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Root systems for two dimensional complex reflection groups

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Abstract

Root systems for all real reflection groups have been known for a long time. A M Cohen (1976) extended the idea of root systems to complex reflection groups: furthermore he explicitly presented root systems for all dimensions greater than two. Here, root systems are given for the two dimensional complex reflection groups which are generated by two reflections.

1 Introduction.

In 1954, Shephard and Todd [13] completely classified the finite complex reflection groups. In 1967 Coxeter [8] gave presentations for all the n dimensional finite complex reflection groups generated by n reflections. He introduced a graphical notation for these groups as follows.

If the presentation has n generators $r_1, \ldots, r_n$ then the graph consists of n nodes. If r_i is of order m, then the number m is written inside the node, although if $m = 2$, then by convention the number 2 may be omitted. If r_i and r_j commute, then the corresponding nodes are not joined. If r_i and r_j do not commute then they are related by the braid relation

$$\underbrace{r_i r_j r_i \ldots}_{e} = \underbrace{r_j r_i r_j \ldots}_{e}$$

with e factors on each side for some $e \in \mathbb{N}$ and the corresponding nodes are joined by an edge of weight e, which is omitted (by convention) when $e = 3$. Coxeter shows that if the graph is not a tree (that is, it contains no cycles), then it contains a subgraph which is a triangle with a number k written inside (as this paper will mainly be involved with groups generated by two generators, we will use s and t to denote the generators (and u if a third generator is involved))

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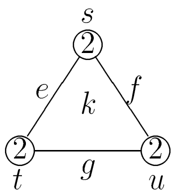


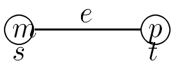
The corresponding group has relations

$$(stut)^k = 1, (st)^e = (su)^f = (tu)^g = 1.$$

Coxeter showed that in all cases, the orders of s, t, u is 2 and that at least two of the edges have weight 3.

Inspired by these graphs and by root systems associated with real reflection groups (see, for example, Bourbaki [1]), in 1976 Cohen [6] defined root graphs and root systems connected with finite complex reflection groups for dimension greater than 2. Our aim in this paper is to obtain root graphs and root systems associated with two dimensional complex reflection groups which are generated by two reflections.

We concern ourselves with a graph with two nodes



with presentation

$$s^m = t^p = 1, \ \underbrace{sts\ldots}_{e} = \underbrace{tst\ldots}_{e}.$$

Coxeter also used the notation $m[e]p$ for this presentation.

Although the classification of such finite reflection groups is of long standing (see, for example, Coxeter [7] and Koster [11]), we give a new classification which simultaneously leads in a natural way to root graphs and root systems for these groups. This is done by first introducing certain polynomials $f_l(x)$, $l \geq 0$ and then determining which of these are admissible, that is, have all their roots in $(0, 1)$. It turns out that the irreducible two dimensional complex reflection groups are in one-one correspondence with these polynomials; furthermore, their roots provide a natural way of forming the corresponding root graphs and root systems.

We remark that if e is odd, then relation

$$\underbrace{sts\ldots}_{e} = \underbrace{tst\ldots}_{e}$$

can be written as

$$(st)^{(e-1)/2}s \ = \ (ts)^{(e-1)/2}t.$$

Thus, the generators s and t are conjugate and therefore of the same order. Hence if e is odd, then $p = m$. We see later that this condition arises naturally in the classification via the above polynomials.

More recently, complex reflection groups have been given more prominence through the work of M. Broué, G. Malle and R. Rouquier [2],[3],[4] in their work on the representation theory for reductive algebraic groups. In particular, they have given presentations for all finite complex reflection groups in the style of those given by Coxeter. Also, they have presented diagrams which generalise Coxeter diagrams in an interesting way. We present our results so that they are consistent with these recent developments.

2 Preliminaries.

In this section, the basic definitions and notation required later are given following the approach in [6],[9].

Let V be a complex vector space of dimension n. A *reflection* in V is a linear transformation of V of finite order with exactly $(n-1)$ eigenvalues equal to 1. A *reflection group* G in V is a group generated by reflections in V. There exists a unitary inner product $(\ ,\ )$ on V invariant under G. A reflection group G is said to be r-dimensional if the dimension of the subspace V^G of points fixed by G is $n-r$. The group G is irreducible if the restriction to a G-invariant complement of V^G in V is irreducible.

A (unitary) *root* of a reflection in V is an eigenvector (of length 1) corresponding to the unique eigenvalue not equal to 1 of the reflection. A (unitary) *root* of G is a (unitary) root of a reflection in G.

Let s be a reflection in V of order $m > 1$. There exists $v \in V, v \neq 0$ and a primitive *mth* root of unity ζ such that

$$s_{v,m}x = x - (1-\zeta)(x,v)(v,v)^{-1}v$$

for all $x \in V$, where $s = s_{v,m}$. If t is any unitary transformation of V, we have

$$ts_{v,m}t^{-1} = s_{tv,m}.$$

Define $\theta_G : V \to \mathbb{N}$ by $\theta_G(v) = |G_W|$, where $W =< v >^{\perp}, v \in V$. The number $\theta_G(v)$ is called the *order* of v (with respect to G).

A *vector graph* is a pair (B, θ), where B is a non-empty finite subset $\mathbb{C}^{\infty}$, such that for all $u, v \in B, |(u,v)| = 1$ if and only if $u = v$ and θ is a map from B to $\mathbb{N} \setminus \{1\}$. We say that B is the set of *vertices* and $\theta(v)$, for $v \in b$, is the *order* of v. Let $\Gamma = (B, \theta)$ be a vector graph. Then, we define dim Γ to be the dimension of the vector space spanned by B, and $W(\Gamma)$ to be the group generated by all the reflections $s_{v,\theta(v)}$ for $v \in B$. The vector graph Γ is called a *root graph* if

(i) dim $\Gamma = |B|$

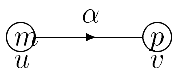
(ii) $W(\Gamma)$ is a finite reflection group.

We say that Γ is *irreducible* if $W(\Gamma)$ is irreducible in dim Γ (or that Γ is connected). Thus in Section 3 we restrict ourselves to the classification of irreducible root graphs. The vector graph Γ is said to be *congruent* to the vector graph $\Gamma' = (B', \theta')$ if there is a $t \in \mathbf{Gl}(\mathbb{C}^\infty)$ such that $\theta'(tv) = \theta(v)$ for $v \in B$ and the elements of B are eigenvectors of t.

In this paper we only concern ourselves with vector graphs with two nodes (since we only consider two-dimensional reflection groups generated by two reflections).

Let $B = \{u, v\}$. To u is assigned the value $\theta(u)$ [and $\theta(v)$ to v] and to the edge is assigned the value (u, v), together with an arrow from u to v.

For example, if $\theta(u) = m$ and $\theta(v) = p$ and $(u, v) = \alpha$, then the vector graph is



We adopt the following conventions.

(i) If $m = 2$, we may omit the number 2.

(ii) If $\alpha \in \mathbb{R}$, the arrow is omitted.

(iii) If $l = m$ and $\alpha = -1/2$, we may omit the value $-1/2$.

A pair (R, f) consisting of

(i) a finite set R of non-zero elements of $\mathbb{C}^\infty$,

(ii) a map $f : R \to \mathbb{N}\backslash\{1\}$ such that for all $u, v \in R$, $s_{u,f(u)}R = R$ and $f(s_{u,f(u)}v) = f(v)$ is called a *pre-root system*. To $\Sigma = (R, f)$ is associated the reflection group $W(\Sigma)$ defined by $W(\Sigma) =< s_{u,f(u)} | u \in R >$.

A pre-root system Σ is called a *root system* if in addition

(iii) $\alpha u \in R$ if and only if $\alpha u \in W(\Sigma)u$ for all $u \in R, \alpha \in \mathbb{C}$.

If (B, θ) is a root graph with $B = \{e_1, \ldots, e_n\}$, then $det((e_i, e_j))$ is a positive real number.

A group G of unitary automorphisms of V is said to be *imprimitive* if V is a direct sum $V = V_1 \oplus \cdots \oplus V_k$ of non-trivial proper subspaces $V_i (1 \leq i \leq k)$ of V such that V_i is invariant under G. If such a direct splitting of V does not exist, G is said to be *primitive*.

Let $\mathcal{S}_n$ be the (symmetric) group of all $n \times n$ permutation matrices and let $A(m, p, n)$ be the set of all diagonal $n \times n$ matrices with $\zeta^{\rho_i}, \rho_i \in \mathbb{Z}$ in the (i, i) position, where ζ is a primitive mth root of unity and $\sum_{i=1}^n \rho_i \equiv 0 \pmod p$. Define $G(m, p, n) = A(m, p, n) \rtimes \mathcal{S}_n$, then the imprimitive groups in V are of the form $G(m, p, n)$, where $p|m$.

Remark 2.1 *(i) $G(m, m, 2)$ is conjugate to $W(I_2(m))$ (notation of [1]), the dihedral group of order $2m$.*

(ii) $G(1, 1, n) = W(A_{n-1}) \cong \mathcal{S}_n$, the Weyl group of type A_{n-1}.

(iii) $G(2,1,n) = W(B_n)$, *the Weyl group of type* B_n.

(iv) $G(2,2,n) = W(D_n)$, *the Weyl group of type* D_n.

(v) If $p \neq 1$, *then* $G(m,p,n)$ *can be defined with* $n+1$ *generating reflections, but for* $p = 1, m$, n *generating reflections are sufficient.*

Thus when $n = 2$, only two of these groups, namely $G(m,m,2)$ and $G(m,1,2)$, are generated by two reflections.

The group $G(m,m,2)$ is generated by reflections of order 2 corresponding to the root graph $\{\frac{1}{\sqrt{2}}(\epsilon_1 - \epsilon_2), \frac{1}{\sqrt{2}}(\epsilon_1 - \zeta\epsilon_2)\}$ and the group $G(m,1,2)$ is generated by reflections of orders 2 and m respectively corresponding to the root graph $\{\frac{1}{\sqrt{2}}(\epsilon_1 - \epsilon_2), \epsilon_2\}$. Thus, the vector graphs are respectively, which root graphs with the following root systems:

$$\Sigma(m,m,2) = (R(m,m,2), f)$$

where $R(m,m,2) = \pm\mu_m\{\frac{1}{\sqrt{2}}(\epsilon_1 - \zeta^k\epsilon_2), 1 \leq k \leq m\}$ and

$$\Sigma(m,1,2) = (R(m,m,2) \cup R_2, f)$$

where $R_2 = \mu_m\{\epsilon_1, \epsilon_2\}$ and $f(R(m,m,2)) = 2$ and $f(R_2) = m$ and where $\mu_m = \{\zeta^k | 1 \leq k \leq m\}$. Here and later, $\{\epsilon_1, \epsilon_2\}$ is the standard basis for $\mathbb{C}^2$.

3 Classification of two-dimensional reflection groups.

In this section, we consider certain polynomials which are used later in the classification.

Let $l, m, p \in \mathbb{N}$ and ζ and ξ be primitive mth and pth roots of unity respectively. A sequence of polynomials $\{f_l(x) := f_{l,m,p}(x) \in \mathbb{C}[x]\}$ are defined as follows:-

Put

$$f_1(x) = f_2(x) = 1.$$

If l is even, $l \geq 2$, put

$$f_{l+1}(x) = x f_l(x) + \frac{\zeta}{(1-\zeta)(1-\xi)} f_{l-1}(x) \quad (3.1)$$

$$f_{l+2}(x) = f_{l+1}(x) + \frac{\xi}{(1-\zeta)(1-\xi)} f_l(x). \quad (3.2)$$

It is easily seen that $f_l(x)$ is a monic polynomial of degree $\lfloor (l-1)/2 \rfloor$, where $\lfloor i \rfloor$ denotes the integer part of i. For smaller values of l, these polynomials are calculated explicitly as these are required later in the classification we give the $f_l(x)$ for $3 \leq l \leq 10$.

$$\begin{aligned}
f_3(x) &= x + \frac{\zeta}{(1-\zeta)(1-\xi)} \\
f_4(x) &= x + \frac{\zeta+\xi}{(1-\zeta)(1-\xi)} \\
f_5(x) &= x^2 + \frac{2\zeta+\xi}{(1-\zeta)(1-\xi)}x + \frac{\zeta^2}{(1-\zeta)^2(1-\xi)^2} \\
f_6(x) &= x^2 + \frac{2(\zeta+\xi)}{(1-\zeta)(1-\xi)}x + \frac{\zeta^2+\zeta\xi+\xi^2}{(1-\zeta)^2(1-\xi)^2} \\
f_7(x) &= x^3 + \frac{3\zeta+2\xi}{(1-\zeta)(1-\xi)}x^2 + \frac{3\zeta^2+2\zeta\xi+\xi^2}{(1-\zeta)^2(1-\xi)^2}x + \frac{\zeta^3}{(1-\zeta)^3(1-\xi)^3} \\
f_8(x) &= x^3 + \frac{3(\zeta+\xi)}{(1-\zeta)(1-\xi)}x^2 + \frac{3\zeta^2+4\zeta\xi+3\xi^2}{(1-\zeta)^2(1-\xi)^2}x + \frac{\zeta^3+\zeta^2\xi+\zeta\xi^2+\xi^3}{(1-\zeta)^3(1-\xi)^3} \\
f_9(x) &= x^4 + \frac{4\zeta+3\xi}{(1-\zeta)(1-\xi)}x^3 + \frac{6\zeta^2+6\zeta\xi+3\xi^2}{(1-\zeta)^2(1-\xi)^2}x^2 + \frac{4\zeta^3+3\zeta^2\xi+2\zeta\xi^2+\xi^3}{(1-\zeta)^3(1-\xi)^3}x + \frac{\zeta^4}{(1-\zeta)^4(1-\xi)^4} \\
f_{10}(x) &= x^4 + \frac{4(\zeta+\xi)}{(1-\zeta)(1-\xi)}x^3 + \frac{6\zeta^2+9\zeta\xi+6\xi^2}{(1-\zeta)^2(1-\xi)^2}x^2 + \frac{4\zeta^3+6\zeta^2\xi+6\zeta\xi^2+4\xi^3}{(1-\zeta)^3(1-\xi)^3}x + \frac{\zeta^4+\zeta^3\xi+\zeta^2\xi^2+\zeta\xi^3+\xi^4}{(1-\zeta)^4(1-\xi)^4}.
\end{aligned}$$

Easy calculations show that

$$\frac{\zeta+\xi}{(1-\zeta)(1-\xi)} = -\frac{1}{2}\left(1 + cot\frac{\pi}{m}cot\frac{\pi}{p}\right) \tag{3.3}$$

$$\frac{\zeta}{(1-\zeta)^2} = -\frac{1}{4}cosec^2\frac{\pi}{m}, \ \frac{\xi}{(1-\xi)^2} = -\frac{1}{4}cosec^2\frac{\pi}{p}. \tag{3.4}$$

Thus, for example,

$$f_4(x) = x - \frac{1}{2}\left(1 + cot\frac{\pi}{m}cot\frac{\pi}{p}\right).$$

We now prove some easy results concerning these polynomials which are required later.

Lemma 3.1 *(i)If l is even, $l \geq 4$, then*

$$f_{l+2}(x) = f_l(x)f_4(x) - \frac{1}{16}cosec^2\frac{\pi}{m}cosec^2\frac{\pi}{p}f_{l-2}(x) \tag{3.5}$$

$$f_{l+3}(x) = x\left(f_l(x)f_4(x) - \frac{1}{16}cosec^2\frac{\pi}{m}cosec^2\frac{\pi}{p}f_{l-2}(x)\right) + \frac{\zeta}{(1-\zeta)(1-\xi)}f_{l+1}(x) \tag{3.6}$$

(ii)If l is even, then $f_l(x) \in \mathbb{R}[x]$.

(iii)If l is odd and $\xi = \zeta$, then $f_l(x) \in \mathbb{R}[x]$.

Proof. From (3.1),(3.2) and (3.4), we obtain

$$\begin{aligned} f_{l+2}(x) &= xf_l(x) + \frac{\zeta}{(1-\zeta)(1-\xi)} f_{l-1}(x) + \frac{\xi}{(1-\zeta)(1-\xi)} f_l(x) \\ &= f_l(x)f_4(x) - \frac{\zeta}{(1-\zeta)(1-\xi)}(f_l(x) - f_{l-1}(x)) \\ &= f_l(x)f_4(x) - \frac{\zeta\xi}{(1-\zeta)^2(1-\xi)^2} f_{l-2}(x) \\ &= f_l(x)f_4(x) - \frac{1}{16} cosec^2\frac{\pi}{m} cosec^2\frac{\pi}{p} f_{l-2}(x) \end{aligned}$$

which proves (3.5). Formula (3.6) now follows directly from (3.2) and (3.4). The proof of (ii) and (iii) follows from (3.3) by induction on l.

Lemma 3.2 *(i) The coefficient of x^{n-1} in $f_{2n+2}(x)$ is*

$$\begin{aligned} \frac{n(\zeta+\xi)}{(1-\zeta)(1-\xi)} &= -\frac{n}{2}\left(\frac{cos(\frac{\pi}{m}-\frac{\pi}{p})}{sin\frac{\pi}{m} sin\frac{\pi}{p}}\right) \\ &= -\frac{n}{2}\left(1 + cot\frac{\pi}{m} cot\frac{\pi}{p}\right). \end{aligned}$$

(ii) If l is odd and if $\xi = \zeta$ the constant term in $f_l(x)$ is

$$(\zeta(1-\zeta)^{-2})^{l-2} = \left(-\frac{1}{4} cosec^2\frac{\pi}{m}\right)^{l-2} \in \mathbb{R}.$$

Proof. The proofs are simple induction arguments and we give a sketch of the proof only.

(i) If $l = 2n$, from (3.5) we have

$$f_{2n+2}(x) = f_{2n}(x)f_4(x) - \frac{1}{16} cosec^2\frac{\pi}{m} cosec^2\frac{\pi}{p} f_{2n-2}(x).$$

Since $f_{2n-2}(x)$ and $f_{2n}(x)$ have degrees $n-2$ and $n-1$ respectively, the result follows by induction on n and noting the form of $f_4(x)$ given above.

(ii) If l is odd and $\xi = \zeta$, then if we put $x = 0$ in (3.6), the result follows by induction.

Lemma 3.3 *If l is even and if $f_l(a) = 0, f_{l-2}(a) > 0$ for some $a \geq 1$, then there exists $b > 1$ such that $f_m(b) = 0$ for even $m > l$.*

Proof. Since

$$f_{l+2}(a) = f_l(a)f_4(a) - \frac{1}{16} cosec^2\frac{\pi}{m} cosec^2\frac{\pi}{p} f_{l-2}(a).$$

and since $f_l(a) = 0$, we have $f_{l+2}(a) < 0$. As the $f_l(x)$ are monic, there exist $a_1 > a$ such that $f_{l+2}(a_1) = 0$ and $f_l(a_1) > 0$. Thus, by induction there exist $b > 1$ such that $f_m(b) = 0$ for even $m > l$.

Lemma 3.4 *If $p = m$ and $f_{l-1}(a) > 0$ and $f_l(a) = 0$ for some $a \geq 1$, then there exist $b > 1$ such that $f_m(b) = 0$ for all $m > l$.*

Proof Whether l is odd or even, since $f_l(a) = 0$, from (3.1) and (3.2) we have

$$f_{l+1}(a) \quad = \quad -\frac{1}{4} cosec^2 \frac{\pi}{m} f_{l-1}(a)$$

and so $f_{l+1}(a) < 0$. Thus there exist $a_1 > a$ with $f_{l+1}(a_1) = 0$ and $f_l(a_1) > 0$. Hence, by induction there exist $b > 1$ such that $f_m(b) = 0$ for all $m > l$.

Motivated by the requirements of the classification problem, we now make the following definition.

Definition 3.5 *If $l, m, p \in \mathbb{N}, m \leq p$, we say the triple $(m, p; l)$ is*

(i) admissible *if all the roots of $f_l(x) = 0$ are in $(0, 1)$,*

(ii) semi-admissible *if all the roots of $f_l(x) = 0$ are in $[0, 1]$ with one root equal to 1,*

(iii) inadmissible *if $f_l(x) = 0$ has at least one root not in $[0, 1]$.*

We now classify the admissible and semi-admissible triples; we do this in a series of lemmas.

Lemma 3.6 *If $(m, p; l)$ is admissible then*

(i) for l odd, $m = p < 6$;

(ii) for l even, $(m, p) \in \{(2, m), (3, 3), (3, 4), (3, 5)\}$, where $m \leq p$.

Proof. (i) If l is odd, then $f(x) \in \mathbb{R}[x]$ only if $p = m$; thus $f_l(x) = 0$ can only have real roots if $p = m$. Put $\xi = \zeta$, then from Lemma 3.2(ii) we have $0 \leq |\frac{1}{4} cosec^2 \frac{\pi}{m}| \leq 1$, which implies that $m < 6$ as required.

(ii) If l is even, then from Lemma 3.2(i), the sum of the roots of $f_l(x) = 0$ is $\frac{k}{2}(1 + cot\frac{\pi}{m} cot\frac{\pi}{p})$, where k is the degree of $f_l(x)$. Since $(m, p; l)$ is admissible, all the roots are in $(0, 1)$ and so

$$\frac{k}{2}\left(1 + cot\frac{\pi}{m} cot\frac{\pi}{p}\right) \quad < \quad k,$$

that is, $tan\frac{\pi}{m} tan\frac{\pi}{p} > 1$, which gives the required result.

Lemma 3.7 *If $(m, m; l - 1)$ is admissible and $(m, m; l)$ is semi-admissible then $(m, m; r)$ is inadmissible for $r > l$.*

Proof. As $(m, m; , l - 1)$ is admissible and $(m, m; l)$ is semi-admissible we have $f_{l-1}(1) > 0$ and $f_l(1) = 0$. The result now follows from Lemma 3.4.

Lemma 3.8 *If $m > 2$, then*

(i) The triple $(m, m; l)$ is admissible if and only if $(m, m; l) = (3, 3; l), (l = 3, 4, 5), (4, 4, 3)$,
$(5, 5; 3)$;

(ii) The triple $(m, m; l)$ is semi-admissible if and only if $(m, m; l) = (3, 3; 6)$, $(4, 4; 4), (6, 6; 3)$

Proof. If $l = 3$, then $f_3(x) = x - \frac{1}{4}cosec^2\frac{\pi}{m}$ and $0 < \frac{1}{4}cosec^2\frac{\pi}{m} < 1$ if and only if $m = 2, 3, 4, 5$. Furthermore, for $m = 6$, $f_3(1) = 0$ and so $(6, 6; 3)$ is semi-admissible.

If $l = 4$, then $f_4(x) = x - \frac{1}{2}cosec^2\frac{\pi}{m}$, and we see that

$$\frac{1}{2}cosec^2\frac{\pi}{m}\begin{cases} < 1 & \text{if m=2 or 3} \\ = 1 & \text{if m=4} \\ > 1 & \text{if } m \geq 5, \end{cases}$$

which proves that $(2, 2; 4)$ and $(3, 3; 4)$ are admissible, $(4, 4; 4)$ is semi-admissible and that $(m, m; 4)$ is inadmissible if $m \geq 5$.

If $l = 5$, then

$$f_5(x) \quad = \quad x^2 + \frac{3\zeta}{(1-\zeta)^2}x + \frac{\zeta^2}{(1-\zeta)^4},$$

whose two roots are $\frac{1}{8}(3 \pm \sqrt{5})cosec^2\frac{\pi}{m}$. Consideration of these roots shows that $(m, m; 5)$ is admissible only if $m = 2, 3$ and is never semi-admissible.

If $l = 6$, then

$$\begin{aligned} f_6(x) \quad &= \quad x^2 + \frac{4\zeta}{(1-\zeta)^2}x + \frac{3\zeta^2}{(1-\zeta)^4} \\ &= \quad \left(x + \frac{\zeta}{(1-\zeta)^2}\right)\left(x + \frac{3\zeta}{(1-\zeta)^2}\right), \end{aligned}$$

and the corresponding roots are $\frac{1}{4}cosec^2\frac{\pi}{m}$ and $\frac{3}{4}cosec^2\frac{\pi}{m}$. Both of these roots are in $(0, 1)$ only for $m = 2$. For $m = 3$, both these roots are in [0,1] with one of these roots equal to 1. Thus $(m, m; 6)$ is admissible if $m = 2$ and semi-admissible if $m = 3$. Furthermore, for $m > 3$, there is always a root outside $[0, 1]$ and so $(m, m; 6)$ is inadmissible for $m > 3$. By Lemma 3.7, $(3, 3; l)$ is inadmissible for $l > 6$. Thus, the proof of this lemma is complete.

Remark 3.9 *Note that $(2, 2; l)$ is admissible for all l.*

We have now classified all the admissible $(m, m; l)$ for both l even and l odd. We now proceed towards the classification of all admissible triples $(m, p; l)$ for l even.

Lemma 3.10 *Let l be even.*

(i) If $(m,p;l-2)$ is admissible and $(m,p;l)$ is either semi-admissible or inadmissible, then $(m,p;k)$ is inadmissible for $k>l$.

(ii) If $(m,p;l)$ is admissible, then so is $(m,p;l-2)$ for $l>2$.

Proof. (i) If $(m,p;l-2)$ is admissible and $(m,p;l)$ is either semi-admissible or inadmissible, then for some $a\geq 1$, $f_{l-2}(a)>0$ and $f_l(a)=0$. Thus, the result follows from Lemma 3.3.

(ii) This is a consequence of (i).

Lemma 3.11 *(i) The triple $(m,p;4), m<p$ is admissible if and only if $(m,p)=(2,m), m>3, (3,4), (3,5)$.*

(ii) The triple $(m,p;4), m<p$ is semi-admissible if and only if $(m,p)=(3,6)$.

Proof. We first note that the only root of $f_4(x)=0$ is $\frac{1}{2}(1+cot\frac{\pi}{m}cot\frac{\pi}{p})$.

(i) We see that $\frac{1}{2}(1+cot\frac{\pi}{m}cot\frac{\pi}{p})<1$ if and only if $tan\frac{\pi}{m}tan\frac{\pi}{p}>1$ which implies that if $m<p, (m,p)=(2,m), (m\geq 3), (3,4), (3,5)$ as required.

(ii) Similarly, $tan\frac{\pi}{m}tan\frac{\pi}{p}=1$ if and only if $(m,p)=(3,6)$.

We now consider the case $l=6$. By Lemma 3.10 and Lemma 3.11, $(m,p;6)$ can only be admissible if $(m,p)=(2,m), (m\geq 3), (3,4), (3,5) (m<p)$. We prove

Lemma 3.12 *(i) The triple $(m,p;6)$ is admissible if and only if $(m,p)=(2,3), (2,4), (2,5)$.*

(ii) The triple $(m,p;6), m<p$ is semi-admissible if and only if $(m,p)=(2,6)$.

Proof. From (3.5), we have

$$f_6(x) \quad = \quad f_4(x)^2-\frac{1}{16}cosec^2\frac{\pi}{m}cosec^2\frac{\pi}{p},$$

from which we deduce that the two roots of $f_6(x)=0$ are

$$\frac{1}{2}\left(1+cot\frac{\pi}{m}cot\frac{\pi}{p}\right)\pm\frac{1}{4}cosec\frac{\pi}{m}cosec\frac{\pi}{p}.$$

If $m=2$, the two roots are $\frac{1}{2}\pm\frac{1}{4}cosec\frac{\pi}{p}$, from which we see that $(2,p;6)$ is admissible only if $p=3,4,5$ and semi-admissible if $p=6$.

If $m=3$, the two roots are $\frac{1}{2}(1+\frac{1}{\sqrt{3}}(cot\frac{\pi}{p}\pm cosec\frac{\pi}{p}))$, from which an easy calculation shows that one of the roots is >1 for all $p\geq 4$.

We now consider the case $l=8$. By Lemma 3.10 and Lemma 3.12, $(m,p;8)$ can only be admissible if $(m,p)=(2,3), (2,4), (2,5)$. We prove

Lemma 3.13 *(i) The triple $(m,p;8)$ is admissible if and only if $(m,p)=(2,3)$.*
(ii) The triple $(m,p;8)$ is semi-admissible if and only if $(m,p)=(2,4)$.

Proof. From (3.5), we see that

$$f_8(x) \quad = \quad f_4(x)\left(f_6(x) - \frac{1}{16}cosec^2\frac{\pi}{m}cosec^2\frac{\pi}{p}\right).$$

We need only concern ourselves with the roots of $f_6(x)-\frac{1}{16}cosec^2\frac{\pi}{m}cosec^2\frac{\pi}{p}=0$. If $m=2$, a simple calculation shows that this reduces to $x^2-x+\frac{1+\xi^2}{4(1-\xi)^2}=0$, from which we deduce that $(m,p;8)$ is admissible if and only if $(m,p)=(2,3)$ and semi-admissible if $(m,p)=(2,4)$.

Similar calculations now show that we have the following lemma.

Lemma 3.14 *(i) The triple $(m,p;10)$ is admissible if and only if $(m,p)=(2,3)$*
(ii) The triple $(m,p;12)$ is inadmissible except for $(m,p)=(2,3)$

Proof. We see this by considering the appropriate entries in Table 2.

We have therefore proved the following theorem.

Theorem 3.15 *(i) The admissible triples $(m,p;l)$ are*
$(2,2;l)(l\geq 2),(3,3;3),(3,3;4),(3,3;5),(4,4;3),(5,5;3),(2,m;4)(m\geq 3),(3,4;4),(3,5;4),(2,3;6),(2,4;6),(2,5;6),(2,3;8),(2,3;10)$.

(ii) The semi-admissible triples $(m,p;l)$ are
$(6,6;3),(4,4;4),(3,6;4),(2,6;6),(3,3;6),(2,4;8),(2,3;12)$.

We now show that this not only leads to a classification of two dimensional complex reflection groups but we also easily obtain the corresponding root graphs and root systems.

Let G be a reflection group with root graph $B=\{u,v\}$, where u and v are unitary roots. Let s and t denote the corresponding reflections whose orders are m and p respectively. Then, it can be proved (see, for example, Koster [11]) that for some positive integer l

$$\underbrace{\ldots sts}_{l}=\underbrace{\ldots tst}_{l} \tag{3.7}$$

From Section 2, since B is linearly independent, we have

$$det\begin{pmatrix}(u,u) & (u,v)\\ (v,u) & (v,v)\end{pmatrix}>0,$$

from which it follows that

$$0 < (u,v)(v,u) = |(u,v)|^2 < 1.$$

Now, put $a = |(u,v)|^2 \in (0,1)$ and let $v' = (1-\xi)(u,v)v$ and let $u^{(l)} = \underbrace{\ldots tstu}_{l}$. Furthermore, let

$$u^{(l)} = \alpha_{(u^{(l)},u)}u + \alpha_{(u^{(l)},v')}v',$$

where $\alpha_{(u^{(l)},u)}, \alpha_{(u^{(l)},v')} \in \mathbf{C}$. If l is even, then $u^{(l)} = \underbrace{tst\ldots stu}_{l}$ and so

$$u^{(l+1)} = su^{(l)} = (1-\zeta)(1-\xi)\left\{\frac{\zeta}{(1-\zeta)(1-\xi)}\alpha_{(u^{(l)},u)} - a\alpha_{(u^{(l)},v')}\right\}u + \alpha_{(u^{(l)},v')}v';$$

thus

$$\alpha_{(u^{(l+1)},u)} = (1-\zeta)(1-\xi)\left\{\frac{\zeta}{(1-\zeta)(1-\xi)}\alpha_{(u^{(l)},u)} - a\alpha_{(u^{(l)},v')}\right\} \tag{3.8}$$

and

$$\alpha_{(u^{(l+1)},v')} = \alpha_{(u^{(l)},v')}. \tag{3.9}$$

Similarly,

$$u^{(l+2)} = tu^{(l+1)} = \alpha_{(u^{(l+1)},u)}u + (\xi\alpha_{(u^{(l)},v')} - \alpha_{(u^{(l+1)},u)})v'$$

from which we obtain

$$\alpha_{(u^{(l+2)},v')} = \xi\alpha_{(u^{(l)},v')} - \alpha_{(u^{(l+1)},u)} \tag{3.10}$$

and

$$\alpha_{(u^{(l+2)},u)} = \alpha_{(u^{(l+1)},u)}. \tag{3.11}$$

Now, for l even $l \geq 2$, define

$$g_{l-1}(a) = \left(\frac{1}{(1-\zeta)(1-\xi)}\right)^{\frac{l-2}{2}} \alpha_{(u^{(l)},u)} \tag{3.12}$$

$$g_l(a) = -\left(\frac{1}{(1-\zeta)(1-\xi)}\right)^{\frac{l-2}{2}} \alpha_{(u^{(l)},v')} \tag{3.13}$$

Then, we can prove the following theorem.

Theorem 3.16 *(i) If l is even, $l \geq 2$, then*

$$g_{l+1}(a) = ag_l(a) + \frac{\zeta}{(1-\zeta)(1-\xi)}g_{l-1}(a) \tag{3.14}$$

$$g_{l+2}(a) = g_{l+1}(a) + \frac{\xi}{(1-\zeta)(1-\xi)}g_l(a), \tag{3.15}$$

where $g_{l-1}(a)$ and $g_l(a)$ are monic polynomials in a.

Proof. The proof is by induction on l. An easy verification shows that $u^{(1)} = u, u^{(2)} = u - v'$ and

$$u^{(3)} = (1-\zeta)(1-\xi)\left(a + \frac{\zeta}{(1-\zeta)(1-\xi)}\right) u - v'.$$

Substitution of (3.12) and (3.13) in (3.8) and (3.11) now gives (3.14), and similarly in (3.9) and (3.10) to give (3.15).

We note that these are precisely the defining relations of the polynomials $f_l(x)$ given in (3.1) and (3.2). We will exploit this now in order to classify two dimensional root graphs and root systems.

Using the relation (3.7), we find that

$$\begin{aligned} u^{(l+1)} &= \underbrace{\ldots tstu}_{l+1} \\ &= \underbrace{\ldots stsu}_{l+1} \\ &= \zeta(\underbrace{\ldots stu}_{l}). \end{aligned}$$

This implies that if l is even,

$$su^{(l)} = \zeta u^{(l)}$$

that is, $u^{(l)}$ is a multiple of u. Hence, the coefficient of v, and thus of v', in $u^{(l)}$ is zero. This means that $\alpha(u^{(l)}, v') = 0$, and so, $g_l(a) = 0$, that is, a is a root of $g_l(x) = 0$, where $a \in (0,1)$. However, *this is precisely the requirement for a triple* $(m, p; l)$ *to be admissible.* Thus, the two dimensional complex reflection groups which are generated by two reflections correspond to the admissible triples listed in Theorem 3.15(i). We have therefore recovered the following well known theorem.

Theorem 3.17 *The two dimensional complex reflection groups which are generated by two reflections correspond to the Coxeter graphs*

$$\textcircled{3}\overset{3}{\text{———}}\textcircled{3} \quad \textcircled{3}\overset{4}{\text{———}}\textcircled{3} \quad \textcircled{3}\overset{5}{\text{———}}\textcircled{3} \quad \textcircled{4}\overset{3}{\text{———}}\textcircled{4} \quad \textcircled{5}\overset{3}{\text{———}}\textcircled{5}$$

$$\textcircled{3}\overset{4}{\text{———}}\textcircled{4} \quad \textcircled{3}\overset{4}{\text{———}}\textcircled{5} \quad \textcircled{2}\overset{6}{\text{———}}\textcircled{3} \quad \textcircled{2}\overset{6}{\text{———}}\textcircled{4} \quad \textcircled{2}\overset{6}{\text{———}}\textcircled{5}$$

$$\textcircled{2}\overset{8}{\text{———}}\textcircled{3} \quad \textcircled{2}\overset{10}{\text{———}}\textcircled{3} \quad \textcircled{2}\overset{l}{\text{———}}\textcircled{2} \quad \textcircled{2}\overset{4}{\text{———}}\textcircled{m}$$

Similarly, using the classification of semi-admissible triples given in Theorem 3.15(ii) we obtain all the infinite two-dimensional complex reflection groups generated by two reflections.

Theorem 3.18 *The infinite two-dimensional complex reflection groups generated by two reflection groups correspond to the Coxeter graphs*

$$\textcircled{6}\overset{3}{\text{———}}\textcircled{6} \quad \textcircled{4}\overset{4}{\text{———}}\textcircled{4} \quad \textcircled{3}\overset{4}{\text{———}}\textcircled{6} \quad \textcircled{2}\overset{6}{\text{———}}\textcircled{6} \quad \textcircled{3}\overset{6}{\text{———}}\textcircled{3}$$

$$(2) \overset{8}{\text{———}} (4) \qquad (2) \overset{12}{\text{———}} (3)$$

However, what is far more significant, the roots $a = |(u,v)|^2$ of $f_l(x) = 0$ corresponding to the admissible triples listed in Theorem 3.15 can be calculated. These are listed in Table 1. In the first column of that table, we denote these groups using the numbering given in the original classification by Shephard and Todd [13]. Furthermore, we can determine a value for (u,v) in each case: this is clearly not unique. In Table 2, we give the order of each group, the number of elements in the corresponding root system and the number of reflections of each order. In Table 3, we choose a (u,v) to be real in each case; indeed, for consistency with what occurs in the case of real reflection groups, we choose the root to be negative. We note, however, that if we had chosen $(u,v) \in \mathbf{C} \setminus \mathbf{R}$, then by replacing u with ζu for a suitable $\zeta \in \mathbf{C}, |\zeta| = 1$, then we obtain a congruent vector graph $\{\zeta u, v\}$, where $(\zeta u, v) \in \mathbf{R}$.

Shephard and Todd type	**admissible triple**	$\mathbf{f_l(x)}$	**roots**
G_4	$(3,3;3)$	$x - \frac{1}{4}cosec^2\frac{\pi}{3}$	$\frac{1}{3}$
G_5	$(3,3;4)$	$x - \frac{1}{2}(1 + cot^2\frac{\pi}{3})$	$\frac{2}{3}$
G_{20}	$(3,3;5)$	$x^2 - x + \frac{1}{16}cosec^4\frac{\pi}{3}$	$\frac{4}{3}cos^2\frac{k\pi}{5}(k = 1,2)$
G_8	$(4,4;3)$	$x - \frac{1}{4}cosec^2\frac{\pi}{4}$	$\frac{1}{2}$
G_{16}	$(5,5;3)$	$x - \frac{1}{4}cosec^2\frac{\pi}{5}$	$\frac{1}{2}(1 + \frac{1}{\sqrt{5}})$
G_{10}	$(3,4;4)$	$x - \frac{1}{2}(1 + cot\frac{\pi}{3}cot\frac{\pi}{4})$	$\frac{1}{2}(1 + \frac{1}{\sqrt{3}})$
G_{18}	$(3,5;4)$	$x - \frac{1}{2}(1 + cot\frac{\pi}{3}cot\frac{\pi}{5})$	$\frac{1}{2}(1 + \frac{1}{\sqrt{3}}cot\frac{\pi}{5})$
G_6	$(2,3;6)$	$x^2 - x + \frac{1}{4}(1 - \frac{1}{4}cosec^2\frac{\pi}{3})$	$\frac{1}{2}(1 \pm \frac{1}{\sqrt{3}})$
G_9	$(2,4;6)$	$x^2 - x + \frac{1}{4}(1 - \frac{1}{4}cosec^2\frac{\pi}{4})$	$\frac{1}{2}(1 \pm \frac{1}{\sqrt{2}})$
G_{17}	$(2,5;6)$	$x^2 - x + \frac{1}{4}(1 - \frac{1}{4}cosec^2\frac{\pi}{5})$	$\frac{1}{2}(1 \pm \frac{1}{2}cosec\frac{\pi}{5})$
G_{14}	$(2,3;8)$	$(x - \frac{1}{2})(x^2 - x + \frac{1}{12})$	$\frac{1}{2}, \frac{1}{2}(1 \pm \sqrt{\frac{2}{3}})$
G_{21}	$(2,3;10)$	$\prod_{k=1}^{2}(x^2 - x + \frac{1}{3}cos^2\frac{k\pi}{5})$	$\frac{1}{2}(1 \pm \frac{2}{\sqrt{3}}cos\frac{k\pi}{5})(k = 1,2)$
$G(m,1,2)$	$(2,m;4)$	$x - \frac{1}{2}(1 + cot\frac{\pi}{2}cot\frac{\pi}{m})$	$\frac{1}{2}$
$G(m,m,2)$	$(2,2;m)$	$\prod_{k=1}^{\lfloor(m-1)/2\rfloor}(x - cos^2\frac{k\pi}{m})$	$cos^2\frac{k\pi}{m}$

Table 1: The polynomials $f_l(x)$ and their roots

In Table 3, a complete list of the (u,v) is given. We give simultaneously in the form

$$\underset{s}{(m)} \overset{e}{\underset{\alpha_n}{\text{———}}} \underset{t}{(p)}$$

Shephard and Todd type	**admissible triple**	**order of $\mathbf{G_n}$**	**number of elements in $\mathbf{R_n}$**	**number of reflections of order**			
				2	3	4	5
G_4	$(3,3;3)$	24	24		8		
G_5	$(3,3;4)$	72	48		16		
G_{20}	$(3,3;5)$	360	120		40		
G_8	$(4,4;3)$	96	48	6		12	
G_{16}	$(5,5;3)$	600	120				48
G_{10}	$(3,4;4)$	288	168	6	16	12	
G_{18}	$(3,5;4)$	1800	960		40		48
G_6	$(2,3;6)$	48	72	6	8		
G_9	$(2,4;6)$	192	144	18		12	
G_{17}	$(2,5;6)$	1200	840	30			48
G_{14}	$(2,3;8)$	144	120	12	16		
G_{21}	$(2,3;10)$	720	600	30	40		

Table 2: Statistics about rootsystems

both the Coxeter graph and the root graph in each case, where $\alpha_n = (u, v)$ for the group G_n. Furthermore, $\alpha'_n = \sqrt{1-\alpha_n^2}$. In addition, we give the corresponding root system R_n, where these are expressed in terms of the positive systems P_n as defined by Hughes [9],[10] and Can [5]. Here μ_n denotes the group of *nth* roots of unity and ω, i, ζ, ξ and η denote respectively primitive cube, fourth, fifth, eighth and twentieth roots of unity.

As mentioned above, these root systems are not unique. More 'symmetric' root systems may be obtained by either selecting a non-real complex value for the inner product α_n or by embedding the root system in $\mathbf{R}^3$. For example,

$$\mu_6 \left\{ \epsilon_1, -\frac{1}{\sqrt{3}}(\omega^k \epsilon_1 + \epsilon_2 + \epsilon_3), 0 \le k \le 2 \right\}$$

is an alternative root system for G_4 and

$$\mu_{12} \left\{ \epsilon_1, \epsilon_2, -\frac{1}{\sqrt{3}}(\omega^k \epsilon_1 + \epsilon_2 + \epsilon_3), \frac{1}{\sqrt{3}}(-\omega^k \epsilon_1 - \omega^k \epsilon_2 + \epsilon_3), 0 \le k \le 2 \right\}$$

is an alternative root system for G_5, where $\{\epsilon_1, \epsilon_2, \epsilon_3\}$ is the natural basis for $\mathbf{C}^3$.

Also, for example,

$$\mu_{10}\left\{\epsilon_1, \epsilon_2, -\frac{1}{1-\zeta}(\zeta^k\epsilon_1 + (\zeta^2+\zeta^4)\epsilon_2), \frac{1}{1-\zeta}((1+\zeta^3)\epsilon_1 - \zeta^k\epsilon_2), 0 \le k \le 4\right\}$$

is an alternative root system for G_{16}.

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Shephard Todd type	diagram root graph	$-\alpha_n$	α'_n	positive system P_n	root system R_n
G_4	$\underset{s}{3} \overset{3}{\underset{\alpha_4}{\text{———}}} \underset{t}{3}$	$\frac{1}{\sqrt{3}}$	$\frac{\sqrt{2}}{\sqrt{3}}$	$\{\epsilon_1, \alpha_4\omega^k\epsilon_1 + \alpha'_4\epsilon_2, 0 \le k \le 2\}$	$\mu_6 P_4$
G_5	$\underset{s}{3} \overset{4}{\underset{\alpha_5}{\text{———}}} \underset{t}{3}$	$\frac{\sqrt{2}}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	$\{\epsilon_1, \epsilon_2, \omega^k\alpha_5\epsilon_1 + \alpha'_5\epsilon_2, \alpha'_5\epsilon_1 - \alpha_5\omega^k\epsilon_2, 0 \le k \le 2\}$	$\mu_6 P_5$
G_{20}	$\underset{s}{3} \overset{5}{\underset{\alpha_{20}}{\text{———}}} \underset{t}{3}$	$\frac{1}{2}(\frac{1+\sqrt{5}}{\sqrt{3}})$	$\frac{1}{2}(\frac{1-\sqrt{5}}{\sqrt{3}})$	$\{\epsilon_1, \epsilon_2, \omega^k\alpha_{20}\epsilon_1 + \alpha'_{20}\epsilon_2, \alpha'_{20}\epsilon_1 - \omega^k\alpha_{20}\epsilon_2,$ $\omega^k e^{i\theta}\alpha_4\epsilon_1 \pm \alpha'_4\epsilon_2, \alpha'_4\epsilon_1 \pm \omega^k e^{i\theta}\alpha_4\epsilon_2,$ $0 \le k \le 2, \ \ \theta = tan^{-1}\alpha_{20}\}$	$\mu_6 P_{20}$
G_8	$\underset{s}{4} \overset{3}{\underset{\alpha_8}{\text{———}}} \underset{t}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$\{\epsilon_1, \epsilon_2, i^k\alpha_8\epsilon_1 + \alpha'_8\epsilon_2, 0 \le k \le 3\}$	$\mu_8 P_8$
G_{16}	$\underset{s}{5} \overset{3}{\underset{\alpha_{16}}{\text{———}}} \underset{t}{5}$	$(\frac{1}{2}(1 + \frac{1}{\sqrt{5}}))^{\frac{1}{2}}$	$(\frac{1}{2}(1 - \frac{1}{\sqrt{5}}))^{\frac{1}{2}}$	$\{\epsilon_1, \epsilon_2, \zeta^k\alpha_{16}\epsilon_1 + \alpha'_{16}\epsilon_2, \alpha'_{16}\epsilon_1 - \zeta^k\alpha_{16}\epsilon_2, 0 \le k \le 4\}$	$\mu_{10} P_{16}$
G_{10}	$\underset{s}{4} \overset{4}{\underset{\alpha_{10}}{\text{———}}} \underset{t}{3}$	$(\frac{1}{2}(1 + \frac{1}{\sqrt{3}}))^{\frac{1}{2}}$	$(\frac{1}{2}(1 - \frac{1}{\sqrt{3}}))^{\frac{1}{2}}$	$\{\omega^k\alpha_{10}\epsilon_1 + \alpha'_{10}\epsilon_2, \alpha'_{10}\epsilon_1 - \alpha_{10}\omega^k\epsilon_2, 0 \le k \le 2\}$	$\mu_{12} P_{10} \cup \mu_{12} P_5$
G_{18}	$\underset{s}{5} \overset{4}{\underset{\alpha_{18}}{\text{———}}} \underset{t}{3}$	$(\frac{1}{2}(1 + \frac{1}{\sqrt{3}} cot \frac{\pi}{5}))^{\frac{1}{2}}$	$(\frac{1}{2}(1 - \frac{1}{\sqrt{3}} cot \frac{\pi}{5}))^{\frac{1}{2}}$	$\{\epsilon_1, \epsilon_2, \zeta^k\alpha_{18}\epsilon_1 + \alpha'_{18}\epsilon_2, \alpha'_{18}\epsilon_1 - \alpha_{18}\zeta^k\epsilon_2,$ $\zeta^k\alpha_{20}\epsilon_1 + \alpha'_{20}\epsilon_2, \alpha'_{20}\epsilon_1 - \alpha_{20}\zeta^k\epsilon_2,$ $\zeta^k\alpha_4\epsilon_1 + \alpha'_4\epsilon_2, \alpha'_4\epsilon_1 - \alpha_4\zeta^k\epsilon_2, 0 \le k \le 4\}$	$\mu_{30} P_{18}$
G_6	$\underset{s}{3} \overset{6}{\underset{\alpha_6}{\text{———}}} \underset{t}{2}$	$(\frac{1}{2}(1 + \frac{1}{\sqrt{3}}))^{\frac{1}{2}}$	$(\frac{1}{2}(1 - \frac{1}{\sqrt{3}}))^{\frac{1}{2}}$	$\{\omega^k\alpha_6\epsilon_1 + \alpha'_6\epsilon_2, \alpha'_6\epsilon_1 - \alpha_6\omega^k\epsilon_2, 0 \le k \le 2\}$	$\mu_{12} P_4 \cup \mu_4 P_6$
G_9	$\underset{s}{4} \overset{6}{\underset{\alpha_9}{\text{———}}} \underset{t}{2}$	$(\frac{1}{2}(1 + \frac{1}{\sqrt{2}}))^{\frac{1}{2}}$	$(\frac{1}{2}(1 - \frac{1}{\sqrt{2}}))^{\frac{1}{2}}$	$\{i^k\alpha_9\epsilon_1 + \alpha'_9\epsilon_2, \alpha'_9\epsilon_1 - \alpha_9 i^k\epsilon_2, i^k\alpha_8\epsilon_1 + \xi\alpha'_8\epsilon_2$ $0 \le k \le 3\}$	$\mu_8 P_8 \cup \mu_8 P_9$
G_{17}	$\underset{s}{5} \overset{6}{\underset{\alpha_{17}}{\text{———}}} \underset{t}{2}$	$(\frac{1}{2}(1 + \alpha_{16}))^{\frac{1}{2}}$	$(\frac{1}{2}(1 - \alpha_{16}))^{\frac{1}{2}}$	$\{\zeta^k\alpha_{17}\epsilon_1 + \alpha'_{17}\epsilon_2, \alpha'_{17}\epsilon_1 - \alpha_{17}\zeta^k\epsilon_2,$ $2\zeta^k\alpha_{17} sin \frac{17\pi}{20}\epsilon_1 + 2\eta\alpha'_{17} sin \frac{13\pi}{20}\epsilon_2,$ $2\zeta^k\alpha'_{17} sin \frac{13\pi}{20}\epsilon_1 - 2\eta\alpha_{17} sin \frac{17\pi}{20}\epsilon_2,$ $\zeta^k\alpha_8\epsilon_1 \pm \eta\alpha'_8\epsilon_2, 0 \le k \le 4\}\}$	$\mu_{20} P_{16} \cup \mu 20 P_{17}$
G_{14}	$\underset{s}{3} \overset{8}{\underset{\alpha_{14}}{\text{———}}} \underset{t}{2}$	$(\frac{1}{2}(1 + \alpha_5))^{\frac{1}{2}}$	$(\frac{1}{2}(1 - \alpha_5))^{\frac{1}{2}}$	$\{\omega^k\alpha_{14}\epsilon_1 + \alpha'_{14}\epsilon_2, \alpha'_{14}\epsilon_1 - \alpha_{14}\omega^k\epsilon_2,$ $(\alpha_{14} + \alpha'_{14})(\omega^k\alpha_8\epsilon_1 \pm i\alpha_8\epsilon_2), 0 \le k \le 2\}$	$\mu_6 P_5 \cup \mu_6 P_{14}$
G_{21}	$\underset{s}{3} \overset{10}{\underset{\alpha_{21}}{\text{———}}} \underset{t}{2}$	$(\frac{1}{2}(1 + \alpha_{20}))^{\frac{1}{2}}$	$(\frac{1}{2}(1 - \alpha_{20}))^{\frac{1}{2}}$	$\{\omega^k\alpha_{21}\epsilon_1 + \alpha'_{21}\epsilon_2, \alpha'_{21}\epsilon_1 - \alpha_{21}\omega^k\epsilon_2,$ $\omega^k\alpha'_{21}\epsilon_1 + \alpha_{21}\epsilon_2, \alpha_{21}\epsilon_1 - \alpha'_{21}\omega^k\epsilon_2,$ $\omega^k\alpha_6\epsilon_1 \pm \alpha'_6\epsilon_2, \alpha'_6\epsilon_1 \pm \alpha_6\omega^k\epsilon_2, \omega^k\alpha_8\epsilon_1 \pm i\alpha'_8\epsilon_2$ $0 \le k \le 2\}$	$\mu_{12} P_{20} \cup \mu_{12} P_{21}$

Table 3: Root systems