) + \overline{u}Y\right)\quad . \tag{99}$$

All one has to do is to prove the summability in $R[[Y]]$. For this we again need the Feynman diagrammatic machinery. Again we need both the notions of pre-Feynman and Feynman diagrams. We let I and J be two fixed finite sets.

Definition 11 *A pre-Feynman diagram sructure of type (I, J) on a finite set E is an ordered collection*

$$\mathcal{E} = (E_{\overline{u}}, E_u, E_{int}, E_{ext}, \pi_H, \pi_Y, \rho_u, \rho_{\overline{u}}) \tag{100}$$

made of the following data.

- *$E_{\overline{u}}, E_u, E_{int}, E_{ext}$ are subsets of E.*
- *π_H, π_Y are (unordered) sets of subsets of E.*
- *ρ_u is a map from I to $E_{ext} \cap E_u$.*
- *$\rho_{\overline{u}}$ is a map from J to $E_{ext} \cap E_{\overline{u}}$.*

We also ask that the previous data satisfy the following constraints.

- *E is the disjoint union of E_u and $E_{\overline{u}}$.*
- *E is the disjoint union of E_{int} and E_{ext}.*
- *$\rho_u : I \to E_{ext} \cap E_u$ and $\rho_{\overline{u}} : J \to E_{ext} \cap E_{\overline{u}}$ are bijective.*
- *$\pi_H \cap \pi_Y = \emptyset$ and $\pi_H \cup \pi_Y$ forms a partition of E_{int}.*
- *For any block $B \in \pi_H$, also called an H-vertex, $\#(B \cap E_{\overline{u}}) = 1$ and $\#(B \cap E_u) \geq 2$*
- *For any block $B \in \pi_Y$, also called a Y-vertex or a leaf, $\#(B \cap E_{\overline{u}}) = 1$ and $B \cap E_u = \emptyset$.*

Definition 12 *A Feynman diagram structure of type (I, J) on a finite set E is a couple $(\mathcal{E}, \mathcal{C})$ made of a pre-Feynman diagram structure*

$$\mathcal{E} = (E_{\overline{u}}, E_u, E_{int}, E_{ext}, \pi_H, \pi_Y, \rho_u, \rho_{\overline{u}}) \tag{101}$$

of type (I, J) on E and a bijective map $\mathcal{C} : E_{\overline{u}} \to E_u$.

Transport of structure is defined in the same obvious manner as in Section III.1, which again provides us with two functors $PreFey$ and Fey which are combinatorial species in the sense of Joyal.

Definition 13 *Let $\mathcal{E}$ be a pre-Feynman diagram structure of type (I, J) on a finite set E, and suppose we are given two assignment maps $\tau_u : I \to [n]$ and $\tau_{\overline{u}} : J \to [n]$. We call an* index attribution *any map $\alpha : E \to [n]$ such that $\alpha|_{E_{ext} \cap E_u} = \tau_u \circ \rho_u^{-1}$ and $\alpha|_{E_{ext} \cap E_{\overline{u}}} = \tau_{\overline{u}} \circ \rho_{\overline{u}}^{-1}$. Given such an index attribution map α and a block $B \in \pi_H$, if $B \cap E_{\overline{u}} = \{\overline{x}\}$ and $B \cap E_u = \{y_1, \ldots, y_p\}$ with $p \geq 2$ we define*

$$H(B, \alpha) \stackrel{\text{def}}{=} H^{[p]}_{\alpha(\overline{x}), \alpha(y_1) \ldots \alpha(y_p)} = -F^{[p]}_{\alpha(\overline{x}), \alpha(y_1) \ldots \alpha(y_p)} \ , \tag{102}$$

which does not depend on the above enumeration of the elements of $B \cap E_u$. Likewise, if $B \in \pi_Y$ is such that $B = \{\overline{x}\}$, with $\overline{x} \in E_{\overline{u}}$, we write

$$Y(B, \alpha) \stackrel{\text{def}}{=} Y_{\alpha(\overline{x})} \quad . \tag{103}$$

We can now define the amplitude *of the pre-Feynman diagram structure $\mathcal{E}$ on E with respect to the assignment maps τ_u and $\tau_{\overline{u}}$ as*

$$\mathcal{A}_{PreFey}(E, \mathcal{E}, \tau_u, \tau_{\overline{u}}) \stackrel{\text{def}}{=} \sum_{\alpha} \left(\prod_{\overline{x} \in E_{\overline{u}}} \overline{u}_{\alpha(\overline{x})} \right) \left(\prod_{x \in E_u} u_{\alpha(x)} \right) \times \left(\prod_{B \in \pi_H} H(B, \alpha) \right) \left(\prod_{B \in \pi_Y} Y(B, \alpha) \right) \ , \tag{104}$$

which belongs to $R[[\overline{u}, u, Y]]$. Again the sum is over all index attribution maps α.

Definition 14 *With the same notation as in the previous definition, to a Feynman diagram structure $\mathcal{F}$ of type (I,J) on E and two assignment maps τ_u and $\tau_{\overline{u}}$ we associate the corresponding* amplitude

$$\mathcal{A}_{Fey}(E,\mathcal{F},\tau_u,\tau_{\overline{u}}) \stackrel{\text{def}}{=} \sum_{\alpha}\left(\prod_{\overline{x}\in E_{\overline{u}}}(A^{-1})_{\alpha(\mathcal{C}_s(\overline{x}))\alpha(\overline{x})}\right)$$
$$\times\left(\prod_{B\in\pi_H} H(B,\alpha)\right)\left(\prod_{B\in\pi_Y} Y(B,\alpha)\right) \quad , \tag{105}$$

where $(A^{-1})_{ij}$ denotes the entries of the covariance matrix $A^{-1}\in GL_n(R)$. The amplitude $\mathcal{A}_{Fey}(E,\mathcal{F},\tau_u,\tau_{\overline{u}})$ is an element in $R[[Y]]$.

Again these amplitudes are obviously invariant by relabelling or transport of structure. One defines as in Section III.1 the notions of automorphism groups of pairs $(E,\mathcal{E})$ and $(E,\mathcal{F})$ with $\mathcal{E}$ a pre-Feynman diagram structure and $\mathcal{F}$ a Feynman diagram structure on E. The following proposition is proved like its sibling from Section III.1.

Proposition 6 *If $\mathcal{E}$ is pre-Feynman diagram structure on E,*

$$\#Aut(E,\mathcal{E}) = \prod_{p\ge 1}\left(m_{H,p}!(p!)^{m_{H,p}}\right) \ \times\ m_Y! \quad , \tag{106}$$

where for each $p\ge 2$, $m_{H,p}$ counts the blocks $B\in\pi_H$ such that $\#(B\cap E_u)=p$ and $m_Y=\#(\pi_Y)$.

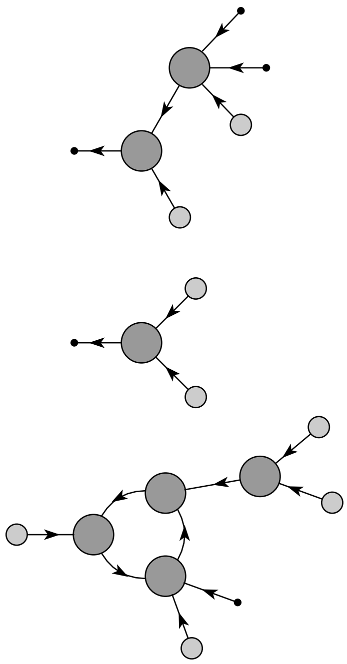
One can also derive by the same arguments as in Proposition 4 the following result.

Proposition 7 *Given two finite sets I and J and two index assignment maps τ_u and $\tau_{\overline{u}}$ one has in the ring $R[[\overline{u},u,Y]]$*

$$\left(\prod_{i\in I} u_{\tau_u(i)}\right)\left(\prod_{j\in J}\overline{u}_{\tau_{\overline{u}}(j)}\right)\exp\left(\overline{u}H(u)+\overline{u}Y\right) = \sum_{[E,\mathcal{E}]}\frac{\mathcal{A}_{preFey}(E,\mathcal{E},\tau_u,\tau_{\overline{u}})}{\#Aut(E,\mathcal{E})} \quad . \tag{107}$$

Before we state the analog of Proposition 5, and in order to take care of issues of summability we have to analyse more closely the Feynman diagram structure appearing here. Given such a structure $\mathcal{F}$ of type (I,J) on E, we can associate to it an ordinary digraph G on the set $\tilde{E}$ defined as the disjoint union of $\tilde{E}_u$ the set of one-element subsets of $E_{ext}\cap E_u$, $\tilde{E_{\overline{u}}}$ the set of one-element subsets of $E_{ext}\cap E_{\overline{u}}$, $\tilde{E}_H=\pi_H$ and $\tilde{E}_Y=\pi_Y$. Therefore $\tilde{E}$ is a partition of E. Now G is the set of ordered pairs (a,b), with $a,b\in\tilde{E}$, such that there exist $x\in a\cap E_{\overline{u}}$ and $y\in b\cap E_u$ such that

$y = \mathcal{C}(x)$. If the link (a, b) is in G we call a its origin and b its end. For example for the Feynman diagram represented by the following picture.



(108)

I corresponds to the 2 half-lines

$$\bullet\!\!\leftarrow$$

called the u-sources; and J corresponds to the 3 half-lines

$$\leftarrow\!\!\bullet$$

called the $\overline{u}$-sources. E is the set of all half-lines and has $2 \times 18 = 36$ elements. We also have $\#(\tilde{E}_H) = 7$, $\#(\tilde{E}_Y) = 8$, $\#(\tilde{E}) = 2 + 7 + 8 + 3 = 20$, and $\#(G) = 18$. It is a simple but tedious matter of going through the previous definitions to verify that the only possible connected components of the digraph G on $\tilde{E}$ are of two types.

Tree-like : A tree where all the links are oriented towards the root that has to be the unique element of $\tilde{E}_u$ in the component. The leaves are either Y-vertices, (elements of $\tilde{E}_Y$) or $\overline{u}$-sources (elements of $\tilde{E}_{\overline{u}}$). The internal vertices of the tree are all H-vertices, i.e., elements of $\tilde{E}_H$, and *have at least two* offsprings. This crucial property is because H has been defined as the *nonlinear* part of $-F$.

Unicyclic : A graph with a unique central oriented circuit of H-vertices on which trees like above are hooked. The latter are oriented towards the circuit, and their leaves are either Y-vertices of $\overline{u}$-sources. Such a graph contains no element of $\tilde{E}_u$.

Remark : Note the analogy with the combinatorial species of endofunctions, which live here on the "functorially" derived *abstract* set $\tilde{E}$. No reference is made to the *concrete* set of indices $[n]$, or to the dimensionality n of the problem, which only appear in the calculation of amplitudes. The need of varying n, in order to realize the manifold $\mathcal{B}$ mentioned in the introduction as an "inductive limit" of finite sets (and thus the set of maps $\mathcal{B} \to \mathcal{T}$ as a "projective limit"), makes the use of Feynman diagrams almost inescapable in QFT.

An easy consequence of the preceding analysis of our Feynman diagram structures, obtained by counting the half-lines and using the fact that the H-vertices have valence at least 3, is the following.

Lemma 2 *A tree-like connected Feynman diagram, which is then necessarily of type (I, J) with $\#(I) = 1$, satisfies*

$$\#(\tilde{E}) \leq 2l \ , \tag{109}$$

where l is the total number of leaves $l \stackrel{\text{def}}{=} \#(\tilde{E}_Y) + \#(\tilde{E}_{\overline{u}})$.

From which one deduces, by adding the above inequalities obtained for each tree growing off the central circuit, the next result.

Lemma 3 *A unicyclic connected Feynman diagram, which is then necessarily of type (I, J) with $I = \emptyset$, satisfies also*

$$\#(\tilde{E}) \leq 2\left(\#(\tilde{E}_Y) + \#(\tilde{E}_{\overline{u}})\right) \ . \tag{110}$$

Finally by adding the inequalities for each connected component one obtains the following.

Lemma 4 *Any Feynman diagram, of arbitrary type (I, J), also satisfies*

$$\#(E) \leq \#(\tilde{E}) \leq 2\left(\#(\tilde{E}_Y) + \#(\tilde{E}_{\overline{u}})\right) = 2\left(\#(\pi_Y) + \#(J)\right) \ . \tag{111}$$

Although quite trivial the above lemmas are crucial in order to ensure that the grading, with respect to which the topology of the ring $R[[Y]]$ is defined, and which is related to Y-vertices only, grows with the complexity of the Feynman diagram. This observation securing the summability and the same argument as in Proposition 5 now entail the following.

Proposition 8 *Let the finite sets I and J and the assignment maps τ_u and $\tau_{\overline{u}}$ be given. Let U be the element of $R[[\overline{u}, u, Y]]$ given by*

$$U = \left(\prod_{i \in I} u_{\tau_u(i)}\right)\left(\prod_{j \in J} \overline{u}_{\tau_{\overline{u}}(j)}\right) \exp\left(\overline{u}H(u) + \overline{u}Y\right) \quad ; \tag{112}$$

then the following identity holds in $R[[Y]]$, both sides being summable

$$\int d\overline{u}du \ e^{-\overline{u}Au} \ U = (\det \ A)^{-1} \sum_{[E,\mathcal{E}]} \sum_{\mathcal{C}} \frac{\mathcal{A}_{Fey}\left(E, (\mathcal{E}, \mathcal{C}), \tau_u, \tau_{\overline{u}}\right)}{\#Aut(E, \mathcal{E})} \quad , \tag{113}$$

where the sum is over equivalence classes of pre-Feynman diagram structures of type (I, J)*,* $(E, \mathcal{E})$ *being an arbitrary representative of such a class.* $\mathcal{C}$ *is summed over contraction schemes i.e., bijective maps* $\mathcal{C} : E_{\overline{u}} \to E_u$*.*

By the same proof as that of Theorem 3, one now arrives at the next result.

Theorem 4 *With the same hypothesis as in the previous proposition one has, both sides being summable in* $R[[Y]]$*,*

$$\int d\overline{u}du\ e^{-\overline{u}Au}\ U = (\det\ A)^{-1} \sum_{[E,\mathcal{F}]} \frac{\mathcal{A}_{Fey}(E, \mathcal{F}, \tau_u, \tau_{\overline{u}})}{\#Aut(E, \mathcal{F})} \quad , \tag{114}$$

where the sum is over equivalence classes of Feynman diagram structures of type (I, J)*, and* $(E, \mathcal{F})$ *denotes an arbitrary class representative.*

We have now completely defined, in a mathematically precise fashion, the numerator and the denominator that appear in Claim 3. They correspond to the situation where $(I, J) = ([1], \emptyset)$ with $\tau_u(1) = i$, and the situation where $(I, J) = (\emptyset, \emptyset)$, respectively. The Feynman diagrams in the former situation can be called, according to physical terminology, *1-point* diagrams. In the latter situation they would rather be called *vacuum* diagrams. Before we end this section, we still have to prove the following precise restatement of Claim 3.

Theorem 5 *The compositional inverse of* $F = (F_i)_{1\le i\le n}$ *satisfies in the ring* $R[[Y]]$ *the equation*

$$(F^{-1})_i(Y) = \frac{\int d\overline{u}du\ e^{-\overline{u}Au}\ u_i e^{\overline{u}H(u)+\overline{u}Y}}{\int d\overline{u}du\ e^{-\overline{u}Au}\ e^{\overline{u}H(u)+\overline{u}Y}} \quad , \tag{115}$$

the denominator being invertible in $R[[Y]]$*.*

We will use the standard statistical mechanics notation $< . >$ for averages and introduce, given the finite sets I and J and their associated assignment maps τ_u and $\tau_{\overline{u}}$, the *unnormalized correlation function*

$$< \left(\prod_{i\in I} u_{\tau_u(i)}\right) \left(\prod_{j\in J} \overline{u}_{\tau_{\overline{u}}(j)}\right) >_U \stackrel{\text{def}}{=}$$

$$(\det\ A) \int d\overline{u}du\ e^{-\overline{u}Au} \left(\prod_{i\in I} u_{\tau_u(i)}\right) \left(\prod_{j\in J} \overline{u}_{\tau_{\overline{u}}(j)}\right) e^{\overline{u}H(u)+\overline{u}Y} \quad . \tag{116}$$

Note that

$$\det\ A = \frac{1}{\int d\overline{u}du\ e^{-\overline{u}Au}\ 1} \tag{117}$$

represents the normalization by its total weight (in order to have a probability measure) of the "Gaussian measure" $d\overline{u}du\ e^{-\overline{u}Au}$. It is not the full "interacting measure" $d\overline{u}du\ e^{-\overline{u}Au+\overline{u}H(u)+\overline{u}Y}$, hence the word "unnormalized". The corresponding *normalized correlation function* is rather

$$<\left(\prod_{i\in I} u_{\tau_u(i)}\right)\left(\prod_{j\in J} \overline{u}_{\tau_{\overline{u}}(j)}\right) >_N \stackrel{\text{def}}{=} \frac{1}{Z} < \left(\prod_{i\in I} u_{\tau_u(i)}\right)\left(\prod_{j\in J} \overline{u}_{\tau_{\overline{u}}(j)}\right) >_U \quad , \tag{118}$$

where the Z is the *partition function* defined by

$$Z \stackrel{\text{def}}{=} < 1 >_U = (\det\ A) \int d\overline{u}du\ e^{-\overline{u}Au}\ e^{\overline{u}H(u)+\overline{u}Y} \quad . \tag{119}$$

It is given by Theorem 4 as a sum over classes of, not necessarily connected, vacuum (i.e., of type $(\emptyset,\emptyset)$) Feynman diagram structures

$$Z = \sum_{\substack{[E,\mathcal{F}] \\ \text{type } (\emptyset,\emptyset)}} \frac{\mathcal{A}_{Fey}(E,\mathcal{F},\tau_u,\tau_{\overline{u}})}{\#Aut(E,\mathcal{F})} \quad . \tag{120}$$

The constant term of Z is given by the contribution of the trivial diagram corresponding to $E=\emptyset$, and is equal to 1 (one can check that our definitions also hold in this degenerate case). As a result, Z i.e., the denominator in Theorem 5 is invertible in $R[[Y]]$. One can also define the subspecies of nontrivial connected vacuum Feynman diagrams $\mathcal{F}$ on a set E by adding to Definition 12, in the case where $I=J=\emptyset$, the condition that $E\neq\emptyset$ and that the associated set $\tilde{E}$ and digraph G are such that G connects $\tilde{E}$. One can then define the *free energy*

$$W \stackrel{\text{def}}{=} \sum_{\substack{[E,\mathcal{F}] \text{ type } (\emptyset,\emptyset) \\ \text{connected } E\neq\emptyset}} \frac{\mathcal{A}_{Fey}(E,\mathcal{F})}{\#Aut(E,\mathcal{F})} \ , \tag{121}$$

which is summable in $R[[Y]]$, as a part of the sum for Z, which is already known to be summable. One can also prove this directly using Lemma 3. Note that there is no longer a need to specify the maps τ_u and $\tau_{\overline{u}}$ whose graphs are empty. Note also that the diagrams appearing in the last equation are each made of a single nonempty unicyclic connected component whose leaves are all Y-vertices. Now it is easy to check the following identity.

Proposition 9

$$Z = \exp(W) \quad . \tag{122}$$

Similar statements for Hurwitz or exponential generating series are quite familiar in combinatorial theory. It boils down to the use of the multinomial theorem, the invariance of amplitudes by relabelling and, most importantly here, their *factorization* over connected components.

In fact, for any fixed type (I,J) one can define in an analogous way, the subspecies of connected Feynman diagram structures of type (I,J), by requiring that the digraph G connects the derived set $\tilde{E}$. This allows, again given the assignment maps τ_u and $\tau_{\overline{u}}$, to define the *connected correlation functions*

$$< \left(\prod_{i\in I} u_{\tau_u(i)}\right)\left(\prod_{j\in J} \overline{u}_{\tau_{\overline{u}}(j)}\right) >_C \stackrel{\rm def}{=} \sum_{\substack{[E,\mathcal{F}]\ \text{type}\ (I,J)\\ \text{connected}}} \frac{\mathcal{A}_{Fey}(E,\mathcal{F},\tau_u,\tau_{\overline{u}})}{\#Aut(E,\mathcal{F})} \quad . \tag{123}$$

These can also be called *cumulants* or *semi-invariants* in conformity with the terminology of mathematical statistics and probability theory. They are also related to the so-called *Ursell functions* in statistical mechanics. Indeed, one has the following result.

Theorem 6

$$< \left(\prod_{i\in I} u_{\tau_u(i)}\right)\left(\prod_{j\in J} \overline{u}_{\tau_{\overline{u}}(j)}\right) >_U =$$

$$Z \times \sum_{\pi} \prod_{(\tilde{I},\tilde{J})\in\pi} < \left(\prod_{i\in \tilde{I}} u_{\tau_u|_{\tilde{I}}(i)}\right)\left(\prod_{j\in \tilde{J}} \overline{u}_{\tau_{\overline{u}}|_{\tilde{J}}(j)}\right) >_C \quad , \tag{124}$$

where the sum is over all (unordered) sets π of pairs $(\tilde{I},\tilde{J})$ such that $\tilde{I}$ and $\tilde{J}$ are not simultaneously empty subsets of I and J respectively, and such that

$$\pi_I \stackrel{\rm def}{=} \{\tilde{I}\subset I \mid \tilde{I}\neq\emptyset \text{ and } \exists \tilde{J}\subset J, (\tilde{I},\tilde{J})\in\pi\} \quad , \tag{125}$$

and

$$\pi_J \stackrel{\rm def}{=} \{\tilde{J}\subset J \mid \tilde{J}\neq\emptyset \text{ and } \exists \tilde{I}\subset I, (\tilde{I},\tilde{J})\in\pi\} \tag{126}$$

are partitions of I and J respectively.

Proof : One starts from the expression given by Theorem 4 for the unnormalized correlation function

$$< \left(\prod_{i\in I} u_{\tau_u(i)}\right)\left(\prod_{j\in J} \overline{u}_{\tau_{\overline{u}}(j)}\right) >_U$$

as a sum over classes $[E,\mathcal{F}]$ of corresponding Feynman diagrams. Given such a diagram, one divides E according to the connected components of $\tilde{E}$ that are determined by the digraph G. We let $E_Z\subset E$ be the union of vacuum connected components (i.e., those which do not intersect the images of ρ_u and $\rho_{\overline{u}}$). For any set of labels $F\subset E$ corresponding to a connected component which does intersect $\rho_u(I)$ and $\rho_{\overline{u}}(J)$, we consider $I_F \stackrel{\rm def}{=} \rho_u^{-1}(\rho_u(I)\cap F)$ and $J_F \stackrel{\rm def}{=} \rho_{\overline{u}}^{-1}(\rho_{\overline{u}}(J)\cap F)$ and we let π be the set of pairs (I_F,J_F) obtained in this way. The set π satisfies the conditions stated in the theorem. For each $(\tilde{I},\tilde{J})\in\pi$, we let $E_{(\tilde{I},\tilde{J})}$ be the unique component F of E such that $\tilde{I}=I_F$ and $\tilde{J}=J_F$. One then canonically deduces from the Feynman diagram structure $\mathcal{F}$ of type (I,J) on E an induced connected diagram structure

$\mathcal{F}_{(\tilde{I},\tilde{J})}$ of type $(\tilde{I},\tilde{J})$ on $E_{(\tilde{I},\tilde{J})}$. One also obtains in the same obvious manner a (not necessarily connected) Feynman diagram structure $\mathcal{F}_Z$ of type $(\emptyset,\emptyset)$ on E_Z. The index assignment maps for a pair $(\tilde{I},\tilde{J}) \in \pi$ are defined from τ_u and $\tau_{\overline{u}}$ by restriction from I to $\tilde{I}$ and from J to $\tilde{J}$ respectively. All one has to do in proving the equality (124) is to notice that one can replace the global sum over $[E,\mathcal{F}]$ by the sum over the set π and independent sums on the classes $[E_{(\tilde{I},\tilde{J})},\mathcal{F}_{(\tilde{I},\tilde{J})}]$ for $(\tilde{I},\tilde{J}) \in \pi$ and the class $[E_Z,\mathcal{F}_Z]$, the amplitudes being factorized over connected components and also the symmetry factors. Indeed one has a canonical group isomorphism

$$Aut(E,\mathcal{F}) \simeq Aut(E_Z,\mathcal{F}_Z) \times \prod_{(\tilde{I},\tilde{J})\in\pi} Aut\left(E_{(\tilde{I},\tilde{J})},\mathcal{F}_{(\tilde{I},\tilde{J})}\right) \quad . \tag{127}$$

■

An immediate consequence is the next result.

Corollary 1 :

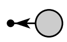
$$\frac{\int d\overline{u}du\ e^{-\overline{u}Au}\ u_i e^{\overline{u}H(u)+\overline{u}Y}}{\int d\overline{u}du\ e^{-\overline{u}Au}\ e^{\overline{u}H(u)+\overline{u}Y}} =< u_i >_C \ , \tag{128}$$

which is a sum over connected Feynman diagrams of type $([1],\emptyset)$, which must be tree-like with leaves exclusively made of Y-vertices.

Proof of Theorem 5 : One starts from

$$< u_i >_C = \sum_{\substack{[E,\mathcal{F}]\ \text{type}\ ([1],\emptyset) \\ \text{connected}}} \frac{\mathcal{A}_{Fey}(E,\mathcal{F},\tau_u,\tau_{\overline{u}})}{\#Aut(E,\mathcal{F})} \quad , \tag{129}$$

with $\tau_u(1) = i$ and $\tau_{\overline{u}}$ empty. In the above sum one distinguishes the simplest term corresponding to the diagram class



(130)

for which $\#(E) = 2$, $\#Aut(E,\mathcal{F}) = 1$ and the amplitude is given by

$$\mathcal{A}_{Fey}(E,\mathcal{F},\tau_u,\tau_{\overline{u}}) = \sum_{j=1}^{n} (A^{-1})_{ij} Y_j \quad . \tag{131}$$

It corresponds to the linear term of the formal inverse $(F^{-1})_i(Y)$. Let Γ denote the sum of the remaining terms for which $\pi_H \neq \emptyset$. For such a term, there is a distinguished H-vertex $B_0 \in \pi_H$, which is closest to the root in $E_u \cap E_{ext}$, and with $p \geq 2$ attached tree-like connected Feynman diagram structures of type $([1],\emptyset)$, which we denote by $(E_1,\mathcal{F}_1),\ldots,(E_p,\mathcal{F}_p)$. Let $\mathcal{C}$ be the set of isomorphism classes $[E,\mathcal{F}]$ of connected Feynman diagram structures of type $([1],\emptyset)$. There is a bijective correspondence between classes $[E,\mathcal{F}]$ appearing in Γ and finitely supported families $(m_c)_{c\in\mathcal{C}}$ of integers $m_c\in\mathbb{N}$ such that $\sum_{c\in\mathcal{C}} m_c\ge 2$, defined by letting m_c count the number of indices q such that $(E_q,\mathcal{F}_q)$ belongs to the class c. Moreover, the cardinality of $Aut(E,\mathcal{F})$ is completely determined by $(m_c)_{c\in\mathcal{C}}$. So is the amplitude of $(E,\mathcal{F})$ which we denote then by $\mathcal{A}((m_c)_{c\in\mathcal{C}})$. One trivially has

$$\#Aut(E,\mathcal{F})=\left(\prod_{c\in\mathcal{C}}\#Aut(c)^{m_c}\right)\left(\prod_{c\in\mathcal{C}}m_c!\right)\quad ; \tag{132}$$

since an isomorphism of the big tree $(E,\mathcal{F})$ can operate inside each of the branches $(E_1,\mathcal{F}_1),\ldots,(E_p,\mathcal{F}_p)$ and can also exchange isomorphic branches. One can therefore write

$$\Gamma=\sum_{p\ge2}\sum_{\substack{(m_c)_{c\in\mathcal{C}}\\ \sum_{c\in\mathcal{C}}m_c=p}}\frac{\mathcal{A}\left((m_c)_{c\in\mathcal{C}}\right)}{\left(\prod_{c\in\mathcal{C}}\#Aut(c)^{m_c}\right)\left(\prod_{c\in\mathcal{C}}m_c!\right)}\ , \tag{133}$$

which by the multinomial theorem amounts to the same thing as summing over sequences $([E_1,\mathcal{F}_1],\ldots,[E_p,\mathcal{F}_p])$ of elements of $\mathcal{C}$. Thus

$$\Gamma=\sum_{p\ge2}\frac{1}{p!}\sum_{([E_1,\mathcal{F}_1],\ldots,[E_p,\mathcal{F}_p])}\frac{\mathcal{A}\left((m_c)_{c\in\mathcal{C}}\right)}{\left(\prod_{c\in\mathcal{C}}\#Aut(c)^{m_c}\right)}\quad , \tag{134}$$

where $(m_c)_{c\in\mathcal{C}}$ is the family of multiplicities defined $([E_1,\mathcal{F}_1],\ldots,[E_p,\mathcal{F}_p])$. This expression becomes

$$\Gamma=\sum_{p\ge2}\frac{1}{p!}\sum_{([E_1,\mathcal{F}_1],\ldots,[E_p,\mathcal{F}_p])}\sum_{j,j_1,\ldots,j_p=1}^{n}(A^{-1})_{ij}H^{[p]}_{j,j_1\ldots j_p}\prod_{q=1}^{p}\left(\frac{\mathcal{A}(E_q,\mathcal{F}_q,j_q)}{\#Aut(E_q,\mathcal{F}_q)}\right)\quad , \tag{135}$$

where the index j_q defines the τ_u assignment map for the subdiagram $(E_q,\mathcal{F}_q)$ also of type $([1],\emptyset)$. Noting that by definition $H^{[p]}_{j,j_1\ldots j_p}=-F^{[p]}_{j,j_1\ldots j_p}$ and using (123), the previous expression recombines into

$$\Gamma=-\sum_{p\ge2}\frac{1}{p!}\sum_{j,j_1,\ldots,j_p=1}^{n}(A^{-1})_{ij}F^{[p]}_{j,j_1\ldots j_p}<u_{j_1}>_C\cdots<u_{j_p}>_C\quad . \tag{136}$$

Therefore

$$<u_i>_C=\sum_{j=1}^{n}(A^{-1})_{ij}Y_j-\sum_{p\ge2}\frac{1}{p!}\sum_{j,j_1,\ldots,j_p=1}^{n}(A^{-1})_{ij}F^{[p]}_{j,j_1\ldots j_p}<u_{j_1}>_C\cdots<u_{j_p}>_C\quad . \tag{137}$$

Multiplying on the left by the matrix $A=(F^{[1]}_{i,j})_{1\le i,j\le n}$ and transposing the sum over p gives

$$F_i(<u>_C)=Y_i\ , \tag{138}$$

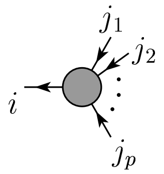
which shows that $<u_i>_C\in R[[Y]]$ is the i-th component of the right compositional inverse, that is simply the inverse, of $F=(F_i)_{1\le i\le n}$, which concludes our proof. ■

III.3 Lagrange-Good inversion

In order to avoid lengthy repetitions of the previous arguments, we will be rather brief in this section, and only detail the new ingredients needed. We work in the ring $R[[X_1,\ldots,X_n]]$. We suppose that we have n power series $(G_i)_{1\le i\le n}$ in n variables defined by their tensor elements $G^{[p]}_{i,j_1\ldots j_p}$ with $p\ge 0$. We define as before the unnormalized correlation functions

$$< \left(\prod_{i\in I} u_{\tau_u(i)}\right)\left(\prod_{j\in J}\overline{u}_{\tau_{\overline{u}}(j)}\right) >_U = \int d\overline{u}du\ e^{-\overline{u}u}\left(\prod_{i\in I} u_{\tau_u(i)}\right)\left(\prod_{j\in J}\overline{u}_{\tau_{\overline{u}}(j)}\right) e^{\overline{u}XG(u)} \tag{139}$$

in the ring $R[[X_1,\ldots,X_n]]$, by extending formal Gaussian integration with the identity matrix as a covariance, from monomials in the u's and $\overline{u}$'s to elements of $R[[\overline{u},u,X]]$, whenever the summation (over the multiindices defining the monomials) converges in $R[[X]]$. One needs almost the same definitions of pre-Feynman and Feynman diagram structures as in Section III.2, except that one has only one type of vertices, which we call XG-vertices, corresponding to a partition π_{XG} of E_{int}. A block $B\in\pi_{XG}$ must have exactly one element in $E_{\overline{u}}$, but any number of elements of E_u is allowed this time, even zero (corresponding to the tree leaves). The contribution of such an XG-vertex in the amplitude of a Feynman diagram is



$$= X_i G^{[p]}_{i,j_1\ldots j_p} \quad . \tag{140}$$

A contraction line corresponds to a factor

$$i \longleftarrow j = \delta_{ij} \tag{141}$$

in the amplitude of a Feynman graph. Apart from this small difference, the treatment is exactly the same as in Section III.2. One has an analog of Corollary 1.

Proposition 10

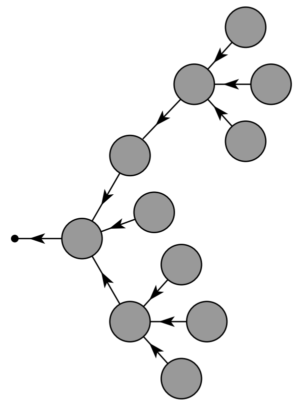
$$\frac{\int d\overline{u}du\ e^{-\overline{u}u}\ u_i e^{\overline{u}XG(u)}}{\int d\overline{u}du\ e^{-\overline{u}u}\ e^{\overline{u}XG(u)}} =< u_i >_C \stackrel{\rm def}{=} \sum_{\substack{[E,\mathcal{F}]\ \text{type}\ ([1],\emptyset)\\ \text{connected}}} \frac{\mathcal{A}_{Fey}(E,\mathcal{F},\tau_u,\tau_{\overline{u}})}{\#Aut(E,\mathcal{F})} \quad , \tag{142}$$

with $\tau_u(1)=i$ and $\tau_{\overline{u}}=\emptyset$, and the amplitude $\mathcal{A}_{Fey}(E,\mathcal{F},\tau_u,\tau_{\overline{u}})$ is defined using the Feynman rules *(140) and (141).*

For example, the amplitude

$$\begin{aligned}\mathcal{A}_{Fey}(E,\mathcal{F},\tau_u,\tau_{\overline{u}}) = &\sum_{\alpha_1,\ldots,\alpha_{10}=1}^{n} X_i G^{[3]}_{i,\alpha_1\alpha_2\alpha_3}\\ &\times\left(X_{\alpha_1}G^{[1]}_{\alpha_1,\alpha_4}X_{\alpha_4}G^{[3]}_{\alpha_4,\alpha_5\alpha_6\alpha_7}X_{\alpha_5}G^{[0]}_{\alpha_5}X_{\alpha_6}G^{[0]}_{\alpha_6}X_{\alpha_7}G^{[0]}_{\alpha_7}\right)\\ &\times\left(X_{\alpha_2}G^{[0]}_{\alpha_2}\right)\left(X_{\alpha_3}G^{[3]}_{\alpha_3,\alpha_8\alpha_9\alpha_{10}}X_{\alpha_8}G^{[0]}_{\alpha_8}X_{\alpha_9}G^{[0]}_{\alpha_9}X_{\alpha_{10}}G^{[0]}_{\alpha_{10}}\right)\end{aligned} \tag{143}$$

is assigned to the Feynman diagram



(144)

whose automorphism group has cardinality $\#Aut(E,\mathcal{F}) = 3! \times 3!$.

The convergence of the Feynman diagram expansions in the ring $R[[X]]$ is ensured by the fact that each vertex (and not only the leaves like in the previous section) increases the grading by one unit. By repeating the same arguments as in the proof of Theorem 5, consisting in identifying the nearest XG-vertex to the root and summing over the sub-trees that are attached to it, it is immediate that the series

$$F_i(X) \stackrel{\text{def}}{=} < u_i >_C \tag{145}$$

is a solution of the implicit equations

$$F_i(X) = X_i G_i(F(X)) \ \text{ for } \ 1 \le i \le n \quad . \tag{146}$$

This gives a rigorous restatement of Claim 4 in Section II.3. One also has an analog of Proposition 9, saying that

$$Z \stackrel{\text{def}}{=} \int d\overline{u}du\ e^{-\overline{u}u} e^{\overline{u}XG(u)} = \exp\left(W\right) \quad , \tag{147}$$

with

$$W \stackrel{\text{def}}{=} \sum_{\substack{[E,\mathcal{F}]\ \text{type}\ (\emptyset,\emptyset) \\ \text{connected}\ E\neq\emptyset}} \frac{\mathcal{A}_{Fey}\left(E,\mathcal{F}\right)}{\#Aut(E,\mathcal{F})} \quad , \tag{148}$$

where the sum is over equivalence classes of nonempty connected vacuum Feynman diagrams. A closer look at these diagrams will allow us to prove the following.

Theorem 7 *Using the notations of Section II.3*

$$Z = \frac{1}{\det\left(I - X\partial G(F)\right)} \quad . \tag{149}$$

The proof of this theorem depends on the following result which deserves to be stated as an independent theorem. The argument must be familiar to the practitioner of combinatorial species but we could not find it stated explicitely as we need it, in the literature. We cannot resist calling it "the principle of variation of ambiguity" and it states a kind of *functoriality* of Feynman diagrammatic perturbation series. Ambiguity refers to the "degree of resolution" of the combinatorial description that we ("the observer") chose and which is like a combinatorialist's "choice of coordinates".

Theorem 8 *Let $\mathcal{M}$ and $\mathcal{N}$ be two combinatorial species in the sense of Joyal [23], related by a natural transformation (or a morphism of functors) ρ. That is for every finite set E we have a (not necessarily bijective) map $\rho_E : \mathcal{M}(E) \to \mathcal{N}(E)$, such that for any bijection $\sigma : E \to F$ between finite sets E and F, one has $\rho_F \circ \mathcal{M}(\sigma) = \mathcal{N}(\sigma) \circ \rho_E$. Suppose we have defined for every pair (E, M), consisting of a finite set E and a structure $M \in \mathcal{M}(E)$ of type $\mathcal{M}$ on E, an amplitude $\mathcal{A}(E, M)$ taking values in a formal power series ring $R[[V]]$, where V denotes any collection of indeterminates and the ground ring R contains $\mathbb{Q}$. Assume that $\mathcal{A}(E, M)$ is constant over equivalence classes, denoted by $[E, M]$, of pairs (E, M) for the relation $(E, M) \sim (E', M')$ if and only if there exists a bijection $\sigma : E \to E'$ with $\mathcal{M}(\sigma)(M) = M'$.*

The conclusion of the theorem is that if the left-hand side of

$$\sum_{[E,M]} \frac{\mathcal{A}(E,M)}{\#Aut(E,M)} = \sum_{[E,N]} \frac{1}{\#Aut(E,N)} \sum_{\substack{M\in\mathcal{M}(E) \\ \rho_E(M)=N}} \mathcal{A}(E,M) \tag{150}$$

converges in $R[[V]]$, then so does the right-hand side and the equality holds. Note that in the left-hand side one sums over classes for the species $\mathcal{M}$, while in the right-hand side one sums over classes for the species $\mathcal{N}$.

Proof : Note that by the equivariance of the transformation ρ, the expression

$$\frac{1}{\#Aut(E,N)} \sum_{\substack{M\in\mathcal{M}(E) \\ \rho_E(M)=N}} \mathcal{A}(E,M)$$

is independent of the pair (E, N) in a given class $[E, N]$ for the species $\mathcal{N}$. Note also that there is no set-theoretic difficulty in speaking of "the set of all equivalence classes" for a species $\mathcal{M}$. Indeed such a set can be easily constructed as a quotient of the disjoint union of the countable family of finite sets $(\mathcal{M}([k]))_{k\in\mathbb{N}}$. Therefore the families of elements of $R[[V]]$ to be summed on both sides of (150) are well-defined. Let us first show that the summability of the left-hand side implies that of the right-hand side. One can define a map $\overline{\rho}$ from the set of equivalence classes of $\mathcal{M}$ to that of $\mathcal{N}$ by

$$\overline{\rho}([E,M]) \stackrel{\text{def}}{=} [E, \rho_E(M)] \quad . \tag{151}$$

Let V^α be a monomial in $R[[V]]$ and denote by $[V^\alpha]\Omega$ the coefficient of V^α in a power series $\Omega \in R[[V]]$. If

$$[V^\alpha]\left(\frac{1}{\#Aut(E,N)} \sum_{\substack{M\in\mathcal{M}(E)\\ \rho_E(M)=N}} \mathcal{A}(E,M)\right) \neq 0 \quad ; \tag{152}$$

then $[E,N]$ is the image by $\overline{\rho}$ of an $[E,M]$ such that $[V^\alpha]\mathcal{A}(E,M) \neq 0$. If the left-hand side converges, there are finitely many such $[E,M]$'s and since $\overline{\rho}$ is a finite-to-one map, there are finitely $[E,N]$'s such that (152) is true.

To prove the equality in (150), one simply needs to check that for any class $[E,N]$ for the species $\mathcal{N}$ the following equality, involving only finite sums, holds:

$$\sum_{\substack{[E,M]\\ \overline{\rho}([E,M])=[E,N]}} \frac{\mathcal{A}(E,M)}{\#Aut(E,M)} = \frac{1}{\#Aut(E,N)} \sum_{\substack{M\in\mathcal{M}(E)\\ \rho_E(M)=N}} \mathcal{A}(E,M) \quad . \tag{153}$$

First fix a representative (E,N) of the concerned $\mathcal{N}$-class. Let $\pi_{\mathcal{N}}$ be the partition of $\mathcal{N}(E)$ into equivalence classes for the relation $N_1 \sim N_2$ defined by the existence of a bijection $\sigma : E \to E$ such that $\mathcal{N}(\sigma)(N_1) = N_2$. Let $\pi_{\mathcal{M}}$ be the analogous partition of $\mathcal{M}(E)$. Let π_ρ be the partition of $\mathcal{M}(E)$ defined by the nonempty inverse images by ρ_E of elements of $\mathcal{N}(E)$. It is clear by functoriality of ρ that $\pi_{\mathcal{M}}$ is finer than π_ρ. Let $\overline{N}$ be the block of $\pi_{\mathcal{N}}$ containing N. One easily checks that

$$\sum_{\substack{[E,M]\\ \overline{\rho}([E,M])=[E,N]}} \frac{\mathcal{A}(E,M)}{\#Aut(E,M)} = \sum_{\substack{B\in\pi_{\mathcal{M}}\\ \rho_E(B)\subset\overline{N}}} \frac{1}{\#(E)!}\frac{\#(E)!}{\#Aut(E,M)}\mathcal{A}(E,M) \quad , \tag{154}$$

where M stands for any element of B. Indeed every class $[E,M]$ with $\overline{\rho}([E,M]) = [E,N]$ corresponds to a $B \in \pi_{\mathcal{M}}$ sent by ρ_E into $\overline{N}$. One also has

$$\frac{\#(E)}{\#Aut(E,M)} = \#(B) \quad ; \tag{155}$$

since B is the orbit of any $M \in B$ for the action of $\mathfrak{S}(E)$, the group of permutations of E, on the set $\mathcal{M}(E)$. Therefore the right hand side of (154) becomes

$$\frac{1}{\#(E)!} \sum_{\substack{M\in\mathcal{M}(E)\\ \rho_E(M)\in\overline{N}}} \mathcal{A}(E,M) = \frac{1}{\#(E)!} \sum_{N'\in\overline{N}} \sum_{\substack{M\in\mathcal{M}(E)\\ \rho_E(M)=N'}} \mathcal{A}(E,M) \quad . \tag{156}$$

Now again by functoriality of ρ,

$$\sum_{\substack{M\in\mathcal{M}(E)\\ \rho_E(M)=N'}} \mathcal{A}(E,M)$$

does not depend on N' in $\overline{N}$, the latter of which is the orbit of N under the action of $\mathfrak{S}(E)$ on $\mathcal{N}(E)$, and therefore has cardinality

$$\frac{\#(E)!}{\#Aut(E,N)} \quad .$$

Thus

$$\frac{1}{\#(E)!} \sum_{\substack{M\in\mathcal{M}(E)\\ \rho_E(M)\in\overline{N}}} \mathcal{A}(E,M) = \frac{1}{\#Aut(E,N)} \sum_{\substack{M\in\mathcal{M}(E)\\ \rho_E(M)=N}} \mathcal{A}(E,M) \quad , \tag{157}$$

from which (153) and the proof of the theorem follow. ∎

Remark : We have already used this principle in two particular instances:

- In Theorem 7, where $\mathcal{M}$ was the species of Feynman diagrams, and $\mathcal{N}$ that of pre-Feynman diagrams. The transformation ρ amounted to forgetting the contraction scheme.
- In (79), where $\mathcal{M}$ was the species of Feynman diagrams, and $\mathcal{N}$ was the vacuous species ($\#(\mathcal{N}(E)) = 1$ for any finite E). Applying ρ meant to forget everything except the cardinality of E.

Proof of Theorem 7 : We start from the expression (148) for W that we rewrite, following the notation of Theorem 8, as

$$W = \sum_{[E,N]} \frac{\mathcal{A}(E,N)}{\#Aut(E,N)} \quad . \tag{158}$$

Here the species $\mathcal{N}$ is that of nontrivial connected Feynman diagram structures of type $(\emptyset,\emptyset)$. The amplitude is the one defined by the Feynman rules (140) and (141). We now introduce a new species $\mathcal{M}$ as follows. For any finite set E, we call an $\mathcal{M}$-structure on E, any couple $(N,\mathcal{O})$ where $N \in \mathcal{N}(E)$ and $\mathcal{O}$ consists of a total ordering $B_1 < \ldots < B_p$ of the XG-vertices appearing in the central circuit of N, and of a total ordering $x_1^q < \ldots < x_{k_q}^q$ of the elements of $B_q \cap E_u$ for each q, $1 \le q \le p$. We require that the order $B_1 < \ldots < B_p$ be compatible with the orientation of the circuit, i.e., $B_1, \ldots, B_p$ is the sequence of vertices obtained by following the orientation of the contraction lines, along the circuit, starting from B_1. Note that it is possible that some $k_q \stackrel{\text{def}}{=} \#(B_q \cap E_u)$ be zero. However one allways has $p \ge 1$. Transport of structure for $\mathcal{M}$ is defined in the obvious covariant way. The morphism of functors ρ is defined by $\rho_E(N,\mathcal{O}) = N$ for any $(N,\mathcal{O}) \in \mathcal{M}(E)$. We also define the amplitude for an $\mathcal{M}$-structure $M = (N,\mathcal{O})$, keeping the previous notations, by

$$\mathcal{A}(E,M) \stackrel{\text{def}}{=} \frac{\mathcal{A}(E,N)}{p \cdot k_1! \cdots k_p!} \quad . \tag{159}$$

Now, after the trivial check that the left hand side of the following equality converges in $R[[X]]$, Theorem 8 implies that

$$\sum_{[E,M]} \frac{\mathcal{A}(E,M)}{\#Aut(E,M)} = \sum_{[E,N]} \frac{1}{\#Aut(E,N)} \sum_{\substack{M\in\mathcal{M}(E)\\ \rho_E(M)=N}} \mathcal{A}(E,M) \quad . \tag{160}$$

But,

$$\sum_{\substack{M\in\mathcal{M}(E)\\ \rho_E(M)=N}} \mathcal{A}(E,M) = \mathcal{A}(E,N) \quad . \tag{161}$$

Indeed, the *product* $p \cdot k_1! \cdots k_p!$ does not depend on $\mathcal{O}$, besides it is equal to the number of these possible orderings $\mathcal{O}$. One can write as a result

$$W = \sum_{[E,M]} \frac{\mathcal{A}(E,\rho_E(M))}{p\cdot k_1!\cdots k_p!} \frac{1}{\#Aut(E,M)} \quad . \tag{162}$$

The point is that the automorphism group of a pair (E,M) is much more manageble since the central circuit has been completely *rigidified*, i.e., all the elements of E that belong to an XG-vertex along the circuit are fixed by automorphisms of (E,M). Indeed, for any q, $1\le q\le p$ and any ν, $1\le\nu\le k_q$, $x_\nu^q \in B_q\cap E_u$ is the new root of a tree-like connected Feynman diagram structure $\mathcal{F}_\nu^q$ of type $([1],\emptyset)$ on a subset E_ν^q of E. The corresponding ρ_u map has $\{x_\nu^q\}$ as an image. An automorphism of (E,M), if restricted to E_ν^q, reduces to an automorphism of $\mathcal{F}_\nu^q$. Therefore

$$\#Aut(E,M) = \prod_{q=1}^{p}\left(\prod_{\nu=1}^{k_q} \#Aut(E_\nu^q,\mathcal{F}_\nu^q)\right) \quad . \tag{163}$$

Besides the amplitude $\mathcal{A}(E,N)$, with $N=\rho_E(M)$ is given by

$$\mathcal{A}(E,N) = \sum_I \mathcal{L}_I \prod_{q=1}^{p}\left(\prod_{\nu=1}^{k_q} \mathcal{A}(E_\nu^q,\mathcal{F}_\nu^q,i_\nu^q)\right) \quad , \tag{164}$$

where the sum is over families $I=(i_\nu^q)_{1\le q\le p,1\le\nu\le k_q}$ of indices in $[n]$. $\mathcal{A}(E_\nu^q,\mathcal{F}_\nu^q,i_\nu^q)$ is the amplitude of the Feynman diagram structure $\mathcal{F}_\nu^q$ of type $([1],\emptyset)$ on E_ν^q with respect to the index assignment map with value i_ν^q. Finally $\mathcal{L}_I$ is the contribution of the *amputated* circuit

$$\mathcal{L}_I \stackrel{\text{def}}{=} \sum_{j_1,\ldots,j_p=1}^{n} \prod_{q=1}^{p}\left(X_{j_q} G^{[k_q+1]}_{j_q,j_{q+1}i_1^q\ldots i_{k_q}^q}\right) \quad , \tag{165}$$

with the convention that $j_{p+1} \stackrel{\text{def}}{=} j_1$. Note also that classes $[E,M]$ are in bijective correspondence with families $([E_\nu^q,\mathcal{F}_\nu^q])_{1\le q\le p,1\le\nu\le k_q}$ of classes of connected Feynman diagram structures of type $([1],\emptyset)$ where all values of $p\ge 1$ and $k_q\ge 0$, for $1\le q\le p$, are allowed. The previous observation, Equations (142) and (145), and the expression of a derivative $\partial_j G_i$ in tensorial notation is enough to show that

$$W = \sum_{p\geq 1} \frac{1}{p} tr[X\partial G\,(F(X))]^p \quad , \tag{166}$$

and, as a result of Jacobi's identity and Equation (147),

$$Z = \exp{(W)} = \frac{1}{\det{(I - X\partial G(F))}} \quad . \tag{167}$$

■

Note that we have an analog of Theorem 6 whose statement and proof are the same in the present context. As a consequence one has the following result.

Theorem 9 *For any monomial $\Omega(F) = F_1^{\alpha_1}\cdots F_n^{\alpha_n}$, the following identity in $R[[X]]$, both sides being well-defined, holds*

$$\Omega(F) \times \frac{1}{\det{(I - X\partial G(F))}} = \int d\overline{u}du\ e^{-\overline{u}u}\ \Omega(u)e^{\overline{u}XG(u)} \quad . \tag{168}$$

It is easily checked that one can expand the $e^{\overline{u}XG(u)}$ and take out the sum to get, in the ring $R[[X]]$:

$$\int d\overline{u}du\ e^{-\overline{u}u}\ \Omega(u)e^{\overline{u}XG(u)} = \sum_{\alpha\in\mathbb{N}^n} \frac{X^\alpha}{\alpha!} \int d\overline{u}du\ e^{-\overline{u}u}\ \overline{u}^\alpha \Omega(u)G(u)^\alpha \quad . \tag{169}$$

Note that

$$\int d\overline{u}du\ e^{-\overline{u}u}\ \overline{u}^\alpha \Omega(u)G(u)^\alpha \tag{170}$$

is in R and can be computed, by going back to the definition of formal Gaussian integration with covariance matrix given by the identity matrix, as

$$\left(\frac{\partial}{\partial u}\right)^\alpha [\Omega(u)G(u)^\alpha]\Big|_{u=0} \quad .$$

This concludes our derivation of the implicit form of the multivariable Lagrange-Good inversion.

IV Comments

1) By now, it should be clear to the reader that we have only scratched the tip of the iceberg. In QFT, there are basically four categories of fields (i.e., types of variables on which one can define a formal Gaussian integration scheme). This division is strangely reminiscent of the distinction between the main families of classical groups. We indeed have :

- Complex Bosonic fields : The variables commute and come with an involution exchanging them in pairs. This is the situation we covered here. Expansions involve digraphs, and Wick's theorem is in terms of *permanents*.
- Complex Fermionic fields : The variables anti-commute and also come with an involution. Graphs are directed but usually involve an extra -1 factor per circuit. Wick's theorem uses *determinants*. In many respects, Fermionic integration intuitively behaves like Bosonic integration in a "negative dimensional space", whatever that means.
- Real Bosonic fields : The variables commute and no involution on them is given. The graphs are undirected. Wick's theorem is in term of *hafnians*, i.e., sums are over perfect matchings instead of permutations. The covariance matrices must be symmetric.
- Real Fermionic fields : The variables anti-commute. No involution is, at least beforehand, given. Covariance matrices must be skew-symmetric, therefore graphs have to be, somewhat artificially, oriented to avoid sign ambiguities in their amplitudes. Wick's theorem involves *Pfaffians*.

Clearly, a similar approach to ours, using combinatorial species, can be developed for all four types of fields; although one has to be careful with Fermions. For instance, we do not know if one can make sense of situations where vertices have an odd number of half-lines or, in the complex case, unequal numbers of incoming and outgoing half-lines. The case of ribbon graphs (see [17] for instance) is covered by the above tentative classification. The GUE random matrix ensemble, for example, belongs to the complex Bosonic case, while the GOE falls in the real Bosonic case.
2) Rules 1 and 2 of our symbolic calculus are rather tautological on the diagrammatic side; but Rule 3 can be understood as a set of *combinatorial conjectures*. Indeed we only proved the correctness of the change of variable formula in a few special cases. It would be a valuable task to explore the extent of its validity. Since determinants are involved in the Jacobian factor, and thus possibly Fermions anyway, it might be a good idea to directly attempt a Feynman diagrammatic statement and proof of its supersymmetric generalization: the Berezin change of variable formula. For someone unfamiliar with this beautiful identity, we recommend consulting : the Appendix A of [28] which is a very clear and concise "formulaire raisonné" of supersymmetry; then the second chapter of [15] to see some examples of calculations and get some exposure to the difficulties due to boundary terms (which however should not intervene for what we have in mind since, to have a Feynman diagram expansion, one needs to integrate over the whole Bosonic space in the presence of a Gaussian weight); and finally [8] for a thorough exposition.
3) Another oddity of the complex Bosonic situation we treated here is that fields or variables come in pairs $\overline{u}, u$. As our starting point was Theorem 1, we have thought of $\overline{u}$ and u as *complex conjugate* of one another, and have designed our notation accordingly. However it seems, with respect to the change of variable formula, that $\overline{u}$ and u can be manipulated as *independent* variables. In fact, Rule 2 which is a

kind of Fourier representation of the Dirac delta function, rather suggests that one should think of $\overline{u}$ and u as *Fourier-dual* variables. Indeed, one can derive the Gurjar-Abhyankar formula for the formal inverse of a system of power series in the latter spirit by a moderate use of the theory of pseudodifferential and Fourier integral operators (see Exercise 3.2 in [21]).
4) There is a definite and quite strange mixture of mathematics with *metamathematics* in Feynman diagrammatic sums. As we mentioned earlier, to describe these expansions in a mathematically precise way, one has to define a "programming language" with its syntactic rules. The sum over diagrams is in fact a sum over "programs" of the "number" (i.e., the amplitude) such a program is meant to compute. Some might think that this is too far-fetched an analogy, and that basic graph theory is enough to accomodate QFT. This is not quite correct. In constructive field theory, the most powerful tools are the so-called *phase-cell* or *multiscale cluster expansions* (see [5, 24, 26] for the current state-of-the-art). These are a kind of *smart* perturbation theory designed to avoid all divergences that appear in the naive perturbative QFT. They make critical use of two extra ingredients: the Heisenberg uncertainty principle ("cluster expansion" refers to the implementation of this idea), and the Wilsonian renormalization group (to which "multiscale" refers). We can assure the reader that the, quite formidable, combinatorial structures that appear in the explicit form of these expansions [3], look much more like "programs" than graphs. Had we known of the theory of species at the time, we would have written what we called "Mayer configurations" (in Chapter 4 of [3]) in this most convenient language. Let us finish, by saying that this intrusion of metamathematics in a problem of mathematical analysis and also its somewhat reckless treatment in the physical literature, rather than the lack of concepts (of which the genius of K. Wilson has provided an ample supply) is the main reason delaying the entry of what we called the "grammar" of QFT into mainstream mathematics. Because of this, we venture to say that, maybe, it is time for professionals to step in: combinatorialists, computer scientists and, why not, mathematical logicians!

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