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Motzkin paths and powers of continued fractions

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Abstract. *We show that the combinatorial description of cumulants by Lehner, in terms of Motzkin paths, can be extended to the description of powers of continued fractions.*

Given an alphabet composed of letters a, $\mathbb{A} = \{a \in \mathbb{A}\}$, let $\Lambda^n(\mathbb{A})$, $n = 0, 1, 2, \ldots$, be the n-th *elementary symmetric function* in $\mathbb{A}$, i.e. the coefficient of x^n in $\lambda_x(A) := \prod_{a\in\mathbb{A}}(1 + ax)$. Let furthemore the *complete functions* $S^n(\mathbb{A})$ be the coefficients of $\sigma_x(\mathbb{A}) = \prod_{a\in\mathbb{A}} 1/(1 - ax)$ and, for any $k \in \mathbb{R}$, let $\Lambda^n(k\mathbb{A})$ be the coefficient of of x^n in $\lambda_x(k\mathbb{A}) := (\lambda_x(\mathbb{A}))^k$. We wish to express the symmetric function $\Lambda^n(k\mathbb{A})$ in terms of the coefficients of the continued fraction expression of

$$\sigma_x(\mathbb{A}) = \sum x^n S^n(\mathbb{A}) := \frac{1}{\lambda_{-x}(\mathbb{A})} = \cfrac{1}{(1-\alpha_0 x) - \cfrac{x^2\xi_1}{(1-\alpha_1 x) - \cfrac{x^2\xi_2}{(1-\alpha_2 x) - \cfrac{x^2\xi_3}{\ddots}}}} \tag{1}$$

i.e. we wish to give the expansion of the k-th power of the continued fraction on the right.

Wronski [Wr] determined the parameters $\alpha_0, \alpha_1, \ldots, \xi_1, \xi_2, \ldots$ as symmetric functions of $\mathbb{A}$ (see [L1] for a translation) and produced an algorithm (essentially Euclidean division) to generate them.

Flajolet ([Fl]; see also [FV], [Vi], [GJ]) has given a combinatorial interpretation of the complete functions $S^n(\mathbb{A})$ in terms of the parameters α_i, ξ_j, that we shall use after recalling the necessary definitions. First, we shall let the cardinality of $\mathbb{A}$ tend toward infinity, so that the elementary symmetric functions are algebraically independent.

Definition. Motzkin paths are paths in the positive integral plane, with East, North-East and South-East steps, starting and finishing at level zero. A *weight* is attributed to each step: α_i for an horizontal step at level i, ξ_i for a South-East step between level i and $i - 1$. The *weight* of the path itself is the product of the weights of its steps. An *irreducible Motzkin path* is a path which does not touch level zero, except at both ends.

One can also write a path as a word, recording the levels attained at each step. For example

$$\begin{array}{cccccccccccccccc}
 & & & & & & & 3 & 3 & 3 & 3 & & & & & \\
 & & & & 2 & & 2 & & & & & 2 & & 2 & & \\
 & & 1 & 1 & & 1 & & & & & & & 1 & & 1 & \\
 & 0 & & & & & & & & & & & & & & 0 \\
 & & & & & & & & & & & & & & & \\
\textit{word} & 0 & 1 & 1 & 2 & 1 & 2 & 3 & 3 & 3 & 3 & 2 & 1 & 2 & 1 & 0 \\
\textit{weight} & & \alpha_1 & & \xi_2 & & & \alpha_3 & \alpha_3 & \alpha_3 & \xi_3 & \xi_2 & & \xi_1 & \xi_0 &
\end{array}$$

It is clear that Motzkin paths can be uniquely factored into irreducible paths, level zero points being the cutting points.

When we write an equality between a function in the α_i, ξ_j and a sum of paths, we shall mean that one has to replace each path by its weight in the latter.

Theorem (Flajolet [Fl]). *For any $n \in \mathbb{N}$, the complete function $S_n(\mathbb{A})$ is equal to the sum of Motzkin paths of length n, the parameters α_i, ξ_j being defined by (1).*

The concatenation product of paths is not commutative. To recover a ring of symmetric functions from the preceding combinatorial interpretation of complete functions, it suffices to take the commutative algebra with the irreducible Motzkin paths as generators (one also has theories of non commutative symmetric functions !). Any symmetric polynomial in $\mathbb{A}$ can now be interpreted as a polynomial function in the irreducible Motzkin paths. We shall write $\equiv$ when allowing commutation of Motzkin paths .

For example, since $1/\sigma_x(\mathbb{A}) = \lambda_{-x}(\mathbb{A})$, then

$$(-1)^{n-1}\Lambda^n(\mathbb{A}) \equiv \text{ sum of all irreducible Motzkin paths of length } n \; . \tag{2}$$

The hook-Schur functions $S_{j+1,1^i}$ are expressible as

$$S_{j+1,1^i} = \Lambda^{i+1}S^j - \Lambda^{i+2}S^{j-1} + \cdots + \pm\Lambda^{i+j+1}S^0 \; . \tag{3}$$

Each term of the sum (3) can be interpreted as the sum of all Motzkin paths of length n with first return to level 0 in position $i+1, i+2, i+3, \ldots$, respectively. Therefore

$$(-1)^i \, S_{j+1,1^i}(\mathbb{A}) \equiv \sum\nolimits_{\vartheta(w)\geq i+1} w \; , \tag{4}$$

where the sum is over all Motzkin paths w of length $i+j+1$, $\vartheta(w)$ denoting the position of the first return to level 0.

A power sum ψ_n is equal to the alternating sum of all hook-Schur functions of degree n (i.e., $\psi_n = S_n - S_{n-1,1} + S_{n-2,1,1} - \cdots$). Therefore, one gets from (4):

$$\psi_n(\mathbb{A}) = \sum\nolimits_{w:\, length(w)=n} \vartheta(w)\, w \; . \tag{5}$$

The Cauchy formula allows to expand elementary or complete functions of a product of two alphabets. In particular, it gives several expansions of the $\Lambda^n(k\mathbb{A})$, $k \in \mathbb{R}$ (these functions can be interpreted as coefficients of the powers of the Lagrange inverse of $x\sigma_x(\mathbb{A})$, see [L2]).

The Cauchy formula says that

$$\Lambda^n(k\mathbb{A}) = \sum_{m_1,m_2,\ldots} \frac{k(k-1)\cdots(k-\ell+1)}{m_1!m_2!\cdots} (\Lambda^1(\mathbb{A}))^{m_1}(\Lambda^2(\mathbb{A}))^{m_2}\cdots \; , \tag{6}$$

where the sum is over all partitions $1^{m_1}2^{m_2}\cdots$ of weight $1m_1 + 2m_2 + \cdots = n$, $\ell = m_1 + m_2 + \cdots$ denoting the length of the partition.

By (2), the products $(\Lambda^1(\mathbb{A}))^{m_1}(\Lambda^2(\mathbb{A}))^{m_2}\cdots$ are the sum of Motzkin paths which touch the ground at positions $1, 2, \ldots, m_1$; $m_1+2, m_1+4, \ldots, m_1+2m_2$; $m_1+2m_2+3, \ldots$

If we enumerate all Motzkin paths instead of the preceding ones, we have to correct by a factor $\ell!/m_1!\,m_2!\cdots$ which tells how many paths can be obtained from a given one by permutation of its irreducible components. Therefore one has :

Theorem. *For any $n \in \mathbb{N}$, any $k \in \mathbb{R}$, one has*

$$\Lambda^n(k\mathbb{A}) \equiv \sum (-1)^{n-\ell} \binom{k}{\ell} w \ , \tag{7}$$

where the sum is over all Motzkin paths of length n, where ℓ is the number of ground points (other than the origin) of the path.

To express cumulants, Lehner [Le] needed the functions $\frac{1}{n-1}\Lambda^n((n-1)\mathbb{A})$ and obtained the above theorem (for $k = n-1$) by direct computation.

For example, for $n = 4$, writing words $w[]$ multiplied by weights instead of paths, one has

$$\Lambda^4(k\mathbb{A}) = \binom{k}{4} w[0,0,0,0,0]\alpha_0^4 - \binom{k}{3}\big(w[0,0,0,1,0] + w[0,0,1,0,0] + w[0,1,0,0,0]\big)\alpha_0^2\xi_1$$
$$+\binom{k}{2} w[0,1,0,1,0]\xi_1^2 + \binom{k}{2}\big(w[0,0,1,1,0] + w[0,1,1,0,0]\big)\alpha_0\alpha_1\xi_1$$
$$-\binom{k}{1} w[0,1,1,1,0]\alpha_1^2\xi_1 - \binom{k}{1} w[0,1,2,1,0]\xi_1\xi_2 \ .$$

We notice that $k = 0$ gives at the limit the logarithm of $\sigma_x(\mathbb{A})$. Indeed, the limit, for $k = 0$, of $\frac{1}{k}\,\Lambda^n(k\mathbb{A})$, $n > 0$, is $\frac{1}{n}\psi_n(\mathbb{A})$. Therefore (7) implies

$$\frac{1}{n}\psi_n(\mathbb{A}) \equiv \sum \frac{1}{\ell}\, w \ , \tag{8}$$

where the statistics differ from (5).

For example, for $n = 4$, formula (5) gives

$$\frac{1}{4}\psi_4(\mathbb{A}) = \frac{1}{4}w[0,0,0,0,0]\alpha_0^4 + \frac{1}{4}w[0,0,1,0,0]\alpha_0^2\xi_1 + \frac{1}{4}w[0,0,1,0,0]\alpha_0^2\xi_1$$
$$+\frac{1}{4}w[0,0,1,1,0]\alpha_0\alpha_1\xi_1 + \frac{2}{4}w[0,1,0,0,0]\alpha_0^2\xi_1 + \frac{2}{4}w[0,1,0,1,0]\xi_1^2$$
$$+\frac{3}{4}w[0,1,1,0,0]\alpha_0\alpha_1\xi_1 + \frac{4}{4}w[0,1,1,1,0]\alpha_1^2\xi_1 + \frac{4}{4}w[0,1,2,1,0]\xi_1\xi_2 \ ,$$

while (8) states that

$$\tfrac{1}{4}\psi_4(\mathbb{A}) = \tfrac{1}{4}w[0,0,0,0,0]\alpha_0^4 + \tfrac{1}{3}\Big(w[0,0,0,1,0] + w[0,0,1,0,0] + w[0,1,0,0,0]\Big)\alpha_0^2\xi_1$$
$$+\tfrac{1}{2}w[0,1,0,1,0]\xi_1^2 + \tfrac{1}{2}\Big(w[0,0,1,1,0] + w[0,1,1,0,0]\Big)\alpha_0\alpha_1\xi_1 + w[0,1,1,1,0]\alpha_1^2\xi_1$$

$$+w[0,1,2,1,0]\xi_1\xi_2\ .$$

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