SIMULTANEOUS MAJ STATISTICS

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Abstract. The generating function for words with several simultaneous *maj* weights is given. New *maj*-like Mahonian statistics result. Some applications to integer partitions are given.

1. Introduction.

The usual *maj* statistic [2] on words w is defined by adding the location of the descents of the word w,

$$maj(w) = \sum_{i:w_i > w_{i+1}} i.$$

This definition presumes that the alphabet for the letters of w have been linearly ordered, for example $2 > 1 > 0$,

$$maj(1102201) = 2 + 5 = 7 = maj_{210}(1102201).$$

However a similar definition can be made assuming any linear ordering σ; here we take $1 > 2 > 0$, $\sigma = 120$, and $2 > 0 > 1$, $\sigma = 201$

$$maj_{120}(1102201) = 2 + 5 = 7, \quad maj_{201}(1102201) = 5 + 6 = 11.$$

In this paper we consider the generating function for several such simultaneous *maj* statistics (see Corollary 1). A more general generating function is given (Theorem 3), and some applications to Mahonian statistics (Corollary 2) and integer partitions (Theorem 4) are stated.

We first give a 3 letter theorem, which motivates the general result (Theorem 3). Let $W(m, n, k)$ be the set of words of length $m + n + k$ with m 0's, n 1's and k 2's.

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Theorem 1. *For any non-negative integers m, n, and k we have*

$$\sum_{w\in W(m,n,k)} x^{maj_{120}(w)} y^{maj_{201}(w)} z^{maj_{012}(w)} = x^{n+k}y^k \begin{bmatrix} m+n+k-1 \\ m-1,\, n,\, k \end{bmatrix}_{xyz} + y^{k+m}z^m \begin{bmatrix} m+n+k-1 \\ m,\, n-1,\, k \end{bmatrix}_{xyz} + z^{m+n}x^n \begin{bmatrix} m+n+k-1 \\ m,\, n,\, k-1 \end{bmatrix}_{xyz}.$$

Proof. We prove a stronger statement, that the three terms in Theorem 1 are the generating functions for the words in $W(m,n,k)$ ending in 0, 1, and 2 respectively.

We proceed by induction on $m+n+k$. If w ends in a 0, the penultimate letter must be either 0, 1 or 2. Using induction we must verify that

$$x^{n+k}y^k \begin{bmatrix} m+n+k-1 \\ m-1,\, n,\, k \end{bmatrix}_{xyz} = x^{n+k}y^k \begin{bmatrix} m+n+k-2 \\ m-2,\, n,\, k \end{bmatrix}_{xyz} + x^{m+n+k-1}y^{m+k-1}z^{m-1} \begin{bmatrix} m+n+k-2 \\ m-1,\, n-1,\, k \end{bmatrix}_{xyz} + (xy)^{m+n+k-1}z^{m+n-1}x^n \begin{bmatrix} m+n+k-2 \\ m-1,\, n,\, k-1 \end{bmatrix}_{xyz},$$

which is the well-known recurrence formula [1] for the xyz-trinomial coefficient.

The other two cases are verified similarly. □

It should be noted that if any two of x, y, z are set equal to 1, then the usual maj generating function as a q-trinomial coefficient results.

2. A 7-variable theorem.

Theorem 1 contains three free variables, x, y and z. In this section we generalize Theorem 1 to Theorem 2, which contains seven free variables. Then we indicate how to specialize Theorem 2 to obtain new explicit classes of Mahonian statistics on words of 0's, 1's, and 2's.

Suppose that the weights of the various possible ascents and descents in position $m+n+k-1$ of a word w of m 0's, n 1's, and k 2's are given by

(wt10) $a_0^{m-1}a_1^n a_2^k$ for a descent 10,
(wt21) $b_0^m b_1^{n-1} b_2^k$ for a descent 21,
(wt20) $c_0^{m-1}c_1^n c_2^k$ for a descent 20,
(wt01) $d_0^m d_1^{n-1} d_2^k$ for an ascent 01,
(wt12) $e_0^m e_1^n e_2^{k-1}$ for an ascent 12,
(wt02) $f_0^m f_1^n f_2^{k-1}$ for an ascent 02.

Also suppose that the generating function for all such words w has the form

$$p_0(n,k)\begin{bmatrix} m+n+k-1 \\ m-1,\ n,\ k \end{bmatrix}_B + p_1(k,m)\begin{bmatrix} m+n+k-1 \\ m,\ n-1,\ k \end{bmatrix}_B + p_2(m,n)\begin{bmatrix} m+n+k-1 \\ m,\ n,\ k-1 \end{bmatrix}_B \tag{2.1}$$

for some base B, and $p_0(n,k) = p_{01}^n p_{02}^k$, $p_1(k,m) = p_{11}^k p_{12}^m$, $p_2(m,n) = p_{21}^m p_{22}^n$. We also assume that the three terms in (2.1) correspond to the w which end in 0, 1, and 2 respectively.

Thus we have 25 free variables

$$\cup_{i=0}^{2}\{a_i, b_i, c_i, d_i, e_i, f_i, p_{i1}, p_{i2}\} \cup \{B\}.$$

These 25 variables are related by the three equations which we require by induction

$$\begin{aligned} p_0(n,k)\begin{bmatrix} m+n+k-1 \\ m-1,\ n,\ k \end{bmatrix}_B &= p_0(n,k)\begin{bmatrix} m+n+k-2 \\ m-2,\ n,\ k \end{bmatrix}_B \\ &+ a_0^{m-1}a_1^n a_2^k\ p_1(k,m-1)\begin{bmatrix} m+n+k-2 \\ m-1,\ n-1,\ k \end{bmatrix}_B \\ &+ c_0^{m-1}c_1^n c_2^k\ p_2(m-1,n)\begin{bmatrix} m+n+k-2 \\ m-1,\ n,\ k-1 \end{bmatrix}_B, \end{aligned} \tag{2.2a}$$

$$\begin{aligned} p_1(k,m)\begin{bmatrix} m+n+k-1 \\ m,\ n-1,\ k \end{bmatrix}_B &= p_1(k,m)\begin{bmatrix} m+n+k-2 \\ m,\ n-2,\ k \end{bmatrix}_B \\ &+ b_0^m b_1^{n-1} b_2^k\ p_2(m,n-1)\begin{bmatrix} m+n+k-2 \\ m,\ n-1,\ k-1 \end{bmatrix}_B \\ &+ d_0^m d_1^{n-1} d_2^k\ p_0(n-1,k)\begin{bmatrix} m+n+k-2 \\ m-1,\ n-1,\ k \end{bmatrix}_B, \end{aligned} \tag{2.2b}$$

$$\begin{aligned} p_2(m,n)\begin{bmatrix} m+n+k-1 \\ m,\ n,\ k-1 \end{bmatrix}_B &= p_2(m,n)\begin{bmatrix} m+n+k-2 \\ m,\ n,\ k-2 \end{bmatrix}_B \\ &+ f_0^m f_1^n f_2^{k-1}\ p_0(n,k-1)\begin{bmatrix} m+n+k-2 \\ m-1,\ n,\ k-1 \end{bmatrix}_B \\ &+ e_0^m e_1^n e_2^{k-1}\ p_1(k-1,m)\begin{bmatrix} m+n+k-2 \\ m,\ n-1,\ k-1 \end{bmatrix}_B. \end{aligned} \tag{2.2c}$$

We do not know the general solution to the equations (2.2a-c). However, we will give the general solution to (2.2a-c) if we make another assumption. If we specify that the coefficient of the second term on the the right side of (2.2a) is B^{m-1} times the coefficient of the first term, and the coefficient of the third term is B^{m+n-1} times the coefficient of the first term, then the B-trinomial recurrence relation verifies (2.2a). These two equations are

$$\begin{aligned} a_0^{m-1}a_1^n a_2^k p_{11}^k p_{12}^{m-1} &= B^{m-1}p_{01}^n p_{02}^k, \\ c_0^{m-1}c_1^n c_2^k p_{21}^{m-1} p_{22}^n &= B^{m+n-1}p_{01}^n p_{02}^k. \end{aligned} \tag{2.3a}$$

Similarly, we assume the B-trinomial recurrence for (2.2b) and (2.2c), which become

$$\begin{aligned} b_0^m b_1^{n-1} b_2^k p_{21}^m p_{22}^{n-1} &= B^{n-1}p_{11}^k p_{12}^m, \\ d_0^m d_1^{n-1} d_2^k p_{01}^{n-1} p_{02}^k &= B^{n+k-1}p_{11}^k p_{12}^m. \end{aligned} \tag{2.3b}$$

and

$$\begin{aligned} f_0^m f_1^n f_2^{k-1} p_{01}^n p_{02}^{k-1} &= B^{k-1}p_{21}^m p_{22}^n, \\ e_0^m e_1^n e_2^{k-1} p_{11}^{k-1} p_{12}^m &= B^{k+m-1}p_{21}^m p_{22}^n. \end{aligned} \tag{2.3c}$$

Since these equations should hold for all m, n and k, each of these 6 equations contains 3 equations (one each in m, n, and k). Thus we have 18 equations in the 25 free variables, which are written in a matrix form, where the first column comes from the equations in (2.3a):

$$\begin{pmatrix} p_{12}a_0 & p_{21}b_0 & f_0 \\ a_1 & p_{22}b_1 & p_{01}f_1 \\ p_{11}a_2 & b_2 & p_{02}f_2 \\ p_{21}c_0 & d_0 & p_{12}e_0 \\ p_{22}c_1 & p_{01}d_1 & e_1 \\ c_2 & p_{02}d_2 & p_{11}e_2 \end{pmatrix} = \begin{pmatrix} B & p_{12} & p_{21} \\ p_{01} & B & p_{22} \\ p_{02} & p_{11} & B \\ B & p_{12} & p_{21}B \\ p_{01}B & B & p_{22} \\ p_{02} & p_{11}B & B \end{pmatrix}.$$

One may find the general solution to these 18 equations, leaving 7 free variables

$$\{a_0, a_1, a_2, b_0, b_1, b_2, B\}.$$

The explicit solutions for the remaining 18 variables are given below. The weights (wt) become (W):

(W10) $a_0^{m-1}a_1^n a_2^k$ for a descent 10,
(W21) $b_0^m b_1^{n-1} b_2^k$ for a descent 21,
(W20) $(a_0b_0)^{m-1}(a_1b_1)^n(a_2b_2)^k$ for a descent 20,
(W01) $(B/a_0)^m(B/a_1)^{n-1}(B/a_2)^k$ for an ascent 01,
(W12) $(B/b_0)^m(B/b_1)^n(B/b_2)^{k-1}$ for an ascent 12,
(W02) $(B/a_0b_0)^m(B/a_1b_1)^n(B/a_2b_2)^{k-1}$ for an ascent 02,

and

$$\begin{aligned} &p_0(n,k) = a_1^n(a_2b_2)^k, \quad p_1(k,m) = b_2^k(B/a_0)^m, \\ &p_2(m,n) = (B/a_0b_0)^m(B/b_1)^n. \end{aligned}$$

Theorem 2. *The generating function of all words $w \in W(m,n,k)$ with weights given by (W) is*

$$a_1^n(a_2b_2)^k \begin{bmatrix} m+n+k-1 \\ m-1,\, n,\, k \end{bmatrix}_B + b_2^k(B/a_0)^m \begin{bmatrix} m+n+k-1 \\ m,\, n-1,\, k \end{bmatrix}_B + (B/a_0b_0)^m(B/b_1)^n \begin{bmatrix} m+n+k-1 \\ m,\, n,\, k-1 \end{bmatrix}_B .$$

Theorem 1 is the special case of Theorem 2 for which $B = xyz$, $a_0 = a_1 = a_2 = x$, and $b_0 = b_1 = b_2 = y$ hold.

There are 7 other versions of Theorem 2. These 8 theorems arise by independently replacing the pair of factors (B^{m-1}, B^{m+n-1}) by (B^{m+k-1}, B^{m-1}) in equation (2.3a), (B^{n-1}, B^{n+k-1}) by (B^{n+m-1}, B^{n-1}) in equation (2.3b), and (B^{k-1}, B^{k+m-1}) by (B^{k+n-1}, B^{k-1}) in (2.3c). The B-trinomial recurrence still holds. For instance if we make a replacement in (2.3a),

$$\text{(2.3a}'\text{)} \qquad \begin{aligned} a_0^{m-1}a_1^n a_2^k p_{11}^k p_{12}^{m-1} &= B^{m+k-1} p_{01}^n p_{02}^k, \\ c_0^{m-1}c_1^n c_2^k p_{21}^{m-1} p_{22}^n &= B^{m-1} p_{01}^n p_{02}^k, \end{aligned}$$

then the explicit solutions to (2.3a$'$) and (2.3b-c) give the weight (W$'$):

(W$'$10) $a_0^{m-1}a_1^n a_2^k$ for a descent 10,
(W$'$21) $b_0^m b_1^{n-1} b_2^k$ for a descent 21,
(W$'$20) $(a_0b_0)^{m-1}(a_1b_1/B)^n(a_2b_2/B)^k$ for a descent 20,
(W$'$01) $(B/a_0)^m(B/a_1)^{n-1}(B^2/a_2)^k$ for an ascent 01,
(W$'$12) $(B/b_0)^m(B/b_1)^n(B/b_2)^{k-1}$ for an ascent 12,
(W$'$02) $(B/a_0b_0)^m(B/a_1b_1)^n(B^2/a_2b_2)^{k-1}$ for an ascent 02,

and the corresponding theorem is the following:

Theorem 2$'$. *The generating function of all words $w \in W(m,n,k)$ with weights given by (W$'$) is*

$$a_1^n(a_2b_2/B)^k \begin{bmatrix} m+n+k-1 \\ m-1,\, n,\, k \end{bmatrix}_B + b_2^k(B/a_0)^m \begin{bmatrix} m+n+k-1 \\ m,\, n-1,\, k \end{bmatrix}_B + (B/a_0b_0)^m(B/b_1)^n \begin{bmatrix} m+n+k-1 \\ m,\, n,\, k-1 \end{bmatrix}_B .$$

We do not state the remaining 6 variations here.

We can find Mahonian statistics by requiring that the generating function in Theorem 2 is the B-trinomial via the B-trinomial recurrence. There are six choices for this recurrence, one for each ordering of the 3 terms. So Theorem 2 gives a total of 6 possible Mahonian statistics, one of which (maj_{012}), is found by setting $a_0 = a_1 = a_2 = b_0 = b_1 = b_2 = 1$. Theorem 2$'$

also gives a total of 6 possible Mahonian statistics, one of which is found by setting $a_0 = a_1 = b_0 = b_1 = b_2 = 1,\ a_2 = B$. Similarly there are 6 possible Mahonian statistics for each of other 6 versions of Theorem 2, for a total of $6 \times 8 = 48$. Six of them are the six possible maj_σ statistics, the remaining 42 come in 7 classes of six each, and they are all variations on maj. Each class of size 6 consists of a maj variation, and 5 others which correspond to 5 non-trivial reorderings of $\{0, 1, 2\}$ of that maj variation. We give below one member of each class, eight in total.

We start with an example from Theorem 2′. If we set $a_0 = a_1 = b_0 = b_1 = b_2 = 1,\ a_2 = B$ in Theorem 2′, the weight (W′) reduces to

(W′10) B^k for a descent 10,
(W′21) 1 for a descent 21,
(W′20) B^{-n} for a descent 20,
(W′01) $B^{m+n+k-1}$ for an ascent 01,
(W′12) $B^{m+n+k-1}$ for an ascent 12,
(W′02) $B^{m+n+k-1}$ for an ascent 02.

Note that the above weight (W′) is a perturbation of maj_{012} involving the descents 10 and 20. We write it as $maj_{012} + s_0$, where s_0 is defined in the following way. We define s_0 by giving the non-zero values at adjacent letters. One then adds these values to find s_0. It is assumed that if w is truncated after the adjacent letters, w has m 0's, n 1's, and k 2's.

$s_0(w)$:

(1) k for an adjacent 10,
(2) $-n$ for an adjacent 20.

For example,

$$s_0(22012110201) = -0 + 3 - 3 = 0.$$

It turns out (we do not write down the details here) that the eight statistics (including maj_{012}) can be defined by three independent perturbations of maj_{012}: s_0, s_1, and s_2. For any subset $A \subset \{0, 1, 2\}$ put

$$s_A(w) = \sum_{i \in A} s_i(w).$$

Then the eight Mahonian statistics are $maj_{012} + s_A$. In fact the set A indicates which replacements are made in (2.3a-c). For instance the above (W′) is $maj_{012} + s_{\{0\}}$ and if we make replacements, say in (2.3b) and (2.3c), then the corresponding statistics will be $maj_{012} + s_{\{1,2\}}$, and so on. We define s_1, s_2 analogously by giving the non-zero values at adjacent letters. One then adds these values to find the statistic. It is assumed that if w is truncated after the adjacent letters, w has m 0's, n 1's, and k 2's.

$s_1(w)$:

(1) m for an adjacent 21,
(2) $-k$ for an adjacent 01.

$s_2(w)$:

(1) n for an adjacent 02,
(2) $-m$ for an adjacent 12.

For example,

$$s_1(22012110201) = -2 + 1 - 4 = -5, \quad s_2(22012110201) = -1 + 3 = 2.$$

Below is a table evaluating maj_{012}, s_0, s_1, and s_2 at the 6 permutations of 012. Note that the maj_{012} generating function is $1 + 2B + 2B^2 + B^3$, which is also the generating function for $maj_{012} + s_A$, for any subset $A \subset \{0, 1, 2\}$.

word	maj_{012}	s_0	s_1	s_2
012	3	0	0	-1
021	1	0	1	0
102	2	0	0	1
120	1	-1	0	0
201	2	0	-1	0
210	0	1	0	0

We repeat that all 48 Mahonian statistics may be found from these 8 by permuting the letters 0, 1, and 2. In this case maj_{012} becomes maj_σ, and each s_i is found by applying σ to 0, 1, and 2 in the definition of s_i.

3. N letters.

In this section we briefly generalize Theorem 2 to words with N letters in Theorem 3. We state the N letter version of Theorem 1 in Corollary 1. There are $N!\,2^N$ Mahonian statistics, which come in 2^N families each of size $N!$. We explicitly give the corresponding 2^N Mahonian statistics in Corollary 2.

Let $W(a_0, a_1, \cdots, a_{N-1})$ be the set of all words w with a_i i's, $0 \le i \le N-1$.

If the words w have N letters instead of 3 letters, then each adjacent pair ij, $i \ne j$, could be weighted by N variables, instead of 3 variables. Also the coefficients p_i, $0 \le i \le N-1$ would have $N-1$ variables. Together with the base B, we have a total of $N(N^2 - N) + N(N-1) + 1 = N^3 - N + 1$ variables. Each of the N recurrences required by induction gives $N(N-1)$ equations in these variables. So $N(N-1) + 1$ variables will be free in the multivariable version of Theorem 2.

In order to fully describe the resulting theorem, some care must be taken with notation.

The $N(N-1) + 1$ free variables may be taken to be the base B along with the N weights of the adjacent pairs $(i+1)i$, for $i = 0, \cdots, N-2$, for which we use the variables

$$(x_{i0}, x_{i1}, \cdots, x_{iN-1}), \quad 0 \le i \le N-2.$$

Suppose that w ends in an adjacent pair ij, $i \neq j$, and that there are n_k k's preceding the last letter j of w. The weight of the pair ij is given by

$$\begin{aligned} &\prod_{k=0}^{N-1} (\prod_{l=j}^{i-1} x_{lk})^{n_k} && \text{if } j < i, \\ &\prod_{k=0}^{N-1} (B/\prod_{l=i}^{j-1} x_{lk})^{n_k} && \text{if } i < j. \end{aligned} \tag{4.2}$$

As usual, we multiply the weights of adjacent pairs to find the weight of the word w.

Theorem 3. *The generating function of all words $w \in W(a_0, a_1, \cdots, a_{N-1})$ with weights given by (4.2) is*

$$\sum_{i=0}^{N-1} p_i(a_0, a_1, \cdots, a_{N-1}) \begin{bmatrix} a_0 + \cdots + a_{N-1} - 1 \\ a_0, \cdots, a_i - 1, \cdots, a_{N-1} \end{bmatrix}_B$$

where

$$p_i(a_0, a_1, \cdots, a_{N-1}) = \left(\prod_{l=0}^{i-1} (B/\prod_{k=1}^{i-l} x_{i-k,l})^{a_l} \right) \left(\prod_{l=i+1}^{N-1} (\prod_{k=0}^{l-i-1} x_{i+k,l})^{a_l} \right).$$

Note that p_i in Theorem 3 is independent of a_i.

The multivariable version of Theorem 1 occurs if

$$x_{i0} = x_{i1} = \cdots = x_{iN-1} = x_i, \quad 0 \leq i \leq N-2,$$

and $B = x_0 x_1 \cdots x_{N-1}$. Then the weights (4.2) become

$$\begin{aligned} &(x_j \cdots x_{i-1})^{n_0 + \cdots + n_{N-1}} && \text{if } j < i, \\ &(x_0 \cdots x_{i-1} x_j \cdots x_{N-1})^{n_0 + \cdots + n_{N-1}} && \text{if } i < j, \end{aligned}$$

and the next corollary holds.

Corollary 1. *We have*

$$\sum_{w \in W(a_0, \cdots, a_{N-1})} \prod_{i=0}^{N-1} x_i^{maj_{i+1 \cdots (N-1) 0 1 \cdots i}(w)} =$$
$$\sum_{i=0}^{N-1} p_i(a_0, a_1, \cdots, a_{N-1}) \begin{bmatrix} a_0 + \cdots + a_{N-1} - 1 \\ a_0, \cdots, a_i - 1, \cdots, a_{N-1} \end{bmatrix}_{x_0 \cdots x_{N-1}}$$

where

$$p_i(a_0, a_1, \cdots, a_{N-1}) = \Big(\prod_{l=0}^{i-1} (x_0 \cdots x_{l-1} x_i \cdots x_{N-1})^{a_l}\Big)\Big(\prod_{l=i+1}^{N-1} (x_i \cdots x_{l-1})^{a_l}\Big).$$

We next give the 2^N Mahonian statistics which follow from Theorem 3. Again they may be classified by perturbations of $maj_{01\cdots N-1}$. For any subset $A \subset \{0, 1, \cdots, N-1\}$, define

$$s_A(w) = \sum_{i \in A} s_i(w).$$

The individual statistics $s_i(w)$ only depend upon the subwords of w ending in i, as in §2. For any given $i \in w$, suppose that i is preceded by n_j j's, $0 \le j \le N-1$. Extend the definition of n_j to be periodic mod N: $n_{j+N} = n_j$ for all j. If the letter preceding i is $i+k$, the contribution to $s_i(w)$ is positive on the circular interval $[i+k+1, i-1]$ and negative on the circular interval $[i+1, i+k-1]$,

$$(n_{i+k+1} + n_{i+k+2} + \cdots + n_{(i-1)}) - (n_{i+1} + n_{i+2} + \cdots + n_{i+k-1}). \tag{3.1}$$

We add the contributions of (3.1) over all $i \in w$ to find $s_i(w)$. There is no contribution if $k = 0$; that is, for a repeated ii. For example,

$$s_1(41241012411312301) = 0 + (-1) + (-3) + (1-2) + (4-2) + (-8) = -11.$$

Corollary 2. *For any set $A \subset \{0, 1, \cdots, N-1\}$, the statistic $maj_{01\cdots N-1} + s_A$ is Mahonian on $W(a_0, a_1, \cdots, a_{N-1})$.*

These Mahonian statistics are examples of *splittable* statistics [3].

One may also allow weights on the adjacent letters 00, 11, and 22 for a more general version of Theorem 3.

4. Applications to partitions.

In this section we apply Theorem 1 and Theorem 3 to integer partitions.

The special case $k = 0$, $z = 1$, $x = y = q$ of Theorem 1 is

$$\sum_{w \in W(m,n,0)} q^{maj_{10}(w) + maj_{01}(w)} = \begin{bmatrix} m+n \\ m \end{bmatrix}_{q^2} \frac{q^m + q^n}{1 + q^{m+n}} := f(m, n, q). \tag{4.1}$$

MacMahon [4, p. 139] previously gave (4.1).

The following generating function (using standard notation found in [1]) follows from (4.1),

$$\sum_{m,n \ge 0} f(m, n, q) \frac{(xq)^m (yq)^n}{(q;q)_{m+n}} = \frac{(xyq^2; q^2)_\infty}{(xq, yq; q)_\infty}. \tag{4.2}$$

One way to see (4.2) is to consider the generating function for pairs of partitions (λ, μ) with distinct parts, weighted by

$$x^{\#\text{ of parts of }\lambda} y^{\#\text{ of parts of }\mu} q^{|\lambda|+|\mu|}$$

which is

$$\prod_{k=1}^{\infty}\left(1+\frac{xq^k}{1-xq^k}+\frac{yq^k}{1-yq^k}\right)=\frac{(xyq^2;q^2)_\infty}{(xq,yq;q)_\infty}.$$

To prove (4.1), we must find a weight preserving bijection ϕ from the set of such (λ, μ), # parts of $\lambda = m$, # parts of $\mu = n$, to the set of ordered pairs (w, γ), where $w \in W(m, n, 0)$, and γ is a partition with $m + n$ parts.

To define w, order the $m + n$ parts of $\lambda \cup \mu$ into a partition θ, and let $w_i = 0$ if $\theta_i \in \lambda$, $w_i = 1$ if $\theta_i \in \mu$. This is well defined since the parts of λ and μ are distinct. To define γ, let t_i be the number of descents or ascents to the right of position i in the word w. Then we let $\gamma = \theta - t$. For example if

$$\lambda = 7742, \quad \mu = 88661,$$

then

$$\theta = 887766421, \qquad w = 110011001, \quad t = 443322110, \qquad \gamma = 444444311.$$

This correspondence is the desired bijection ϕ.

The natural analog of ϕ on triples (λ, μ, θ) without pairwise common parts produces a word $w \in W(m, n, k)$ and a partition γ. The q-statistic on the word w again counts all ascents and descents of w by their positions. However, in Theorem 1, we see that the six possible ascents/descents in w are weighted differently by position:

$$\begin{aligned}
&01 \text{ by } yz,\\
&02 \text{ by } z,\\
&10 \text{ by } x,\\
&12 \text{ by } xz,\\
&20 \text{ by } xy,\\
&21 \text{ by } y.
\end{aligned}$$

So if we choose $x = q^a$, $y = q^b$, $z = q^c$, an occurrence of 01 in positions j and $j + 1$ of w contributes a weight of $q^{j(b+c)}$. This in turn implies that the bijection ϕ must be modified so that the part in λ corresponding to w_j must be at least $b + c$ larger than the part in μ corresponding to w_{j+1}. We need six different inequalities for the six possible juxtapositions of parts. Let $\phi_{a,b,c}$ be the modified bijection.

For example, if $m = k = 2$, $n = 1$, $a = 2$, $b = c = 1$, then the juxtaposed parts sizes must differ by

$$\begin{aligned} &2 \text{ for } \lambda\mu,\\ &1 \text{ for } \lambda\theta,\\ &2 \text{ for } \mu\lambda,\\ &3 \text{ for } \mu\theta,\\ &3 \text{ for } \theta\lambda,\\ &1 \text{ for } \theta\mu. \end{aligned}$$

The three possible triples (λ, μ, θ) whose weight is q^{12} are given below, along with result of the bijection $\phi_{2,1,1}$:

$$\begin{aligned} (22, 6, 11) &\to (10022, 31111),\\ (32, 5, 11) &\to (10022, 22111),\\ (43, 1, 22) &\to (00221, 21111). \end{aligned}$$

Corollary 3. *Let a, b and c be positive integers. The generating function for all triples of partitions (λ, μ, θ) without pairwise common parts, such that λ has m parts, μ has n parts, and θ has k parts, and any adjacent parts in the partition $\lambda \cup \mu \cup \theta$ of type*

1. *$\lambda\mu$ differ by $b + c$,*
2. *$\lambda\theta$ differ by c,*
3. *$\mu\lambda$ differ by a,*
4. *$\mu\theta$ differ by $a + c$,*
5. *$\theta\lambda$ differ by $a + b$,*
6. *$\theta\mu$ differ by b,*

is given by

$$\frac{q^{m+n+k}}{(q;q)_{m+n+k}}\Bigg(q^{a(n+k)+bk}\begin{bmatrix} m+n+k-1 \\ m-1,\, n,\, k\end{bmatrix}_{q^{a+b+c}} +$$
$$q^{b(m+k)+cm}\begin{bmatrix} m+n+k-1 \\ m,\, n-1,\, k\end{bmatrix}_{q^{a+b+c}} + q^{c(n+m)+an}\begin{bmatrix} m+n+k-1 \\ m,\, n,\, k-1\end{bmatrix}_{q^{a+b+c}}\Bigg).$$

In Theorem 3, if all $x_i = q$, the following theorem results. All subscripts are taken $\bmod N$.

Theorem 4. *The generating function for all N-tuples of integer partitions $(\lambda_1, \cdots, \lambda_N)$ without pairwise common parts, such that*

(a) *λ_i has a_i parts, $1 \le i \le N$,*
(b) *if the partition $\lambda_1 \cup \lambda_2 \cup \cdots \cup \lambda_N$ has adjacent parts bc, for $b \in \lambda_i$ and $c \in \lambda_j$, then $b - c \ge (i - j) \mod N$,*

is given by

$$\frac{q^f}{(q;q)_f}\begin{bmatrix} a_1+\cdots+a_N \\ a_1,\cdots,a_N\end{bmatrix}_{q^N}\frac{\sum_{i=1}^N q^{e_i}}{\sum_{i=0}^{N-1} q^{if}},$$

where $f=a_1+a_2+\cdots+a_N$, *and* $e_i=a_i+2a_{i+1}+\cdots+(N-1)a_{i+N-2}$.

5. Remarks.

MacMahon [5, §30] defined a statistic related to maj, denoted here by MAJ, which weights each descent by the amount of the descent. For example,

$$MAJ(20211201)=2*1+1*3+2*6=17,$$

because the descent 20 in positions $1,6$ are weighted by $2-0=2$, while the descent 21 in position 3 is weighted by $2-1=1$. Let MIN denote the analogous statistic using the ascents. Then MacMahon alludes [5, §40] to the following theorem for words with three letters.

Theorem 5. *For any non-negative integers m, n, and k we have*

$$\begin{aligned}\sum_{w\in W(m,n,k)} x^{MAJ(w)}y^{MIN(w)} = {}& x^{n+2k}\begin{bmatrix} m+n+k-1 \\ n\end{bmatrix}_{xy}\begin{bmatrix} m+k-1 \\ m-1\end{bmatrix}_{(xy)^2} \\ &+y^{m-k}\begin{bmatrix} m+n+k-1 \\ n-1\end{bmatrix}_{xy}\begin{bmatrix} m+k \\ m\end{bmatrix}_{(xy)^2}\frac{(xy)^{2k}+(xy)^{m+k}}{1+(xy)^{m+k}} \\ &+y^{2m+n}\begin{bmatrix} m+n+k-1 \\ n\end{bmatrix}_{xy}\begin{bmatrix} m+k-1 \\ m\end{bmatrix}_{(xy)^2}.\end{aligned}$$

If $x=y$, $y=1$ or $x=1$, the three terms in Theorem 5 sum to a single product (see [5, §38, §40]). The proof of Theorem 5 is identical to the proof of Theorem 1. We do not know a multivariable version of Theorem 5.

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