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ORTHOGONAL POLYNOMIALS ASSOCIATED WITH ROOT SYSTEMS

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Abstract. Let R and S be two irreducible root systems spanning the same vector space and having the same Weyl group W, such that S (but not necessarily R) is reduced. For each such pair (R, S) we construct a family of W-invariant orthogonal polynomials in several variables, whose coefficients are rational functions of parameters $q, t_1, t_2, \ldots, t_r$, where r $(= 1, 2$ or $3)$ is the number of W-orbits in R. For particular values of these parameters, these polynomials give the values of zonal spherical functions on real and p-adic symmetric spaces. Also when $R = S$ is of type A_n, they conincide with the symmetric polynomials described in I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, 2nd edition, Oxford University Press (1995), Chapter VI.

Foreword

The text that follows this Foreword is that of my 1987 preprint with the above title. It is now in many ways a period piece, and I have thought it best to reproduce it unchanged. I am grateful to Tom Koornwinder and Christian Krattenthaler for arranging for its publication in the Séminaire Lotharingien de Combinatoire.

I should add that the subject has advanced considerably in the intervening years. In particular, the conjectures in §12 below are now theorems. For a sketch of these later developments the reader may refer to my booklet *"Symmetric functions and orthogonal polynomials"*, University Lecture Series Vol. 12, American Mathematical Society (1998), and the references to the literature given there.

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Introduction

The orthogonal polynomials which are the subject of this paper are Laurent polynomials in several variables. To be a little more precise, they are elements of the group algebra A of the weight lattice P of a root system R, invariant under the action of the Weyl group of R, and they depend rationally on two parameters q and t (*). They are indexed by the dominant weights and they are pairwise orthogonal with respect to a certain weight function $\triangle$, to be defined later.

For particular values of the parameters q and t, these polynomials reduce to familiar objects:

(i) when $q = t$ they are independent of q and are the Weyl characters for the root system R.

(ii) when $t = 1$ they are again independent of q, and are the elements of A corresponding to the orbits of the Weyl group in the weight lattice P.

(iii) when $q = 0$ they are (up to a scalar factor) the polynomials that give the values of zonal spherical functions on a semisimple p-adic Lie group G relative to a maximal compact subgroup K, such that the restricted root system of (G, K) is the dual root system $R^\vee$. Here the value of the parameter t is the reciprocal of the cardinality of the residue field of the local field over which G is defined.

(iv) finally, when q and t both tend to 1, in such a way that $(t-1)/(q-1)$ tends to a definite limit k, then (for certain values of k) our polynomials give the values of zonal spherical functions on a real (compact or non-compact) symmetric space G/K arising from finite-dimensional representations of G that have a K-fixed vector $\neq 0$. Here the root system R is the restricted root system of G/K, and the parameter k is half the root multiplicity (assumed for the purposes of this description to be the same for all restricted roots).

Thus these two-parameter families of orthogonal polynomials constitute a sort of bridge between harmonic analysis on real symmetric spaces and on their p-adic analogues. It is perhaps natural to ask, in view of recent developments (quantum groups, etc.), whether there is a group-like object depending on two parameters q and t that lies behind this theory; but on this question we have nothing to say. We would only remark that such a hypothetical object would have to partake of the properties of a p-adic Lie group when $q = 0$, and of a real Lie group in the limiting case $(q, t) \to (1, 1)$ described in (iv) above.

All this is in fact a simplified description of the theory. The context in which we shall work is that of an "admissible pair" (R, S) of root systems: this means that R and S are finite root systems in the same vector space, having the same Weyl group W and such that S (but not necessarily R) is reduced. In this context we define parameters q_α and t_α for each root $\alpha \in R$, such that $q_\alpha = q_\beta$ and $t_\alpha = t_\beta$ if α and

(*) This is a simplified description for the purposes of this introduction.

β are in the same W-orbit. This is described, and the appropriate notation established, in Sections 1 and 2. The weight function $\triangle$ and the accompanying scalar product are defined in Section 3. The main result of the paper is Theorem (4.1), which asserts the existence of a family of orthogonal polynomials P_λ associated with a given admissible pair (R, S).

The proof of the theorem consists in constructing a suitable self-adjoint linear operator E with distinct eigenvalues; the polynomials P_λ are the eigenfunctions of E, suitably normalized. In fact we need two constructions for such a linear operator. The first of these is described in Section 5, and works whenever the root system S (assumed irreducible) possesses a minuscule weight, that is to say provided that S is not of type E_8, F_4 or G_2. The second construction, described in Section 6, is based on the premise, familiar to experts in standard monomial theory, that the next best thing to a minuscule weight is a quasi-minuscule weight, and produces an operator E with the desired properties in the cases not covered by the previous construction.

In Sections 8–11 we consider the particular cases corresponding to (i)-(iv) above. We also consider, in Section 9, the case where R is of rank 1. If R is of type A_1, the polynomials P_λ are essentially the q-ultraspherical polynomials [1], whereas if R is of type BC_1 the P_λ reduce to a particular case of the orthogonal polynomials of Askey and Wilson [2]. Also, if R is of type A_n $(n \geq 1)$ the P_λ are essentially the symmetric functions that are the subject of Chapter VI of [11].

Finally, in Section 12 we put forward two conjectures relating to the polynomials P_λ. They involve a common generalization of Harish-Chandra's c-function and its p-adic counterpart, and one of the conjectures includes as a special case the constant term conjectures of [10] and [13].

§1

Let V be a real vector space of finite dimension, endowed with a positive-definite symmetric bilinear form $\langle u, v\rangle$. We shall write $|v| = \langle v, v\rangle^{1/2}$ for $v \in V$, and

$$v^\vee = 2v/|v|^2$$

if $v \neq 0$. If R is a root system in V, we denote by $R^\vee$ the dual root system $\{\alpha^\vee : \alpha \in R\}$.

Let R and S be root systems in V (and spanning V). The pair (R, S) will be said to be *admissible* if R and S have the same Weyl group W, and S (but not necessarily R) is reduced.

Suppose that (R, S) is admissible. Then the set of hyperplanes in V orthogonal to the roots is the same for both R and S, and hence (as S is reduced) there exists for each $\alpha \in R$ a unique positive real number u_α such that

$$\alpha_* = u_\alpha^{-1}\alpha \in S,$$

and the mapping $f : R \to S$ defined by $f(\alpha) = \alpha_*$ is surjective.

Let $\alpha \in R$, $w \in W$ and let $\beta = w\alpha$. Then $w(\alpha_*) \in S$, and is a positive scalar multiple of β, so that $w(\alpha_*) = \beta_* = (w\alpha)_*$. Hence the mapping f commutes with the action of W, and $u_\alpha = u_\beta$ whenever α, β lie in the same W-orbit in R. Moreover if R is not reduced and α, $2\alpha \in R$, we have

$$u_{2\alpha} = 2u_\alpha, \tag{1.1}$$

since $(2\alpha)_* = \alpha_*$.

From now on we shall assume that R (and therefore also S) is *irreducible*. Another pair (R', S') of root systems in V will be said to be *similar* to (R, S) if there exist positive real numbers a, b such that $R' = aR$ and $S' = bS$. The effect of passing from (R, S) to a similar pair is simply to multiply each u_α by the same positive scalar factor.

The classification of irreducible admissible pairs (R, S) up to similarity is easily described. There are three cases to consider.

(i) R is reduced and $S = R$, so that $u_\alpha = 1$ for each $\alpha \in R$.

(ii) R is reduced, with two root-lengths, and $S = R^\vee$. Then $u_\alpha = \frac{1}{2}|\alpha|^2$ for each $\alpha \in R$. We may assume that $|\alpha|^2 = 2$ for each short root $\alpha \in R$, and then we have $u_\alpha = 1$ if $\alpha \in R$ is short, and $u_\alpha = m$ if α is long, where $m = 2$ if R is of type B_n, C_n or F_4, and $m = 3$ if R is of type G_2.

(iii) R is not reduced, hence is of type BC_n $(n \geq 1)$. Let

$$R_1 = \{\alpha \in R : \tfrac{1}{2}\alpha \notin R\}\ ,\ R_2 = \{\alpha \in R : 2\alpha \notin R\}, \tag{1.2}$$

so that R_1 and R_2 are reduced root systems of types B_n, C_n respectively if $n \geq 2$ (if $n = 1$ they are both of type A_1). Up to similarity, there are two possibilities for S when $n \geq 2$, namely $S = R_1$ and $S = \frac{1}{2}R_2$ (which coincide when $n = 1$). In both cases $u_\alpha = 1$ or 2 for each $\alpha \in R$ (it is for this reason that we chose $\frac{1}{2}R_2$ rather than R_2).

Thus the function $\alpha \mapsto u_\alpha$ on R, when appropriately normalized, is either constant and equal to 1, or else takes just two values $\{1, 2\}$ or $\{1, 3\}$. We shall assume this normalization henceforth.

Remark. The classification of irreducible admissible pairs (R, S) up to similarity is closely related to (but not identical with) the classification of irreducible affine root systems as defined in [9], or equivalently of "echelonnages" as defined in [3].

The polynomials which are the subject of this paper will involve parameters q and t_α, $\alpha \in R$, such that $t_\alpha = t_\beta$ if $|\alpha| = |\beta|$. It would be possible to regard these parameters as independent indeterminates over $\mathbb{Z}$, but it will be more useful to think of them as real variables. So let q be a real number such that $0 \leq q < 1$, and for each $\alpha \in R$ let

$$q_\alpha = q^{u_\alpha},$$

so that $q_{w\alpha} = q_\alpha$ for each $w \in W$, and the set $\{q_\alpha : \alpha \in R\}$ is either $\{q\}$ or $\{q, q^2\}$ or $\{q, q^3\}$. From (1.1) we have

$$q_{2\alpha} = q_\alpha^2$$

if α, $2\alpha \in R$.

Next, for each $\alpha \in R$ let t_α be a real number ≥ 0, such that $t_\alpha = t_\beta$ if $|\alpha| = |\beta|$. If $\alpha \in V$ but $\alpha \notin R$ we set $t_\alpha = 1$. Furthermore, let $k_\alpha = (\log t_\alpha)/(\log q_\alpha)$ if $q \neq 0$ and $t_\alpha \neq 0$, so that

$$t_\alpha = q_\alpha^{k_\alpha}.$$

If $\alpha \notin R$ we have $k_\alpha = 0$.

Finally, let $\mathbb{Z}[t]$ (respectively $\mathbb{Z}[q,t]$) denote the ring of polynomials in the t_α and $t_{2\alpha}^{1/2}$ (respectively and q) with integer coefficients, and let $\mathbb{Q}(q,t)$ denote the field of fractions of $\mathbb{Z}[q,t]$, i.e., the field of rational functions of q and the t_α, $t_{2\alpha}^{1/2}$.

§2

Let (R, S) be an irreducible admissible pair of root systems in V. Let $\{\alpha_1, \ldots, \alpha_n\}$ be a basis (or set of simple roots) of R, and let R^+ denote the set of positive roots determined by this basis. Let

$$Q = \sum_{i=1}^{n} \mathbb{Z}\alpha_i, \qquad Q^+ = \sum_{i=1}^{n} \mathbb{N}\alpha_i$$

be respectively the root lattice of R and its positive octant. Furthermore let P and P^{++} be respectively the weight lattice of R and the cone of dominant weights. We have $Q \subset P$ (but $Q^+ \not\subset P^{++}$ if $n > 1$). If R is not reduced, then Q is the root lattice of R_1 (defined in (1.2)) and P is the weight lattice of R_2.

We define a partial order on P by

$$\lambda \geq \mu \text{ if and only } \lambda - \mu \in Q^+. \tag{2.1}$$

Let A denote the group algebra over $\mathbb{R}$ of the free Abelian group P. For each $\lambda \in P$, let e^λ denote the corresponding element of A, so that $e^\lambda \cdot e^\mu = e^{\lambda+\mu}$, $(e^\lambda)^{-1} = e^{-\lambda}$ and $e^0 = 1$, the identity element of A. The e^λ, $\lambda \in P$, form an $\mathbb{R}$-basis of A.

The Weyl group W of R acts on P and hence also on A: $w(e^\lambda) = e^{w\lambda}$ for $w \in W$ and $\lambda \in P$. Let A^W denote the subalgebra of W-invariant elements of A.

Since each W-orbit in P meets P^{++} in exactly one point, it follows that the "monomial symmetric functions"

$$m_\lambda = \sum_{\mu \in W\lambda} e^\mu \qquad (\lambda \in P^{++})$$

form an $\mathbb{R}$-basis of A^W. Another basis is provided by the Weyl characters: let

$$R_2^+ = \{\alpha \in R^+ : 2\alpha \notin R\},$$

$$\rho = \frac{1}{2} \sum_{\alpha \in R_2^+} \alpha, \tag{2.2}$$

$$\delta = \prod_{\alpha \in R_2^+} (e^{\alpha/2} - e^{-\alpha/2}) = e^\rho \prod_{\alpha \in R_2^+} (1 - e^{-\alpha}). \tag{2.3}$$

Then $w\delta = \varepsilon(w)\delta$ for each $w \in W$, where $\varepsilon(w) = \det(w) = \pm 1$.

For each $\lambda \in P$ let

$$\chi_\lambda = \delta^{-1} \sum_{w \in W} \varepsilon(w) e^{w(\lambda+\rho)}. \tag{2.4}$$

Then $\chi_\lambda \in A^W$ for all $\lambda \in P$, and the χ_λ with $\lambda \in P^{++}$ form an $\mathbb{R}$-basis of A^W. Moreover, we have

$$\chi_\lambda = m_\lambda + \text{lower terms} \qquad (\lambda \in P^{++})$$

where by "lower terms" we mean a linear combination of the m_μ such that $\mu \in P^{++}$ and $\mu < \lambda$ (for the partial ordering (2.1)).

If $\lambda \notin P^{++}$, then either $\chi_\lambda = 0$ or else there exists $\mu \in P^{++}$ and $w \in W$ such that $\mu + \rho = w(\lambda + \rho)$, and in this case $\chi_\lambda = \varepsilon(w)\chi_\mu$.

Let $f \in A$, say

$$f = \sum_{\lambda \in P} f_\lambda e^\lambda$$

with only finitely many nonzero coefficients f_λ. We shall regard f as a function on V as follows: if $x \in V$, then

$$f(x) = \sum f_\lambda q^{\langle \lambda, x \rangle}. \tag{2.5}$$

For f as above, define

$$\bar{f} = \sum f_\lambda e^{-\lambda}$$

so that $\bar{f}(x) = f(-x)$ for all $x \in V$. Also let

$$[f]_1 = \text{ constant term of } f = f_0.$$

Clearly we have

$$[f]_1 = [\bar{f}]_1 = [wf]_1, \tag{2.6}$$

for all $w \in W$.

Next, for each $\mu \in V$ we define $T_\mu f$ by

$$(T_\mu f)(x) = f(x+\mu) \tag{2.7}$$

so that

$$T_\mu f = \sum f_\lambda \, q^{\langle \lambda, \mu \rangle} e^\lambda.$$

Each T_μ is an $\mathbb{R}$-algebra automorphism of A, with inverse $T_{-\mu}$. We have

$$w T_\mu w^{-1} = T_{w\mu} \tag{2.8}$$

for each $w \in W$, and

$$\overline{T_\mu f} = T_{-\mu} \bar{f}. \tag{2.9}$$

Finally, let $g = \sum g_\lambda e^\lambda$ be another element of A. Then

$$[\bar{f} T_\mu g]_1 = [\bar{g} T_\mu f]_1. \tag{2.10}$$

For both sides are equal to $\sum f_\lambda g_\lambda q^{\langle \lambda, \mu \rangle}$.

§3

We shall now define a scalar product on the algebra A. For this purpose we introduce the notation

$$(x;q)_\infty = \prod_{i=0}^{\infty} (1 - xq^i)$$

and for each $k \in \mathbb{R}$

$$(x;q)_k = (x;q)_\infty / (xq^k;q)_\infty. \tag{3.1}$$

In particular, if $k \in \mathbb{N}$ we have

$$(x;q)_k = \prod_{i=0}^{k-1} (1 - xq^i). \tag{3.2}$$

Now let (R, S) be an irreducible admissible pair of root systems with common Weyl group W, and let

$$\begin{aligned} \triangle = \triangle(q,t) &= \prod_{\alpha \in R} \frac{(t_{2\alpha}^{1/2} e^\alpha; q_\alpha)_\infty}{(t_\alpha t_{2\alpha}^{1/2} e^\alpha; q_\alpha)_\infty} \\ &= \prod_{\alpha \in R} (t_{2\alpha}^{1/2} e^\alpha; q_\alpha)_{k_\alpha} \end{aligned} \tag{3.3}$$

by (3.1). If the k_α are all integers ≥ 0, then by (3.2) the product $\triangle$ is a finite product of factors of the form $1 - q_\alpha^i t_{2\alpha}^{1/2} e^\alpha$, and is clearly W-invariant, hence is an element of A^W. In this case we define the scalar product of two elements $f, g \in A$ to be

$$\langle f, g\rangle = |W|^{-1}[f\bar{g}\triangle]_1, \tag{3.4}$$

i.e., the constant term of the Laurent polynomial $f\bar{g}\triangle$, divided by the order of W.

For arbitrary values of the parameters k_α we proceed as follows. Let $Q^\vee$ be the root lattice of the dual root system $R^\vee$, and let $T = V/Q^\vee$. Then each e^λ, $\lambda \in P$, may be regarded as a character of the torus T by the rule $e^\lambda(\mathring{x}) = e^{2\pi i\langle\lambda, x\rangle}$, where $\mathring{x} \in T$ is the image of $x \in V$. By linearity, this enables us to regard each element of A as a continuous function on T.

Consider now the product

$$(t_\alpha t_{2\alpha}^{1/2} e^\alpha(\mathring{x}); q_\alpha)_\infty = \prod_{r=0}^{\infty} \left(1 - q_\alpha^{k_\alpha + k_{2\alpha} + r} e^\alpha(\mathring{x})\right),$$

where $\alpha \in R$ and $\mathring{x} \in T$. This product converges uniformly on T to a continuous function (since $0 \leq q_\alpha < 1$) which does not vanish on T provided that $k_\alpha + k_{2\alpha} \notin -\mathbb{N}$. Likewise the product $(t_{2\alpha}^{1/2} e^\alpha; q_\alpha)_\infty$ represents a continuous function on T, and therefore $\triangle$ defined by (3.3) is a continuous function on T provided that

$$k_\alpha + k_{2\alpha} \notin -\mathbb{N}$$

for all $\alpha \in R$ (where $k_{2\alpha} = 0$ if $2\alpha \notin R$). Hence $\triangle$ may be expanded as a convergent Fourier series on the torus T, say

$$\triangle = \sum_{\lambda \in P} a_\lambda e^\lambda, \tag{3.5}$$

where

$$a_\lambda = \int_T e^{-\lambda} \triangle,$$

the integration being with respect to normalized Haar measure on T.

We now define the scalar product of $f, g \in A$ to be

$$\langle f, g\rangle = \langle f, g\rangle_{q,t} = |W|^{-1} \int_T f\bar{g}\triangle. \tag{3.6}$$

When the k_α are non-negative integers, this definition agrees with the previous one (3.4), since $\int_T e^\lambda = \delta_{0\lambda}$ for $\lambda \in L$.

Let

$$\triangle^+ = \prod_{\alpha \in R^+} (t_{2\alpha}^{1/2} e^{\alpha}; q_\alpha)_{k_\alpha}, \tag{3.7}$$

so that $\triangle = \triangle^+ \cdot \overline{\triangle^+}$. From (3.6) it follows that

$$\langle f, g \rangle = |W|^{-1} \int_T (f\triangle^+)(\overline{g\triangle^+})$$

and hence that the scalar product is symmetric and positive definite.

We shall next derive another expression for the scalar product (3.6) restricted to A^W. For each $w \in W$ let

$$R(w) = R^+ \cap -wR^+, \qquad t_w = \prod_{\alpha \in R(w)} t_\alpha, \tag{3.8}$$

$$W(t) = \sum_{w \in W} t_w. \tag{3.9}$$

Also let

$$\Pi = \prod_{\alpha \in R^+} \frac{1 - t_\alpha t_{2\alpha}^{1/2} e^{-\alpha}}{1 - t_{2\alpha}^{1/2} e^{-\alpha}}$$

and

$$\triangle' = \triangle \Pi = \prod_{\alpha \in R^+} (t_{2\alpha}^{1/2} e^{\alpha}; q_\alpha)_{k_\alpha} (t_{2\alpha}^{1/2} q_\alpha e^{-\alpha}; q_\alpha)_{k_\alpha}. \tag{3.10}$$

From [8] we have the identity

$$\sum_{w \in W} w\Pi = W(t)$$

so that

$$W(t)\triangle = \sum_{w \in W} w\triangle'. \tag{3.11}$$

Now let $f, g \in A^W$. Then we have

$$\begin{aligned} W(t)\langle f, g \rangle &= \frac{W(t)}{|W|} \int_T f\bar{g}\triangle \\ &= |W|^{-1} \sum_{w \in W} \int_T f\bar{g} \cdot w\triangle' \\ &= \int_T f\bar{g}\triangle' \end{aligned}$$

since f and g are W-invariant. Hence

$$\langle f,g\rangle = W(t)^{-1}\int_T f\bar{g}\triangle' \tag{3.12}$$

for $f,g\in A^W$.

Remark. If we choose to regard the parameters q and t_α as indeterminates over $\mathbb{Z}$ rather than as real numbers, we can expand $\triangle'$ as a formal Laurent series. For this purpose let $\varphi=\sum m_i\alpha_i$ be the highest root of R and let

$$x_0 = qe^{-\varphi},\quad x_i = e^{\alpha_i}\quad (1\le i\le n).$$

Then $q = x_0e^{\varphi} = x_0x_1^{m_1}\cdots x_n^{m_n}$ is a monomial in the x's, and it follows that each of the products $q_\alpha^i e^\alpha$, $q_\alpha^{i+1}e^{-\alpha}$, where $\alpha\in R^+$ and $i\ge 0$, is also a monomial in the x's, since $\varphi\ge\alpha$ for each root $\alpha\in R^+$. Moreover the total degrees of these monomials tend to ∞ as $i\to\infty$, and therefore $\triangle'$ can be expanded as a formal power series in $x_0,x_1,\dots,x_n$, say

$$\triangle' = \sum_r b_r(t)x^r,$$

where the sum is over all $r=(r_0,\dots,r_n)\in\mathbb{N}^{n+1}$, and $x^r = x_0^{r_0}\cdots x_n^{r_n}$, and the coefficients $b_r(t)$ lie in the ring $\mathbb{Z}[t]$ of polynomials in the t_α and $t_{2\alpha}^{1/2}$ with integer coefficients.

In terms of the original variables we have

$$x^r = q^{r_0}\exp\left(\sum_{i=1}^n r_i\alpha_i - r_0\varphi\right),$$

and therefore for each $\lambda\in Q$ the coefficient of e^λ in $\triangle'$ is

$$a'_\lambda(q,t) = \sum_{r_0} q^{r_0}b_r(t), \tag{3.13}$$

where the vector $r=(r_0,r_1,\dots,r_n)$ is determined from λ and r_0 by the equation

$$\sum_{i=1}^n r_i\alpha_i = \lambda + r_0\varphi, \tag{3.14}$$

and the sum in (3.13) is over all integers $r_0\ge 0$ such that the r_i determined by (3.14) are all ≥ 0. Thus we have

$$\triangle' = \sum_{\lambda\in Q} a'_\lambda(q,t)e^\lambda$$

a formal Laurent series with coefficients in $\mathbb{Z}[t][[q]]$.

The identity (3.10) now gives

$$\begin{aligned}\triangle &= W(t)^{-1} \sum_{w \in W} w\triangle' \\ &= W(t)^{-1} \sum_{w,\lambda} a'_\lambda(q,t) e^{w\lambda}\end{aligned}$$

so that

$$\triangle = \sum_{\lambda \in Q} a_\lambda(q,t) e^\lambda, \tag{3.15}$$

where

$$a_\lambda(q,t) = W(t)^{-1} \sum_{w \in W} a'_{w\lambda}(q,t).$$

The expression (3.15) is the expansion of $\triangle$ as a W-invariant formal Laurent series, with coefficients in the ring of formal power series $\mathbb{Q}(t)[[q]]$, where $\mathbb{Q}(t)$ is the field of fractions of the ring $\mathbb{Z}[t]$. This expansion is of course the same thing as the Fourier series (3.5).

If $f, g \in A$, the constant term in $f\bar{g}\triangle$ is now well-defined, being a finite linear combination of the coefficients $a_\lambda(q,t)$, and we have $\langle f, g\rangle = |W|^{-1}[f\bar{g}\triangle]_1$ as in (3.4).

§4

We can now state the main result of this paper.

Theorem (4.1). *For each irreducible admissible pair (R, S) of root systems there exists a unique basis $(P_\lambda)_{\lambda \in P^{++}}$ of A^W such that*

(i) $P_\lambda = m_\lambda + \sum_{\substack{\mu < \lambda \\ \mu \in P^{++}}} u_{\lambda\mu}(q,t) m_\mu$
with coefficients $a_{\lambda\mu}(q,t) \in \mathbb{Q}(q,t)$;

(ii) $\langle P_\lambda, P_\mu\rangle = 0$ *if $\lambda \neq \mu$.*

It is clear that the P_λ, if they exist, are unique. If the partial order (2.1) on P^{++} were a total order, the existence of the P_λ would follow directly from the Gram-Schmidt orthogonalization process. However, the partial order (2.1) is not a total order (unless rank $R = 1$), and we should therefore have to choose a compatible total order on P^{++} before applying Gram-Schmidt. The content of (4.1) is that however we extend the partial order to a total order, we end up with the same basis of A^W.

Theorem (4.1) will be a consequence of the following proposition:

Proposition (4.2). *For each irreducible admissible pair (R, S) there exists a linear operator $E : A^W \to A^W$ with the following three properties:*

(i) *E is self adjoint, i.e., $\langle Ef, g\rangle = \langle f, Eg\rangle$ for all $f, g \in A^W$.*

(ii) *We have*

$$Em_\lambda = \sum_{\substack{\mu\le\lambda \\ \mu\in P^{++}}} c_{\lambda\mu} m_\mu$$

for each $\lambda \in P^{++}$, with coefficients $c_{\lambda\mu} \in q^{a(\lambda)}\mathbb{Z}[q, t]$, where $a : P \to \mathbb{Q}$ is a homomorphism such that $a(Q) \subset \mathbb{Z}$.

(iii) *If $\lambda \neq \mu$, then $c_{\lambda\lambda} \neq c_{\mu\mu}$; i.e., the eigenvalues of E are distinct.*

Granted the existence of E with these properties, let

$$E_\lambda = \prod_{\substack{\mu<\lambda \\ \mu\in P^{++}}} \frac{E - c_{\mu\mu}}{c_{\lambda\lambda} - c_{\mu\mu}}$$

for $\lambda \in P^{++}$. Then the elements $P_\lambda = E_\lambda m_\lambda$ of A^W satisfy the conditions of (4.1). Indeed, it is clear from (4.2)(ii) that the P_λ satisfy (4.1)(i). (The fractional exponents $a(\lambda)$ cause no trouble, because $a(\lambda) - a(\mu) \in \mathbb{Z}$ if $\lambda > \mu$.) On the other hand, let M_λ be the subspace of A^W spanned by the m_μ such that $\mu \le \lambda$; then M_λ is finite-dimensional and stable under E, and the minimal polynomial of E restricted to M_λ is $\prod_{\mu\le\lambda}(X - c_{\mu\mu})$, since the $c_{\mu\mu}$ are all distinct. Hence $(E - c_{\lambda\lambda})E_\lambda = 0$ on M_λ, and therefore

$$EP_\lambda = EE_\lambda m_\lambda = c_{\lambda\lambda}E_\lambda m_\lambda = c_{\lambda\lambda}P_\lambda.$$

If now $\lambda \neq \mu$ we have

$$\begin{aligned} c_{\lambda\lambda}\langle P_\lambda, P_\mu\rangle &= \langle EP_\lambda, P_\mu\rangle = \langle P_\lambda, EP_\mu\rangle \\ &= c_{\mu\mu}\langle P_\lambda, P_\mu\rangle \end{aligned}$$

by the self-adjointness of E, and hence $\langle P_\lambda, P_\mu\rangle = 0$ by (4.2)(iii). ∎

In the next two sections we shall construct for each irreducible admissible pair (R, S) an operator E satisfying the conditions of (4.2). Our first construction, in §5, works whenever the root system $S^\vee$ has a minuscule fundamental weight (equivalent conditions are that $P \neq Q$, or that R is not of type E_8, F_4 or G_2). In §6 we shall give another construction which workes in these excluded cases.

§5

In this section we shall assume that $S^\vee$ possesses a minuscule fundamental weight, i.e., that there exists a vector $\pi \in V$ such that $\langle \pi, \alpha_*\rangle$ takes just two values 0 and 1 as α runs through R^+. We have then (2.7)

$$T_\pi e^\alpha = q^{\langle\pi,\alpha\rangle}e^\alpha = q_\alpha^{\langle\pi,\alpha_*\rangle}e^\alpha$$

so that

$$T_\pi e^\alpha = \begin{cases} q_\alpha e^\alpha & \text{if } \langle \pi, \alpha_* \rangle = 1, \\ e^\alpha & \text{if } \langle \pi, \alpha_* \rangle = 0. \end{cases} \tag{5.1}$$

Now let

$$\begin{aligned} \Phi_\pi &= (T_\pi \triangle^+)/\triangle^+ \\ &= \prod_{\substack{\alpha \in R^+ \\ \langle \pi, \alpha_* \rangle = 1}} \frac{1 - t_\alpha t_{2\alpha}^{1/2} e^\alpha}{1 - t_{2\alpha}^{1/2} e^\alpha} \end{aligned} \tag{5.2}$$

by (5.1) and the definition (3.7) of $\triangle^+$.

We define an operator E_π on A as follows:

$$E_\pi f = \sum_{w \in W} w(\Phi_\pi \cdot T_\pi f). \tag{5.3}$$

Let us first show that E_π is self-adjoint (on the assumption that it maps A into A, which we shall justify shortly). Since

$$E_\pi f = \sum_{w \in W} \frac{w(T_\pi(\triangle^+ f))}{w \triangle^+},$$

and since $\triangle = w\triangle = w\triangle^+ \cdot \overline{w\triangle^+}$ for each $w \in W$, we have

$$\begin{aligned} \langle E_\pi f, g \rangle &= |W|^{-1} \sum_{w \in W} \left[w(T_\pi(\triangle^+ f)) \cdot \overline{w(\triangle^+ g)} \right]_1 \\ &= \left[T_\pi(\triangle^+ f) \cdot \overline{\triangle^+ g} \right]_1 \end{aligned}$$

by (2.6), and by (2.10) this expression is symmetrical in f and g. Hence

$$\langle E_\pi f, g \rangle = \langle E_\pi g, f \rangle = \langle f, E_\pi g \rangle. \tag{5.4}$$

To show that E_π maps A into A, we need to express Φ_π in a more convenient form. If $\alpha \in R^+$ and $\frac{1}{2}\alpha \in R^+$, the corresponding factors in the product (5.2) combine to give

$$\begin{aligned} \frac{(1 - t_\alpha e^\alpha)\left(1 - t_{\alpha/2} t_\alpha^{1/2} e^{\alpha/2}\right)}{(1 - e^\alpha)\left(1 - t_\alpha^{1/2} e^{\alpha/2}\right)} &= \frac{\left(1 + t_\alpha^{1/2} e^{\alpha/2}\right)\left(1 - t_{\alpha/2} t_\alpha^{1/2} e^{\alpha/2}\right)}{1 - e^\alpha} \\ &= \frac{1 + (1 - t_{\alpha/2}) t_\alpha^{1/2} e^{\alpha/2} - t_{\alpha/2} t_\alpha e^\alpha}{1 - e^\alpha}. \end{aligned}$$

If $\frac{1}{2}\alpha \notin R^+$ (and $2\alpha \notin R^+$), this is still correct, since then $t_{\alpha/2} = 1$. It follows that

$$\Phi_\pi = \prod_{\alpha \in R_2^+} \frac{1 + \left(1 - t_{\alpha/2}^{\langle \pi, \alpha_* \rangle}\right) t_\alpha^{\langle \pi, \alpha_* \rangle/2} e^{\alpha/2} - (t_{\alpha/2} t_\alpha)^{\langle \pi, \alpha_* \rangle} e^\alpha}{1 - e^\alpha}, \tag{5.5}$$

where as before $R_2^+ = \{\alpha \in R^+ : 2\alpha \notin R\}$.

Since $t_\alpha^{\langle \pi, \alpha_* \rangle} = q^{k_\alpha \langle \pi, \alpha \rangle}$, we have

$$\prod_{\alpha \in R^+} t_\alpha^{\langle \pi, \alpha_* \rangle} = q^{2\langle \pi, \rho_k \rangle}, \tag{5.6}$$

where

$$\rho_k = \frac{1}{2} \sum_{\alpha \in R^+} k_\alpha \alpha. \tag{5.7}$$

Hence the product (5.5) for Φ_π can be rewritten in the form

$$\Phi_\pi = \delta^{-1} e^\rho q^{2\langle \pi, \rho_k \rangle} \Psi,$$

where δ and ρ are as in (2.2) and (2.3), and Ψ is the product

$$\Psi = \prod_{\alpha \in R_2^+} \left(1 + \left(t_{\alpha/2}^{-\langle \pi, \alpha_* \rangle} - 1\right) t_\alpha^{-\langle \pi, \alpha_* \rangle/2} e^{-\alpha/2} - (t_{\alpha/2} t_\alpha)^{-\langle \pi, \alpha * \rangle} e^{-\alpha}\right).$$

If we multiply out this product, we shall obtain

$$\Phi_\pi = \delta^{-1} q^{2\langle \pi, \rho_k \rangle} \sum_X \varphi_X(t) e^{\rho - \sigma(X)}, \tag{5.8}$$

summed over all subsets X of R^+ such that

$$\alpha \in X \Rightarrow 2\alpha \notin X, \tag{5.9}$$

with the following notation:

$$\begin{aligned} \sigma(x) &= \sum_{\alpha \in X} \alpha, \\ \varphi_X(t) &= \prod_{\alpha \in X} \varphi_\alpha(t), \end{aligned} \tag{5.10}$$

$$\varphi_\alpha(t) = \begin{cases} -(t_{\alpha/2} t_\alpha)^{-\langle \pi, \alpha_* \rangle} & \text{if } 2\alpha \notin R, \\ \left(t_\alpha^{-\langle \pi, \alpha_* \rangle} - 1\right) t_{2\alpha}^{-\langle \pi, \alpha_* \rangle/2} & \text{if } 2\alpha \in R. \end{cases} \tag{5.11}$$

We can now calculate $E_\pi e^\mu$, where $\mu \in P$. Since $w(T_\pi e^\mu) = q^{\langle \pi,\mu\rangle} e^{w\mu}$, and since $w\delta = \varepsilon(w)\delta$ for each $w \in W$, where $\varepsilon(w) = \det(w) = \pm 1$, we shall obtain from (5.8)

$$\begin{aligned} E_\pi e^\mu &= \delta^{-1} q^{\langle \pi, 2\rho_k + \mu\rangle} \sum_X \varphi_X(t) \sum_{w\in W} \varepsilon(w) e^{w(\mu+\rho-\sigma(X))} \\ &= q^{\langle \pi, 2\rho_k+\mu\rangle} \sum_X \varphi_X(t) \chi_{\mu-\sigma(X)} \end{aligned}$$

from which it follows that E_π maps A into A^W.

Now let $\lambda \in P^{++}$. Then we have

$$\begin{aligned} E_\pi m_\lambda &= \sum_{\mu\in W\lambda} E_\pi e^\mu \\ &= \sum_X \varphi_X(t) \sum_{\mu \in W\lambda} q^{\langle \pi, 2\rho_k+\mu\rangle} \chi_{\mu-\sigma(X)}. \end{aligned} \tag{5.12}$$

In this sum, either $\chi_{\mu-\sigma(X)} = 0$ or else there exists $w \in W$ and $\nu \in P^{++}$ such that

$$\nu + \rho = w(\mu + \rho - \sigma(X)) \tag{5.13}$$

in which case $\chi_{\mu-\sigma(X)} = \varepsilon(w)\chi_\nu$. But $\rho - \sigma(X)$ is of the form

$$\rho - \sigma(X) = \frac{1}{2} \sum_{\alpha \in R_2^+} \varepsilon_\alpha \alpha,$$

where each coefficient ε_α is ± 1 or 0, hence $w(\rho - \sigma(X))$ is of the same form, and therefore

$$w(\rho - \sigma(X)) = \rho - \sigma(Y) \tag{5.14}$$

for same subset Y of R^+ such that $\alpha \in Y \Rightarrow 2\alpha \notin Y$. From (5.13) and (5.14) it follows that

$$\nu = w\mu - \sigma(Y) \le w\mu \le \lambda \tag{5.15}$$

and hence that $E_\pi m_\lambda$ is a linear combination of the χ_ν such that $\nu \in P^{++}$ and $\nu \le \lambda$. Hence we have

$$E_\pi m_\lambda = \sum_{\substack{\nu \le \lambda \\ \nu \in P^{++}}} b_{\lambda\nu} \chi_\nu,$$

where from (5.12) the coefficient $b_{\lambda\nu}$ is given by

$$b_{\lambda\nu} = \sum \varepsilon(w) q^{\langle \pi, 2\rho_k+\mu\rangle} \varphi_X(t)$$

summed over triples (X, μ, w) where $X \subset R^+$, $\mu \in W\lambda$ and $w \in W$ satisfy (5.9) and (5.13). From (5.6) and the definition (5.10) of $\varphi_X(t)$ it follows that $q^{\langle \pi, 2\rho_k \rangle}\varphi_X(t) \in \mathbb{Z}[t]$, the ring of polynomials over $\mathbb{Z}$ generated by the t_α and $t_{2\alpha}^{1/2}$. As to the scalar product $\langle \pi, \mu \rangle$, we have

$$\langle \pi, \mu \rangle = \langle \pi, w_0\lambda \rangle + \langle \pi, \theta \rangle$$

where w_0 is the longest element of W and $\theta = \mu - w_0\lambda \in Q^+$. Now $\langle \pi, \alpha_* \rangle = 0$ or 1 for each $\alpha \in R^+$, and hence $\langle \pi, \alpha \rangle = 0$ or u_α for $\alpha \in R^+$. Hence $\langle \pi, \theta \rangle$ is a non-negative integer and therefore $b_{\lambda\nu} \in q^{\langle \pi, w_0\lambda \rangle}\mathbb{Z}[q, t]$. The exponent $\langle \pi, w_0\lambda \rangle$ need not be an integer, but it is a rational number.

From this it follows that

$$E_\pi m_\lambda = \sum_{\substack{\nu \leq \lambda \\ \nu \in P^{++}}} c_{\lambda\nu}(\pi) m_\nu \tag{5.16}$$

with coefficients $c_{\lambda\nu}(\pi) \in q^{\langle \pi, w_0\lambda \rangle}\mathbb{Z}[q, t]$.

We must now calculate the leading coefficient $c_{\lambda\lambda}(\pi)$ in (5.16). From (5.15) it follows that $\nu = \lambda$ if and only if Y is empty and $w\mu = \lambda$, that is to say if and only if $\mu = w^{-1}\lambda$ and $w(\rho - \sigma(X)) = \rho$, or equivalently $\sigma(X) = \rho - w^{-1}\rho$. But this implies [8] that $X = R_2(w) = R_2^+ \cap -wR_2^+$. Hence the coefficient of m_λ in (5.16) is

$$c_{\lambda\lambda}(\pi) = \sum_{w \in W} \varepsilon(w)\varphi_{R_2(w)} q^{\langle \pi, 2\rho_k + w^{-1}\lambda \rangle}.$$

Since $\varepsilon(w) = (-1)^{|R_2(w)|}$, we obtain from (5.10) and (5.11)

$$\begin{aligned} \varepsilon(w)\varphi_{R_2(w)}(t) &= \prod_{\alpha \in R_2(w)} (t_{\alpha/2}t_\alpha)^{-\langle \pi, \alpha_* \rangle} \\ &= \prod_{\alpha \in R(w)} t_\alpha^{-\langle \pi, \alpha_* \rangle} \\ &= q^{\langle \pi, w^{-1}\rho_k - \rho_k \rangle} \end{aligned}$$

since $t_\alpha^{-\langle \pi, \alpha_* \rangle} = q^{-\langle \pi, k_\alpha \alpha \rangle}$. Hence

$$\begin{aligned} c_{\lambda\lambda}(\pi) &= q^{\langle \pi, \rho_k \rangle} \sum_{w \in W} q^{\langle w\pi, \lambda + \rho_k \rangle} \\ &= q^{\langle \pi, \rho_k \rangle} \widetilde{m}_\pi(\lambda + \rho_k) \end{aligned} \tag{5.17}$$

where $\widetilde{m}_\pi = \sum_{w \in W} e^{w\pi}$.

It remains to examine whether the eigenvalues $c_{\lambda\lambda}(\pi)$ of E_π are all distinct as λ runs through P^{++}, for a suitable choice of the minuscule weight π. It will appear that this is so in all cases except D_n, $n \geq 4$ (and of course excepting E_8, F_4 and G_2, where there is no minuscule weight).

Let

$$p_r(x) = \sum_{w \in W} \langle x, w\pi \rangle^r \qquad (x \in V).$$

The p_r, $r \geq 1$, are W-invariant polynomial functions on V.

(5.18). *Suppose that S is not of type D_n $(n \geq 4)$, and that if S is of type A_n the minuscule weight π is the fundamental weight corresponding to an end node of the Dynkin diagram. Then the p_r generate the $\mathbb{R}$-algebra of W-invariant polynomial functions on V, and hence separate the W-orbits in V.*

This is easily verified for S of type A, B or C. For E_6 and E_7 see [12].

Assume now that the hypotheses of (5.18) are satisfied, and that $\lambda, \mu \in P^{++}$ are such that $c_{\lambda\lambda}(\pi) = c_{\mu\mu}(\pi)$, i.e., that

$$\sum_{w \in W} q^{\langle \lambda + \rho_k, w\pi \rangle} = \sum_{w \in W} q^{\langle \mu + \rho_k, w\pi \rangle}.$$

By operating on both sides with $(q\partial/\partial q)^r$ and then setting $q = 1$ and $t_\alpha = 1$ for each $\alpha \in R$, we obtain $p_r(\lambda) = p_r(\mu)$ for all $r \geq 1$. Hence by (5.18) λ and μ are in the same W-orbit, and therefore $\lambda = \mu$. It follows that the eigenvalues $c_{\lambda\lambda}(\pi)$ of E_π are all distinct.

There remains the case where $S(= R)$ is of type D_n. Let $\varepsilon, \ldots, \varepsilon_n$ be an orthonormal basis of V; we may then assume that R^+ consists of the vectors $\varepsilon_i \pm \varepsilon_j$ with $i < j$. Then P^{++} consists of the vectors $\lambda = \sum \lambda_i \varepsilon_i$ for which the λ_i are all integers or all half-integers, and $\lambda_1 \geq \cdots \geq \lambda_{n-1} \geq |\lambda_n|$. The fundamental weights

$$\pi_1 = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_n), \quad \pi_2 = \pi_1 - \varepsilon_n$$

are both minuscule, and the W-orbit of π_1 (respectively π_2) consists of all sums $\frac{1}{2} \sum \pm \varepsilon_i$ containing an even (respectively odd) number of minus signs. Hence the formula (5.17) gives

$$c_{\lambda\lambda}(\pi_1) \pm c_{\lambda\lambda}(\pi_2) = n! \prod_{i=1}^{n} \left(q^{\lambda_i} t^{n-i} \pm q^{-\lambda_i} \right)$$

where $t = t_\alpha, \alpha \in R$.

Choose an integer $N > \frac{1}{2}n(n-1)$. Then the eigenvalues of the operator

$$E = \frac{1}{n!} \left(\left(t^N + 1\right) E_{\pi_1} + \left(t^N - 1\right) E_{\pi_2} \right) \tag{5.19}$$

are from above

$$c_\lambda = t^N \prod_{i=1}^{n} (q^{\lambda_i} t^{n-i} + q^{-\lambda_i}) + \prod_{i=1}^{n} (q^{\lambda_i} t^{n-i} - q^{-\lambda_i}).$$

Now suppose that $\lambda, \mu \in P^{++}$ are such that $c_\lambda = c_\mu$. From our choice of N it follows that

$$\prod_{i=1}^{n} (q^{\lambda_i} t^{n-i} + q^{-\lambda_i}) = \prod_{i=1}^{n} (q^{\mu_i} t^{n-i} + q^{-\mu_i}), \tag{5.20}$$

$$\prod_{i=1}^{n} (q^{\lambda_i} t^{n-i} - q^{-\lambda_i}) = \prod_{i=1}^{n} (q^{\mu_i} t^{n-i} - q^{-\mu_i}). \tag{5.21}$$

From (5.20) we conclude that $\lambda_i = \mu_i$ $(1 \leq i \leq n-1)$ and $\lambda_n = \pm\mu_n$. If $\lambda_n \neq 0$, then (5.21) shows that $\lambda_n = \mu_n$, and hence $\lambda = \mu$. So the eigenvalues c_λ of E are all distinct.

To recapitulate, let $E : A^W \to A^W$ be the operator E_π defined by (5.3) when R is not of type D_n, and by (5.19) when R is of type D_n. By (5.4), (5.16) and the discussion above, it follows the E satisfies the three conditions of (4.2).

§6

In the cases where there is no minuscule weight available, another construction is needed. We shall assume in this section that R is reduced, so that either $S = R$ or $S = R^\vee$, from the classification in §1.

Let $\varphi \in R^+$ be such that φ_* is the highest root of S, and let $\pi = (\varphi_*)^\vee = u_\varphi \varphi^\vee$. For each $\alpha \in R^+$ the Cauchy-Schwarz inequality gives

$$0 \leq \langle \pi, \alpha_* \rangle = \frac{2\langle \varphi_*, \alpha_* \rangle}{|\varphi_*|^2} \leq \frac{2|\alpha_*|}{|\varphi_*|} \leq 2$$

with equality if and only if $\alpha_* = \varphi_*$. Since $\langle \pi, \alpha_* \rangle$ is an integer, it follows that for each $\alpha \in R^+$

$$\langle \pi, \alpha_* \rangle = \begin{cases} 0 \text{ or } 1 & \text{if } \alpha \neq \varphi, \\ 2 & \text{if } \alpha = \varphi. \end{cases} \tag{6.1}$$

Thus π just fails to be a minuscule weight.

Remark. In fact $\varphi_* = \varphi$, so that $u_\varphi = 1$ and $q_\varphi = q$. This is clear if $S = R$, whereas if $S = R^\vee$ (and R has two root-lengths) φ is the highest short root of R, so that $u_\varphi = 1$ in this case also. Hence $\pi = \varphi^\vee$.

Let

$$\Phi_\pi = (T_\pi \triangle^+)/\triangle^+$$

as in §5, and define an operator F_π on A^W as follows:

$$F_\pi f = \sum_{w\in W} w(\Phi_\pi \cdot U_\pi f)$$

where $U_\pi = T_\pi - 1$.

Let us first show that F_π is self-adjoint. If $f, g \in A^W$ we have

$$\begin{aligned}\langle F_\pi f, g\rangle &= |W|^{-1}\sum_{w\in W}\left[w((T_\pi\triangle^+)(U_\pi f))\cdot\overline{w(\triangle^+ g)}\right]_1\\ &= \left[(T_\pi\triangle^+)(T_\pi f - f)\overline{\triangle^+ g}\right]_1\\ &= \left[T_\pi(\triangle^+ f)\cdot\overline{\triangle^+ g}\right]_1 - \left[(T_\pi\triangle^+)\overline{\triangle^+}\cdot f\bar{g}\right]_1\\ &= A(f,g) - B(f,g),\end{aligned}$$

say. We have $A(f,g) = A(g,f)$ by (2.10). As to $B(f,g)$, let $G = (T_\pi\triangle^+)\overline{\triangle^+}$ and let w_0 be the longest element of the Weyl group W. Then $w_0\pi = -\pi$, and $w_0\triangle^+ = \overline{\triangle^+}$, so that

$$w_0 G = \big(T_{-\pi}\overline{\triangle^+}\big)\triangle^+ = \overline{T_\pi\triangle^+}\triangle^+ = \overline{G}$$

and therefore

$$\begin{aligned}B(f,g) &= [Gf\bar{g}]_1 = [(w_0G)f\bar{g}]_1\\ &= [\overline{G}f\bar{g}]_1 = [G\bar{f}g]_1 = B(g,f).\end{aligned}$$

It follows that

$$\langle F_\pi f, g\rangle = \langle F_\pi g, f\rangle = \langle f, F_\pi g\rangle \tag{6.2}$$

and hence that F_π is self-adjoint.

From (6.1) we have, since $q_\varphi = q$,

$$\begin{aligned}\Phi_\pi &= \frac{1-t_\varphi e^\varphi}{1-e^\varphi}\cdot\frac{1-qt_\varphi e^\varphi}{1-qe^\varphi}\prod_{\substack{\alpha\in R^+\\ \alpha\neq\varphi}}\frac{1-t_\alpha^{\langle\pi,\alpha_*\rangle}e^\alpha}{1-e^\alpha}\\ &= q^{2\langle\pi,\rho_k\rangle}\frac{(1-t_\varphi^{-1}e^{-\varphi})(1-q^{-1}t_\varphi^{-1}e^{-\varphi})}{(1-e^{-\varphi})(1-q^{-1}e^{-\varphi})}\prod_{\substack{\alpha\in R^+\\ \alpha\neq\varphi}}\frac{1-t_\alpha^{-\langle\pi,\alpha_*\rangle}e^{-\alpha}}{1-e^{-\alpha}}\\ &= \delta^{-1}e^\rho q^{2\langle\pi,\rho_k\rangle}\frac{(1-t_\varphi^{-1}e^{-\varphi})(1-q^{-1}t_\varphi^{-1}e^{-\varphi})}{1-q^{-1}e^{-\varphi}}\prod_{\substack{\alpha\in R^+\\ \alpha\neq\varphi}}\left(1-t_\alpha^{-\langle\pi,\alpha_*\rangle}e^{-\alpha}\right)\end{aligned}$$

and therefore

$$\Phi_\pi = \delta^{-1} q^{2\langle \pi, \rho_k \rangle} \frac{(1 - t_\varphi^{-1} e^{-\varphi})(1 - q^{-1} t_\varphi^{-1} e^{-\varphi})}{1 - q^{-1} e^{-\varphi}} \sum_X \psi_X(t) e^{\rho - \sigma(X)} \tag{6.3}$$

summed over all subsets X of R^+ such that $\varphi \notin X$, where

$$\sigma(X) = \sum_{\alpha \in X} \alpha,$$

and

$$\psi_X(t) = (-1)^{|X|} \prod_{\alpha \in X} t_\alpha^{-\langle \pi, \alpha_* \rangle}. \tag{6.4}$$

Now let $\lambda \in P^{++}$ and $\mu \in W\lambda$. Since $T_\pi e^\mu = q^{\langle \mu, \pi \rangle} e^\mu = q^{\langle \mu, \varphi^\vee \rangle} e^\mu$, we have

$$U_\pi m_\lambda = \sum_{\mu \in W\lambda} \left(q^{\langle \mu, \varphi^\vee \rangle} - 1 \right) e^\mu. \tag{6.5}$$

Any $\mu \in W\lambda$ such that $\langle \mu, \varphi^\vee \rangle = 0$ will contribute nothing to this sum. The remaining elements of the orbit $W\lambda$ fall into pairs $\{\mu, w_\varphi \mu\}$ where $\langle \mu, \varphi^\vee \rangle > 0$ and w_φ is the reflection associated with φ. Hence we may rewrite (6.5) in the form

$$U_\pi m_\lambda = \sum_{\substack{\mu \in W\lambda \\ \langle \mu, \varphi^\vee \rangle > 0}} \left(q^{\langle \mu, \varphi^\vee \rangle} - 1 \right) e^\mu \left(1 - (q e^\varphi)^{-\langle \mu, \varphi^\vee \rangle} \right)$$

from which it follows that

$$\frac{U_\pi m_\lambda}{1 - q^{-1} e^{-\varphi}} = \sum_{\substack{\mu \in W\lambda \\ \langle \mu, \varphi^\vee \rangle > 0}} \left(q^{\langle \mu, \varphi^\vee \rangle} - 1 \right) \sum_{j=0}^{\langle \mu, \varphi^\vee \rangle - 1} q^{-j} e^{\mu - j\varphi}. \tag{6.6}$$

From (6.3) and (6.6) we obtain

$$\begin{aligned} \Phi_\pi \cdot U_\pi m_\lambda = \delta^{-1} q^{2\langle \pi, \rho_k \rangle} & \sum_{X, \mu} \psi_X(t) e^{\rho - \sigma(X)} (1 - t_\varphi^{-1} e^{-\varphi}) \\ & \times (1 - q^{-1} t_\varphi^{-1} e^{-\varphi}) \left(q^{\langle \mu, \varphi^\vee \rangle} - 1 \right) \prod_{j=0}^{\langle \mu, \varphi^\vee \rangle - 1} q^{-j} e^{\mu - j\varphi}. \end{aligned} \tag{6.7}$$

This is a sum of terms of the form $a \delta^{-1} e^\eta$, $\eta \in P$. Since $\sum_{w \in W} w(\delta^{-1} e^\eta) \in A^W$, it follows from (6.7) that $F_\pi m_\lambda \in A^W$, and hence that F_π maps A^W into A^W.

Moreover, the terms $a\delta^{-1}e^{\eta}$ that occur in (6.7) are such that

$$\eta = \rho - \sigma(X) + \mu - j\varphi$$

where $0 \le j \le \langle \mu, \varphi^\vee \rangle + 1$ and $\langle \mu, \varphi^\vee \rangle \ge 1$ and $X \subset R^+ - \{\varphi\}$. If η is not regular (i.e., if $W_\eta \neq 1$, where W_η is the subgroup of W that fixes η) it will contribute nothing to (6.7). If on the other hand η is regular, then we have $w\eta = \xi + \rho$ for some $\xi \in P^{++}$ and some $w \in W$, so that

$$\xi + \rho = w(\rho - \sigma(X)) + w(\mu - j\varphi). \tag{6.8}$$

There are two cases to consider.

(i) Suppose that $w\varphi = \alpha \in R^+$. Since $\rho - \sigma(X)$ is of the form $\frac{1}{2}\sum_{\alpha \in R^+} \varepsilon_\alpha \alpha$, where each ε_α is ± 1, it follows that $w(\rho - \sigma(X))$ is of the same form, hence that

$$w(\rho - \sigma(X)) = \rho - \sigma(Y)$$

for some subset Y of R^+. Hence

$$\xi = w\mu - j\alpha - \sigma(Y) \le w\mu \le \lambda \tag{6.9}$$

and therefore each such term $a\delta^{-1}e^{\eta}$ in (6.7) contributes $\varepsilon(w)a\chi_\xi$, where $\xi \le \lambda$, to $F_\pi m_\lambda$.

Moreover we have equality in (6.9) if and only if $j = 0$, the subset Y is empty, and $w\mu = \lambda$, i.e., $\mu = w^{-1}\lambda$ and $\sigma(X) = \rho - w^{-1}\rho$, so that $X = R(w) = R^+ \cap -wR^+$. The coefficient a of $\delta^{-1}e^{\eta} = \delta^{-1}e^{w^{-1}(\lambda+\rho)}$ in (6.7) is then

$$a = \psi_{R(w)}(t) q^{2\langle \pi, \rho_k \rangle} \left(q^{\langle \lambda, w\pi \rangle} - 1 \right) \tag{6.10}$$

(since $q_\varphi^{\langle \mu, \varphi^\vee \rangle} = q^{\langle w^{-1}\lambda, \pi \rangle} = q^{\langle \lambda, w\pi \rangle}$). From (6.4) we have

$$\psi_{R(w)}(t) = \varepsilon(w) \prod_{\alpha \in R(w)} t_\alpha^{-\langle \pi, \alpha_* \rangle}$$

and since $t_\alpha^{-\langle \pi, \alpha_* \rangle} = q^{-\langle \pi, k_\alpha \alpha \rangle}$ it follows that

$$\psi_{R(w)}(t) = \varepsilon(w) q^{-\langle \pi, \rho_k - w^{-1}\rho_k \rangle}. \tag{6.11}$$

From (6.10) and (6.11) the coefficient of $\delta^{-1}e^{w^{-1}(\lambda+\rho)}$ in (6.7) is therefore

$$a = \varepsilon(w) q^{\langle \pi, \rho_k \rangle} \left(q^{\langle w\pi, \lambda + \rho_k \rangle} - q^{\langle w\pi, \rho_k \rangle} \right) \tag{6.12}$$

and the corresponding contribution to $F_\pi m_\lambda$ is

$$q^{\langle \pi, \rho_k \rangle} \left(q^{\langle w\pi, \lambda + \rho_k \rangle} - q^{\langle w\pi, \rho_k \rangle} \right) \chi_\lambda. \tag{6.13}$$

(ii) Suppose now that $w\varphi = -\alpha$, where $\alpha \in R^+$, and let $\nu = w_\varphi \mu = \mu - \langle \mu, \varphi^\vee \rangle \varphi$. Then

$$\mu - j\varphi = \nu - \varphi + j'\varphi,$$

where $j' = \langle \mu, \varphi^\vee \rangle + 1 - j$, so that $0 \leq j' \leq \langle \mu, \varphi^\vee \rangle + 1$. Hence (6.8) now takes the form

$$\xi + \rho = w(\rho - \sigma(X) - \varphi) + w(\nu + j'\varphi).$$

Since $\varphi \notin X$, we have $w(\rho - \sigma(X) - \varphi) = \rho - \sigma(Y)$ for some $Y \subset R^+$, and therefore

$$\xi = w\nu - \sigma(Y) - j'\alpha \leq w\nu \leq \lambda. \tag{6.14}$$

So again each term $a\delta^{-1}e^\eta$ in (6.7) contributes $\varepsilon(w)a\chi_\xi$, where $\xi \leq \lambda$, to $F_\pi m_\lambda$.

Moreover, we have equality in (6.14) if and only if $j' = 0$, the subset Y is empty, and $w\nu = \lambda$, i.e., $\nu = w^{-1}\lambda$ and $\sigma(X) + \varphi = \rho - w^{-1}\rho$, so that $X \cup \{\varphi\} = R(w)$ and $\mu = w_\varphi w^{-1}\lambda$ and $j = \langle \mu, \varphi^\vee \rangle + 1$. Hence the coefficient of $\delta^{-1}e^\eta = \delta^{-1}e^{w^{-1}(\lambda+\rho)}$ in $\Phi_\pi \cdot U_\pi m_\lambda$ is now

$$a = \psi_{R(w)-\{\varphi\}}(t) q^{2\langle \pi, \rho_k \rangle} t_\varphi^{-2} \cdot q^{-\langle \mu, \varphi^\vee \rangle} \left(q^{\langle \mu, \varphi^\vee \rangle} - 1 \right).$$

Since $-\langle \mu, \varphi^\vee \rangle = -\langle w_\varphi w^{-1}\lambda, \varphi^\vee \rangle = \langle w^{-1}\lambda, \varphi^\vee \rangle = \langle \lambda, w\varphi^\vee \rangle$ we have

$$a = -t_\varphi^{-2} \psi_{R(w)-\{\varphi\}}(t) q^{2\langle \pi, \rho_k \rangle} \left(q^{\langle \lambda, w\pi \rangle} - 1 \right).$$

Moreover, from (6.4),

$$\begin{aligned} -t_\varphi^{-2} \psi_{R(w)-\{\varphi\}}(t) &= \varepsilon(w) t_\varphi^{-2} \prod_{\substack{\alpha \in R(w) \\ \alpha \neq \varphi}} t_\alpha^{-\langle \pi, \alpha_* \rangle} \\ &= \varepsilon(w) \prod_{\alpha \in R(w)} t_\alpha^{-\langle \pi, \alpha_* \rangle} \\ &= \varepsilon(w) q^{-\langle \pi, \rho_k - w^{-1}\rho_k \rangle} \end{aligned}$$

as in (6.11). So finally the coefficient of $\delta^{-1}e^{w^{-1}(\lambda+\rho)}$ in (6.7) is given by the same expression (6.12) as before, and the corresponding contribution to $F_\pi m_\lambda$ is again given by (6.13).

To recapitulate, these calculations show that $F_\pi m_\lambda$ is a linear combination of the Weyl characters χ_ξ such that $\xi \in P^{++}$ and $\xi \leq \lambda$, and an inspection of (6.7) shows that the coefficient of χ_ξ in $F_\pi m_\lambda$ lies in $q^{-\langle\lambda,\varphi^\vee\rangle}\mathbb{Z}[q,t]$. Moreover the coefficient of χ_λ is

$$q^{\langle\pi,\rho_k\rangle}\sum_{w\in W}\left(q^{\langle w\pi,\lambda+\rho_k\rangle}-q^{\langle w\pi,\rho_k\rangle}\right)=q^{\langle\pi,\rho_k\rangle}(\widetilde{m}_\pi(\lambda+\rho_k)-\widetilde{m}_\pi(\rho_k)) \tag{6.15}$$

where $\widetilde{m}_\pi=\sum_{w\in W}e^{w\pi}=|W_\pi|m_\pi$. Hence we have

$$F_\pi m_\lambda=\sum_{\substack{\mu\leq\lambda\\ \mu\in P^{++}}}c_{\lambda\mu}(\pi)m_\mu \tag{6.16}$$

with $c_{\lambda\lambda}(\pi)$ given by (6.15), and $c_{\lambda\mu}(\pi)\in q^{-\langle\lambda,\varphi^\vee\rangle}\mathbb{Z}[q,t]$.

To complete the proof of (4.2), it remains to establish that the $c_{\lambda\lambda}(\pi)$, $\lambda\in P^{++}$, are all distinct when R is of type E_8, F_4, or G_2. For this purpose we argue as in the last part of §5: if $\lambda,\mu\in P^{++}$ are such that $c_{\lambda\lambda}(\pi)=c_{\mu\mu}(\pi)$, then by operating with $(q\partial/\partial q)^r$ and then setting $q=1$ we shall obtain

$$\sum_{w\in W}\langle w\pi,\lambda\rangle^r=\sum_{w\in W}\langle w\pi,\mu\rangle^r$$

for each $r\geq 1$. But in each case π is either the highest root or the highest short root of R, and it is known [12] that when R is of type E_8, F_4 or G_2 the polynomial functions on V

$$p_r(x)=\sum_{w\in W}\langle w\pi,x\rangle^r \qquad (r\geq 1)$$

generate the $\mathbb{R}$-algebra of W-invariant polynomial functions in V, and therefore separate the W-orbits in V. Hence λ and μ lie in the same W-orbit, and so $\lambda=\mu$.

By (6.2), (6.16) and the above discussion, it follows that the linear operator F_π on A^W satisfies the three conditions of (4.2) when R is of type E_8, F_4 or G_2. This completes the proof of (4.2) and hence of Theorem (4.1).

§**7**

Let us say that a linear operator $L:A^W\to A^W$ is *triangular* if

$$Lm_\lambda=\alpha_\lambda m_\lambda+\text{lower terms}$$

for each $\lambda\in P^{++}$, that is to say if the matrix of L relative to the basis (m_λ) of A^W is triangular.

(7.1). *If L is triangular and self-adjoint, the P_λ are eigenfunctions of L.*

Proof. Since L is triangular, we have

$$LP_\lambda = \sum_{\substack{\mu \leq \lambda \\ \mu \in P^{++}}} \alpha_{\lambda\mu} P_\mu$$

with coefficients $a_{\lambda\mu}$ given by

$$a_{\lambda\mu}|P_\mu|^2 = \langle LP_\lambda, P_\mu \rangle = \langle P_\lambda, LP_\mu \rangle$$

since L is self-adjoint. But

$$LP_\mu = \sum_{\substack{\nu \leq \mu \\ \nu \in P^{++}}} a_{\mu\nu} P_\nu,$$

so that $\langle P_\lambda, LP_\mu \rangle = 0$ unless $\mu = \lambda$. Hence $a_{\lambda\mu} = 0$ unless $\mu = \lambda$, which proves (7.1).

From (7.1) it follows that all self-adjoint triangular linear operators on A^W are simultaneously diagonalized by the P_λ, and therefore commute with each other.

Consider in particular the case in which R is of type A_n. Then all the fundamental weights π_i $(1 \leq i \leq n)$ are minuscule, and hence the construction of §5 furnishes n self-adjoint triangular linear operators E_{π_i} $(1 \leq i \leq n)$ on A^W, which by the above remark commute with each other. Moreover the eigenvalues of E_{π_i} are (up to a scalar factor) $m_{\pi_i}(\lambda + \rho_k)$. Since the m_{π_i} generate the algebra A^W, it follows that for each $f \in A^W$ there is a unique linear operator $L_f \in \mathbb{R}[E_{\pi_1}, \ldots, E_{\pi_n}]$ such that

$$L_f P_\lambda = f(\lambda + \rho_k) P_\lambda \tag{7.2}$$

for all $\lambda \in P^{++}$. Moreover, since each E_{π_i} is a linear combination of the translation operators $T_{w\pi_i}$, $w \in W$, it follows that L_f is of the form

$$L_f = \sum_{\mu \in P} a_{\mu,f}(q,t) T_\mu \tag{7.3}$$

with coefficients $a_{\mu,f}(q,t)$ which are rational functions of q and t.

For an arbitrary admissible pair (R, S) of root systems, we can of course define L_f for each $f \in A^W$ by (7.2). But except in the case A_n just mentioned, I do not know whether L_f is expressible in the form (7.3).

§8

In the following sections we shall consider some particular cases.

(i) Suppose first that $k_\alpha = 0$ for each $\alpha \in R$, i.e., $t_\alpha = 1$. Then $\triangle = 1$, so that the scalar product is now

$$\langle f, g\rangle = |W|^{-1}[f\bar{g}]_1.$$

It follows that

$$P_\lambda = m_\lambda \tag{8.1}$$

for all $\lambda \in P^{++}$, and hence that

$$\langle P_\lambda, P_\mu\rangle = |W_\lambda|^{-1}\delta_{\lambda\mu} \tag{8.2}$$

where W_λ is the subgroup of W that fixes λ.

(ii) Next, suppose that R is reduced and that $k_\alpha = 1$ for each $\alpha \in R$, so that $t_\alpha = q_\alpha$. Then

$$\triangle = \prod_{\alpha \in R}(1 - e^\alpha) = \delta\bar{\delta}$$

and the scalar product is now

$$\langle f, g\rangle = |W|^{-1}[f\delta \cdot \overline{g\delta}]_1.$$

Hence in this case we have

$$P_\lambda = \chi_\lambda \tag{8.3}$$

for all $\lambda \in P^{++}$, and

$$\langle P_\lambda, P_\mu\rangle = \delta_{\lambda\mu}. \tag{8.4}$$

§9

In this section we shall consider the cases where R has rank 1, hence is of type A_1 or BC_1.

(i) Suppose first that R is of type BC_1. In this case it terms out that the polynomials P_λ are a particular case of the orthogonal polynomials defined by Askey and Wilson [2]. They define

$$\begin{aligned} &p_n(x; a, b, c, d \mid q) \\ &\quad = a^{-n}(ab; q)_n\,(ac; q)_n\,(ad; q)_n \cdot {}_4\varphi_3\left[\begin{matrix} q^{-n}, q^{n-1}abcd, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad \end{matrix}; q, q\right] \end{aligned} \tag{9.1}$$

where $x = \cos\theta$, the parameters a, b, c, d and q lie in the interval $(-1, 1)$ of $\mathbb{R}$, and ${}_4\varphi_3$ is the usual notation for a q-hypergeometric series. Thus p_n is a Laurent polynomial in $e^{i\theta}$ with leading term $(abcdq^{n-1}; q)_n e^{ni\theta}$, and in fact is (despite its appearance) symmetrical in all four parameters a, b, c, d. For our purposes it is more convenient to take $e^{i\theta}$ rather than $\cos\theta$ as the argument, and we prefer to have the leading coefficient equal to 1; so we define

$$\tilde{p}_n(e^{i\theta}; a, b, c, d \mid q) = (abcdq^{n-1}; q)_n^{-1}\, p_n(\cos\theta; a, b, c, d \mid q). \tag{9.2}$$

Then Theorem (2.2) of [2] takes the form

$$\frac{1}{2\pi}\int_0^\pi \tilde{p}_m\, \tilde{p}_n \left|f(e^{i\theta})\right|^2 d\theta = \delta_{mn}\tilde{h}_n \tag{9.3}$$

where

$$f(u) = (u^2; q)_\infty / (au; q)_\infty (bu; q)_\infty (cu; q)_\infty (du; q)_\infty \tag{9.4}$$

and

$$\tilde{h}_n = \frac{(Aq^{2n-1}; q)_\infty\, (Aq^{2n}; q)_\infty}{(Aq^{n-1}; q)_\infty\, (q^{n+1}; q)_\infty}\, \Pi\, (abq^n; q)_\infty^{-1} \tag{9.5}$$

where $A = abcd$ and Π means the product of the six terms such as $(abq^n; q)_\infty^{-1}$.

Now let R be a root system of type BC_1, with $R^+ = \{\alpha, 2\alpha\}$ and $S = \{\pm\alpha\}$. Then $q_\alpha = q$ and $q_{2\alpha} = q^2$. We shall write k_1, k_2, t_1, t_2 for $k_\alpha, k_{2\alpha}, t_\alpha, t_{2\alpha}$ respectively, so that $t_1 = q^{k_1}$ and $t_2 = q^{2k_2}$. We have $P^{++} = \mathbb{N}\alpha$.

Let

$$(a, b, c, d) = (q^{1/2}, -q^{1/2}, t_1 t_2^{1/2}, -t_2^{1/2}). \tag{9.6}$$

With this choice of parameters, $f(u)$ takes the form

$$f(u) = \frac{(t_2^{1/2}u; q)_\infty\, (u^2; q^2)_\infty}{(t_1 t_2^{1/2}u; q)_\infty\, (t_2 u^2; q^2)_\infty}$$

so that $\triangle = f(e^\alpha) f(e^{-\alpha})$. It now follows from (9.1), (9.2) and (9.3) that for $n \geq 0$

$$P_{n\alpha} = \tilde{p}_n(e^\alpha; q^{1/2}, -q^{1/2}, t_1 t_2^{1/2}, -t_2^{1/2} \mid q) \tag{9.7}$$

and that $|P_{n\alpha}|^2$ is given by (9.5) when the parameters a, b, c, d are as in (9.6). After some reduction we find that

$$|P_{n\alpha}|^2 = \frac{(q^{2n}t_1 t_2; q)_{k_1}\, (q^{2n}t_1^2 t_2; q^2)_{k_2}}{(q^{2n+1}t_2; q)_{k_1}\, (q^{2n+2}; q^2)_{k_2}} \tag{9.8}$$

For later reference we shall calculate $P_{n\alpha}(\rho_k^*)$, where $\rho_k^* = \frac{1}{2}(k_1 + k_2)\alpha^\vee$, so that $e^\alpha(\rho_k^*) = q^{k_1+k_2} = t_1 t_2^{1/2}$. The original formula (9.1) for p_n shows that when $e^{i\theta}$ is replaced by a, the series ${}_4\varphi_3$ reduces to its first term, which is 1. By the symmetry of the parameters a, b, c, d it follows that when $e^{i\theta}$ is replaced by c in p_n, the result is

$$c^{-n}(ac; q)_n\,(bc; q)_n\,(cd; q)_n.$$

In view of (9.6) and (9.7), this observation enables us to calculate $P_{n\alpha}(\rho_k^*)$; after some reduction, we obtain

$$P_{n\alpha}(\rho_k^*) = (t_1 t_2^{1/2})^{-n}\frac{(q^{2n}t_1t_2; q)_{k_1}\,(q^{2n}t_1^2t_2; q^2)_{k_2}}{(t_1t_2; q)_{k_1}\,(t_1^2t_2; q^2)_{k_2}}. \tag{9.9}$$

(ii) Suppose now that R is of type A_1, with positive root α, so that $P^{++} = \frac{1}{2}\mathbb{N}\alpha$. In this case the P_λ are essentially the q-ultraspherical polynomials of Askey and Ismail [1]. If $\lambda = \frac{1}{2}n\alpha$, where $n \geq 0$, we have

$$P_\lambda = \frac{(q; q)_n}{(t; q)_n}\varphi_n(e^{\alpha/2}) \tag{9.10}$$

where $t = t_\alpha$ and

$$\varphi_n(x) = \sum_{i+j=n}\frac{(t; q)_i\,(t; q)_j}{(q; q)_i\,(q; q)_j}x^{i-j} \tag{9.11}$$

(so that $\varphi_n(e^{i\theta}) = C_n(\cos\theta; t \mid q)$ in the notation of [1]). The φ_n have the generating function

$$F(x, u) = \sum_{n\geq 0}\varphi_n(x)u^n = 1/(xu; q)_k(x^{-1}u; q)_k \tag{9.12}$$

(where $k = k_\alpha$, so that $t = q^k$), as follows from (9.11) and the q-binomial theorem.

One way of establishing (9.10) is to verify that the φ_n defined by (9.11) are eigenfunctions of the operator E_π of §5, which in the present situation takes the form

$$(E_\pi f)(x) = \frac{tx - x^{-1}}{x - x^{-1}}f(q^{1/2}x) + \frac{x - tx^{-1}}{x - x^{-1}}f(q^{1/2}x^{-1}) \tag{9.13}$$

for a Laurent polynomial $f(x) \in \mathbb{R}[x, x^{-1}]$. More precisely, we have to verify that

$$E_\pi\varphi_n = (tq^{n/2} + q^{-n/2})\varphi_n$$

or equivalently (9.12) that

$$(E_\pi F)(x, u) = tF(x, q^{1/2}u) + F(x, q^{-1/2}u).$$

But this is straightforward to verify from (9.12) and (9.13).

From [1] we have

$$|P_{n\alpha/2}|^2 = (q^n t;q)_k/(q^{n+1};q)_k \tag{9.14}$$

and

$$P_{n\alpha/2}(\tfrac{1}{2}k\alpha^\vee) = t^{-n/2}(q^n t;q)_k/(t;q)_k; \tag{9.15}$$

this latter formula is equivalent to $\varphi_n(t^{1/2}) = t^{-n/2}(t^2;q)_n/(q;q)_n$, which follows from (9.12) since

$$\begin{aligned} F(t^{1/2}, ut^{1/2}) &= 1/(u;q)_k(tu;q)_k \\ &= 1/(u;q)_{2k} \\ &= \sum_{n\geq 0} \frac{(t^2;q)_n}{(q;q)_n} u^n. \end{aligned}$$

§10

We consider next the case where $q = 0$, the t_α being arbitrary. Then we have

$$\triangle = \prod_{\alpha\in R} \frac{1 - t_{2\alpha}^{1/2} e^\alpha}{1 - t_{2\alpha}^{1/2} t_\alpha e^\alpha}.$$

In this case there is an explicit formula for P_λ:

$$P_\lambda = W_\lambda(t)^{-1} \sum_{w\in W} w\left(e^\lambda \prod_{\alpha\in R^+} \frac{1 - t_\alpha t_{2\alpha}^{1/2} e^{-\alpha}}{1 - t_{2\alpha}^{1/2} e^{-\alpha}}\right) \tag{10.1}$$

where W_λ is the subgroup of W that fixes $\lambda \in P^{++}$, and

$$W_\lambda(t) = \sum_{w\in W_\lambda} t_w$$

with t_w as defined in (3.8).

To prove (10.1), we shall first show that $P_\lambda = m_\lambda$ + lower terms, and then that $\langle P_\lambda, P_\mu\rangle = 0$ if $\lambda \neq \mu$.

Let

$$\begin{aligned} \Phi_\lambda &= e^\lambda \prod_{\alpha\in R^+} \frac{1 - t_\alpha t_{2\alpha}^{1/2} e^{-\alpha}}{1 - t_{2\alpha}^{1/2} e^{-\alpha}} \\ &= \delta^{-1} e^{\lambda+\rho} \prod_{\alpha\in R_2^+} \left(1 + (1 - t_{\alpha/2}) t_\alpha^{1/2} e^{-\alpha/2} - t_{\alpha/2} t_\alpha e^{-\alpha}\right) \end{aligned}$$

as in §5, where $R_2^+ = \{\alpha \in R^+ : 2\alpha \notin R\}$ and δ, ρ are as defined in (2.2), (2.3). On multiplying out this product we shall obtain

$$\Phi_\lambda = \delta^{-1} \sum_X \varphi_X(t) e^{\lambda+\rho-\sigma(X)} \tag{10.2}$$

summed over all subsets X of R^+ such that $\alpha \in X \Rightarrow 2\alpha \notin X$, where $\sigma(X) = \sum_{\alpha\in X} \alpha$ and

$$\begin{aligned} \varphi_X(t) &= \prod_{\alpha\in X} \varphi_\alpha(t), \\ \varphi_\alpha(t) &= \begin{cases} -t_{\alpha/2}t_\alpha & \text{if } 2\alpha \notin R, \\ (1-t_\alpha)t_{2\alpha}^{1/2} & \text{if } 2\alpha \in R. \end{cases} \end{aligned} \tag{10.3}$$

Let

$$Q_\lambda = \sum_{w\in W} w\Phi_\lambda.$$

Then it follows from (10.2) that

$$Q_\lambda = \sum_X \varphi_X(t)\chi_{\lambda-\sigma(X)} \tag{10.4}$$

summed over subsets $X \subset R^+$ as above. If $\chi_{\lambda-\sigma(X)} \neq 0$, there exists $w \in W$ and $\mu \in P^{++}$ such that

$$\mu + \rho = w(\lambda + \rho - \sigma(X))$$

and we have $\chi_{\lambda-\sigma(X)} = \varepsilon(w)\chi_\mu$. Now (5.14) $w(\rho - \sigma(X)) = \rho - \sigma(Y)$ for some subset Y of R^+ such that $\alpha \in Y \Rightarrow 2\alpha \notin Y$. Hence $\mu = w\lambda - \sigma(Y) \leq w\lambda \leq \lambda$, and so it follows from (10.4) that Q_λ is a linear combination of the χ_μ such that $\mu \in P^{++}$ and $\mu \leq \lambda$.

Moreover, we have $\mu = \lambda$ if and only if $w\lambda = \lambda$ and $\sigma(Y) = 0$, that is to say if and only if $w \in W_\lambda$ and $\sigma(X) = \rho - w^{-1}\rho$, which implies that $X = R_2(w)$ and hence (10.3)

$$\begin{aligned} \varphi_X(t) &= \prod_{\alpha\in R_2(w)} (-t_{\alpha/2}t_\alpha) \\ &= \varepsilon(w) \prod_{\alpha\in R(w)} t_\alpha = \varepsilon(w)t_w. \end{aligned}$$

Hence the coefficient of χ_λ in Q_λ is

$$\sum_{w\in W_\lambda} t_w = W_\lambda(t)$$

and therefore P_λ as defined by (10.1) is of the form m_λ + lower terms.

It remains to prove that $\langle P_\lambda, P_\mu\rangle = 0$ if $\lambda \neq \mu$. We may assume that $\lambda \not\leq \mu$. We have

$$Q_\lambda \triangle = \sum_{w\in W} w\Psi_\lambda$$

where

$$\begin{aligned}\Psi_\lambda = \triangle\Phi_\lambda = e^\lambda\triangle^+ &= e^\lambda \prod_{\alpha\in R^+} \frac{1-t_{2\alpha}^{1/2}e^\alpha}{1-t_\alpha t_{2\alpha}^{1/2}e^\alpha} \\ &= e^\lambda \prod_{\alpha\in R^+}\left(1+(t_\alpha-1)\sum_{r\geq 1} t_\alpha^{r-1}t_{2\alpha}^{r/2}e^{r\alpha}\right) \\ &= \sum_{\mu\in Q^+} a_\mu e^{\lambda+\mu}\end{aligned}$$

say, with $a_0 = 1$. Hence

$$Q_\lambda\triangle = \sum_{\mu\in Q^+} a_\mu \sum_{w\in W} e^{w(\lambda+\mu)}.$$

If $\lambda+\mu = w_1\pi$, where $\pi\in P^{++}$ and $w_1 \in W$, then we have $\pi \geq w_1\pi = \lambda+\mu\geq\lambda$, with equality only if $\mu = 0$. Hence

$$Q_\lambda\triangle = |W_\lambda| m_\lambda + \text{higher terms};$$

and since $\lambda\not\leq\mu$ this sum has no terms in common with

$$P_\mu = m_\mu + \text{lower terms}.$$

Hence $\langle Q_\lambda, P_\mu\rangle = |W|^{-1}[\bar{P}_\mu Q_\lambda\triangle]_1 = 0$. This completes the proof of (10.1).

Moreover, these calulations show that

$$\langle Q_\lambda, P_\lambda\rangle = \frac{|W_\lambda|}{|W|}[m_\lambda\overline{m_\lambda}]_1 = 1$$

so that

$$|P_\lambda|^2 = W_\lambda(t)^{-1}. \tag{10.5}$$

The formula (10.1) is essentially the formula of [7], Theorem (4.1.2) for the zonal spherical function on a p-adic Lie group. More precisely, let G be a simply-connected group of p-adic type, as defined in [7], and let K be a special maximal compact subgroup of G, such that the root system Σ_1 of [7], (3.1) is the *dual* $R^\vee$ of R. The root structure of G attaches a positive integer $q_{\alpha^\vee}$ to each root $\alpha^\vee \in R^\vee$, and we take $t_\alpha = q_{\alpha^\vee}^{-1}$. The double cosets of K in G are indexed by the elements of $Q^{++} = P^{++} \cap Q$, and the zonal spherical functions ω_S on G relative to K are parametrized by the $\mathbb{C}$-algebra homomorphisms $s : \mathbb{C}[Q]^W \to \mathbb{C}$.

For each $\lambda \in Q^{++}$ let g_λ be a representative of the corresponding double coset of K in G. Then the formula for the zonal spherical function is

$$\omega_s(g_\lambda^{-1}) = u_\lambda(t) s(P_\lambda)$$

with P_λ as in (10.1) and

$$u_\lambda(t) = \frac{W_\lambda(t)}{W(t)} \prod_{\alpha \in R^+} t_\alpha^{\langle \lambda, \alpha^\vee \rangle / 2}.$$

Moreover, $\triangle$ is essentially the Plancherel measure on the space of positive definite zonel spherical functions on G relative to K.

§11

In this section we shall consider the "limiting case" as $q \to 1$, the parameters k_α remaining fixed. We shall assume that

(11.1) $$k_\alpha \geq 0$$

for all $\alpha \in R$.

Let

$$\triangle_k = \prod_{\alpha \in R} (1 - e^\alpha)^{k_\alpha}$$

considered (as in §3) as a continuous function on the torus $T = V/Q^\vee$. For $f, g \in A$ we define

(11.2) $$\langle f, g \rangle_k = |W|^{-1} \int_T f \bar{g} \triangle_k$$

using $\triangle_k$ in place of $\triangle(q,t)$. As before, this scalar product on A is symmetric and positive definite.

Suppose that, in addition to (11.1), we have

(11.3) $$k_\alpha + 2k_{2\alpha} \geq 1$$

for all $\alpha \in R$ (so that $k_\alpha \geq 1$ if $2\alpha \notin R$). Then

(11.4) $$\lim_{q \to 1} \triangle(q,t) = \triangle_k$$

uniformly on T.

This is a consequence of the following fact [6]: if $r, s \in \mathbb{R}$ and $z \in \mathbb{C}$ then

$$\lim_{q\to 1} \frac{(q^r z; q)_\infty}{(q^s z; q)_\infty} = (1-z)^{s-r}$$

uniformly on the disc $|z| \leq 1$, provided that $r \leq s$ and $r+s \geq 1$. If we take $q = q_\alpha$, $r = k_{2\alpha}$, $s = k_\alpha + k_{2\alpha}$ we obtain

$$\lim_{q\to 1} (t_{2\alpha}^{1/2} e^\alpha; q_\alpha)_{k_\alpha} = (1-e^\alpha)^{k_\alpha}$$

uniformly on T, provided that (11.1) and (11.3) hold. Taking the product over all $\alpha \in R$, we obtain (11.4).

Until further notice we shall assume (11.3) as well as (11.1).

Let

$$f(q) = \sum_{\lambda \in P} f_\lambda(q) e^\lambda$$

be an element of A depending on $q \in (0,1)$. If $f_\lambda(q) \to f_\lambda$ as $q \to 1$ for each $\lambda \in P$, we shall write

$$\lim_{q\to 1} f(q) = f$$

where $f = \sum f_\lambda e^\lambda$.

Suppose also that

$$\lim_{q\to 1} g(q) = g$$

in A. Then

$$\lim_{q\to 1} \langle f(q), g(q) \rangle_{q,t} = \langle f, g \rangle_k. \tag{11.5}$$

By linearity it is enough to prove this when $f(q) = f_\lambda(q) e^\lambda$, and $g(q) = g_\mu(q) e^\mu$. We have then

$$\langle f(q), g(q) \rangle_{q,t} = |W|^{-1} f_\lambda(q) g_\mu(q) \int_T e^{\lambda-\mu} \triangle(q,t)$$

which by (11.4) tends to the limit

$$|W|^{-1} f_\lambda g_\mu \int_T e^{\lambda-\mu} \triangle_k = \langle f, g \rangle_k$$

as $q \to 1$.

Consider now the behaviour of

$$P_\lambda(q,t) = \sum_{\substack{\mu \leq \lambda \\ \mu \in P^{++}}} u_{\lambda\mu}(q,t) m_\mu \qquad (\lambda \in P^{++}) \tag{11.6}$$

as $q \to 1$. We claim that

$$\lim_{q\to 1} P_\lambda(q,t) \textit{ exists for each } \lambda \in P^{++} : \tag{11.7}$$

in other words, that each of the coefficients $u_{\lambda\mu}(q,t)$ tends to a finite limit as $q \to 1$.

We shall prove this by induction on λ. When $\lambda = 0$ there is nothing to prove, since $P_0(q,t) = 1$; so assume that $\lambda \neq 0$ and that

$$P_\mu(k) = \lim_{q\to 1} P_\mu(q,t) \tag{11.8}$$

exists for all $\mu \in P^{++}$ such that $\mu < \lambda$.

The equations (11.6) can be inverted to give say

$$m_\lambda = \sum_{\substack{\mu\leq\lambda \\ \mu\in P^{++}}} v_{\lambda\mu}(q,t)P_\mu(q,t) \tag{11.9}$$

with $v_{\lambda\lambda} = 1$. The v's are cofactors of the (unipotent) matrix formed by the u's, hence are polynomials in the u's, and conversely the u's are polynomials in the v's. From (11.9) and the orthogonality of the P's we have

$$v_{\lambda\mu}(q,t) = \frac{\langle m_\lambda, P_\mu(q,t)\rangle_{q,t}}{|P_\mu(q,t)|^2_{q,t}}.$$

Hence, by (11.5) and (11.8), we have

$$\lim_{q\to 1} v_{\lambda\mu}(q,t) = \frac{\langle m_\lambda, P_\mu(k)\rangle_k}{|P_\mu(k)|^2_k}$$

whenever $\mu < \lambda$. But, as we have just remarked, $u_{\lambda\mu}(q,t)$ is a polynomial in the v's with integer coefficients. Hence

$$u_{\lambda\mu}(k) = \lim_{q\to 1} u_{\lambda\mu}(q,t)$$

exists for each $\mu < \lambda$, and (11.7) is proved.

Since $u_{\lambda\mu}(q,t) \in \mathbb{C}(q,t)$, hence is a rational function of q and the q^{k_α}, its limit as $q \to 1$ may be computed by differentiating its numerator and denominator sufficiently often, and then setting $q = 1$. This shows that $u_{\lambda\mu}(k)$ is rational function of the k_α.

We now define

$$\begin{aligned} P_\lambda(k) &= \sum_{\mu\leq\lambda} u_{\lambda\mu}(k)m_\mu \\ &= \lim_{q\to 1} P_\lambda(q,t). \end{aligned} \tag{11.10}$$

By (11.5) we have

$$(11.11) \qquad \langle P_\lambda(k), P_\mu(k)\rangle_k = 0$$

if $\lambda \neq \mu$, and the properties (11.10) and (11.11) characterize the polynomials $P_\lambda(k)$.

The existence of these polynomials has been established by Heckman and Opdam [4, 14] by other methods.

Define linear operators $\Box$ and $D_\alpha(\alpha \in R)$ on A by

$$\Box e^\lambda = |\lambda|^2 e^\lambda, \quad D_\alpha e^\lambda = \langle \lambda, \alpha\rangle e^\lambda$$

and as in [14] let

$$(11.12) \qquad L(k) = \Box + \frac{1}{2}\sum_{\alpha\in R} k_\alpha \frac{1+e^{-\alpha}}{1-e^{-\alpha}} D_\alpha.$$

An equivalent definition is

$$(11.13) \qquad L(k)f = \delta_k^{-1/2}\Box\left(\delta_k^{1/2} f\right) - \left(\delta_k^{-1/2}\Box\delta_k^{1/2}\right) f$$

where

$$\delta_k = \prod_{\alpha\in R^+} (e^{\alpha/2} - e^{-\alpha/2})^{2k_\alpha}.$$

From (11.12), it is clear that $wL(k)w^{-1} = L(k)$ for all $w \in W$, and that if $f \in A$ then $g = \delta L(k) f \in A$, where δ is as defined in §2. If $f \in A^W$, it follows that g is skew-symmetric with respect to W, and hence $L(k)f = \delta^{-1}g \in A^W$. Thus $L(k)$ maps A^W into A^W, and a simple calculation shows that

$$L(k)m_\lambda = \langle \lambda, \lambda + 2\rho_k\rangle m_\lambda + \text{lower terms}$$

for $\lambda \in P^{++}$, where ρ_k is given by (5.7). Moreover, since $\triangle = \delta_k^{1/2}\overline{\delta_k^{1/2}}$, it follows from (11.13) that $L(k)$ is self-adjoint for the scalar product (11.2). Hence as in (7.1) we conclude that the P_λ are eigenfunctions of $L(k)$, and more precisely that

$$(11.14) \qquad L(k)P_\lambda(k) = \langle \lambda, \lambda + 2\rho_k\rangle P_\lambda(k)$$

for $\lambda \in P^{++}$.

From (11.12) and (11.14) we derive the following recurrence relation for the coefficients $u_{\lambda\mu}(k)$ in (11.10):

$$(11.15) \qquad (|\lambda + \rho_k|^2 - |\mu + \rho_k|^2) u_{\lambda\mu}(k) = 2\sum_{\alpha\in R^+}\sum_{r\geq 1} k_\alpha \langle \mu + r\alpha, \alpha\rangle u_{\lambda,\mu+r\alpha}(k).$$

Here $\lambda, \mu \in P^{++}$, $\mu < \lambda$, and $u_{\lambda\mu} = u_{\lambda, w\mu}$ for all $w \in W$. Let $\nu = \lambda - \mu \in Q^+$, then

$$|\lambda + \rho_k|^2 - |\mu + \rho_k|^2 = \langle \nu, \lambda + \mu + 2\rho_k \rangle.$$

We have $\langle \nu, \lambda + \mu \rangle > 0$; also

$$\langle \rho_k, \alpha_i^\vee \rangle = k_{\alpha_i} + 2k_{2\alpha_i}$$

for a simple root $\alpha_i \in R$, from which it follows that $\langle \rho_k, \nu \rangle$ is a positive linear combination of the k_α. Hence if the k_α are all ≥ 0, as we are assuming throughout (11.1), we have

$$|\lambda + \rho_k|^2 - |\mu + \rho_k|^2 > 0$$

whenever $\lambda, \mu \in P^{++}$ and $\lambda > \mu$. It follows now from the recurrence formula (11.15) by induction on $\nu = \lambda - \mu$ that the coefficients $u_{\lambda\mu}(k)$ are *positive*, and more precisely that they are rational functions of the k's in which both numerator and denominator are polynomials in the k's with positive integral coefficients. So finally we can drop the restriction (11.3); the polynomials P_λ are well-defined provided that (11.1) holds.

Now let G/K be a non-compact symmetric space, and $G = KAN$ an Iwasawa decomposition of the semisimple Lie group G. Let $\mathfrak{g}, \mathfrak{k}, \mathfrak{a}$ be the Lie algebras of G, K, A respectively, let $\mathfrak{a}^*$ be the vector space dual to $\mathfrak{a}$, and $\Sigma \subset \mathfrak{a}^*$ the restricted root system of G/K. For each $\beta \in \Sigma$ let m_β denote the multiplicity of β. We shall take

$$V = \mathfrak{a}^*, \quad R = 2\Sigma, \quad k_\alpha = \tfrac{1}{2} m_{\alpha/2} \tag{11.16}$$

for each $\alpha \in R$. Then $L(k)$ is the radial part of the Laplacian on G/K.

Let $\mathfrak{h} \supset \mathfrak{a}$ be a Cartan subalgebra of $\mathfrak{g}$; then $\mathfrak{h} = \mathfrak{a} \oplus (\mathfrak{h} \cap \mathfrak{k})$, and we put $\mathfrak{h}_{\mathbb{R}} = \mathfrak{a} \oplus i(\mathfrak{h} \cap \mathfrak{k})$. Let M be a finite-dimensional irreducible representation space for G, with highest weight $\lambda \in \mathfrak{h}_{\mathbb{R}}^*$. Then ([5], p. 535) M is spherical, i.e., has a nonzero vector fixed by K, if and only if λ vanishes on $i(\mathfrak{h} \cap \mathfrak{k})$ and (regarded as an element of $V = \mathfrak{a}^*$) we have $\langle \lambda, \alpha^\vee \rangle \in \mathbb{N}$ for all $\alpha \in R^+$, i.e., $\lambda \in P^{++}$.

The recursion formula (11.15) conincides with Harish-Chandra's recursion formula ([5], p. 427) for the coefficients of the zonal spherical function ω_λ defined by M. Hence for $X \in \mathfrak{a}^+$, the positive Weyl chamber in $\mathfrak{a}$, we have

$$\omega_\lambda(\exp X) = P_\lambda(X)/P_\lambda(0) \tag{11.17}$$

where each e^λ ($\lambda \in P$) is now to be regarded as a function on $\mathfrak{a} = V^*$ by the rule $e^\lambda(X) = e^{\lambda(X)}$, the latter e being the classical exponential function.

If G/K is compact, the formula (11.17) is the same, except that now $X \in i\mathfrak{a}$ in place of $\mathfrak{a}^+$.

Note the contrast with the p-adic situation of §10. There the restricted root system was $R^\vee$, and the weights $\lambda \in P^{++}$ indexed the double cosets of K in G. Here, on the other hand, the restricted root system is (similar to) R, and the $\lambda \in P^{++}$ index the finite-dimensional spherical representations.

§12

We shall conclude with some conjectures. In order to state them concisely, we introduce the q-gamma function, defined for $0 < q < 1$ by

$$\Gamma_q(x) = \frac{(q;q)_{x-1}}{(1-q)^{x-1}} = \frac{(q;q)_\infty \, (1-q)^{1-x}}{(q^x;q)_\infty},$$

where $x \in \mathbb{R}$ but $x \neq 0, -1, -2, \ldots$, at which points Γ_q has simple poles.
We have

$$\Gamma_q(x+1) = \frac{1-q^x}{1-q} \Gamma_q(x)$$

and hence

$$\lim_{k \to 0} \frac{\Gamma_q(kx)}{\Gamma_q(ky)} = \lim_{k \to 0} \frac{1-q^{ky}}{1-q^{kx}} \frac{\Gamma_q(kx+1)}{\Gamma_q(ky+1)} = \frac{y}{x}. \tag{12.1}$$

As $q \to 1$, $\Gamma_q(x) \to \Gamma(x)$ for each x, and therefore we shall write $\Gamma_1(x)$ for the ordinary gamma function.

It will be convenient also to introduce

$$\Gamma_q^*(x) = 1/\Gamma_q(1-x). \tag{12.2}$$

When $q = 1$ we have

$$\Gamma_1^*(x) = \frac{\sin \pi x}{\pi} \Gamma_1(x) \tag{12.3}$$

but there is no particularly simple relationship between $\Gamma_q^*(x)$ and $\Gamma_q(x)$ for general values of q.

Now define, for $\lambda \in V$ and $\alpha \in R^+$,

$$c_\alpha(\lambda; q_\alpha) \;=\; \frac{\Gamma_{q_\alpha}\big(\langle \lambda, \alpha^\vee \rangle + \frac{1}{2} k_{\alpha/2}\big)}{\Gamma_{q_\alpha}\big(\langle \lambda, \alpha^\vee \rangle + \frac{1}{2} k_{\alpha/2} + k_\alpha\big)} \tag{12.4}$$

(where as usual $k_{\alpha/2} = 0$ if $\frac{1}{2}\alpha \notin R$), and

$$c(\lambda) = c(\lambda; q, t) = \prod_{\alpha \in R^+} c_\alpha(\lambda; q_\alpha). \tag{12.5}$$

Also define $c_\alpha^*(\lambda; q_\alpha)$ and $c^*(\lambda)$ by using Γ^* in place of Γ in (12.4) and (12.5).

We can now state

Conjecture (12.6). *For all* $\lambda \in P^{++}$

$$|P_\lambda|^2 = \frac{c^*(-\lambda-\rho_k)}{c(\lambda+\rho_k)}.$$

Suppose in particular that the k_α are non-negative integers. Then (12.6) takes the form

$$|P_\lambda|^2 = \prod_{\alpha\in R^+}\prod_{i=0}^{k_\alpha-1} \frac{1-q_\alpha^{\langle\lambda+\rho_k,\alpha^\vee\rangle+\frac{1}{2}k_{\alpha/2}+i}}{1-q_\alpha^{\langle\lambda+\rho_k,\alpha^\vee\rangle-\frac{1}{2}k_{\alpha/2}-i}}. \tag{12.6$'$}$$

Even when $\lambda=0$ (so that $P_\lambda=1$) there is something to be proved here. Indeed, when R is reduced and $\lambda=0$, the conjecture (12.6$'$) reduces to the constant term conjectures of [10] and [13]. For conjecture A' of [13] asserts that the constant term of the product

$$\prod_{\alpha\in R^+}\prod_{i=0}^{k_\alpha-1}(1-q_\alpha^i e^\alpha)(1-q_\alpha^{i+1}e^{-\alpha}) \tag{12.7}$$

should be equal to

$$\prod_{\alpha\in R}\frac{(q_\alpha;q_\alpha)_{|\langle\rho_k,\alpha^\vee\rangle+k_\alpha|}}{(q_\alpha;q_\alpha)_{|\langle\rho_k,\alpha^\vee\rangle|}}. \tag{12.8}$$

Now the product (12.7) is precisely the product $\triangle'$ defined in (3.10), and by (3.12) the constant term of $\triangle'$ is $W(t)|1|^2$, which by (12.6$'$) is equal to

$$W(t)\prod_{\alpha\in R^+}\frac{(q_\alpha;q_\alpha)_{\langle\rho_k,\alpha^\vee\rangle+k_\alpha-1}\,(q_\alpha;q_\alpha)_{\langle\rho_k,\alpha^\vee\rangle-k_\alpha}}{(q_\alpha;q_\alpha)_{\langle\rho_k,\alpha^\vee\rangle-1}\,(q_\alpha;q_\alpha)_{\langle\rho_k,\alpha^\vee\rangle}}. \tag{12.9}$$

On the other hand we have

$$W(t)=\prod_{\alpha\in R^+}\frac{1-q_\alpha^{\langle\rho_k,\alpha^\vee\rangle+k_\alpha}}{1-q_\alpha^{\langle\rho_k,\alpha^\vee\rangle}},$$

by ([8], 2.4 nr) applied to the root system $S^\vee$. Hence (12.9) is equal to (12.8), which proves our assertion.

If however R is of type BC_n, there are two choices for S when $n\geq 2$, so that our conjecture (12.6$'$) when $\lambda=0$ contains two distinct constant-term conjectures related to the root system BC_n. Neither of these is obviously equivalent to Morris's Conjecture A for BC_n ([13], 3.4).

Recall next that each $f\in A$ is regarded as a function on V, by the rule $e^\lambda(x)=q^{\langle\lambda,x\rangle}$ ($x\in V$, $\lambda\in P$). Also let

$$\alpha^*=(\alpha_*)^\vee=u_\alpha\alpha^\vee$$

for each $\alpha\in R$, and

$$\rho_k^*=\frac{1}{2}\sum_{\alpha\in R^+}k_\alpha\alpha^*.$$

Conjecture (12.10). *For $\lambda \in P^{++}$,*

$$P_\lambda(\rho_k^*) = q^{-\langle \lambda, \rho_k^* \rangle} c(\rho_k)/c(\lambda + \rho_k).$$

Both conjectures (12.6) and (12.10) are true in each of the situations considered in Sections 8–11:

(i) When R is reduced and the k_α are all equal to 1, we have $P_\lambda = \chi_\lambda$ (8.3), and (12.6) reduces to $|\chi_\lambda|^2 = 1$, i.e., to (8.4). As to (12.10), we have $\rho_k^* = \frac{1}{2} \sum_{\alpha \in R} \alpha^* = \rho^*$ say, and

$$P_\lambda(\rho_k^*) = \chi_\lambda(\rho^*) = \frac{\sum_{w \in W} \varepsilon(w)\, q^{\langle w(\lambda+\rho), \rho^* \rangle}}{\sum_{w \in W} \varepsilon(w)\, q^{\langle w\rho, \rho^* \rangle}}.$$

By Weyl's denominator formula for the root system $S^\vee$, this factorizes to give

$$q^{-\langle \lambda, \rho^* \rangle} \prod_{\alpha \in R^+} \frac{1 - q_\alpha^{\langle \lambda+\rho, \alpha^\vee \rangle}}{1 - q_\alpha^{\langle \rho, \alpha^\vee \rangle}}$$

in agreement with (12.10).

(ii) When the k_α are all zero we have $P_\lambda = m_\lambda$ and $|P_\lambda|^2 = |W_\lambda|^{-1}$. On the other hand, it follows from our definitions that when the k_α are all zero we have $c_\alpha^*(-\lambda - \rho_k; q_\alpha) = 1$ for all $\alpha \in R^+$ and $c_\alpha(\lambda + \rho_k; q_\alpha) = 1$ for all $\alpha \in R^+$ such that $\langle \lambda, \alpha^\vee \rangle \neq 0$. When $\langle \lambda, \alpha^\vee \rangle = 0$ we have to interpret $c_\alpha(\lambda + \rho_k; q_\alpha)$ by means of the limit relation (12.1); with $k_\alpha = k$ for all α this leads to

$$\lim_{k \to 0} \frac{c^*(-\lambda - \rho_k)}{c(\lambda + \rho_k)} = \prod_{\substack{\alpha \in R^+ \\ \langle \lambda, \alpha^\vee \rangle = 0}} \frac{\langle \rho, \alpha^\vee \rangle + e_\alpha}{\langle \rho, \alpha^\vee \rangle + e_\alpha + 1}$$

where $e_\alpha = \frac{1}{2}$ if $\frac{1}{2}\alpha \in R$, and $e_\alpha = 0$ otherwise; and this product is equal to $|W_\lambda|^{-1}$. This checks (12.6) in this case, and (12.10) is analogous (both sides are equal to $|W|/|W_\lambda|$).

(iii) When R is of rank 1 (§9) the formulas (9.8), (9.9), (9.14) and (9.15) show that both conjectures are true.

(iv) When R is of type A_n, the polynomials P_λ are essentially the same as the symmetric functions $P_\lambda(x; q, t)$ studied in [11], Chapter VI. Both conjectures are true in this case, and are proved in *loc. cit.*

(v) In the situation of §10 we must express everything in terms of the t_α before setting $q = 0$. We have

$$q_\alpha^{\langle \lambda + \rho_k, \alpha^\vee \rangle} = q_\alpha^{\langle \lambda, \alpha^\vee \rangle} q^{\langle \rho_k, \alpha^* \rangle}$$

and $\langle \rho_k, \alpha^* \rangle = \frac{1}{2} \sum_{\beta \in R^+} k_\beta u_\beta \langle \beta_*, \alpha^* \rangle$, so that

$$q^{\langle \rho_k, \alpha^* \rangle} = \prod_{\beta \in R^+} t_\beta^{\langle \beta_*, \alpha^* \rangle / 2} = \mathbf{t}^{\mathrm{ht}(\alpha^*)}$$

in the notation of [8]. It follows that $c^*(-\lambda - \rho_k) = 1$ when $q = 0$, and that

$$c(\lambda + \rho_k) = \prod_{\substack{\alpha \in R^+ \\ \langle \lambda, \alpha^\vee \rangle = 0}} \frac{1 - t_{\alpha/2} t_\alpha \mathbf{t}^{\mathrm{ht}(\alpha^*)}}{1 - t_{\alpha/2} \mathbf{t}^{\mathrm{ht}(\alpha^*)}}$$

which from the results of [8] is easily seen to be equal to the polynomial $W_\lambda(t)$. Hence in the present situation the right-hand side of (12.6) is equal to $W_\lambda(t)^{-1}$, which by (10.5) is equal to $|P_\lambda|^2$.

Next consider (12.10). We have

$$e^\alpha(\rho_k^*) = q^{\langle \alpha, \rho_k^* \rangle} = \prod_{\beta \in R^+} t_\beta^{\langle \beta^\vee, \alpha \rangle / 2} = \mathbf{t}^{\mathrm{ht}(\beta)}$$

and in the formula (10.1) for P_λ, when we evaluate at ρ_k^*, all the terms will vanish except that corresponding to w_0, the longest element of W. Consequently we obtain

$$(e^\lambda P_\lambda)(\rho_k^*) = W(t)/W_\lambda(t)$$

which from above is also equal to $c(\rho_k)/c(\lambda + \rho_k)$, thus verifying (12.10) in this case.

(vi) Finally, in the "limiting case" $q \to 1$ considered in §11, Heckman [4] has proved that $|P_\lambda|_k^2/|1|_k^2$ has the value predicted by (12.6), and recently Opdam [15] has evaluated $|1|_k^2$. These results confirm that (12.6) is true in the limiting case.

As to (12.10), it follows from the work of Harish-Chandra ([5] Chapter V) that in the symmetric space situation (11.16) the zonal spherical function ω_λ for $\lambda \in P^{++}$ is given by

$$\omega_\lambda(\exp X) = \frac{c(\lambda + \rho_k)}{c(\rho_k)} P_\lambda(X) \qquad (X \in \mathfrak{a}^+).$$

Hence by comparison with (11.17) we have

$$P_\lambda(0) = c(\rho_k)/c(\lambda + \rho_k) \tag{12.11}$$

which proves (12.10) in this case. Recently Opdam ([15], Cor. 5.2) has proved (12.11) for arbitrary values of the k_α.

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