*Séminaire Lotharingien de Combinatoire* **45** (2001), Article B45f

# MOMENTS OF INERTIA ASSOCIATED WITH THE LOZENGE TILINGS OF A HEXAGON

ILSE FISCHER

Abstract. Consider the probability that an arbitrary chosen lozenge tiling of the hexagon with side lengths $a$, $b$, $c$, $a$, $b$, $c$ contains the horizontal lozenge with lowest vertex $(x, y)$ as if it described the distribution of mass in the plane. We compute the horizontal and the vertical moments of inertia with respect to this distribution. This solves a problem by Propp [1, Problem 7].

## 1. Introduction

Let $a$, $b$ and $c$ be positive integers and consider a hexagon with side lengths $a$,$b$,$c$,$a$,$b$,$c$ whose angles are 120° (see Figure 1). The subject of our interest is lozenge tilings of such a hexagon using lozenges with all sides of length 1 and angles of 60° and 120°. Figure 2 shows an example of a lozenge tiling of a hexagon with $a = 3$, $b = 5$ and $c = 4$.

We introduce the following oblique angled coordinate system: Its origin is located in one of the two vertices, where sides of length $b$ and $c$ meet, and the axes are induced by those two sides (see Figure 3). The units are chosen such that the side lengths of the considered hexagon are $\sqrt{2}a$, $b$, $c$, $\sqrt{2}a$, $b$, $c$ in this coordinate system. (That is to say, the two triangles in Figure 3 with vertices in the origin form the unit 'square'.)

Let $P_{a,b,c}(x, y)$ denote the probability that an arbitrary chosen lozenge tiling of the hexagon with side lengths $a$,$b$,$c$,$a$,$b$,$c$ contains the horizontal lozenge with lowest vertex $(x, y)$ in the oblique angled coordinate system. Note that $S = ((a + b)/2, (a + c)/2)$ is the centre of the hexagon in question. Consider the probability $P_{a,b,c}(x, y)$ as if it described the distribution of mass. In [1, Problem 7] Propp suggests to compute the horizontal moment of inertia with respect to $S$

$$\sum_{y=0}^{a+c-1} \sum_{x=1}^{a+b-1} P_{a,b,c}(x, y) \left( x - \frac{a+b}{2} \right)^2$$

and the vertical moment of inertia with respect to $S$

$$\sum_{y=0}^{a+c-1} \sum_{x=1}^{a+b-1} P_{a,b,c}(x, y) \left( 2 \left( y - \frac{a+c-1}{2} \right) - \left( x - \frac{a+b}{2} \right) \right)^2$$

of this distribution. (Note that $2(y - \frac{a+c-1}{2}) - (x - \frac{a+b}{2}) = 0$ is the horizontal line in the coordinates of the oblique angled system which contains the lowest vertex of the horizontal lozenge in the centre of the hexagon.) Our theorem is the following.

**Theorem 1.** *Let $a$, $b$, $c$ be positive integers and let $P_{a,b,c}(x, y)$ denote the probability that an arbitrary chosen lozenge tiling of a hexagon with side lengths $a$, $b$, $c$, $a$, $b$, $c$*

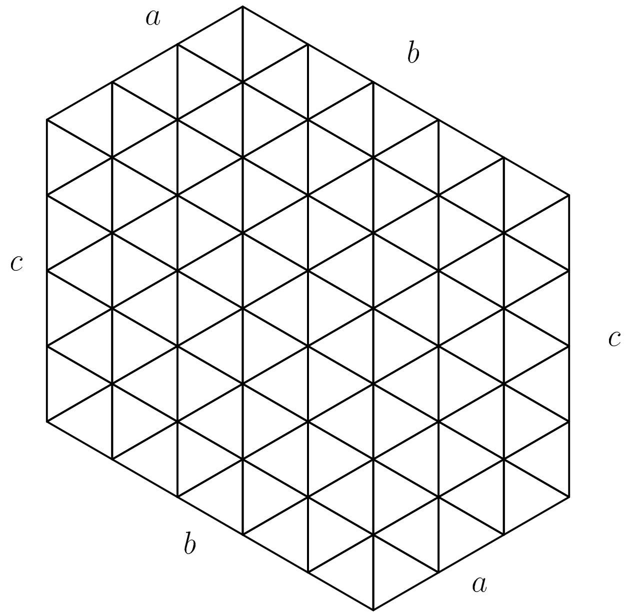


FIGURE 1

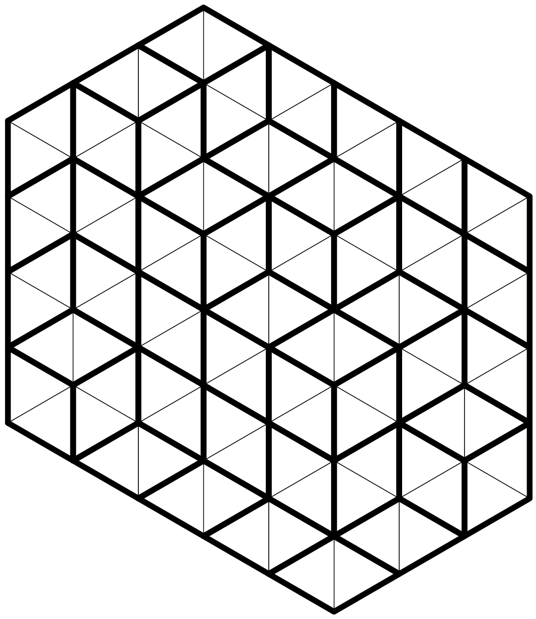

FIGURE 2

*contains the horizontal lozenge with lowest vertex* $(x, y)$. *Then the horizontal moment of inertia with respect to* $S$ *is equal to*

$$\sum_{x=1}^{a+b-1} \sum_{y=0}^{a+c-1} P_{a,b,c}(x,y) \left( x - \frac{a+b}{2} \right)^2 = \frac{1}{12} ab(a^2 + b^2 - 2) \tag{1.1}$$

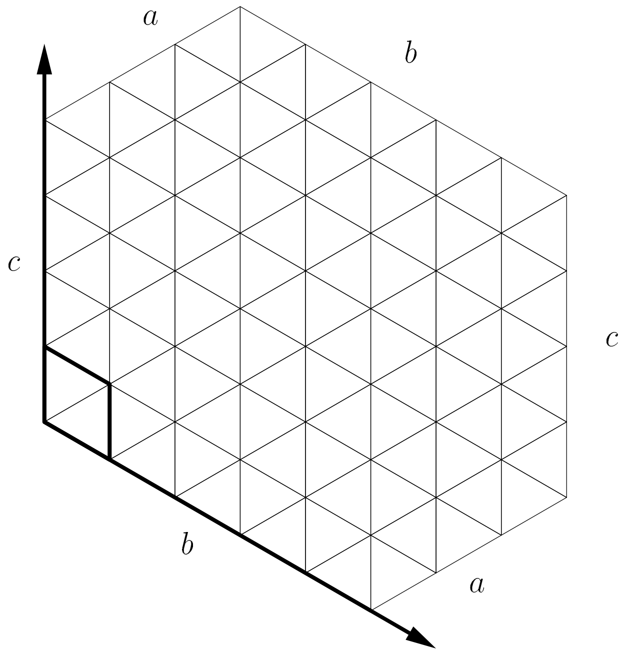


FIGURE 3

*and the vertical moment of inertia with respect to $S$ is equal to*

$$\sum_{y=0}^{a+c-1} \sum_{x=1}^{a+b-1} P_{a,b,c}(x,y) \left( 2\left( y - \frac{a+c-1}{2} \right) - \left( x - \frac{a+b}{2} \right) \right)^2 =$$

$$= \frac{1}{12} ab(a^2 + b^2 - 2 + 4c^2 + 4ac + 4bc). \quad (1.2)$$

In fact Propp [1] has already noticed that the computation of the horizontal moment of inertia is easy and states the formula for the case $a = b = c = n$. Furthermore, he concludes that the vertical moment of inertia is more difficult to compute for the first few values for the case $a = b = c = n$ do not seem to predict the formula to be a polynomial of degree 4 as it is the case for the horizontal moment of inertia. However, this conclusion is based on a miscalculation. The vertical moment of inertia for the case $a = b = c = 2$ is equal to 18 (not to 20) and thus we obtain the polynomial $7n^4/6 - n^2/6$ for the vertical moment of inertia if $a = b = c = n$. (See (1.2).) Nevertheless the computation of the vertical moment of inertia seems to be more involved compared to the computation of the horizontal moment of inertia.

The following section is devoted to the proof of Theorem 1. Our combinatorial proof is based on the correspondance between lozenge tilings of a hexagon with side lengths $a$, $b$, $c$, $a$, $b$, $c$ and plane partitions in an $a \times b \times c$ box, i.e. plane partitions of shape $b^a$ and with entries between 0 and $c$. Thus we can avoid to use an explicit expression for the probability $P_{a,b,c}(x,y)$. In Lemma 2 we observe that we are in fact able to compute the inner sum of (1.1). In order to obtain (1.2), we split up the left hand side of (1.2) into three double sums and again we compute their inner sums after possibly interchanging the summation order (Lemma 1 - 3). In Lemma 5 we demonstrate that

Lemma 3 and thus the computation of the vertical moment of inertia is trivial if we assume $a = b$.

## 2. Five lemmas and the proof of the Theorem

**Lemma 1.** *Let $a$, $b$, $c$ be positive integers and $0 \le y_0 \le a + c - 1$. Furthermore let $P_{a,b,c}(x, y)$ denote the probability that an arbitrary chosen lozenge tiling of the hexagon with side lengths $a$, $b$, $c$, $a$, $b$, $c$ contains the horizontal lozenge with lowest vertex $(x, y)$ in the oblique angled coordinate system. Then*

$$\sum_{x=1}^{a+b-1} P_{a,b,c}(x, y_0) = \frac{ab}{a+c}.$$

The main ingredient for the proof is the symmetry of the Schur function.

*Proof.* For a fixed $0 \le y_0 \le a + c - 1$ the sum in question is just the expected value for the number of horizontal lozenges with its lowest vertex on the line $y = y_0$ in the oblique angled coordinate system. First we show that the sum does not depend on $y_0$. For a fixed plane partition in an $a \times b \times c$ box, let $N_i(y_0)$ denote the multiplicity of the entry $y_0$ in the $i$th row of the plane partition. Then the number of horizontal lozenges with its lowest vertex on the line $y = y_0$ in a given lozenge tiling is equal to

$$\sum_{i=1}^{a} N_i(y_0 - a + i)$$

in the corresponding plane partition. If we add $a - i$ to every entry in the $i$th row $(1 \le i \le a)$ of the plane partition we obtain a plane partition with strictly decreasing columns. Then the sum above is just the number of $y_0$'s in this plane partition. The content $(\mu_i)_{i \ge 0}$ of a plane partition is the sequence with

$$\mu_i = \text{number of i's in the plane partition.}$$

Fix a sequence $(\mu_i)_{i \ge 0}$ and an integer $j \ge 0$. Let $\nu_i = \mu_i$ if $i \ne j, j+1$, $\nu_j = \mu_{j+1}$ and $\nu_{j+1} = \mu_j$. By the bijection in [2, page 152] the number of plane partitions with decreasing rows, strictly decreasing columns and content $(\mu_i)_{i \ge 0}$ is equal to the number of such plane partitions with content $(\nu_i)_{i \ge 0}$. Thus the expected value in question is independent of $y_0$. Now the assertion follows since the total number of horizontal lozenges is equal to $ab$, i.e.

$$\sum_{y=0}^{a+c-1} \sum_{x=1}^{a+b-1} P_{a,b,c}(x, y) = ab.$$

$\square$

In our next lemma we compute the inner sum of (1.1).

**Lemma 2.** *Let $a$, $b$, $c$ be positive integers and $1 \le x_0 \le a + b - 1$. Furthermore let $P_{a,b,c}(x_0, y_0)$ denote the probability that an arbitrary chosen lozenge tiling of the hexagon* *with side lengths $a$, $b$, $c$, $a$, $b$, $c$ contains the horizontal lozenge with lowest vertex $(x, y)$ in the oblique angled coordinate system. Then*

$$\sum_{y=0}^{a+c-1} P_{a,b,c}(x_0, y) = \begin{cases} x_0 & 1 \leq x_0 \leq \min(a,b) \\ \min(a,b) & \min(a,b) \leq x_0 \leq \max(a,b) \\ a+b-x_0 & \max(a,b) \leq x_0 \leq a+b-1 \end{cases} .$$

*Proof of Lemma 2:* For a fixed $1 \leq x_0 \leq a+b-1$ the sum in question is just the expected value for the number of horizontal lozenges with its lowest vertex on the vertical line $x = x_0$. But this number does not depend on the lozenge tiling. $\square$

**Lemma 3.** *Let $a$, $b$, $c$ be positive integers and $1 \leq x_0 \leq a+b-1$. Furthermore let $P_{a,b,c}(x, y)$ denote the probability that an arbitrary chosen lozenge tiling of the hexagon with side lengths $a$, $b$, $c$, $a$, $b$, $c$ contains the horizontal lozenge with lowest vertex $(x, y)$ in the oblique angled coordinate system. Then*

$$\sum_{y=0}^{a+c-1} P_{a,b,c}(x_0, y) \left(y - \frac{a+c-1}{2}\right)$$
$$= \begin{cases} \frac{(-a^2-ab-ac+bc+ax_0+bx_0)x_0}{2(a+b)} & 1 \leq x_0 \leq a \\ \frac{ac(a+b-2x_0)}{2(a+b)} & a \leq x_0 \leq b \\ \frac{(-b^2-ab-bc+ac+ax_0+bx_0)(a+b-x_0)}{2(a+b)} & b \leq x_0 \leq a+b-1 \end{cases}$$

*if $a \leq b$ and*

$$\sum_{y=0}^{a+c-1} P_{a,b,c}(x_0, y) \left(y - \frac{a+c-1}{2}\right)$$
$$= \begin{cases} \frac{(-a^2-ab-ac+bc+ax_0+bx_0)x_0}{2(a+b)} & 1 \leq x_0 \leq b \\ \frac{-b(a+b+c)(a+b-2x_0)}{2(a+b)} & b \leq x_0 \leq a \\ \frac{(-b^2-ab-bc+ac+ax_0+bx_0)(a+b-x_0)}{2(a+b)} & a \leq x_0 \leq a+b-1 \end{cases}$$

*if $a \geq b$.*

For the proof of Lemma 3 we need the following definition of an $(n; k_1, k_2, \dots, k_a)$-array and the following lemma. Roughly speaking an $(n; k_1, k_2, \dots, k_a)$-array is the bottom-left part of a plane partition $T$ in an $a \times b \times c$-box with $T(i, i+n) = k_i$, which is dissected along the set of cells $\{(i, i+n), 1 \leq i \leq a\}$.

**Definition 1.** *For every two positive integers $a$ and $n$ define the following set of cells:*

$$F_{a,n} = \{(i,j) | 1 \leq i \leq a, 1 \leq j \leq i+n\}$$

*(See Figure 4.) Let $(k_1, k_2, \dots, k_a)$ be a decreasing sequence of positive integers smaller or equal to $c$. Then an $(n; k_1, k_2, \dots, k_a)$-array is an assignment $T$ of integers to the cells in $F_{a,n}$ such that*

(i) $T(i, i+n) = k_i$ *for* $1 \leq i \leq a$,
(ii) $T(i, j) \leq c$ *for all* $(i, j) \in F_{a,n}$ *and*
(iii) *rows and columns are decreasing.*

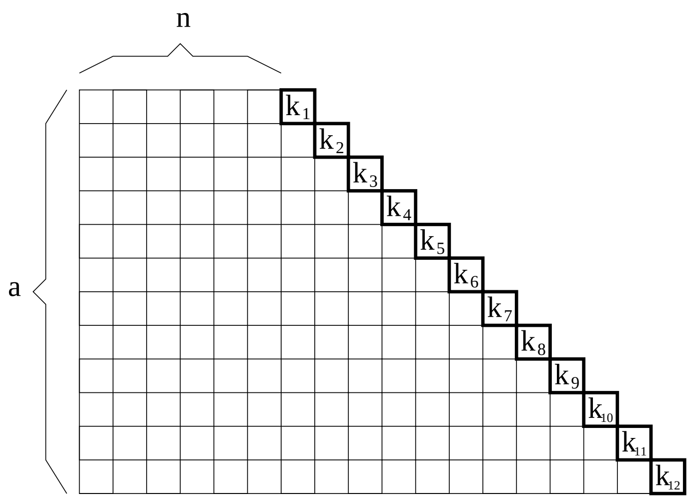


FIGURE 4. $F_{a,n}$ for $a = 12$ and $n = 6$.

*The norm of an* $(n; k_1, k_2, \ldots, k_a)$*-array* $T$ *is defined as*

$$n(T) = \sum_{(i,j) \in F_{a,n}} T(i,j).$$

*Let* $A_n(k_1, k_2, \ldots, k_a)$ *denote the number of* $(n; k_1, k_2, \ldots, k_a)$*-arrays and*

$$S_n(k_1, k_2, \ldots, k_a) = \sum_T \sum_{\substack{(i,j) \in F_{a,n}, \\ i+n \neq j}} T(i,j),$$

*where the outer sum is taken over all* $(n; k_1, k_2, \ldots, k_a)$*-arrays* $T$.

**Lemma 4.** *Let* $a$ *and* $n$ *be positive integers and let* $(k_1, k_2, \ldots, k_a)$ *be a decreasing sequence of positive integers smaller or equal to* $c$. *Then*

$$\frac{S_n(k_1, k_2, \ldots, k_a)}{A_n(k_1, k_2, \ldots, k_a)} = \frac{1}{2} \left( n\,a\,c + (a+n-1) \sum_{i=1}^{a} k_i \right).$$

*Proof:* First we describe a bijection between $(n; k_1, k_2, \ldots, k_a)$-assignments and semistandard tableaux of shape $(c - k_a, c - k_{a-1}, \ldots, c - k_1)$ with entries between 1 and $a + n$.

Let $T$ be an arbitrary $(n; k_1, k_2, \ldots, k_a)$-assignment. In order to obtain the corresponding semistandard tableau replace every entry $T(i,j)$ of $T$ with $c - T(i,j)$. We obtain an assignment with increasing rows and columns. Every row of this assignment is a partition, where the parts are written in reverse order. We conjugate these partitions and write them right justified and in increasing order among each other. Clearly the rows are weakly increasing but the columns are stricly increasing. If we rotate this assignment by 180° we obtain a reverse semistandard tableau of shape $(c - k_a, c - k_{a-1}, \ldots, c - k_1)$ with entries between 1 and $a + n$. Finally we replace every entry $e$ of this reverse semistandard tableau by $a + n + 1 - e$ and obtain the desired semistandard tableau $T'$. This procedure is obviously reversible.

The norms of the two corresponding objects are related as follows

$$n(T)=c\left(a\,n+\frac{(a+1)\,a}{2}\right)-(a+n+1)\left(a\,c-\sum_{i=1}^{a}k_i\right)+n(T'),$$

where $n(T')$ is the sum of the parts of $T'$.

Note that the statement in the lemma is equivalent to

$$\frac{\sum\limits_{T}n(T)}{\sum\limits_{T}1}=\frac{1}{2}\left(n\,a\,c+(a+n+1)\sum_{i=1}^{a}k_i\right),$$

where the sums are taken over all $(n;k_1,k_2,\dots,k_a)$-arrays $T$. Thus we have to show that

$$\frac{\sum\limits_{T'}n(T')}{\sum\limits_{T'}1}=\frac{1}{2}(a\,c-\sum_{i=1}^{a}k_i)(a+n+1),$$

where the sums are taken over all semistandard tableaux $T'$ of shape $(c-k_a,c-k_{a-1},\dots,c-k_1)$ with entries between 1 and $a+n$ or, equivalently, that

$$\frac{\sum\limits_{T'}n(T')}{\sum\limits_{T'}1}=\frac{1}{2}\left(\sum_{i=1}^{r}\lambda_i\right)(a+1),$$

where the sums are taken over all semistandard tableaux $T'$ of shape $(\lambda_1,\lambda_2,\dots,\lambda_r)$ with entries between 1 and $a$.

By Stanley's hook-content formula [3, Theorem 15.3] we have

$$\sum_{T'}q^{n(T')}=q^{\sum\limits_{i=1}^{r}i\lambda_i}\prod_{\rho\in\lambda}\frac{1-q^{a+c_\rho}}{1-q^{h_\rho}},\tag{2.1}$$

where the sum is taken over all semistandard tableaux $T'$ of shape $\lambda:=(\lambda_1,\lambda_2,\dots,\lambda_r)$ with entries between 1 and $a$ and the product is taken over all cells $\rho$ in the Ferrer diagramm of shape $\lambda$. Furthermore $h_\rho$ denotes the hooklength and $c_\rho$ the content of cell $\rho$. We observe that

$$\frac{\sum\limits_{T'}n(T')}{\sum\limits_{T'}1}=\lim_{q\to1}\frac{\left(\sum\limits_{T'}q^{n(T')}\right)'}{\sum\limits_{T'}q^{n(T')}}.$$

With the help of (2.1) we compute the derivative of $\sum_{T'} q^{n(T')}$.

$$\begin{aligned}\left(\sum_{T'} q^{n(T')}\right)' &= \left(\sum_{i=1}^{r} i\lambda_i\right) q^{(\sum_{i=1}^{r} i\lambda_i)-1} \prod_{\rho\in\lambda} \frac{1-q^{a+c_\rho}}{1-q^{h_\rho}} + q^{\sum_{i=1}^{r} i\lambda_i} \\ &\quad \times \sum_{\rho\in\lambda} \frac{-(a+c_\rho)(q^{a+c_\rho-1})(1-q^{h_\rho})+(1-q^{a+c_\rho})(h_\rho q^{h_\rho-1})}{(1-q^{h_\rho})(1-q^{a+c_\rho})} \\ &\quad \times \prod_{\rho'\in\lambda} \frac{1-q^{a+c_{\rho'}}}{1-q^{h_{\rho'}}}\end{aligned}$$

Thus

$$\begin{aligned}\lim_{q\to 1} \frac{(\sum_{T'} q^{n(T')})'}{\sum_{T'} q^{n(T')}} &= \sum_{i=1}^{r} i\lambda_i + \frac{1}{2}\sum_{\rho\in\lambda}(a+c_\rho-h_\rho) \\ &= \sum_{i=1}^{r} i\lambda_i + \frac{a}{2}\sum_{i=1}^{r}\lambda_i + \frac{1}{2}\sum_{j=1}^{\lambda_1} j\lambda'_j - \frac{1}{2}\sum_{i=1}^{r} i\lambda_i \\ &\quad -\frac{1}{2}\left(\sum_{i=1}^{r}\frac{(\lambda_i+1)\lambda_i}{2} + \sum_{j=1}^{\lambda_1}\frac{\lambda'_j(\lambda'_j-1)}{2}\right) \\ &= \frac{(a+1)}{2}\sum_{i=1}^{r}\lambda_i,\end{aligned}$$

for

$$\sum_{i=1}^{r} i\lambda_i = \sum_{j=1}^{\lambda_1}\frac{(\lambda'_j+1)\lambda'_j}{2}$$

are two ways to express the norm of the tableau of shape $\lambda$ and with constant entry $i$ in every cell in the $i$th row and

$$\sum_{j=1}^{\lambda_1} j\lambda'_j = \sum_{i=1}^{r}\frac{(\lambda_i+1)\lambda_i}{2}$$

are two way to express the norm of the tableau of shape $\lambda$ and with constant entry $j$ in every cell in the $j$th column. This concludes the proof of the lemma. □

In the following $E(i,j)$ denotes the expected values of the entry in cell $(i,j)$ of a plane partition in an $a\times b\times c$ box.

*Proof of Lemma 3.* If $a \leq b$ we have

$$\sum_{y=0}^{a+c-1} P_{a,b,c}(x,y)\ y = \begin{cases} \sum\limits_{i=1+a-x}^{a} (E(i,x-a+i)+a-i) & 1 \leq x \leq a \\ \sum\limits_{i=1}^{a} (E(i,x-a+i)+a-i) & a \leq x \leq b \\ \sum\limits_{i=1}^{a+b-x} (E(i,x-a+i)+a-i) & b \leq x \leq a+b-1 \end{cases}$$

and if $b \leq a$ we have

$$\sum_{y=0}^{a+c-1} P_{a,b,c}(x,y)\ y = \begin{cases} \sum\limits_{i=1+a-x}^{a} (E(i,x-a+i)+a-i) & 1 \leq x \leq b \\ \sum\limits_{i=1+a-x}^{a+b-x} (E(i,x-a+i)+a-i) & b \leq x \leq a \\ \sum\limits_{i=1}^{a+b-x} (E(i,x-a+i)+a-i) & a \leq x \leq a+b-1 \end{cases}.$$

Thus, by Lemma 2, we have to show that

$$\sum_{i=1+a-x}^{a} E(i,x-a+i) = \frac{b\,c\,x}{a+b}$$

for $1 \leq x \leq \min(a,b)$ and

$$\sum_{i=1}^{a+b-x} E(i,x-a+i) = \frac{a\,c\,(a+b-x)}{a+b}$$

for $\max(a,b) \leq x \leq a+b-1$. Furthermore, if $a \leq b$ we have to show that

$$\sum_{i=1}^{a} E(i,x-a+i) = \frac{a\,c\,(a+b-x)}{a+b}$$

for $a \leq x \leq b$ and if $b \leq a$ we have to show that

$$\sum_{i=1+a-x}^{a+b-x} E(i,x-a+i) = \frac{b\,c\,x}{a+b}$$

for $b \leq x \leq a$.

In the following we will only consider the case $a \leq b$ for the other case is similar. Let

$$D(x) = \begin{cases} \sum\limits_{i=1+a-x}^{a} E(i,x-a+i) & 1 \leq x \leq a \\ \sum\limits_{i=1}^{a} E(i,x-a+i) & a \leq x \leq b \\ \sum\limits_{i=1}^{a+b-x} E(i,x-a+i) & b \leq x \leq a+b-1 \end{cases}.$$

Let $1 \leq x \leq a$ and $G_x$ be the following set of cells

$$G_x = \{(i,j) | 1 \leq i \leq a, 1 \leq j \leq b\} \setminus \{(i,y-a+i) | 1 \leq y < x, 1+a-y \leq i \leq a\}.$$

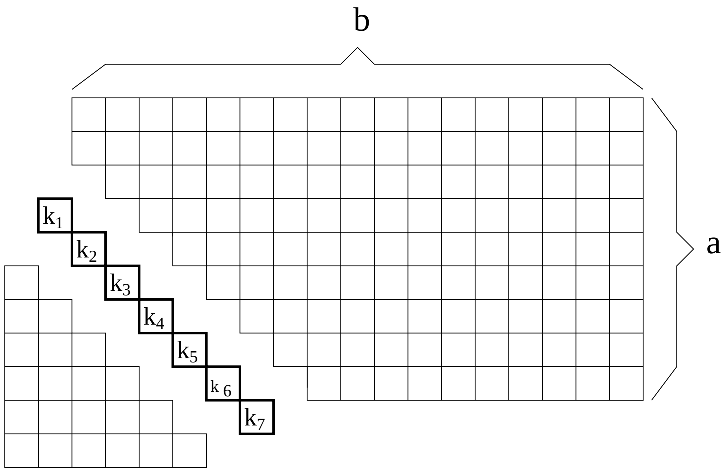


FIGURE 5

(See Figure 5.) An $x$-partial plane partition in an $a \times b \times c$ box is an assignment of the cells in $G_x$ with integers between 0 and $c$ such that rows and columns are decreasing. Let $(k_1, k_2, \ldots, k_x)$ be a decreasing sequence of integers in $\{0, 1, \ldots, c\}$ and let $U(k_1, k_2, \ldots, k_x)$ be the number of $x$-partial plane partitions $T$ with $T(i, x-a+i) = k_{x-a+i}$ for all $i$. Furthermore let $N(a, b, c)$ denote the number of plane partitions in an $a \times b \times c$ box. Then

$$D(x) = \frac{1}{N(a,b,c)} \sum_{(k_1,k_2,\ldots,k_x)} A_0(k_1, k_2, \ldots, k_x)\ U(k_1, k_2, \ldots, k_x) \sum_{i=1}^{x} k_i$$

for $1 \le x \le a$. (See Figure 5.)

Thus

$$\begin{aligned} \frac{1}{x} D(x) &= \frac{1}{N(a,b,c)} \sum_{(k_1,k_2,\ldots,k_x)} \frac{2}{x(x-1)} S_0(k_1, k_2, \ldots, k_x)\ U(k_1, k_2, \ldots, k_x) \\ &= \frac{1}{\binom{x}{2}} \sum_{y=1}^{x-1} D(y) \end{aligned}$$

for $1 \le x \le a$ by Lemma 4. By induction we obtain that the fraction $\frac{D(x)}{x}$ is independent of $x$ for $1 \le x \le a$. By symmetry $\frac{D(x)}{a+b-x}$ is independent of $x$ for $b \le x \le a+b-1$. Thus

$$D(x) = x\, D(1) \tag{2.2}$$

for $1 \le x \le a$ and

$$D(x) = (a+b-x)\, D(a+b-1) \tag{2.3}$$

for $b \le x \le a+b-1$.

Let $a \le x \le b$ and $G_x$ be the following set of cells.

$$\begin{aligned} G_x = \{(i,j) | 1 \le i \le a, 1 \le j \le b\} \backslash \\ \{(i, y-a+i) | (1 \le y < a, 1+a-y \le i \le a) \vee (a \le y < x, 1 \le i \le a)\} \end{aligned}$$

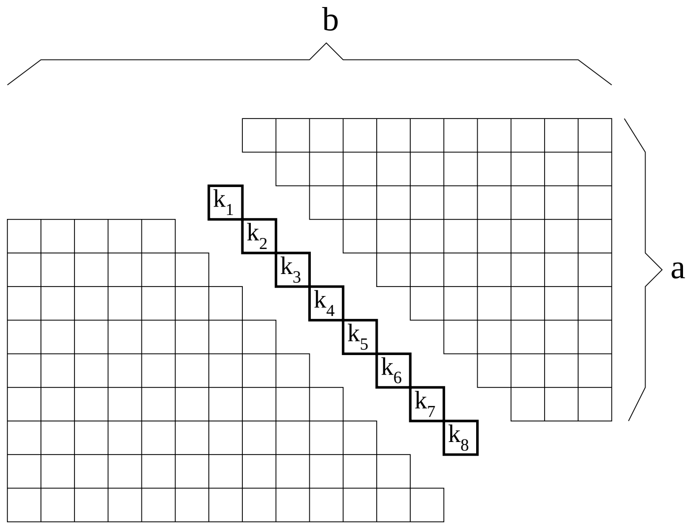


FIGURE 6

(See Figure 6.) An $x$-partial plane partition is defined in the same way as above. Let $(k_1, k_2, \ldots, k_a)$ be a decreasing sequence of integers in $\{0, 1, \ldots, c\}$ and $U(k_1, k_2, \ldots, k_a)$ the number of $x$-partial plane partitions $T$ with $T(i, x-a+i) = k_i$ for $1 \le i \le a$.

Then, by Lemma 4, we have (see Figure 6)

$$
\begin{aligned}
D(x) &= \frac{1}{N(a,b,c)} \sum_{(k_1,k_2,\ldots,k_x)} A_{x-a}(k_1,k_2,\ldots,k_a)\, U(k_1,k_2,\ldots,k_a) \sum_{i=1}^{a} k_i \\
&= \frac{1}{N(a,b,c)} \sum_{(k_1,k_2,\ldots,k_x)} \frac{2S_{x-a}(k_1,k_2,\ldots,k_a) - A_{x-a}(k_1,k_2,\ldots,k_a)\,(x-a)\,a\,c}{(x-1)} \\
&\quad \times U(k_1,k_2,\ldots,k_a) \\
&= \frac{2}{x-1} \sum_{y=1}^{x-1} D(y) - \frac{(x-a)\,a\,c}{(x-1)} \\
&= \frac{2}{x-1} \sum_{y=a+1}^{x-1} D(y) + \frac{D(1)\,a\,(a+1)}{x-1} - \frac{(x-a)\,a\,c}{x-1}
\end{aligned}
$$

and therefore, by induction,

$$D(x) = x\,D(1) - (x-a)\,c \tag{2.4}$$

for $a \le x \le b$. On one hand we have

$$D(b) = b\,D(1) - (b-a)\,c$$

by (2.4). But on the other we have

$$D(b) = a\,D(a+b-1)$$

by (2.3). Furthermore we have

$$D(1) + D(a+b-1) = E(a,1) + E(1,b) = c$$

by the involution on the set of plane partitions in an $a \times b \times c$ box with $T(i,j) = c - T(a+1-i, a+1-j)$, where $T(i,j)$ denotes the entry of the cell $(i,j)$. Thus

$$D(1) = \frac{bc}{a+b}$$

and

$$D(a+b-1) = \frac{ac}{a+b}$$

and the assertion follows by (2.2), (2.3) and (2.4).

The proof of the following lemma shows that Lemma 3 is nearly obvious if we assume $a = b$.

**Lemma 5.** *Let $m$, $n$ be positive integers and $1 \le x_0 \le 2n-1$. Then*

$$\sum_{y=0}^{n+m-1} P_{n,n,m}(x_0,y)\left(y - \frac{n+m-1}{2}\right) = \begin{cases} \frac{1}{2}(x_0-n)x_0 & 1 \le x_0 \le n \\ \frac{1}{2}(x_0-n)(2n-x_0) & n \le x_0 \le 2n \end{cases}.$$

*Proof of Lemma 5:* By Lemma 2 we have

$$\sum_{y=0}^{n+m-1} P_{n,n,m}(x_0,y)\left(-\frac{n+m-1}{2}\right) = \begin{cases} \frac{1-n-m}{2}\, x_0 & 1 \le x_0 \le n \\ \frac{1-n-m}{2}\, (2n-x_0) & n \le x_0 \le 2n-1 \end{cases}.$$

Thus we have to show that

$$\sum_{y=0}^{n+m-1} P_{n,n,m}(x_0,y)\; y = \begin{cases} \frac{x_0+m-1}{2}\, x_0 & 1 \le x_0 \le n \\ \frac{x_0+m-1}{2}\, (2n-x_0) & n \le x_0 \le 2n-1 \end{cases}.$$

This sum is equal to

$$\sum_{y=0}^{n+m-1} P_{n,n,m}(x_0,y)\; y = \begin{cases} \sum\limits_{i=n+1-x_0}^{n} (E(i, x_0-n+i) + n - i) & 1 \le x_0 \le n \\ \sum\limits_{i=1}^{2n-x_0} (E(i, x_0-n+i) + n - i) & n \le x_0 \le 2n-1 \end{cases},$$

where $E(i,j)$ is the expected value of the entry in cell $(i,j)$ in a plane partition in an $n \times n \times m$ box. Therefore we have show that

$$\sum_{i=n+1-x_0}^{n} E(i, x_0-n+i) = \frac{1}{2}\, x_0\, m \quad \text{if} \quad 1 \le x_0 \le n$$

and

$$\sum_{i=1}^{2n-x_0} n - i = \frac{1}{2}\,(2n-x_0)\, m \quad \text{if} \quad n \le x_0 \le 2n-1.$$

But this is obvious since $E(i,j) + E(n+1-j, n+1-i) = m$ by the involution on the set of plane partitions in an $n \times n \times m$ box with $T(i,j) = m - T(n+1-j, n+1-i)$, where $T(i,j)$ denotes the entry of cell $(i,j)$. □

Finally we combine Lemma 1 – 3 in order to prove Theorem 1.

*Proof of Theorem 1:* In order to calculate the horizontal moment of inertia we use Lemma 2.

$$
\begin{aligned}
&\sum_{x=1}^{a+b-1}\left(\sum_{y=0}^{a+c-1} P_{a,b,c}(x,y)\right)\left(x-\frac{a+b}{2}\right)^2 \\
&\qquad= \sum_{x=1}^{\min(a,b)} x\left(x-\frac{a+b}{2}\right)^2 + \sum_{x=\min(a,b)+1}^{\max(a,b)-1} \min(a,b)\left(x-\frac{a+b}{2}\right)^2 \\
&\qquad\qquad+ \sum_{x=\max(a,b)}^{a+b-1} (a+b-x)\left(x-\frac{a+b}{2}\right)^2 \\
&\qquad\qquad\qquad= \frac{1}{12}ab(a^2+b^2-2)
\end{aligned}
$$

The vertical moment of inertia splits into 3 sums.

$$
\begin{aligned}
&\sum_{y=0}^{a+c-1}\sum_{x=1}^{a+b-1} P_{a,b,c}(x,y)\left(2\left(y-\frac{a+c-1}{2}\right)-\left(x-\frac{a+b}{2}\right)\right)^2 \\
&\qquad= 4\sum_{y=0}^{a+c-1}\left(\sum_{x=1}^{a+b-1} P_{a,b,c}(x,y)\right)\left(y-\frac{a+c-1}{2}\right)^2 \\
&\qquad - 4\sum_{x=1}^{a+b-1}\left(\sum_{y=0}^{a+c-1} P_{a,b,c}(x,y)\left(y-\frac{a+c-1}{2}\right)\right)\left(x-\frac{a+b}{2}\right) \\
&\qquad\qquad + \sum_{x=1}^{a+b-1}\left(\sum_{y=0}^{a+c-1} P_{a,b,c}(x,y)\right)\left(x-\frac{a+b}{2}\right)^2 \qquad (2.5)
\end{aligned}
$$

The last sum is equal to the horizontal moment of inertia.

In order to compute the first sum in (2.5) we use Lemma 1.

$$
\begin{aligned}
&4\sum_{y=0}^{a+c-1}\left(\sum_{x=1}^{a+b-1} P_{a,b,c}(x,y)\right)\left(y-\frac{a+c-1}{2}\right)^2 \\
&\qquad= 4\sum_{y=0}^{a+c-1}\frac{ab}{a+c}\left(y-\frac{a+c-1}{2}\right)^2 \\
&\qquad\qquad= \frac{1}{3}ab(a+c-1)(a+c+1)
\end{aligned}
$$

For the second sum in (2.5) we use Lemma 3.

$$
\begin{aligned}
&4\sum_{x=1}^{a+b-1}\left(\sum_{y=0}^{a+c-1} P_{a,b,c}(x,y)\left(y-\frac{a+c-1}{2}\right)\right)\left(x-\frac{a+b}{2}\right) \\
&\qquad= \frac{1}{3}ab(-1+a^2+ac-bc)
\end{aligned}
$$

Thus

$$\sum_{y=0}^{a+c-1} \sum_{x=1}^{a+b-1} P_{a,b,c}(x,y) \left( 2 \left( y - \frac{a+c-1}{2} \right) - \left( x - \frac{a+b}{2} \right) \right)^2 = \frac{1}{3} abc(a+b+c) + \sum_{x=1}^{a+b-1} \left( \sum_{y=0}^{a+c-1} P_{a,b,c}(x,y) \right) \left( x - \frac{a+b}{2} \right)^2$$

and the assertion follows. □

## References

[1] J. Propp, Enumeration of Matchings: Problems and Progress, *New Perspectives in Geometric Combinatorics*, Edited by L. Billera, A. Bjöner, C. Greene, R. Simeon, R. Stanley, Mathematical Sciences Research Institute Publications **38**, Cambridge University Press, 1999.

[2] B. Sagan, The Symmetric Group: Representations, Combinatorial Algorithms and Symmetric Functions, Wadsworth and Brooks/Cole, Mathematics Series, 1991.

[3] R. P. Stanley, *Theory and applications of plane partitions: Part 2*, Stud. Appl. Math. **50** (1971), 259 - 279.

Universität Klagenfurt, Universitätsstrasse 65-67, A-9020 Klagenfurt, Austria.
*E-mail address*: `ilse.fischer@uni-klu.ac.at`