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Random mappings, forests, and subsets associated with Abel-Cayley-Hurwitz multinomial expansions *

Jim Pitman

Department of Statistics
University of California
367 Evans Hall # 3860
Berkeley, CA 94720-3860

Abstract

Various random combinatorial objects, such as mappings, trees, forests, and subsets of a finite set, are constructed with probability distributions related to the binomial and multinomial expansions due to Abel, Cayley and Hurwitz. Relations between these combinatorial objects, such as Joyal's bijection between mappings and marked rooted trees, have interesting probabilistic interpretations, and applications to the asymptotic structure of large random trees and mappings. An extension of Hurwitz's binomial formula is associated with the probability distribution of the random set of vertices of a fringe subtree in a random forest whose distribution is defined by terms of a multinomial expansion over rooted labeled forests.

Contents

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1 Introduction

Recent research at the interface of probability and combinatorics [70, 24, 6, 5, 7, 3] has exposed a rich probabilistic structure associated with the binomial and multinomial expansions due to Abel [1], Cayley [25] and Hurwitz [43]. The probabilistic meaning of these expansions is brought out by consideration of suitably constructed random subsets, trees, forests, and mappings, and by study of the relations between these various random combinatorial objects. The purpose of this paper is to introduce this subject to combinatorialists with some interest in probability theory, and to probabilists with some interest in combinatorics. Combinatorialists may prefer to look first at the condensed version [72] of this paper, written in more combinatorial language. Section 1.1 of this paper provides a correspondence between polynomial identities of Hurwitz type presented in [72] and their probabilistic expressions in this paper. These Hurwitz identities were all first discovered as byproducts of the solution of natural probabilistic problems discussed here. Probabilists may find motivation in the application of some of the results of this paper to the theory of measure-valued and partition-valued coalescent processes and the asymptotic structure of large random trees and random mappings, as considered in [4, 5, 6, 24, 33, 70, 68, 14, 65, 7, 3].

This paper is organized as follows. Section 2 explains how a simple formula of Burtin [23] for random mappings is equivalent to a very useful form of Cayley's multinomial expansion over trees, called here the *forest volume formula.* This formula yields some enumerations of rooted labeled forests which are the basis of everything that follows. Section 3 presents some constructions and properties of a *p-forest of k trees*, that is a random forest of k rooted trees labeled by some finite set S, whose distribution is proportional to terms in the Cayley expansion of $(\sum_{s\in S} p_s)^{n-k}$ over such forests, where p is some arbitrary probability distribution on S. Properties of such a p-forest, first considered in [70] in connection with the construction of a coalescent process, are closely related to properties of a random *p-mapping* defined by assigning each point s of S an image in S with distribution p, independently as s varies, as studied in [23, 78, 48, 65, 7]. Section 4 describes the distribution of the subtree spanning a finite number of vertices of a *p-tree*, that is a p-forest with a single tree component. The result of this section was applied in [5] to describe features of the asymptotic structure of p-trees with a large number of vertices. Section 4.1 shows how results for p-trees can be transferred to p-mappings via Joyal's bijection [49, p. 16] whereby a p-tree with an independent mark with distribution p corresponds to a p-mapping. As shown in [7], this allows a large number of asymptotic results for p-mappings to be deduced from corresponding results for p-trees obtained in [24]. Section 5 shows how the distributions of various random subsets derived from p-trees and p-forests are related to Hurwitz's multinomial expansions [43]. This leads in Section 5.5 to consideration of percolation probabilities defined by p-trees and p-forests. These probabilities are related to extensions and variations of Hurwitz's expansions described in [72]. Section 6 shows that for any subset B of S, the restriction to B of a p-forest is a $p(\cdot|B)$-forest, where $p(\cdot|B)$ is the probability distribution p conditioned on B. Finally, Section 6.3 shows how the structure of p-forests is preserved under an operation of independent deletion of edges, which generalizes that considered in [70].

1.1 Correspondence with results of [72]

Polynomial identity in [72]	Probabilistic expression in this paper
(2)	Theorem 11 (i)-(iii) and Theorem 12
(3)	Theorem 11 (iv) and Proposition 15 (ii)
(4)	Proposition 18 for $k = 1$.
(9)	Lemma 3 (i)
(11)	Proposition 15 (i)
(18)	Theorem 4
(19) and (20)	(16)
(24)	Lemma 23
(26)	Proposition 18
(27)	Theorem 7
(28)	Proposition 14
(29)	(33)

2 Random mappings and the forest volume formula

First, a brief review of probabilistic terms used in this paper. A *probability distribution* on a finite set S is a non-negative real-valued function $p := (p_s, s \in S)$ with $\sum_{s\in S} p_s = 1$. The definition of p is extended to subsets A of S by $p(A) := p_A := \sum_{s\in A} p_s$. Throughout the paper, P denotes a probability distribution on a suitable finite set Ω. A function $X : \Omega \to S$ is called a *random element of* S. The *distribution of* X is the probability distribution p on S defined by

$$p_s := P(X = s) := P(\{\omega \in \Omega : X(\omega) = s\}) \qquad (s \in S).$$

If elements of S are for instance subsets of another set, or trees, or mappings, a random element X of S may called a *random set*, a *random tree*, or a *random mapping*, as the case may be, whether or not the distribution of X is *uniform*, meaning $P(X = s) = 1/|S|$ for all $s \in S$, where $|S|$ is the number of elements of S. Subsets of Ω are called *events*. For an event $B \subseteq \Omega$ with $P(B) > 0$ and a random element X of S, the *conditional distribution of* X *given* B is the probability distribution on S defined by

$$P(X = s \,|\, B) := P(\{\omega \in B : X(\omega) = s\})/P(B) \qquad (s \in S).$$

For X with distribution p, and $A \subseteq S$ with $p_A > 0$, the distribution of X given $X \in A$ may be denoted $p(\cdot|A)$. So $p(s|A) = p_s \chi(s \in A)/p_A$, where $\chi(\cdots)$ is the indicator function which has value 1 if $\cdots$ and 0 else. For further background, and definitions of other probabilistic terms such as independence and expectation, see [34] or [67].

Random Mappings. Given some probability distribution p on a finite set S, let $M := (M_s, s \in S)$ be a *p-mapping* of S. That is to say, the $M_s, s \in S$ are independent random variables with common probability distribution p, defined on some probability space (Ω, P). See [27, 79, 35, 36] for combinatorial background. There is a large probabilistic literature on the stucture of random mappings for uniform p. See e.g. [56, 4, 41, 64, 3] and papers cited there. The case when all of the p_s but one are equal is studied in [83, 63, 18]. Random p-mappings for general p are studied in [23, 78, 48, 65, 7]. See also [23, 31, 44, 45, 16, 17, 19, 50, 13, 74, 40] regarding various other models for random mappings.

For each subset B of S^S, the probability

$$P(M \in B) = \sum_{m \in B} \prod_{s \in S} p_{m_s} \tag{1}$$

is the usual *enumerator polynomial* of B in variables $p_s, s \in S$, as discussed in [27, p. 72], but with the constraints $p_s \geq 0$ and $\sum_s p_s = 1$. A formula for $P(M \in B)$ as a function of $p = (p_s, s \in S)$ therefore amounts to evaluation of an enumerator polynomial. Such probabilistic evaluations are typically simpler than a general expression for the enumerator polynomial of B, because the use of p_s subject to $\sum_{s \in S} p_s = 1$ instead of general variables x_s typically eliminates some factors of $x_S := \sum_{s \in S} x_s$. But such factors can always be recovered by *scaling*: take $p_s = x_s/x_S$ in the probabilistic identity, and multiply both sides by $x_S^{|S|}$ where $|S|$ is the number of elements of S.

To give an example of this translation between probabilistic and combinatorial language, let $\mathcal{D}(M) \subseteq S \times S$ be the usual functional digraph associated with a mapping M of S. So $\mathcal{D}(M) := \{(s, M_s), s \in S\}$ has a directed edge $s \to M_s$ for each $s \in S$. According to a result of Burtin [23], if M is a p-mapping then

$$P[\text{no cycle of } \mathcal{D}(M) \text{ is contained in } S - R\,] = p_R \quad (R \subseteq S). \tag{2}$$

In combinatorial language: the enumerator of the set

$$\{M \in S^S : \text{no cycle of } \mathcal{D}(M) \text{ is contained in } S - R\}$$

is the polynomial $x_R x_S^{|S|-1}$ in variables $x_s, s \in S$. See [12] for six different proofs of (2) with $|R| = 1$. The result for general R follows from this special case by replacing M by $\phi \circ M$ where $\phi : S \to (S - R) \cup \{r\}$ collapses R onto some $r \in R$, and leaves $S - R$ fixed.

Rooted forests. Let $\hat{\mathcal{D}}_R(M)$ denote the digraph with vertex set S derived from $\mathcal{D}(M)$ by first deleting the edges $r \to M_r$ for all $r \in R$, then replacing each of the remaining edges $s \to M_s$ for $s \in S - R$ by its reversal $M_s \to s$. Obviously

$$[\text{no cycle of } \mathcal{D}(M) \text{ is contained in } S - R] \Leftrightarrow [\hat{\mathcal{D}}_R(M) \in \boldsymbol{F}_{S,R}] \tag{3}$$

where $\boldsymbol{F}_{S,R}$ is the set of all forests F of rooted trees labeled by S, whose set of root vertices is R, with edges of F directed away from the roots. Call the set

$$F_s := \{x : s \to x \text{ is an edge of } F\}$$

the *set of children of* s *in the forest* F. For each $F \in \boldsymbol{F}_{S,R}$ the $F_s, s \in S$ form a collection of disjoint, possibly empty sets with union $S - R$, and $s \notin F_s$ for all s. Note that $|F_s|$ is the *out-degree* of vertex s in the forest F, that each $F \in \boldsymbol{F}_{S,R}$ has $|R|$ tree components, some of which may be trivial (i.e. a root vertex with no edges), and that the total number of edges of F is $\sum_s |F_s| = |S| - |R|$. For each subset B of the set

$$\boldsymbol{F}_S := \cup_{R \subseteq S} \boldsymbol{F}_{S,R}$$

of all rooted forests labeled by S, the enumerator polynomial in variables $x_s, s \in S$

$$V_S[F \in B] := V_S[F \in B](x_s, s \in S) := \sum_{F \in B} \prod_{s \in S} x_s^{|F_s|} \tag{4}$$

is called here the *volume of* B, to emphasise that $B \mapsto V_S[F \in B]$ is a measure on subsets B of $\boldsymbol{F}_S$, for each fixed choice of $(x_s, s \in S)$ with $x_s \geq 0$. This notion of *forest volumes* includes both the probabilistic interpretations developed in [70, 68, 69] and in this paper, and Kelmans' notion of the forest volume of a graph [54, 52, 53]. The previous formula $x_R x_S^{|S|-1}$ for the mapping enumerator corresponding to (2) amounts, by cancellation of a common factor of $x_S^{|R|}$, to the following generalization of Cayley's [25] multinomial expansion over trees, which is used repeatedly throughout this paper:

The forest volume formula.

$$V_S[\text{roots}(F) = R] = x_R\, x_S^{|S|-|R|-1} \qquad \text{where } x_B := \sum_{s \in B} x_s \tag{5}$$

and roots(F) denotes the set of root vertices of a forest $F \in \boldsymbol{F}_S$.

Cayley [25] gave the special case of (5) for $|R| = 1$, call it the *tree volume formula*, as well as the special case $x_s \equiv 1$ of the forest volume formula, that is

$$|\boldsymbol{F}_{S,R}| = |R|\, |S|^{|S|-|R|-1}.$$

See [72] for some alternate derivations and history of the forest volume formula and its consequences. For each subset R of S with $|R| = k \leq n = |S|$ and each vector $c := (c_s, s \in S)$ of non-negative integers with $\sum_s c_s = n - k$, the identity of coefficients of $\Pi_s x_s^{c_s}$ in (5) reads

$$|\{F \in \boldsymbol{F}_{S,R} : |F_s| = c_s \text{ for all } s \in S\}| = \frac{c_R\,(n-k-1)!}{\prod_{s\in S} c_s!} \tag{6}$$

which summed over R with $|R| = k$ gives the number of forests with out-degree vector c:

$$|\{F \in \boldsymbol{F}_S : |F_s| = c_s \text{ for all } s \in S\}| = \frac{(n-1)_{n-k}}{\prod_{s\in S} c_s!} \tag{7}$$

with the notation for falling factorials $(x)_m := \prod_{i=0}^{m-1}(x-i)$. Formula (7) is the identity of coefficients of $\Pi_s x_s^{c_s}$ in the result of summing the forest volume formula (5) over all subsets R of S of size k, which gives the volume of all forests of k trees labeled by S with $|S| = n$, as found in [70, (39)], [69, Theorem 1.6], [82, Theorem 5.3.4]:

$$V_S[F \text{ with } k \text{ tree components}] = \binom{n-1}{k-1} x_S^{n-k}. \tag{8}$$

There is also the following generalization of (8), obtained in [68, Lemma 13] and [72, Theorem 3], which reduces to (8) when G is the trivial forest with n singleton components and no edges: for each $G \in \boldsymbol{F}_S$ with g tree components

$$V_S[F \text{ with } k \text{ trees and } F \supseteq G] = \binom{g-1}{k-1} x_S^{g-k} \prod_{s\in S} x_s^{|G_s|}. \tag{9}$$

Yet another extension of the forest volume formula can be made as follows. Write $r \overset{F}{\rightsquigarrow} s$ if there is a directed path from r to s in F, that is a sequence of one or more edges $r \overset{F}{\to} \cdots \overset{F}{\to} s$, where $s_1 \overset{F}{\to} s_2$ means $(s_1, s_2) \in F$. Then for all $R \subset S$, and each fixed choice of an $r \in R$ and an $s \in S - R$,

$$V_S[\text{roots}(F) = R \text{ and } r \overset{F}{\rightsquigarrow} s] = x_r \, x_S^{|S|-|R|-1}. \tag{10}$$

Since for each fixed $s \in S - R$ and each $F \in \boldsymbol{F}_{S,R}$ there is is a unique $r \in R$ with $r \overset{F}{\rightsquigarrow} s$, the forest volume formula (5) is recovered from (10) by summation over $r \in R$. A probabilistic formulation and proof of (10) are given later around (32). See also [72, Theorem 2] for a combinatorial proof of (10).

3 Random Forests

Let $\mathcal{F}_k$ denote a random forest of k trees labeled by S with $|S| = n$, say $\mathcal{F}_k = (\mathcal{F}_{k,s}, s \in S)$ where $\mathcal{F}_{k,s}$ is the random set of children of s in the forest $\mathcal{F}_k$. Since the random vector of out-degree counts

$$\text{counts}(\mathcal{F}_k) := (|\mathcal{F}_{k,s}|, s \in S)$$

is subject to the constraint $\sum_s |\mathcal{F}_{k,s}| = n - k$, the expectation of $|\mathcal{F}_{k,s}|$ must equal $(n-k)p_s$ for some probability distribution p on S. For given $(p_s, s \in S)$, the simplest way to construct such a random forest $\mathcal{F}_k$ is to suppose that $\mathcal{F}_k$ satisfies conditions (i) and (ii) of the following proposition, which extends [70, (36) and (38)]:

Proposition 1 *For each probability distribution p on S with $|S| = n$, the probability distribution of a random forest $\mathcal{F}_k$ with k trees, call it a p-*forest *of k* trees*, is defined by the formula*

$$P(\mathcal{F}_k = F) = \binom{n-1}{k-1}^{-1} \prod_{s \in S} p_s^{|F_s|} \tag{11}$$

for every forest F of k trees labeled by S. A random forest $\mathcal{F}_k$ has this distribution if and only if both

(i) *the distribution of the out-degree count vector is multinomial with parameters* $n-k$ *and* $(p_s, s \in S)$*, meaning that for each vector of counts* $c = (c_s, s \in S)$ *with* $\sum_s c_s = n - k$*,*

$$P(\text{counts}(\mathcal{F}_k) = c) = \frac{(n-k)!}{\prod_{s \in S} c_s!} \prod_{s \in S} p_s^{c_s} \tag{12}$$

and
(ii) *for each such vector of counts* c *the conditional distribution of* $\mathcal{F}_k$ *given* $\text{counts}(\mathcal{F}_k) = c$ *is uniform on the set of all forests with the given out-degrees, as enumerated in* (7)*.*

Proof. The fact that the probabilities (11) sum to 1, over all F with k tree components, is a probabilistic expression of (8) which was noted in [70, Theorem 11]. The equivalence of (i) and (ii) with (11) follows easily from the enumeration (7). □

If p is uniform on S, a p-forest of k trees has uniform distribution on the set of all rooted forests of k trees labeled by S. See [70, §4] for a review of exact combinatorial results and asymptotic distributions known in this case, and their applications to random graphs. See also [10, 57, 66, 59] concerning other models of random trees and forests and their applications.

Constructions of p-trees. Call a p-forest of one tree a *p-tree*. So a p-tree is a random element of the set $\boldsymbol{T}_S$ of all rooted trees labeled by S. The following two constructions of p-trees are based on a sequence of independent random variables $X_0, X_1, X_2, \ldots$ with common distribution p on S with $|S| = n$.
(i) Let $T : S^{n-1} \to \boldsymbol{T}_S$ be the bijection defined by a Prüfer code [75, 27, 62] such that $T(s_1, \ldots, s_{n-1})$ is a tree in which the number of children of s equals the number of j such that $s_j = s$, for every $s \in S$. Then $T(X_1, \ldots, X_{n-1})$ is evidently a p-tree.
(ii) [21, Theorem 1],[58, §6.1] Assuming $p_s > 0$ for every $s \in S$, the random set

$$\{(X_{j-1}, X_j) : j \geq 1, X_j \notin \{X_0, \ldots, X_{j-1}\}\} \subseteq S \times S$$

defines a p-tree. See [24] for applications of this construction, which is related to the classical birthday problem.

See also (19) and (23) in Section 3.2, which for $R = \{r\}$ show how to construct a p-tree conditioned to have root r by appropriate conditioning of a p-mapping. The effect of conditioning a p-tree on its root is discussed further in Lemma 5.

Coalescent construction of p-forests. To review a construction from [70], define a coalescing sequence of forests $\mathcal{F}(0), \mathcal{F}(1), \ldots, \mathcal{F}(n-1)$ as follows, by adding edges one by one in such a way that $\mathcal{F}(j)$ has j edges (and hence $n-j$ tree components) for each $1 \le j \le n-1$. Let $\mathcal{F}(0)$ be the trivial forest labeled by S with no edges. Given that $\mathcal{F}(0), \ldots, \mathcal{F}(j-1)$ have been defined for some $1 \le j \le n-1$, define $\mathcal{F}(j)$ by adding the edge (X_j, Y_j) to $\mathcal{F}(j-1)$, where the X_j are independent with distribution p on S, and given X_j and the (X_i, Y_i) for $1 \le i < j$, the random variable Y_j has uniform distribution on the set of $n-j$ roots of tree components of $\mathcal{F}(j-1)$ other than the component containing X_j. Then $\mathcal{F}(j)$ is a p-forest of $n-j$ trees for every $0 \le j \le n-1$. In particular, $\mathcal{F}(n-1)$ is a p-tree.

As remarked in [70], this coalescent construction can be reversed to recover a p-forest of k trees by deleting $k-1$ edges picked uniformly at random from the $|S|-1$ edges of a p-tree. See Section 6.3 for a generalization.

3.1 Distribution of the roots of a p-forest

For $1 \le k < n := |S|$ let S_k be the set of all subsets R of S with $|R| = k$. It is easily seen that for each a non-negative function $w = (w_s, s \in S)$ with $w_S > 0$, the formula

$$P(\mathcal{R}_k = R) = \binom{n-1}{k-1}^{-1} \frac{w_R}{w_S} \qquad (R \in S_k) \tag{13}$$

defines the probability distribution of a random element $\mathcal{R}_k$ of S_k, call it *the w-distribution on S_k*. For each fixed subset A of S with $|A| = a$, formula (13) implies

$$P(\mathcal{R}_k \supseteq A) = \frac{\binom{n-a}{k-a} w_A + \binom{n-a-1}{k-a-1} (w_S - w_A)}{\binom{n-1}{k-1} w_S}. \tag{14}$$

Proposition 2 *Let* $\operatorname{roots}(\mathcal{F}_k)$ *be the random set of k root vertices of a p-forest of k trees labeled by S with $|S| = n$. Then*
(i) *the distribution of* $\operatorname{roots}(\mathcal{F}_k)$ *is the p-distribution on S_k; in particular, the root of a p-tree has distribution p on S;*

(ii) *the conditional distribution of* roots$(\mathcal{F}_k)$ *given* counts$(\mathcal{F}_k) = c$ *is the* c*-distribution on* S_k*, for each count vector* c *with* $c_S = n - k$*.*

Proof. This is just a probabilistic translation of the forest volume formulae (5), (6) and (7), obtained by using the definition (11) of the distribution of $\mathcal{F}_k$ and canceling some factorials. □

In particular, according to part (i) of the proposition and (14) for $A = \{r\}$, for each $r \in S$

$$P[r \in \text{roots}(\mathcal{F}_k)] = \frac{(n-k)p_r}{n-1} + \frac{k-1}{n-1}. \tag{15}$$

Formula (10), derived later in Theorem 4, allows the two terms on the right side of (15) to be interpreted as follows: for each fixed $s \in S$ with $s \neq r$, these terms are

$$P[r \overset{\mathcal{F}_k}{\rightsquigarrow} s \text{ and } r \in \text{roots}(\mathcal{F}_k)] \text{ and } P[r \overset{\mathcal{F}_k}{\not\rightsquigarrow} s \text{ and } r \in \text{roots}(\mathcal{F}_k)] \tag{16}$$

respectively. See [72, §3] for an alternative proof and further discussion. Similarly, part (ii) of the proposition and (14) yield

$$P[r \in \text{roots}(\mathcal{F}_k) \,|\, \text{counts}(\mathcal{F}_k) = c] = \frac{c_r}{n-1} + \frac{k-1}{n-1} \tag{17}$$

where the two terms can again be interpreted like (16). According to part (ii) of Proposition 1, the conditional probability displayed in (17) is just the fraction of forests F of k trees labeled by S with the given out-degrees $(c_s, s \in S)$ such that r is the root of some tree component of F. As a check, (15) can be recovered from (17) and the multinomial distribution of counts$(\mathcal{F}_k)$ described in Proposition 1, because c_r in (17) is the given value of the binomial$(n-k, p_r)$ variable $|\mathcal{F}_{k,r}|$ whose expectation is $(n-k)p_r$.

3.2 Conditioning on the set of roots.

For a subset R of S with $p_R > 0$, call a random forest $\mathcal{F}_R$ a *p-forest with roots* R if $\mathcal{F}_R$ is distributed like a p-forest of $|R|$ trees conditioned to have root set R. That is, according to the forest volume formula (5) and (11)

$$P(\mathcal{F}_R = F) = p_R^{-1} \prod_{s \in S} p_s^{|F_s|} \qquad (F \in \boldsymbol{F}_{S,R}). \tag{18}$$

Two alternative constructions from p-mappings can be given as follows.

Conditioning a p-mapping to have no cycles within a given set Recall from around (3) that $\mathcal{D}(M)$ is the usual functional digraph with vertex set S associated with a mapping M of S, and that $\hat{\mathcal{D}}_R(M)$ for $R \subseteq S$ is obtained from $\mathcal{D}(M)$ by first deleting all edges leading out of R, then reversing all remaining edges. As a consequence of (3), for $R \subseteq S$ with $p_R > 0$, if M is a p-mapping then

$$\hat{\mathcal{D}}_R(M) \text{ is a } p\text{-forest with roots } R, \text{ given } \hat{\mathcal{D}}_R(M) \in \boldsymbol{F}_{S,R}. \tag{19}$$

Conditioning a p-mapping on its set of cyclic points For a mapping M of S and $v \in S$ define the set of *predecessors of v induced by M* by

$$\text{pred}(v, M) := \{s \in S : M_s^i = v \text{ for some } i \geq 1\} \tag{20}$$

where $s \mapsto M_s^i$ is the ith iterate of M. The set of all cyclic points of M is

$$\text{cyclic}(M) := \{s \in S : s \in \text{pred}(s, M)\}.$$

The usual *forest derived from M* is $\mathcal{F}(M) := \hat{\mathcal{D}}_{\text{cyclic}(M)}(M)$. So $\mathcal{F}(M)$ is a forest of rooted trees labeled by S, with edges directed away from

$$\text{roots}(\mathcal{F}(M)) := \text{cyclic}(M). \tag{21}$$

If $\text{cyclic}(M) = R$, then M is determined by its restriction M^R to R, which is a permutation of R, and its forest $\mathcal{F}(M)$ with $\text{roots}(\mathcal{F}(M)) = R$. So for each forest $F \in \boldsymbol{F}_{S,R}$ and each permutation π of R,

$$P(\mathcal{F}(M) = F, M^R = \pi) = \left(\prod_{s \in S} p_s^{|F_s|}\right) \prod_{r \in R} p_r. \tag{22}$$

Hence for each non-empty subset R of S,

$$\mathcal{F}(M) \text{ is a } p\text{-forest with roots } R, \text{ given } \text{cyclic}(M) = R; \tag{23}$$

$$M^R \text{ is a uniform random permutation of } R, \text{ given } \text{cyclic}(M) = R; \tag{24}$$

$$\mathcal{F}(M) \text{ and } M^R \text{ are conditionally independent, given } \text{cyclic}(M) = R. \tag{25}$$

Note that the two constructions (19) and (23) of a p-forest with roots R are quite different. In (19) the digraph $\hat{\mathcal{D}}_R(M)$ may contain some edges from cycles of $\mathcal{D}(M)$, which cannot appear in $\mathcal{F}(M)$.

By summing (22) over all possible π, the distribution of $\mathcal{F}(M)$ for a p-mapping M is given by

$$P[\mathcal{F}(M) = F] = |\text{roots}(F)|! \left(\prod_{r \in \text{roots}(F)} p_r \right) \left(\prod_{s \in S} p_s^{|F_s|} \right) \tag{26}$$

where F ranges over the set $\boldsymbol{F}_S$ of all $(|S|+1)^{|S|-1}$ rooted forests labeled by S. By application of (21), (26) and the forest volume formula (5), the distribution of the random subset cyclic(M) derived from a p-mapping of S is determined by the formula

$$P(\text{cyclic}(M) = R) = |R|!\, p_R \prod_{r \in R} p_r \qquad (R \subseteq S). \tag{27}$$

Hence for $1 \leq k \leq |S|$ the probability that a p-mapping M of S has exactly k cyclic points is

$$P(|\text{cyclic}(M)| = k) = k! \sum_{|R|=k} p_R \prod_{r \in R} p_r \tag{28}$$

where the sum is over all subsets R of S with $|R| = k$. Jaworski [48, Theorem 2] found an alternative expression for the same probability which can be recast as

$$P(|\text{cyclic}(M)| \geq k) = k! \sum_{|R|=k} \prod_{r \in R} p_r. \tag{29}$$

As a check, either of these formulae (28) and (29) can be deduced from the other. See [24] regarding the asymptotic behaviour of this distribution for large $|S|$, and [65, 7] for related asymptotic results about p-mappings.

Descriptions of a p-forest with roots R. For a forest F labeled by S with roots$(F) = R$, and $v \in S - R$ let $M_v(F) \in S$ be the *mother of v in F*, that is the unique $s \in S$ such that $s \xrightarrow{F} v$. For $A \subseteq S$ the *restriction of F to A* is the forest F^A labeled by A whose set of edges is the intersection with $A \times A$ of the set of edges of F. The following lemma summarizes some basic distributional properties of a p-forest with root set R, which follow easily from these definitions, (11) and (18):

Lemma 3 *Let $\mathcal{H}_1 := \cup_{r\in R}\mathcal{F}_{R,r}$, that is the random set of all vertices of height 1 (children of the roots) in a p-forest $\mathcal{F}_R$ with roots R. Then*
(i) *the distribution of $\mathcal{H}_1$ is given by the formula*

$$P(\mathcal{H}_1 = B) = p_B\, p_{S-R}^{|S-R-B|-1} p_R^{|B|-1} \qquad (B \subseteq S-R) \tag{30}$$

(ii) *for each non-empty $B \subseteq S-R$, the restricted forest $\mathcal{F}_R^{S-R}$ conditioned on $\mathcal{H}_1 = B$ is a $p(\cdot\,|\,S-R)$-forest labeled by $S-R$ with roots B.*
(iii) *Conditionally given $\mathcal{H}_1 = B$, the restricted forest $\mathcal{F}_R^{S-R}$ is independent of the random variables $M_b(\mathcal{F}_R), b \in B$, which are independent with common distribution $p(\cdot\,|\,R)$.*
(iv)

$$\textit{the distribution of } |\mathcal{H}_1| - 1 \textit{ is binomial}(|S| - |R| - 1, p_R) \tag{31}$$

(v) *given $|\mathcal{H}_1| = k$ the restricted forest $\mathcal{F}_R^{S-R}$ is a $p(\cdot\,|\,S-R)$-forest of k trees labeled by $S-R$, with* $\text{roots}(\mathcal{F}_R^{S-R}) = \mathcal{H}_1$.

The following consequence of the previous lemma is the basis for the calculation of various oriented percolation probabilities in Section 5.5. The notation $r \overset{\mathcal{F}_R}{\rightsquigarrow} s$ was defined around (10).

Theorem 4 *For $\mathcal{F}_R$ a p-forest labeled by S with roots $R \subset S$,*

$$P(r \overset{\mathcal{F}_R}{\rightsquigarrow} s) = p_r/p_R \qquad (r \in R, s \in S-R) \tag{32}$$

and for all such r and s the event $(r \overset{\mathcal{F}_R}{\rightsquigarrow} s)$ is independent of the restriction of $\mathcal{F}_R$ to $S-R$.

Proof. Given $\mathcal{H}_1 = B$ say let $X \in B$ be the root of the subtree containing s in the restriction of $\mathcal{F}_R$ to $S-R$. There is a path from r to s in $\mathcal{F}_R$ if and only if $M_X = r$ where $M_X \in R$ is the mother of X in $\mathcal{F}_R$. But according to part (iv) of Lemma 3, given the restricted forest $\mathcal{F}_R^{S-R}$, which together with s determines X, the random variables M_b for $b \in B$ are independent with common distribution $p(\cdot\,|\,R)$. Therefore, the conditional distribution of M_X given $\mathcal{F}_R^{S-R}$ is $p(\cdot\,|\,R)$, as claimed. $\square$

Distribution of level sets. For a random forest F labeled by S let $\mathcal{H}_h(F)$ denote the random subset of S defined by the vertices of F at height h from the root. So $\mathcal{H}_0(F) = \text{roots}(F)$, and for each $h \geq 1$ the set $\mathcal{H}_h(F)$ is the set of all children of vertices in $\mathcal{H}_{h-1}(F)$. Repeated application of Lemma 3 gives a simple formula for the joint distribution of $(\mathcal{H}_i(\mathcal{F}_R), 1 \leq i \leq h)$ for any fixed h. In particular, for $\mathcal{F}_R$ a p-forest with roots R, for each sequence of m non-empty subsets $(B_h, 1 \leq h \leq m)$ whose union is $S - R$,

$$P(\mathcal{H}_h(\mathcal{F}_R) = B_h \text{ for all } 1 \leq h \leq m) = p_R^{|B_1|-1} \prod_{h=2}^{m} p_{B_{h-1}}^{|B_h|}. \tag{33}$$

This is a generalization of a formula of Katz [51] for p uniform and $\mathcal{F}_R$ derived by conditioning $\mathcal{F}(M)$ on $\text{cyclic}(M) = R$ for M a uniform random mapping. See [32, 71] regarding asymptotics of the *height profile* defined by counts of vertices at various levels in a uniform random forest.

4 Spanning Subtrees

Following the approach of Aldous [9, 10, 11] to the asymptotic structure of large random trees, the problem arises of describing the distribution of the subtree spanned by some subset B of the set of vertices of a p-tree $\mathcal{T}$. This problem is most simply treated in terms of the unrooted tree derived from $\mathcal{T}$, whose basic properties are summarized by the following lemma.

Lemma 5 *Let $\mathcal{U}$ be the unrooted tree obtained by ignoring the direction of edges in a random rooted tree $\mathcal{T}$ labeled by S. Then the following two conditions are equivalent:*
(i) *The rooted tree $\mathcal{T}$ is a p-tree, meaning*

$$P(\mathcal{T} = T) = \prod_{s \in S} p_s^{|T_s|} \qquad (T \in \boldsymbol{T}_S) \tag{34}$$

where $\boldsymbol{T}_S$ is the set of $|S|^{|S|-1}$ rooted trees T labeled by S, with edges directed away from $\text{root}(T)$.
(ii) *The distribution of the unrooted tree $\mathcal{U}$ is given by the formula*

$$P(\mathcal{U} = U) = \prod_{s \in S} p_s^{D_s U - 1} \qquad (U \in \boldsymbol{U}_S), \tag{35}$$

where $\boldsymbol{U}_S$ *is the set of* $|S|^{|S|-2}$ *unrooted trees labeled by* S *and*

$$D_s U := |\{v : s \stackrel{U}{\longleftrightarrow} v\}|$$

is the degree of s *in* $U \in \boldsymbol{U}_S$*, and* $\mathcal{U}$ *is independent of* $\mathrm{root}(\mathcal{T})$ *which has distribution* p*:*

$$P[\mathrm{root}(\mathcal{T}) = r] = p_r \qquad\qquad (r \in S). \tag{36}$$

Proof. This follows easily from the tree volume formula, that is (5) for $|R| = 1$, using the well known bijection between $\boldsymbol{U}_S$ and the set $\boldsymbol{T}_{S,r}$ of all trees $T \in \boldsymbol{T}_S$ with $\mathrm{root}(T) = r$, for any fixed $r \in S$. □

The fact that the probabilities in (35) sum to 1 over all $U \in \boldsymbol{U}_S$ amounts by scaling to *Cayley's multinomial expansion over unrooted trees* [25, 76, 70]

$$\sum_{U \in \boldsymbol{U}_S} \prod_{s \in S} x_s^{D_s U - 1} = \left(\sum_{s \in S} x_s \right)^{|S|-2} \tag{37}$$

which for $x_s \equiv 1$ reduces to the Cayley's formula $|\boldsymbol{U}_S| = |S|^{|S|-2}$.

Let $\mathcal{U}$ be an *unrooted* p*-tree labeled by* S, meaning that $\mathcal{U}$ has distribution (35). For an undirected graph G with vertex set S, the probability $P(\mathcal{U} \subseteq G)$ is the sum of probabilities (35) over all spanning trees U of G. Kelmans [54] obtained some results about this polynomial in $(p_s, s \in S)$, which he called the *spanning tree volume* of the vertex-weighted graph (G, p). See also [52] for some generalizations which can be interpreted in terms of random rooted forests, as indicated in [72]. Part (iii) of the following theorem yields a formula for $P(\mathcal{U} \supseteq G)$ for the only graphs G for which this probability is non-zero, that is unrooted forests G. Parts (i) and (ii) of the theorem are the key to the construction in [5] of a model for random trees with edge lengths related to the asymptotics of p-trees with a large number of vertices.

Theorem 6 *Let* $\mathcal{U}$ *be an unrooted* p*-tree labeled by* S*.*
(i) *For each unrooted tree* U *with vertex set* $V(U) \subseteq S$*,*

$$P(\mathcal{U} \supseteq U) = p_{V(U)} \prod_{v \in V(U)} p_v^{D_v U - 1} \tag{38}$$

where $D_v U$ *is the degree of vertex* v *in the tree* U*.*

(ii) *Let B be a subset of S of size two or more, and let $\mathcal{U}_B$ denote the subtree of $\mathcal{U}$ spanning B. Then for every unrooted tree U labeled by $V(U)$ with $B \subseteq V(U) \subseteq S$, such that the set of vertices of U of degree one is contained in B, $P(\mathcal{U}_B = U) = P(\mathcal{U} \supseteq U)$ as given in (38).*
(iii) *For each sequence of unrooted trees $U_i, 1 \leq i \leq m$ with disjoint sets of vertices $V(U_i) \subseteq S$, the events $(\mathcal{U} \supseteq U_i)$ are mutually independent:*

$$P(\cap_{i=1}^{n}(\mathcal{U} \supseteq U_i)) = \prod_{i=1}^{m} P(\mathcal{U} \supseteq U_i). \tag{39}$$

Proof. Fix a tree U with $V(U) = R \subseteq S$. Given that $\mathcal{U} \supseteq U$, let $\mathcal{F}_R$ denote the forest with roots$(\mathcal{F}_R) = R$ derived from $\mathcal{U}$ by first deleting all the edges of U, (which are contained in $R \times R$) then directing the remaining edges away from R. In view of the obvious way that $\mathcal{U}$ can then be recovered from U and $\mathcal{F}_R$, it is easily checked that for each $F \in \boldsymbol{F}_{S,R}$

$$P(\mathcal{U} \supseteq U, \mathcal{F}_R = F) = \left(\prod_{s \in S} p_s^{|F_s|} \right) \prod_{v \in R} p_v^{D_v U - 1} \tag{40}$$

and (38) follows by summation over all $F \in \boldsymbol{F}_{S,R}$, using the forest volume formula (5) and $p_S = 1$. This proves (i), and (ii) follows easily. An alternate proof of (i) can be given by appealing to formula (9), and this argument yields (iii) as well. □

As the notation of the above proof is intended to suggest, formula (40) implies that for each tree U with $V(U) = R \subseteq S$,

$$\mathcal{F}_R \text{ is a } p\text{-forest with roots } R, \text{ given } \mathcal{U} \supseteq U, \tag{41}$$

hence also $\mathcal{F}_R$ is a p-forest with roots R given that the restriction of $\mathcal{U}$ to R is a tree. Compare with the alternate constructions from p-mappings given by (19) and (23).

Corollary 7 *For distinct $u, v \in S$ let $\mathcal{R}_{u,v} := V(\mathcal{U}_{\{u,v\}})$ be the random set of vertices along the path in a p-tree $\mathcal{U}$ from u to v, including u and v. Then the distribution of $\mathcal{R}_{u,v}$ is determined by the formula*

$$P(\mathcal{R}_{u,v} = R) = (|R| - 2)!\, p_R \prod_{r \in R - \{u,v\}} p_r \qquad (\{u,v\} \subseteq R \subseteq S). \tag{42}$$

Proof. For $B = \{u, v\}$ with $u \neq v$, the subtree $\mathcal{U}_{\{u,v\}}$ is determined by its set of vertices $\mathcal{R}_{u,v}$ and the order of these vertices along the path from u to v in $\mathcal{U}$. For each $R \supseteq \{u, v\}$ with $|R| = k$, and each permutation $(r_1, \ldots, r_k) : \{1, \ldots, k\} \to R$ with $r_1 = u$ and $r_k = v$, formula (40) shows that the path from u to v in $\mathcal{U}$ equals $(r_1, \ldots, r_k)$ with probability $p_R \prod_{v \in R - \{u,v\}} p_v$. Since there are $(k-2)!$ possible paths, each with this same probability, (42) follows. $\square$

Since $u \overset{\mathcal{U}}{\longleftrightarrow} v$ if and only if $\mathcal{R}_{u,v} = \{u, v\}$, the particular case of (42) with $R = \{u, v\}$ gives for $u \neq v$

$$P(u \overset{\mathcal{U}}{\longleftrightarrow} v) = p_u + p_v. \tag{43}$$

If $\mathcal{U}$ is defined by unrooting a rooted p-tree $\mathcal{T}$, then obviously

$$P(u \overset{\mathcal{U}}{\longleftrightarrow} v) = P(u \overset{\mathcal{T}}{\to} v) + P(v \overset{\mathcal{T}}{\to} u) \tag{44}$$

so (43) is implied by the simpler formula

$$P(u \overset{\mathcal{T}}{\to} v) = p_u \tag{45}$$

which can be read from (9). A later formula (91) gives the generalization of (45) for a p-forest $\mathcal{F}$ instead of a p-tree $\mathcal{T}$. Another extension of (43), which can be read from (38) and (37), is the following formula, valid for arbitrary $V \subseteq S$:

$$P(\text{the restriction of } \mathcal{U} \text{ to } V \text{ is a tree}) = p_V^{|V|-1}, \tag{46}$$

and given that this event occurs, the restricted tree is a $p(\cdot \,|\, V)$-tree. See Section 6 for more about restrictions of p-trees and p-forests. As a check, for uniform p formula (43) reduces to the well known result [60],[62, Th. 6.1] that for $n \geq 2$ the number of unrooted trees labeled by a set of n vertices which contain a particular edge is $2n^{n-3}$. As remarked by Stone [84] this number can be computed in another way to yield an instance of one of Abel's binomial identities [37]. A variation of this argument, indicated in [72], yields Hurwitz's generalization of Abel's identity stated as (55) in the next section, and its multivariate form expressed probabilistically in Proposition 15.

Some useful variations of formula (42) can be formulated as follows, in terms of a rooted p-tree $\mathcal{T}$. See also [24] for closely related formulae.

Corollary 8 *For $T \in \boldsymbol{T}_S$ and $v \in S$ let $\mathcal{R}(T,v)$ denote the range of the directed path in T from* root(T) *to v. In particular, $\mathcal{R}(T,v) = \{v\}$ if* root$(T) = v$. *Let $\mathcal{T}$ be a p-tree with vertex set S. Then*
(i) *for each fixed $v \in S$*

$$P[\mathcal{R}(\mathcal{T},v) = R] = (|R|-1)!\, p_R \prod_{r \in R - \{v\}} p_r \qquad (v \in R \subseteq S); \tag{47}$$

(ii) *if V is a random vertex with distribution p on S, independent of $\mathcal{T}$, then*

$$P[\mathcal{R}(\mathcal{T},V) = R] = |R|!\, p_R \prod_{r \in R} p_r \qquad (R \subseteq S). \tag{48}$$

Proof. (i) Formula (47) can be deduced like (42) from a variant of (40) with rooted trees, or obtained as follows by application of Lemma 5 and Corollary 7. Let $\mathcal{U}$ be the unrooted p-tree derived from $\mathcal{T}$, and let $\mathcal{R}_{u,v}$ be as in Corollary 7. Then for $|R| \geq 2$

$$P(\mathcal{R}(\mathcal{T},v) = R) = \sum_{u \in R-\{v\}} P(\mathrm{root}(\mathcal{T}) = u, \mathcal{R}_{u,v} = R)$$

$$= \sum_{u \in R-\{v\}} p_u P(\mathcal{R}_{u,v} = R)$$

since root$(\mathcal{T})$ has distibution p, and root$(\mathcal{T})$ is independent of $\mathcal{U}$ and hence of $\mathcal{R}_{u,v}$, by Lemma 5. Formula (47) now follows from (42). Moreover, (47) holds also in the case $|R| = 1$, that is for $R = \{v\}$, since it then reduces to the previous result (35) that root$(\mathcal{T})$ has distribution p.
(ii) By independence of $\mathcal{T}$ and V,

$$P[\mathcal{R}(\mathcal{T},V) = R] = \sum_{v \in R} p_v P[\mathcal{R}(\mathcal{T},v) = R]$$

and (48) follows from (47). □

Compare (48) and (27) to see that the range $\mathcal{R}(\mathcal{T},V)$ of the path in a p-tree from its root to an independent p-distributed vertex V has the same distribution as the random set of cyclic points of a p-mapping M:

$$\mathcal{R}(\mathcal{T},V) \stackrel{d}{=} \mathrm{cyclic}(M) \tag{49}$$

and hence

$$|\mathcal{R}(\mathcal{T}, V)| \stackrel{d}{=} |\text{cyclic}(M)|. \tag{50}$$

So formulae (28) and (29) for the distribution of $|\text{cyclic}(M)|$, and the asymptotic results derived from these formulae in [24], apply to $\mathcal{R}(\mathcal{T}, V)$ as well as to cyclic(M).

4.1 Joyal's bijection between marked rooted trees and mappings

The coincidence in distribution (49), and numerous further coincidences involving the distributions of functionals of p-trees and p-mappings, are explained by Joyal's bijection $J : (\boldsymbol{T}_S \times S) \to S$, which is constructed as follows. First, for each subset R of S with $|R| = k \geq 1$, set up a bijective correspondence between the $k!$ permutations

$$(r_1, \ldots, r_k) : [k] \to R = \{r_1, \ldots, r_k\}$$

and the $k!$ permutations $\pi : R \to R$, say

$$\pi(s) = \pi_{(r_1,\ldots,r_k)}(s) \in R = \{r_1, \ldots, r_k\}.$$

One way to do this is to declare $\pi(s) = r_j$ if s is the jth smallest element of R with respect to some total ordering of S. But other choices of the π corresponding to $(r_1, \ldots, r_k)$ may be useful, as indicated later. Given such a correspondence, for $T \in \boldsymbol{T}_S, v \in S$ define a mapping $M = J(T, v) \in S^S$ by letting M_s be the mother of s in T, except if v lies on the path $(r_1, \ldots, r_k)$ say in T from $r_1 = \text{root}(T)$ to $r_k = v$, in which case

$$M_s = \pi_{(r_1,\ldots,r_k)}(s) \in \{r_1, \ldots, r_k\}.$$

Joyal [49, p. 16] observed that J sets up a bijection between $\boldsymbol{T}_S \times S$ and S^S, and that Cayley's formula $|\boldsymbol{T}_S| = |S|^{|S|-1|}$ is an immediate consequence. By construction, if $M = J(T, v)$ then the set of cyclic points of M is the range of the directed path in T from root(T) to v:

$$\text{cyclic}(M) = \mathcal{R}(T, v), \tag{51}$$

and furthermore the forest derived from M is

$$\mathcal{F}(M) = T - \{\text{edges of } T \text{ on the path in } T \text{ from root}(T) \text{ to } v\}. \tag{52}$$

The coincidence in distribution (49) is now explained by the following proposition.

Proposition 9 *Let $\mathcal{T}$ be a p-tree with vertex set S, and V an S-valued random variable independent of $\mathcal{T}$, with distribution p on S. Then $M := J(\mathcal{T}, V)$ is a p-mapping of S.*

Proof. Since M is determined by its forest $\mathcal{F}(M)$ and the its action as a permutation of $\text{roots}(\mathcal{F}(M)) = \text{cyclic}(M)$, it suffices to check that the joint distribution of $\mathcal{F}(M)$ and the restriction M to $\text{cyclic}(M)$ is given by the same formula (22) as if M were a p-mapping. But this formula (22) is readily verified by a variation of formula (40) for rooted trees, using (51) and (52), where the factor of p_v required in the formula appears from $P(V = v) = p_v$. □

Proposition 9 provides a powerful method for transferring results on the asymptotic structure of p-trees to corresponding results for p-mappings. See [7].

5 Hurwitz distributions

Hurwitz [43] studied sums of the form

$$H_n^{\gamma,\delta} := H_n^{\gamma,\delta}(x, y; z_s, s \in [n]) := \sum_{A \subseteq [n]} (x + z_A)^{|A|+\gamma}(y + z_{\bar{A}})^{|\bar{A}|+\delta} \tag{53}$$

for integers γ and δ, where the sum is over all 2^n subsets A of $[n]$, and $\bar{A} := [n] - A$. Hurwitz used recurrences to obtain the identities

$$xH_n^{-1,0} = yH_n^{0,-1} = (x + y + z_{[n]})^n, \tag{54}$$

$$xyH_n^{-1,-1} = (x + y)(x + y + z_{[n]})^{n-1} \tag{55}$$

which follows easily from (54), and

$$H_n^{0,0} = \sum_{A \subseteq [n]} |A|!\left(\prod_{s \in A} z_s\right)(x + y + z_{[n]})^{|\bar{A}|}. \tag{56}$$

As noted by Hurwitz, for $z_s \equiv 1$ these formulae yield evaluations of corresponding *Abel sums* [1]

$$A_n^{\gamma,\delta}(x, y) := \sum_{a=0}^{n} \binom{n}{a} (x + a)^{a+\gamma}(y + n - a)^{n-a+\delta}. \tag{57}$$

Strehl [86] explains how Hurwitz was led to such identities via the combinatorial problem, which arose in the theory of Riemann surfaces [42], of counting the number of ways a given permutation can be written as a product of a minimal number of transpositions which generate the full symmetric group. For various combinatorial interpretations of these identities and related formulae see [2, 47, 55, 36, 22, 77, 80, 85, 87].

Definition 10 Let p be a probability distribution on the interval of integers $[0, n+1] := \{0, 1, \dots, n, n+1\}$. Say that a random subset V of $[n]$ has the *Hurwitz distribution of index* (γ, δ) *with parameters* $p_0, p_1, \dots, p_{n+1}$, abbreviated $H_n^{\gamma,\delta}(p)$, if $P(V = A)$ is proportional to the Ath term of the Hurwitz sum $H_n^{\gamma,\delta}(p) = H_n^{\gamma,\delta}(p_0, p_{n+1}; p_s, s \in [n])$ defined by (53) as A ranges over subsets of $[n]$. That is to say

$$P(V = A) = H_n^{\gamma,\delta}(p)^{-1}(p_0 + p_A)^{|A|+\gamma}(p_{n+1} + p_{\bar{A}})^{|\bar{A}|+\delta} \qquad (A \subseteq [n]). \quad (58)$$

where in particular, according to (54) and (55),

$$H_n^{-1,0}(p) = \frac{1}{p_0} \quad \text{and} \quad H_n^{-1,-1}(p) = \frac{p_0 + p_{n+1}}{p_0 p_{n+1}} \tag{59}$$

Call the distribution of $|V|$ on $[0, n]$ induced by such a random subset V of $[n]$ the $H_n^{\gamma,\delta}(p)$*-binomial distribution.*

These formulae should be interpreted by continuity in the limit cases when either p_0 or p_{n+1} equals 0. In the *Abel case*

$$p_0 = x/\Sigma; \ p_{n+1} = y/\Sigma; \ p_i = 1/\Sigma \text{ for } i \in [n] \tag{60}$$

where $\Sigma := x + y + n$ for arbitrary $x, y \geq 0$, the $H_n^{\gamma,\delta}(p)$-binomial distribution on $[0, n]$ is obtained by normalization of the terms of the corresponding Abel sum $A_n^{\gamma,\delta}(x, y)$ defined by (57). Call this the *Abel-binomial distribution* or $A_n^{\gamma,\delta}(x, y)$-binomial distribution to indicate the parameters. The Abel-binomial distributions $A_n^{-1,-1}(x, y)$ and $A_n^{-1,0}(x, y)$ are known in the statistical literature as *quasi-binomial distributions* [30, 29, 28, 26].

5.1 Constructions from p-mappings

Recall from (20) that $\text{pred}(v, M)$ is the set of predecessors of v induced by a mapping M. The following theorem summarizes some natural constructions from random mappings of random subsets of $[n]$ with Hurwitz distributions: See also Berg and Mutafchiev [18] for a closely related appearance of Abel-binomial distributions in connection with random mappings.

Theorem 11 *Let M be a p-mapping of $[0, n+1]$. Then each of the following three random subsets of $[n]$ has the Hurwitz distribution $H_n^{-1,0}(p)$:*
(i) (Françon [36, p. 339]) *assuming $p_0 p_{n+1} > 0$, the random set* $\mathrm{pred}(0, M)$ *conditionally given that both* 0 *and* $n+1$ *are fixed points of M;*
(ii) (Jaworski [48, Theorem 3]) *assuming $p_{n+1} = 0$, the random set* $[n] \cap \mathrm{pred}(0, M)$*;*
(iii) *assuming $p_{n+1} > 0$, the random set* $\mathrm{pred}(0, M)$ *conditionally given that $n+1$ is the unique cyclic point of M.*
Moreover, assuming $p_0 p_{n+1} > 0$, a random subset of $[n]$ with the Hurwitz distribution $H_n^{-1,-1}(p)$ is obtained from (iv) (Françon [36, Prop. 3.5]) *assuming $p_0 p_{n+1} > 0$, the random set* $\mathrm{pred}(0, M)$ *conditionally given that both* 0 *and* $n+1$ *are the unique fixed points of M.*

Proof. Parts (i), (ii) and (iv) can be read from the sources cited. Part (iii) is equivalent to the result formulated and proved for a p-tree in Theorem 12 below. All parts are easily checked using the forest volume formula (5). $\square$

Before discussing the translation of (iii) in terms of a p-tree, it is worth noting some striking differences between the first three cases of Theorem 11. Each connected component C of $\mathcal{D}(M)$ contains a unique cycle C_0, and C decomposes further into a collection of tree components of $\mathcal{F}(M)$ whose set of roots is C_0. In case (i) of the proposition, the conditioning forces $\mathrm{pred}(0, M) \cup \{0\}$ to be a connected component of $\mathcal{D}(M)$ which is a single tree component of $\mathcal{F}(M)$ rooted at 0, while $n+1$ is forced to be the unique cyclic point of another component of $\mathcal{D}(M)$. In case (ii) the set $\mathrm{pred}(0, M) \cup \{0\}$ may be either the union of a tree component of $\mathcal{F}(M)$ and a cycle of arbitrary size, or just a subtree of a tree component, according to whether or not $0 \in \mathrm{cyclic}(M)$. In case (iii) the conditioning forces $\mathcal{F}(M)$ to be a tree rooted at $n+1$, and $\mathrm{pred}(0, M) \cup \{0\}$ is a fringe subtree of this tree, as discussed below. As discussed in [15], it is possible to pass between the various cases of Theorem 11 using Joyal's bijection between random mapppings and marked trees, as described in Section 4.

5.2 Constructions from p-trees

For a forest F labeled by S let $V_s(F) := \{v \in S - \{s\} : s \overset{\mathbf{t}}{\rightsquigarrow} v\}$ denote the set of non-root vertices of the *fringe subtree of T rooted at s*, that is the tree $T(s)$ labeled by $\{s\} \cup V_s(T)$ whose edge relation is the restriction to $\{s\} \cup V_s(T)$ of the edge relation of T. See [8] for background and further references to fringe subtrees. If T is a tree component of the forest $\mathcal{F}(M)$ derived from a mapping M, so T is rooted at some vertex $r \in \text{cyclic}(M)$, then for each non-root vertex s of T the set $\text{pred}(s, M)$ of predecessors of s induced by M is identical to $V_s(T)$. As remarked below (23), conditioning a p-mapping M to have a unique cyclic point r makes $\mathcal{F}(M)$ a p-tree with root r. Case (iii) of Theorem 11 can thus be reformulated as follows in terms of trees instead of mappings:

Theorem 12 *Let $\mathcal{T}_{n+1}$ be a p-tree labeled by $[0, n+1]$ with root $n+1$, where p is a probability distribution on $[0, n+1]$ with $p_{n+1} > 0$. Then the random set $V_0(\mathcal{T}_{n+1})$ of non-root vertices of the fringe subtree of $\mathcal{T}_{n+1}$ rooted at 0 has the Hurwitz distribution $H_n^{-1,0}(p)$ on subsets of $[n]$. That is, for all $A \subseteq [n]$, with $\bar{A} := [n] - A$.*

$$P(V_0(\mathcal{T}_{n+1}) = A) = p_0(p_0 + p_A)^{|A|-1}(p_{n+1} + p_{\bar{A}})^{|\bar{A}|}. \tag{61}$$

Proof. Fix an arbitrary subset A of $[n]$. The probability $P(V_0(\mathcal{T}_{n+1}) = A)$ is the sum of the probabilities $P(\mathcal{T}_{n+1} = T)$ over all T such that (i) the restriction of T to $A \cup \{0\}$ is a tree with root 0, and (ii) the restriction of T to $\bar{A} \cup \{0\} \cup \{n+1\}$ is a tree with root $n+1$ which has 0 as a leaf. Each of these probabilities factorizes into one product involving $(p_s, s \in A \cup \{0\})$ and another involving $(p_s, s \in \bar{A} \cup \{n+1\})$. The sum of products is therefore the product of two sums which can be evaluated using the tree volume formula (5), first with $r = 0$ and $S = A \cup \{0\}$, then with $r = n+1$ and $S = \bar{A} \cup \{0\} \cup \{n+1\}$ with $x_0 = 0$. □

The same technique yields the following variation of Theorem 12:

Theorem 13 *Let $V_0(\mathcal{F}_k) \subseteq [n]$ be the set of non-root vertices of the fringe subtree of $\mathcal{F}_k$ rooted at 0, for $\mathcal{F}_k$ a p-forest of k trees labeled by $[0, n]$. Then for $A \subseteq [n]$, with $\bar{A} := [n] - A$,*

$$P(V_0(\mathcal{F}_k) = A) = \binom{n}{k-1}^{-1} p_0(p_0 + p_A)^{|A|-1} \binom{|\bar{A}|}{k-1} p_{\bar{A}}^{|\bar{A}|-(k-1)}. \tag{62}$$

Proof. This is similar to the proof of the previous proposition. Fix $A \subseteq [n]$. A forest F has $V_0(F) = A$ if and only if (i) the restriction of F to $A \cup \{0\}$ is a tree with root 0, and (ii) the restriction of F to $\bar{A} \cup \{0\}$ is a forest of k trees with 0 as a leaf vertex. The relevant sum of products is therefore factorizes into a product of two sums, the first of whichwhich can be evaluated using the forest volume formula (5) with $R = \{0\}$, $S = A \cup \{0\}$, and the second of which yields to (8) with $S = \bar{A} \cup \{0\}$ and $x_0 = 0$. $\square$

5.3 Distribution of tree components

Formulae for the distributions of variously defined tree components of a p-forest follow easily from the forest volume formula. The next two propositions are typical examples.

Proposition 14 *Let p be a probability distribution on $[0, n]$, and let $2 \leq k \leq n$. For $\mathcal{F}_k$ with the distribution induced by p on forests of k trees labeled by $[0, n]$, let $W_0(\mathcal{F}_k) \subseteq [n]$ be the random set of all vertices other than 0 in the tree component of $\mathcal{F}_k$ containing 0. Then for $A \subseteq [n]$, with $\bar{A} := [n] - A$,*

$$P(W_0(\mathcal{F}_k) = A) = \binom{n}{k-1}^{-1} \binom{|\bar{A}| - 1}{k-2} (p_0 + p_A)^{|A|} p_{\bar{A}}^{|\bar{A}|-(k-1)} \qquad (A \subseteq [n]) \tag{63}$$

The first part of the following proposition spells out the probabilistic interpretation of Hurwitz's multinomial theorem [43, VI] [72, (16)] in terms of a p-forest with roots R, as defined by (18).

Proposition 15 *Let $\mathcal{F}_R$ be a p-forest with roots R and vertex set $R \cup [n]$, with R disjoint from $[n]$. For $r \in R$ let $V_r(\mathcal{F}_R)$ be the random subset of $[n]$ defined by the non-root vertices of the tree component of $\mathcal{F}_R$ containing r. Then*
(i) *for each of $|R|^n$ possible choices of disjoint subsets $(B_r, r \in R)$ whose union is $[n]$*

$$P(V_r(\mathcal{F}_R) = B_r \text{ for all } r \in R) = p_R^{-1} \prod_{r \in R} p_r (p_r + p_{B_r})^{|B_r|-1} \tag{64}$$

(ii) *for each subset B of R the random set $V_B(\mathcal{F}_R) := \cup_{r \in B} V_r(\mathcal{F}_R)$ has the Hurwitz distribution $H_n^{-1,-1}(p^B)$ on subsets of $[n]$, where $p_0^B = p_B, p_{n+1}^B = p_{R-B}$, and $p_s^B = p_s$ for $s \in [n]$.*

For $V_B(\mathcal{F}_R)$ defined as in the previous Proposition, there is the remarkably simple formula

$$E(|V_B(\mathcal{F}_R)|) = np_B/p_R \tag{65}$$

because $V_B(\mathcal{F}_R)$ is the sum of the indicator variables $\chi(r \overset{\mathcal{F}_R}{\rightsquigarrow} s)$ over all $r \in B$ and $s \in [n]$, so formula (32) can be applied to compute:

$$E(|V_B(\mathcal{F}_R)|) = \sum_{r \in B} \sum_{s=1}^{n} P(r \overset{\mathcal{F}_R}{\rightsquigarrow} s) = \sum_{r \in B} np_r/p_R = np_B/p_R. \tag{66}$$

On the other hand, Proposition 15(ii) shows that (65) amounts to:

Proposition 16 *The mean of the $H_n^{-1,-1}(p)$-binomial distribution of $|V_{n,p}|$, where $V_{n,p}$ is a random subset of $[n]$ with the Hurwitz $H_n^{-1,-1}(p)$ distribution, is*

$$E\left(|V_{n,p}|\right) = n\left(\frac{p_0}{p_0 + p_{n+1}}\right). \tag{67}$$

This formula can also be checked as follows. Differentiate Hurwitz's formula (54) with respect to x to obtain

$$\sum_{A \subseteq [n]} y\,|A|\,(x + z_A)^{|A|-1}(y + z_{\bar{A}})^{|\bar{A}|-1} = n(x + y + z_{[n]})^{n-1}. \tag{68}$$

From the definition (10) of the $H_n^{-1,-1}(p)$ distribution,

$$E(|V_{n,p}|) = \sum_{A \subseteq [n]} \frac{|A|\,p_0\,p_{n+1}}{(p_0 + p_{n+1})}(p_0 + p_A)^{|A|-1}(p_{n+1} + p_{\bar{A}})^{|\bar{A}|-1}$$

and (67) follows by application of (68) with $x = p_0, y = p_{n+1}$ and $z_s = p_s$ for $s \in [n]$. $\square$

Formula (67) is a generalization of the known result [26] that the Abel $A_n^{-1,-1}(x,y)$-binomial distribution has mean $nx/(x+y)$. The proof of (67) just indicated via (66) provides a probabilistic explanation for this otherwise mysterious exception to the general rule that moments of Abel-binomial distributions are not simple functions of the parameters. See for instance [26] where a complicated expression is obtained for the second factorial moment of the $A_n^{-1,-1}(x,y)$-binomial distribution. In view of this difficulty in the

Abel case, it does not seem possible to simplify the Hurwitz sums for higher moments of the $H_n^{-1,-1}(p)$-binomial distribution. For the $H_n^{0,-1}(p)$-binomial distribution, the Hurwitz sum for the mean does not simplify even in the Abel case.

5.4 A Hurwitz multinomial distribution

Riordan [77] considers multinomial forms of Abel's binomial theorem. See Berg and Mutafchiev [18] for the appearance of an Abel-trinomial distribution in the context of random mappings. The following definition is motivated by Proposition 15.

Definition 17 For a probability distribution p on $[n] \cup R$ with $p_R > 0$, where R is a finite set disjoint from $[n]$, say that a random vector of non-negative integers $\boldsymbol{N}_R := (N_r, r \in R)$ has the *Hurwitz*(p)*-multinomial distribution* if for all vectors of non-negative integers $\boldsymbol{n}_R := (n_r, r \in R)$ with $\sum_r n_r = n$

$$P(\boldsymbol{N}_R = \boldsymbol{n}_R) = p_R^{-1} \sum_{(B_r)} \prod_{r \in R} p_r (p_r + p_{B_r})^{n_r - 1} \tag{69}$$

where the sum is over all $n!/(\prod_r n_r!)$ possible choices of disjoint subsets B_r of $[n]$ whose union is $[n]$ with $|B_r| = n_r, r \in R$.

According to Proposition 15, a random vector $\boldsymbol{N}_R$ with this distribution is obtained by letting N_r be the size of the tree rooted at r in a p-forest with roots R and vertex set $[n] \cup R$. The usual multinomial distribution with parameters n and $(p_r, r \in R)$ corresponds to the case when $p_s = 0$ for all $s \in [n]$. Then in the forest $\mathcal{F}_R$, each vertex $s \in [n]$ is a leaf attached to a root $M_s \in R$ where the M_s are independent with common distribution p. According to Lemma 3, in the general model the restricted forest $\mathcal{F}_R^{[n]}$ clusters the elements of $[n]$ into a random number K of subtrees such that $K - 1$ has binomial$(n-1, p_R)$ distribution. Given K the forest $\mathcal{F}_R^{[n]}$ is a $p(\cdot \,|\, [n])$-forest of K trees labeled by $[n]$, and each of these subtrees is attached to a root picked independently from R according to $p(\cdot \,|\, R)$. The size N_r of the tree rooted at r is then the sum of the sizes of those subtrees of $\mathcal{F}_R^{[n]}$ that happen to have r chosen as their root. From this construction of a random vector $\boldsymbol{N}_R$ with the Hurwitz(p)-multinomial distribution it follows without calculation that this family of multivariate distributions shares with the usual family of multinomial distributions the following basic rule for merging of categories. That is, if Ψ is a map from R to Q say, and $\boldsymbol{N}_Q$ is derived from $\boldsymbol{N}_R$ by merging categories according to Ψ, so the qth component of $\boldsymbol{N}_Q$ is the sum of N_r over r with $\Psi(r) = q$, then $\boldsymbol{N}_Q$ has the Hurwitz(p')-multinomial distribution, where p' is the probability distribution on $Q \cup [n]$ defined by $p'_s = p_s$ if $s \in [n]$ and p'_q is the sum of p_r over r with $\Psi(r) = q$.

5.5 Percolation probabilities

Write $u \overset{\mathbf{f}}{\sim} v$ if there is a path from u to v in the undirected graph obtained by ignoring edge directions in a rooted forest F, that is if $T_u(F) = T_v(F)$, where $T_v(F)$ denotes the set of vertices of the tree component of F containing v. Write $u \overset{\mathbf{f}}{\not\sim} v$ if there is no such path, meaning T_u and T_v are disjoint. This Section treats the problem of finding expressions for the *percolation probability* $P(s \overset{\mathcal{F}_k}{\sim} v)$ and the *oriented percolation probability* $P(s \overset{\mathcal{F}_k}{\rightsquigarrow} v)$ for two vertices s and v of a p-forest $\mathcal{F}_k$. See [23, 73, 46] for closely related studies of such percolation probabilities for the digraph of a random mapping, and [39] for a study of such problems for other models of random forests, and applications to reliability of networks.

Unriented percolation. By a suitable relabeling, it suffices to consider $P(0 \overset{\mathcal{F}_k}{\sim} n+1)$ in the case $S := [0, n+1]$ and $2 \le k \le n+1$. By an argument similar to the proof of Proposition 14,

$$P(0 \overset{\mathcal{F}_k}{\sim} n+1) = \sum_{A \subset [n]} \binom{n+1}{k-1}^{-1} (p_0 + p_{n+1} + p_A)^{|A|+1} \binom{|\bar{A}|-1}{k-2} p_{\bar{A}}^{|\bar{A}|-k+1} \quad (70)$$

where Ath term is $P(T_0(\mathcal{F}_k) = T_{n+1}(\mathcal{F}_k) = \{0\} \cup \{n+1\} \cup A)$. Similarly

$$P(0 \overset{\mathcal{F}_k}{\not\sim} n+1) = \sum_{A \subset [n]} \binom{n+1}{k-1}^{-1} (p_0 + p_A)^{|A|} \binom{|\bar{A}|}{k-2} (p_{n+1} + p_{\bar{A}})^{|\bar{A}|-k+2} \quad (71)$$

where the Ath term is $P(T_0(\mathcal{F}_k) = \{0\} \cup A)$. Another expression for the same probability is obtained by switching p_0 and p_{n+1}, since the Ath term is then $P(V_{n+1}(\mathcal{F}_k) = \{n+1\} \cup A)$. The consequent equality of polynomials in $p_s, s \in S$ is a non-trivial identity, even in the Abel case (60). So is the equality between either of these expressions and 1 minus the right hand expression in (70), where 1 must be replaced by $(\sum_{i=0}^{n+1} p_s)^{n-k+2}$ to obtain the general polynomial identity.

Oriented percolation. By a relabeling of vertices, the problem of finding $P(s \overset{\mathcal{F}_k}{\rightsquigarrow} v)$ for two arbitrary vertices s and v of a p-forest $\mathcal{F}_k$ is reduced to the case when $S = [0, n+1]$, $s = 0$ and $v = n+1$, as in the following proposition.

Proposition 18 *Let $\mathcal{F}_k$ be a p-forest of k trees labeled by $[0, n+1]$. Then*

$$P(0 \overset{\mathcal{F}_k}{\rightsquigarrow} n+1) = \sum_{A \subseteq [n]} \frac{(|\bar{A}|)_{k-1}}{(n+1)_{k-1}} p_0 (p_0 + p_A)^{|A|} (p_{n+1} + p_{\bar{A}})^{|\bar{A}|-(k-1)} \tag{72}$$

where the Ath term equals $P(0 \overset{\mathcal{F}_k}{\rightsquigarrow} n+1, V_k = A)$ for V_k the random set of all $v \in [n]$ such that there exists a directed path from 0 to v in $\mathcal{F}_k$ that does not pass via $n+1$. Also

$$P(0 \overset{\mathcal{F}_k}{\rightsquigarrow} n+1) = \sum_{A \subseteq [n]} \frac{(|\bar{A}|)_{k-1}}{(n+1)_{k-1}} \, |A|! \, p_0 \prod_{s \in A} p_s \tag{73}$$

where the Ath term equals $P(0 \overset{\mathcal{F}_k}{\rightsquigarrow} n+1, \mathcal{L}_k = A)$ for $\mathcal{L}_k$ the random set of all $v \in [n]$ such that v lies on the path which joins 0 to the root of its tree component in $\mathcal{F}_k$.

Proof. These formulae are obtained by application of the forest volume formula, with the help of Theorem 4. See [72] for details. □

For $k = 1$, Proposition 18 yields Hurwitz's expression (56) for $H_n^{0,0}$, along with the following probabilistic interpretation: for $\mathcal{T}_n$ a p-tree labeled by $[0, n+1]$

$$P(0 \overset{\mathcal{T}_n}{\rightsquigarrow} n+1) = p_0 H_n^{0,0}(p_0, p_{n+1}; p_j, j \in [n]). \tag{74}$$

In the Abel case (60) with $x = y = 1$ this implies that $A_n^{0,0}(1,1)$ is the number of rooted trees labeled by $[0, n+1]$ in which there is a directed path from u to v, for arbitrary distinct $u, v \in [0, n+1]$. It is easy to deduce from this the result of Moon [61, Theorem 1], that for $\mathcal{T}_n$ with uniform distribution on rooted trees labeled by $[0, n+1]$, the conditional expectation of the size $|\{v : 0 \overset{\mathcal{T}_n}{\rightsquigarrow} v\}|$ of the fringe subtree of $\mathcal{T}_n$ with root 0, given that $\mathcal{T}_n$ has some root other than 0, is $A_n^{0,0}(1,1)/(n+2)^n$. As observed by Moon, this conditional expectation is also the expected distance in T between any two distinct vertices $u, v \in S$, which is asymptotically equivalent to $\sqrt{\pi n/2}$ for large n. See [14] for a study of the asymptotic behaviour for large n of the distribution of the size of the fringe tree $\{v : 0 \stackrel{T}{\leadsto} v\}$ for T distributed according to a non-uniform forest volume distribution on trees labeled by S, and [5, 24] for further study of the asymptotics of large random trees of this kind.

6 Restrictions of p-forests

Call a random forest $\mathcal{F}$ with a random number of trees a *p-forest* if $\mathcal{F}$ is a p-forest of k trees conditionally given that $\mathcal{F}$ has k trees. Put another way, a random element $\mathcal{F}$ of the set $\boldsymbol{F}_S$ of all forests of rooted trees labeled by S is a p-forest if and only if the distribution of $\mathcal{F}$ is given by the formula

$$P(\mathcal{F} = F) = w_{|F|} \prod_{s \in S} p_s^{|F_s|} \qquad (F \in \boldsymbol{F}_S) \tag{75}$$

for some sequence of weights $(w_m, 1 \leq m \leq |S| - 1)$, where $|F| = \sum_s |F_s|$ is the number of edges of F.

Note. By Proposition 2 (i) and (27), the forest $\mathcal{F}(M)$ derived from a p-mapping is a p-forest if and only if p is uniform on S. However, Corollary 20 below shows how a $p(\cdot|B)$-forest for $B \subset S$ can be obtained by suitable conditioning of a p-mapping.

The following *restriction theorem* is proved in the next subsection.

Theorem 19 *For B a non-empty subset of S and $\mathcal{F}$ a p-forest labeled by S, the restriction $\mathcal{F}^B$ of $\mathcal{F}$ to B is a $p(\cdot \,|\, B)$-forest. The distribution of $|\mathcal{F}^B|$ on $0, \ldots, |B|-1$ is determined by p_B and the distribution of the random number $|\mathcal{F}|$ of edges of $\mathcal{F}$ by the falling factorial moments*

$$E(|\mathcal{F}^B|)_r = E(|\mathcal{F}|)_r \frac{(|B|-1)_r}{(n-1)_r} p_B^r \qquad (r = 1, 2, \ldots). \tag{76}$$

In particular, if $|\mathcal{F}|$ has binomial $(|S|-1, q_0)$ distribution for some $q_0 \in [0,1]$, then $|\mathcal{F}^B|$ has binomial $(|B|-1, q_0\, p_B)$ distribution.

To be more explicit, the distribution of $|\mathcal{F}^B|$ is determined by the factorial moments (76) via the well known sieve formula [20, p. 17]

$$P(|\mathcal{F}^B| = \ell) = \sum_{r=\ell}^{|B|-1} \binom{r}{\ell} (-1)^{r-\ell} \frac{E(|\mathcal{F}^B|)_r}{r!} \quad (0 \le \ell \le |B|-1). \tag{77}$$

Before the proof of Theorem 19, here is a corollary obtained by simply combining this theorem and Lemma 3, followed by some examples to illustrate formula (77). The corollary shows in particular how a q-forest labeled by B can be derived from a p-tree with root $0 \notin B$, for any distribution p on a superset S of $B \cup \{0\}$ such that $q = p(\cdot|B)$.

Corollary 20 *For R and B disjoint non-empty subsets of a finite set S, with $p_R > 0$, the restriction to B of p-forest with roots R is a $p(\cdot\,|\,B)$-forest with a binomial$(|B|-1, p_B)$ number of edges. In particular, if $\mathcal{F}(M)$ is the forest derived from a p-mapping M of S, then conditionally given* cyclic$(M) = R$ *the restriction of $\mathcal{F}(M)$ to B is a $p(\cdot\,|\,B)$-forest with a binomial$(|B|-1, p_B)$ number of edges for each $B \subseteq S - R$.*

Examples. According to (76) and (77), assuming that $\mathcal{F} = \mathcal{F}_k$ has a fixed number k of tree components, so $E(|\mathcal{F}|)_r = (n-k)_r$, for each B with $|B| = b$ the restriction of $\mathcal{F}_k$ to B is a tree with probability

$$P(|\mathcal{F}_k^B| = b-1) = \frac{(n-k)_{b-1}}{(n-1)_{b-1}}\, p_B^{b-1}. \tag{78}$$

The restriction has two tree components with probability

$$P(|\mathcal{F}_k^B| = b-2) = (b-1)\left(\frac{(n-k)_{b-2}}{(n-1)_{b-2}}\, p_B^{b-2} - \frac{(n-k)_{b-1}}{(n-1)_{b-1}}\, p_B^{b-1}\right) \tag{79}$$

and so on. For p uniform, $p_B = b/n$, and the above probabilities are fractions of the total number $\binom{n-1}{k-1} n^{n-k}$ of forests of k rooted trees labeled by $[n]$. To illustrate with (78), the number of forests of k trees labeled by $[n]$ whose restriction to $[b]$ is a tree is

$$\frac{(n-k)_{b-1}}{(n-1)_{b-1}} \left(\frac{b}{n}\right)^{b-1} \binom{n-1}{k-1} n^{n-k} = b^{b-1} \binom{n-b}{k-1} n^{1+n-b-k}. \tag{80}$$

Since b^{b-1} is the number of rooted trees labeled by $[b]$, (80) agrees with the formula of Stanley [81, Ex. 2.11.a] for the number of forests of k trees labeled by $[n]$ which contain a particular forest of $1+n-b$ trees, applied to any of the b^{b-1} forests with one tree component equal to $[b]$ and $n-b$ singleton roots. The following proof of Theorem 19 involves formula (9), which is a generalization of the formula of Stanley just mentioned.

6.1 Proof of the restriction theorem

Lemma 21 *Let $\mathcal{F}$ be a p-forest with restriction $\mathcal{F}^B$ to $B \subseteq S$ with $|B| = b$. Then for each $G \in \boldsymbol{F}_B$ and each vector of non-negative counts $(c_i, i \in B)$ with $P(|\mathcal{F}_i| = c_i \text{ for all } i \in B) > 0$*

$$P(\mathcal{F}^B = G \mid \ |\mathcal{F}_i| = c_i \text{ for all } i \in B) = \frac{(n-1-c_B)_{b-|G|-1}}{(n-1)_{b-1}} \prod_{i \in B} (c_i)_{|G_i|}. \quad (81)$$

Proof. Assume for convenience that $S = [n] := \{1, \ldots, n\}$ and $B = [b]$ for some $b \in [n]$. It is easily seen, as in the proof of [70, Thm. 1.6], that conditionally given $|\mathcal{F}_i| = c_i$ for all $i \in [n]$, the random set $\mathcal{F}_1$ of children of 1 has uniform distribution over all subsets of size c_1 of $\{2, \ldots, n\}$, and for each $2 \leq i < n$ given also the subsets $\mathcal{F}_j$ of $[n]$ for all $j < i$, the random set $\mathcal{F}_i$ has uniform distribution over all subsets of size c_i of some subset of $[n]$ of size $n - 1 - c_1 - \cdots - c_{i-1}$, this subset of $[n]$ being determined by the $\mathcal{F}_j$ for $j < i$ and the constraint that $\mathcal{F}$ is a forest. The event $\mathcal{F}^{[b]} = G$ is identical to the event that $\mathcal{F}_i \cap B = G_i$ for all $i \in [b]$. So conditionally given $|\mathcal{F}_i| = c_i$ for all $i \in [b]$ there are

$$\prod_{m=1}^{b} \binom{n-1-c_{[m-1]}}{c_m} = \frac{(n-1)!}{(n-1-c_{[b]})! \prod_{i=1}^{b} c_i!} \quad (82)$$

equally likely possible choices of the sets $\mathcal{F}_i$ for $i \in [b]$. The number of these choices that make the event $(\mathcal{F}^{[b]} = G)$ occur is

$$\prod_{m=1}^{b} \binom{n-b-c_{[m-1]}-a_{[m-1]}}{c_m - a_m} = \frac{(n-b)!}{(n-b-c_{[b]}-a_{[b]}))! \prod_{i=1}^{b} (c_i - a_i)!} \quad (83)$$

where $a_i := |G_i|$, and the ratio of (83) to (82) simplifies to yield (81). To check the left-hand formula in (83), observe that given choices of the $\mathcal{F}_i$ have been made for $i < m$ in such a way that $|\mathcal{F}_i| = c_i$ and $\mathcal{F}_i \cap [b] = G_i$ for all $i < m$, the choice of the set $\mathcal{F}_m$ of size c_m is subject firstly to the constraint that $\mathcal{F}$

is a forest, and secondly to the constraint that $\mathcal{F}_m \cap [b] = G_m$. This means that there $c_m - a_m$ elements of $[n] - [b]$ to be chosen. The forest constraint forbids the choice of any of the $c_{[m-1]}$ children of vertices $1, \ldots, m-1$. But due to previous choices, $a_{[m-1]}$ of these forbidden vertices are contained in $[b]$, so there are $c_{[m-1]} - a_{[m-1]}$ forbidden vertices within $[n] - [b]$, and the $c_m - a_m$ vertices of $\mathcal{F}_m \cap ([n] - [b])$ are chosen from an allowed set of $n - b - c_{[m-1]} - a_{[m-1]}$ vertices. Therefore, no matter what the $\mathcal{F}_i$ for $i < m$ such that $|\mathcal{F}_i| = c_i$ and $\mathcal{F}_i \cap [b] = G_i$ for all $i < m$, the number of possible choices of $\mathcal{F}_m$ such that $\mathcal{F}_m \cap [b] = G_m$ is the mth factor on the left side of (83). □

For the rest of this section let C_B denote the total number of children of all vertices in B in the p-forest $\mathcal{F}$:

$$C_B := |\mathcal{F} \cap (B \times S)| = \sum_{s \in B} |\mathcal{F}_s|.$$

Lemma 22 *For each $G \in \boldsymbol{F}_B$ with j tree components and each c with $P(C_B = c) > 0$,*

$$P(\mathcal{F}^{[b]} = G \,|\, C_B = c) = \frac{(n-1-c)_{j-1}}{(n-1)_{b-1}} (c)_{b-j} \prod_{s \in B} \left(\frac{p_s}{p_B} \right)^{|G_s|}. \tag{84}$$

Proof. Again, take $S = [n], B = [b]$, and let $C_i := |\mathcal{F}_i|$ for $i \in [n]$. By application of (81),

$$P(\mathcal{F}^{[b]} = G \,|\, C_B = c) = \frac{(n-1-c)_{j-1}}{(n-1)_{b-1}} E_c \left(\prod_{i=1}^{b} (C_i)_{|G_i|} \right) \tag{85}$$

where E_c denotes expectation relative to the conditional law of $(C_1, \ldots, C_b)$ given $C_B = c$, which by Proposition 1 is a multinomial distribution with parameters c and $(p_1/p_B, \ldots, p_b/p_B)$. But this expectation can be evaluated by a calculation with the generating function of the multinomial distribution, and the result is (84). □

Lemma 23 *For $\mathcal{F}$ a p-forest labeled by S, and G a rooted forest labeled by S with r edges,*

$$P(\mathcal{F} \supseteq G) = \frac{E(|\mathcal{F}|)_r}{(|S|-1)_r} \prod_{s \in S} p_s^{|G_s|}. \tag{86}$$

Proof. This can be read from formula (9). □

Completion of the proof. Lemma 22 shows that for each $c \in [n-1]$ the conditional distribution of $\mathcal{F}^{[b]}$ given $C_B = c$ is that of a $p(\cdot \mid B)$-forest, hence so is the unconditional distribution of $\mathcal{F}^{[b]}$. To verify formula (76), recall that for indicator random variables $X_i, i \in I$ and $r = 0, 1, 2, \ldots$ there is the formula

$$E\binom{\sum_{i \in I} X_i}{r} = \sum_{J \subseteq I : |J| = r} P(\cap_{j \in J}(X_j = 1)). \tag{87}$$

Since

$$|\mathcal{F}^B| = \sum_{(s,t) \in B \times B} 1(s \xrightarrow{\mathcal{F}} t) \tag{88}$$

formula (87) gives for $r = 1, 2, \ldots, b-1$

$$E\binom{|\mathcal{F}^B|}{r} = \sum_{G \subseteq B \times B : |G| = r} P(\mathcal{F} \supseteq G). \tag{89}$$

The probability $P(\mathcal{F} \supseteq G)$ is zero unless G is a rooted forest labeled with r edges, in which case this probability is given by (86). Thus

$$E\binom{|\mathcal{F}^B|}{r} = \sum_{G \in \boldsymbol{F}_B : |G| = r} \frac{E(|\mathcal{F}|)_r}{(|S|-1)_r} \prod_{s \in B} p_s^{|G_s|} = \frac{E(|\mathcal{F}|)_r}{(|S|-1)_r} \binom{b-1}{r} p_B^r \tag{90}$$

where the second equality is due to (8). The claim in the binomial case follows easily because the rth factorial moment of the binomial $(n-1, q_0)$ distribution is $(n)_r q_0^r$. $\square$

6.2 Variations

To illustrate formula (86), for any two distinct s and s' in S, the probability that a p-forest $\mathcal{F}$ contains a particular edge (s, s') is

$$P(s \xrightarrow{\mathcal{F}} s') = \frac{E|\mathcal{F}|}{(|S|-1)} p_s. \tag{91}$$

For distinct t and t' in S, with $(s, s') \neq (t', t)$ and $s' \neq t'$, the probability that $\mathcal{F}$ contains both (s, s') and (t, t') is

$$P((s \xrightarrow{\mathcal{F}} s') \cap (t \xrightarrow{\mathcal{F}} t')) = \frac{E(|\mathcal{F}|(|\mathcal{F}|-1))}{(|S|-1)(|S|-2)} p_s p_t. \tag{92}$$

In particular, for such (s,s') and (t,t') the events $(s \stackrel{\mathcal{F}}{\to} s')$ and $(t \stackrel{\mathcal{F}}{\to} t')$ are independent if $\mathcal{F}$ is a p-tree, and negatively correlated if $\mathcal{F}$ is a p-forest of k trees for $k \geq 2$.

The formula (86) for $P(\mathcal{F} \supseteq G)$ may be compared to a determinantal formula of Pemantle [66, Th. 4.2] for such a probability for $\mathcal{F}$ a uniform random spanning tree of a graph. In contrast to (86), there is no simple general formula for $P(\mathcal{F} \subseteq G)$ for a general directed graph $G \subseteq S \times S$. For instance, in the simplest case when p is uniform on S and $\mathcal{F} = \mathcal{T}$ say is a tree, then $P(\mathcal{T} \subseteq G) = t(G)/|S|^{|S|-1}$ where $t(G)$ is the number of spanning subtrees of G. Some references to the classical problem of evaluating $t(G)$ are [62, Chapter 6] and [Ex. 2.11.a][81]. See also [52] for a technique for finding $P(\mathcal{F} \subseteq G)$ for graphs G with special structure.

Lemma 22 has another consequence which is worth recording. Recall that for $1 \leq n \leq N$ and $0 \leq K \leq N$ the *hypergeometric* (n, N, K) *distribution* is the distribution of the number of good elements that appear in a random subset of size n picked from a set of K good elements and $N-K$ bad elements [34].

Proposition 24 *For $\mathcal{F}$ a p-forest labeled by S with $|S| = n$, and*

$$C_B\mathcal{F} := \sum_{s \in B} |\mathcal{F}_s|,$$

(i) *the distribution of C_B given $|\mathcal{F}| = m$ is* binomial (m, p_B)*;*
(ii) *given $|\mathcal{F}|$ and $C_B = c$, the distribution of $|\mathcal{F}^B|$ is* hypergeometic $(b-1, n-1, c)$*; in particular,*
(iii) *if the number of edges of $\mathcal{F}$ has binomial$(n-1,q)$ distribution for some $q \in [0,1]$, then the number $|\mathcal{F}^B|$ of edges of $\mathcal{F}$ in $B \times B$, and the number of edges of $\mathcal{F}$ in $B \times B^c$ are independent, with binomial$(b-1, qp_B)$ and binomial$(n-b, qp_B)$ distributions respectively.*

Proof. The first part can be read from Proposition 1. Sum the expression (84) over all forests $G \in \boldsymbol{F}(B)$ with ℓ edges and simplify using (8) to see that

$$\begin{aligned} P(|\mathcal{F}^B| = \ell \,|\, C_B = c) &= \frac{(n-1-c)_{b-\ell-1}(c)_\ell}{(n-1)_{b-1}} \binom{b-1}{\ell} \\ &= \binom{c}{\ell}\binom{n-1-c}{b-1-\ell}\binom{n-1}{b-1}^{-1} \end{aligned}$$

which yields (ii). Part (iii) is a standard consequence of (i) and (ii). $\square$

An alternative proof of the factorial moment formula (76) can be based on Proposition (24). By standard applications of (87), for $S_{n,p}$ with binomial(n,p) distribution and $H_{n,N,G}$ with hypergeometric(n,N,G) distribution there are the formulae

$$E\binom{S_{n,p}}{r} = \binom{n}{r} p^r; \qquad E\binom{H_{n,N,G}}{r} = \binom{n}{r}\frac{(G)_r}{(N)_r}. \tag{93}$$

By application of these formulae and Proposition 24, for $\mathcal{F}$ with m edges the binomial moments of N_B are

$$E\binom{N_B}{r} = E\left(E\left[\binom{N_B}{r}\middle| C_B\right]\right) = \frac{(b-1)_r}{(n-1)_r}E\binom{C_B}{r} = \frac{(b-1)_r}{(n-1)_r}\binom{m}{r}p_B^r$$

and (76) follows.

Proposition 24 implies the following alternate expression for the distribution of $|\mathcal{F}^{[b]}|$:

$$P(|\mathcal{F}^{[b]}| = \ell) = \binom{n-1}{b-1}^{-1} E\left[\binom{n-1-C_B}{b-\ell-1}\binom{C_B}{\ell}\right] \tag{94}$$

where C_B has binomial (m, p_B) distribution given that $|\mathcal{F}| = m$. Compare (94), (77) and (93) to see that the following moment identity (95) must hold for a binomially distributed random variable Y, with some restrictions on x:

$$E\left[\binom{x-Y}{a}\binom{Y}{b}\right] = \sum_{j=0}^{a}(-1)^j\binom{b+j}{j}\binom{x-b-j}{a-j}E\binom{Y}{b+j} \tag{95}$$

It follows easily that this formula must hold for any random variable Y with all moments finite, for all real x and all non-negative integers a and b. This can be checked as follows. By linearity of the expectation operator E, it suffices to check the formula for a constant random variable Y, say $Y = y$ for some real y. Then the formula reduces easily to

$$\binom{x-y}{a} = \sum_{j=0}^{a}(-1)^j\binom{x-b-j}{a-j}\binom{y-b}{j}. \tag{96}$$

Replace $x-b$ by x and $y-b$ by $-z$ to see that this amounts to

$$\binom{x+z}{a} = \sum_{j=0}^{a} \binom{x-j}{a-j} \binom{z+j-1}{j} \tag{97}$$

for all real x and z, which is a known identity for binomial coefficients (replace n by a, x by $z-1$ and y by $x-a$ in Gould [38][(3.2)]). These identities (96) and (97), when written in terms of the hypergeometric function ${}_2F_1$, can be read from the Chu-Vandermonde summation formula.

6.3 Thinning of p-forests

Let $\mathcal{T}$ be a p-tree with n vertices. For $0 \le q \le 1$ call $\mathcal{F}$ a *q-thinning* of $\mathcal{T}$ if given $\mathcal{T}$ the forest $\mathcal{F}$ is derived from $\mathcal{T}$ by retaining each of the $n-1$ edges of $\mathcal{T}$ independently with probability q. Then, as shown in [70], $\mathcal{F}$ is a p-forest whose number of edges has binomial $(n-1, q)$ distribution. Compare with the conclusion of Corollary 20 to deduce

Corollary 25 *The restriction to B of a q-thinning of a p-tree is a $p(\cdot|B)$-forest with the same distribution as a qp_B-thinning of a $p(\cdot \,|\, B)$-tree.*

Even for p uniform and $q = 1$ this result does not seem evident without calculation. Neither does the independence property in part (iii) of Proposition 24 seem obvious even in this case. In the same vein, there is also the following generalization of results in [70]:

Theorem 26 *Suppose that $\mathcal{F}$ is p-forest. Given $\mathcal{F}$, let each edge $s \xrightarrow{\mathcal{F}} t$ be marked red with probability r_s, independently as (s,t) ranges over all directed edges of $\mathcal{F}$. Let $\mathcal{F}_{\rm red}$ denote the forest of red edges so obtained, and let $p_* := \sum_{s \in S} p_s r_s$. Then $\mathcal{F}_{\rm red}$ is a p'-forest, where $p'_s := p_s r_s / p_*$, and given $\mathcal{F}$ has m edges the number of edges of $\mathcal{F}_{\rm red}$ has a binomial(m, p_*) distribution.*

Proof. This is established by a straightforward calculation using formula (9). □

In particular, if $\mathcal{F}$ is a random tree with uniform distribution on the set of all rooted trees labeled by S, then $\mathcal{F}_{\rm red}$ obtained by the above construction is a p'-forest with p'_s proportional to r_s.

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