A q-PRODUCT TUTORIAL FOR A q-SERIES MAPLE PACKAGE

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ABSTRACT. This is a tutorial for using a new q-series Maple package. The package includes facilities for conversion between q-series and q-products and finding algebraic relations between q-series. Andrews found an algorithm for converting a q-series into a product. We provide an implementation. As an application we are able to effectively find finite q-product factorisations when they exist thus answering a question of Andrews. We provide other applications involving factorisations into theta-functions and eta-products.

Dedicated to George E. Andrews on the occasion of his 60th Birthday

CONTENTS

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1. Introduction

In the study of q-series one is quite often interested in identifying generating functions as infinite products. The classic example is the Rogers-Ramanujan identity

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-1})(1-q^{5n-4})}.$$

Here we have used the notation in (2.2). It can be shown that the left-side of this identity is the generating function for partitions whose parts differ by at least two. The identity is equivalent to saying the number of such partitions of n is equinumerous with partitions of n into parts congruent to $\pm 1 \pmod 5$.

The main goals of the package are to provide facility for handling the following problems.

1. Conversion of a given q-series into an "infinite" product.
2. Factorization of a given rational function into a finite q-product if one exists.
3. Find algebraic relations (if they exist) among the q-series in a given list.

A q-product has the form

$$\prod_{j=1}^{N} (1-q^j)^{b_j}, \tag{1.1}$$

where the b_j are integers.

In [4, §10.7], George Andrews also considered Problems 1 and 2, and asked for an easily accessible implementation. We provide implementations as well as considering factorisations into theta-products and eta-products. The package provides some basic functions for computing q-series expansions of eta functions, theta functions, Gaussian polynomials and q-products. It also has a function for sifting out coefficients of a q-series. It also has the basic infinite product identities: the triple product identity, the quintuple product identity and Winquist's identity.

1.1. Installation instructions.

The `qseries` package can be downloaded via the WWW. First use your favorite browser to access the URL:

`http://www.math.ufl.edu/~frank/qmaple.html` then follow the directions on that page. There are two versions: one for UNIX and one for WINDOWS.

2. Basic functions

We describe the basic functions in the package which are used to build q-series.

2.1. Finite q-products.

2.1.1. *Rising q-factorial.* `aqprod(a,q,n)` returns the product

$$(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}). \tag{2.2}$$

We also use the notation

$$(a;q)_\infty = \prod_{n=1}^{\infty}(1-aq^{n-1}).$$

2.1.2. *Gaussian polynomials.* When $0 \le m \le n$, `qbin(q,m,n)` returns the Gaussian polynomial (or q-binomial coefficient)

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{(q)_n}{(q)_m(q)_{n-m}},$$

otherwise it returns 0.

2.2. Infinite products.

2.2.1. *Dedekind eta products.* Suppose $\Re\tau > 0$, and $q = \exp(2\pi i\tau)$. The Dedekind eta function [27, p. 121] is defined by

$$\begin{aligned}\eta(\tau) &= \exp(\pi i\tau/12)\prod_{n=1}^{\infty}(1-\exp(2\pi i n\tau))\\ &= q^{1/24}\prod_{n=1}^{\infty}(1-q^n).\end{aligned}$$

`etaq(q,k,T)` returns the q-series expansion (up to q^T) of the eta product

$$\prod_{n=1}^{\infty}(1-q^{kn}).$$

This corresponds to the eta function $\eta(k\tau)$ except for a power of q. Eta products occur frequently in the study of q-series. For example, the generating function for $p(n)$, the number of partitions of n, can be written as

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{\prod_{n=1}^{\infty}(1-q^n)}.$$

See [1, pp. 3–4]. The generating function for the number of partitions of n that are p-cores [19], $a_p(n)$, can be written

$$\sum_{n=0}^{\infty} a_p(n) q^n = \prod_{n=1}^{\infty} \frac{(1-q^{pn})^p}{(1-q^n)}.$$

Recently, Granville and Ono [21] were able to prove a long-standing conjecture in group representation theory using elementary and function-theoretic properties of the eta product above.

2.2.2. *Theta functions.* Jacobi [24, Vol I, pp. 497–538] defined four theta functions $\theta_i(z,q)$, $i = 1,2,3,4$. See also [42, Ch. XXI]. Each theta function can be written in terms of the others using a simple change of variables. For this reason, it is common to define

$$\theta(z,q) := \sum_{n=-\infty}^{\infty} z^n q^{n^2}.$$

`theta(z,q,T)` returns the truncated theta-series

$$\sum_{i=-T}^{T} z^i q^{i^2}.$$

The case $z = 1$ of Jacobi's theta functions occurs quite frequently. We define

$$\theta_2(q) := \sum_{n=-\infty}^{\infty} q^{(n+1)^2/2}$$

$$\theta_3(q) := \sum_{n=-\infty}^{\infty} q^{n^2}$$

$$\theta_4(q) := \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2}$$

`theta2(q,t)`, `theta3(q,t)`, `theta4(q,t)` (resp.) returns the q-series expansion to order $O\left(q^T\right)$ of $\theta_2(q)$, $\theta_3(q)$, $\theta_4(q)$ respectively. Let a, and b be positive integers and suppose $|q| < 1$. Infinite products of the form

$$(q^a; q^b)_\infty (q^{b-a}; q^b)_\infty$$

occur quite frequently in the theory of partitions and q-series. For example the right side of the Rogers-Ramanujan identity is the reciprocal of the product with $(a,b) = (1,5)$. In (3.4) we will see how the function `jacprodmake` can be used to identify such products.

3. Product Conversion

In [1, p. 233], [4, §10.7] there is a very nice and useful algorithm for converting a q-series into an infinite product. Any given q-series may be written formally as an infinite product

$$1 + \sum_{n=1}^{\infty} b_n q^n = \prod_{n=1}^{\infty} (1 - q^n)^{-a_n}.$$

Here we assume the b_n are integers. By taking the logarithmic derivative of both sides we can obtain the recurrence

$$n b_n = \sum_{j=1}^{n} b_{n-j} \sum_{d|j} d a_d.$$

Letting $a_n = 1$ we obtain the well-known special case

$$np(n) = \sum_{j=1}^{n} p(n-j)\sigma(j).$$

We can also easily construct a recurrence for the a_n from the recurrence above.

The function `prodmake` is an implementation of Andrews' algorithm. Other related functions are `etamake` and `jacprodmake`.

3.1. prodmake.

`prodmake(f,q,T)` converts the q-series f into an infinite product that agrees with f to $O(q^T)$. Let's take a look at the left side of the Rogers-Ramanujan identity.

```
> with(qseries):
> x:=1:
> for n from 1 to 8 do
        x := x + q^(n*n)/aqprod(q,q,n):
  od:
> x := series(x,q,50);
```

$$x := 1 + q + q^2 + q^3 + 2q^4 + 2q^5 + 3q^6 + 3q^7 + 4q^8 + 5q^9 + 6q^{10} + 7q^{11} + 9q^{12} + 10q^{13} + 12q^{14} + 14q^{15} + 17q^{16} + 19q^{17} + 23q^{18} + 26q^{19} + 31q^{20} + 35q^{21} + 41q^{22} + 46q^{23} + 54q^{24} + 61q^{25} + 70q^{26} + 79q^{27} + 91q^{28} + 102q^{29} + 117q^{30} + 131q^{31} + 149q^{32} + 167q^{33} + 189q^{34} + 211q^{35} + 239q^{36} + 266q^{37} + 299q^{38} + 333q^{39} + 374q^{40} + 415q^{41} + 465q^{42} + 515q^{43} + 575q^{44} + 637q^{45} + 709q^{46} + 783q^{47} + 871q^{48} + 961q^{49} + O(q^{50})$$

```
> prodmake(x,q,40);
```

$$1/\left((1-q)(1-q^4)(1-q^6)(1-q^9)(1-q^{11})(1-q^{14})(1-q^{16})(1-q^{19})(1-q^{21})(1-q^{24})(1-q^{26})(1-q^{29})(1-q^{31})(1-q^{34})(1-q^{36})(1-q^{39})\right)$$

We have rediscovered the right side of the Rogers-Ramanujan identity!

Exercise 1. Find (and prove) a product form for the q-series

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_{2n}}.$$

The identity you find is originally due to Rogers [34](p.330). See also Andrews [2](pp.38–39) for a list of some related papers.

3.2. `qfactor`. The function `qfactor` is a version of `prodmake`. `qfactor(f,T)` attempts to write a rational function f in q as a q-product, i.e., as a product of terms of the form $(1 - q^i)$. The second argument T is optional. It specifies an an upper bound for the exponents of q that occur in the product. If T is not specified it is given a default value of $4d+3$ where d is the maximum of the degree in q of the numerator and denominator. The algorithm is quite simple. First the function is factored as usual, and then it uses `prodmake` to do further factorisation into q-products. Thus even if only part of the function can be written as a q-product qfactor is able to find it.

As an example we consider some rational functions $T(r,h)$ introduced by Andrews [4](p.14) to explain Rogers's [34] first proof of the Rogers-Ramanujan identities. The $T(r,n)$ are defined recursively as follows:

$$T(r,0) = 1, \tag{3.3}$$

$$T(r,1) = 0, \tag{3.4}$$

$$T(r,N) = -\sum_{1\le 2j\le N} \begin{bmatrix} r+2j \\ j \end{bmatrix} T(r+2j, N-2j). \tag{3.5}$$

```
> T:=proc(r,j)
>    option remember;
>    local x,k;
>    x:=0;
>    if j=0 or j=1 then
>      RETURN((j-1)^2):
>    else
>       for k from 1 to floor(j/2) do
>           x:=x-qbin(q,k,r+2*k)*T(r+2*k,j-2*k);
>        od:
>        RETURN(expand(x));
>    fi:
```

```
> end:
>  t8:=T(8,8);
```

$$t8 := 3\,q^9 + 21\,q^{16} + 36\,q^{25} + 9\,q^{36} + q^6 + q^7 + 2\,q^8 + 5\,q^{10} + 6\,q^{11} + 9\,q^{12} + 11\,q^{13} + 15\,q^{14} + 17\,q^{15} + 33\,q^{28} + 34\,q^{27} + 37\,q^{26} + 38\,q^{24} + 36\,q^{23} + 37\,q^{22} + 34\,q^{21} + 33\,q^{20} + 29\,q^{19} + 28\,q^{18} + 23\,q^{17} + 5\,q^{38} + 6\,q^{37} + 11\,q^{35} + 15\,q^{34} + 17\,q^{33} + 21\,q^{32} + 23\,q^{31} + 28\,q^{30} + 29\,q^{29} + 3\,q^{39} + q^{42} + q^{41} + 2\,q^{40}$$

```
>  factor(t8);
```

$$q^6\left(q^4+q^3+q^2+q+1\right)\left(q^4-q^3+q^2-q+1\right)\left(q^{10}+q^9+q^8+q^7+q^6+q^5+q^4+q^3+q^2+q+1\right)\left(q^4+1\right)\left(q^6+q^3+1\right)\left(q^8+1\right)$$

```
>  qfactor(t8,20);
```

$$\frac{(1-q^9)\,(1-q^{10})\,(1-q^{11})\,(1-q^{16})\,q^6}{(1-q)\,(1-q^2)\,(1-q^3)\,(1-q^4)}$$

Observe how we used **factor** to factor **t8** into cyclotomic polynomials. However, **qfactor** was able to factor **t8** as a q-product. We see that

$$T(8,8) = \frac{(q^9;q)_3(1-q^{16})q^6}{(q;q)_4}.$$

> **Exercise 2.** Use **qfactor** to factorize $T(r,n)$ for different values of r and n. Then write $T(r,n)$ (defined above) as a q-product for general r and n.

For our next example we examine the sum

$$\sum_{k=-\infty}^{\infty}(-1)^k q^{k(3k+1)/2}\begin{bmatrix} b+c \\ c+k \end{bmatrix}\begin{bmatrix} c+a \\ a+k \end{bmatrix}\begin{bmatrix} a+b \\ b+k \end{bmatrix}.$$

```
> dixson:=proc(a,b,c,q)
>    local x,k,y;
>    x:=0:  y:=min(a,b,c):
>    for k from -y to y do
>      x:=x+(-1)^k*q^(k*(3*k+1)/2)*
>      qbin(q,c+k,b+c)*qbin(q,a+k,c+a)*qbin(q,b+k,a+b);
>    od:
>    RETURN(x):
> end:
```

```
> dx := expand(dixson(5,5,5,q)):
```

```
> qfactor(dx,20);
```

$$(1-q^6)(1-q^7)(1-q^8)(1-q^9)(1-q^{10})(1-q^{11})(1-q^{12})\\(1-q^{13})(1-q^{14})(1-q^{15})/((1-q)^2(1-q^2)^2(1-q^3)^2(1-q^4)^2\\(1-q^5)^2)$$

We find that

$$\sum_{k=-\infty}^{\infty}(-1)^k q^{k(3k+1)/2}\begin{bmatrix}10\\5+k\end{bmatrix}^3=\frac{(q^6;q)_{10}}{(q;q)_5^2}. \tag{3.6}$$

Exercise 3. Write the sum

$$\sum_{k=-\infty}^{\infty}(-1)^k q^{k(3k+1)/2}\begin{bmatrix}2a\\a+k\end{bmatrix}^3$$

as a q-product for general integral a. The identity you obtain is a special case of [4](Eq.(4.24), p.38).

3.3. `etamake`. Recall from (2.2.1) that `etaq` is the function to use for computing q-expansions of eta products. If one wants to apply the theory of modular forms to q-series it is quite useful to determine whether a given q-series is a product of eta functions. The function in the package for doing this conversion is `etamake`. `etamake(f,q,T)` will write the given q-series f as a product of eta functions which agrees with f up to q^T. As an example, let's see how we can write the theta functions as eta products.

```
> theta2(q,100)/q^(1/4);
```

$$2q^{132}+2q^{110}+2q^{90}+2q^{72}+2q^{56}+2q^{42}+2q^{30}+2q^{20}+2q^{12}+2q^6\\+2q^2+2+q^{156}$$

```
> etamake(",q,100);
```

$$2\frac{\eta(4\tau)^2}{q^{1/4}\eta(2\tau)}$$

```
> theta3(q,100);
```

$$2q^{121}+2q^{100}+2q^{81}+2q^{64}+2q^{49}+2q^{36}+2q^{25}+2q^{16}+2q^9+2q^4\\+2q+1$$

```
> etamake(",q,100);
```

$$\frac{\eta(2\tau)^5}{\eta(4\tau)^2\eta(\tau)^2}$$

```
> theta4(q,100);
```

$$-2\,q^{121}+2\,q^{100}-2\,q^{81}+2\,q^{64}-2\,q^{49}+2\,q^{36}-2\,q^{25}+2\,q^{16}-2\,q^{9}+2\,q^{4}-2\,q+1$$

```
> etamake(",q,100);
```

$$\frac{\eta(\,\tau\,)^2}{\eta(\,2\,\tau\,)}$$

We are led to the well-known identities:

$$\theta_2(q) = 2\frac{\eta(4\tau)^2}{\eta(2\tau)},$$
$$\theta_3(q) = \frac{\eta(2\tau)^5}{\eta(4\tau)^2\eta(\tau)^2},$$
$$\theta_4(q) = \frac{\eta(\tau)^2}{\eta(2\tau)}.$$

The idea of the algorithm is quite simple. Given a q-series f (say with leading coefficient 1) one just keeps recursively multiplying by powers of the right eta function until the desired terms agree. For example, suppose we are given a q-series

$$f = 1 + cq^k + \cdots .$$

Then the next step is to multiply by `etaq(q,k,T)^(-c)`.

Exercise 4. Define the q-series

$$a(q) := \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} q^{n^2+nm+m^2}$$
$$b(q) := \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} \omega^{n-m}\, q^{n^2+nm+m^2}$$
$$c(q) := \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} q^{(n+1/3)^2+(n+1/3)(m+1/3)+(m+1/3)^2}$$

where $\omega = \exp(2\pi i/3)$. Two of the three functions above can be written as eta products. Can you find them?
Hint: It would be wise to define

```
> omega := RootOf(z^2 + z + 1 = 0);
```

See [12] for the answer and much more.

3.4. `jacprodmake`. In (2.2.2) we observed that the right side of the Rogers-Ramanujan identity could be written in terms of a Jacobi product. The function `jacprodmake` converts a q-series into a Jacobi-type product if one exists. Given a q-series f, `jacprodmake(f,q,T)` attempts to convert f into a product of theta functions that agrees with f to order $O(q^T)$. Each theta-function has the form $\mathrm{JAC}(a,b,\infty)$, where a, b are integers and $0 \le a < b$. If $0 < a$, then $\mathrm{JAC}(a,b,\infty)$ corresponds to the theta-product

$$(q^a;q^b)_\infty(q^{b-a};q^b)_\infty(q^b;q^b)_\infty.$$

We call this a theta product because it is $\theta(-q^{(b-2a)/2},q^{b/2})$. The `jacprodmake` function is really a variant of `prodmake`. It involves using `prodmake` to compute the sequence of exponents and then searching for periodicity.

If $a=0$, then $\mathrm{JAC}(0,b,\infty)$ corresponds to the eta-product

$$(q^b;q^b)_\infty.$$

We note that this product can also be thought of as a theta-product since it can be written

$$(q^b;q^b)_\infty = (q^b;q^{3b})_\infty(q^{2b};q^{3b})_\infty(q^{3b};q^{3b})_\infty.$$

Let's re-examine the Rogers-Ramanujan identity.

```
> with(qseries):
> x:=1:
> for n from 1 to 8 do
>     x:=x+q^(n*n)/aqprod(q,q,n):
> od:
> x:=series(x,q,50):
> y:=jacprodmake(x,q,40);
```

$$y := \frac{JAC(0,5,\infty)}{JAC(1,5,\infty)}$$

```
> z:=jac2prod(y);
```

$$z := \frac{1}{(q,q^5)_\infty(q^4,q^5)_\infty}$$

Note that we were able to observe that the left side of the Rogers-Ramanujan identity (at least up through q^{40}) can be written as a quotient of theta functions. We used the function `jac2prod`, to simplify the result and get it into a more recognizable form. The function `jac2prod(jacexpr)` converts a product of theta functions into q-product form; i.e., as a product of functions of the form $(q^a;q^b)_\infty$. Here `jacexpr` is a product (or quotient) of terms $\mathrm{JAC}(i,j,\infty)$, where i, j are integers and $0 \le i < j$.

A related function is `jac2series`. This converts a Jacobi-type product into a form better for computing its q-series. It simply replaces each Jacobi-type product with its corresponding theta-series.

```
> with(qseries):
> x:=0:
> for n from 0 to 10 do
  x := x + q^(n*(n+1)/2)*aqprod(-q,q,n)/aqprod(q,q,2*n+1):
  od:
> x:=series(x,q,50):
> jacprodmake(x,q,50);
```

$$\mathrm{JAC}(0,\,14,\,\infty)^6 \Big/ (\mathrm{JAC}(1,\,14,\,\infty)^2\,\mathrm{JAC}(3,\,14,\,\infty)\,\mathrm{JAC}(4,\,14,\,\infty)\\ \mathrm{JAC}(5,\,14,\,\infty)\,\mathrm{JAC}(6,\,14,\,\infty)\sqrt{\frac{\mathrm{JAC}(7,\,14,\,\infty)}{\mathrm{JAC}(0,\,14,\,\infty)}})$$

```
> jac2series(",500);
```

$$(q^{364} - q^{210} + q^{98} - q^{28} + 1 - q^{14} + q^{70} - q^{168} + q^{308} - q^{490})^6 \Big/ ((\\ -q^{621} + q^{496} - q^{385} + q^{288} - q^{205} + q^{136} - q^{81} + q^{40} - q^{13} + 1 - q\\ + q^{16} - q^{45} + q^{88} - q^{145} + q^{216} - q^{301} + q^{400} - q^{513})^2(-q^{603} + q^{480}\\ - q^{371} + q^{276} - q^{195} + q^{128} - q^{75} + q^{36} - q^{11} + 1 - q^3 + q^{20} - q^{51}\\ + q^{96} - q^{155} + q^{228} - q^{315} + q^{416} - q^{531})(-q^{594} + q^{472} - q^{364} + q^{270}\\ - q^{190} + q^{124} - q^{72} + q^{34} - q^{10} + 1 - q^4 + q^{22} - q^{54} + q^{100} - q^{160}\\ + q^{234} - q^{322} + q^{424} - q^{540})(-q^{585} + q^{464} - q^{357} + q^{264} - q^{185} + q^{120}\\ - q^{69} + q^{32} - q^9 + 1 - q^5 + q^{24} - q^{57} + q^{104} - q^{165} + q^{240} - q^{329}\\ + q^{432} - q^{549})(-q^{576} + q^{456} - q^{350} + q^{258} - q^{180} + q^{116} - q^{66} + q^{30}\\ - q^8 + 1 - q^6 + q^{26} - q^{60} + q^{108} - q^{170} + q^{246} - q^{336} + q^{440} - q^{558})\\ ((-2\,q^{567} + 2\,q^{448} - 2\,q^{343} + 2\,q^{252} - 2\,q^{175} + 2\,q^{112} - 2\,q^{63} + 2\,q^{28}\\ - 2\,q^7 + 1)/(\\ q^{364} - q^{210} + q^{98} - q^{28} + 1 - q^{14} + q^{70} - q^{168} + q^{308} - q^{490}))^{1/2})$$

It seems that the q-series

$$\sum_{n\geq 0} \frac{(-q;q)_n q^{n(n+1)/2}}{(q;q)_{2n+1}}$$

can be written as Jacobi-type product. Assuming that this is the case we used `jac2series` to write this q-series in terms of theta-series at least up to q^{500}. This should provide an efficient method for computing the q-series expansion and also for computing the function at particular values of q.

Exercise 5. Use `jacprodmake` and `jac2series` to compute the q-series expansion of

$$\sum_{n\geq 0}\frac{(-q;q)_n q^{n(3n+1)/2}}{(q;q)_{2n+1}}$$

up to q^{1000}, assuming it is Jacobi-type product. Can you identify the infinite product? This function occurs in Slater's list [36](Eq.(46), p.156).

4. The Search for Relations

The functions for finding relations between q-series are `findhom`, `findhomcombo`, `findnonhom`, `findnonhomcombo`, and `findpoly`.

4.1. findhom.

If the q-series one is concerned with are modular forms of a particular weight, then theoretically these functions will satisfy homogeneous polynomial relations. See [18, p. 263], [9] for more details and examples. The function `findhom(L,q,n,topshift)` returns a set of potential homogeneous relations of order n among the q-series in the list L. The value of `topshift` is usually taken to be zero. However if it appears that spurious relations are being generated then a higher value of `topshift` should be taken.

The idea is to convert this into a linear algebra problem. This program generates a list of monomials of degree n of the functions in the given list of q-series L. The q-expansion (up to a certain point) of each monomial is found and converted into a row vector of a matrix. The set of relations is then found by computing the kernel of the transpose of this matrix. As an example, we now consider relations between the theta functions $\theta_3(q)$, $\theta_4(q)$, $\theta_3(q^2)$, and $\theta_4(q^2)$.

```
> with(qseries):
> findhom([theta3(q,100),theta4(q,100),theta3(q^2,100),
theta4(q^2,100)],q,1,0);
```

```
# of terms , 25
-----RELATIONS-----of order---, 1
```

$$\{\{\}\}$$

```
> findhom([theta3(q,100),theta4(q,100),theta3(q^2,100),
theta4(q^2,100)],q,2,0);
```

```
# of terms , 31
-----RELATIONS-----of order---, 2
```

$$\left\{{X_1}^2+{X_2}^2-2\,{X_3}^2, -X_1X_2+{X_4}^2\right\}$$

From the session above we see that there is no linear relation between the functions $\theta_3(q)$, $\theta_4(q)$, $\theta_3(q^2)$ and $\theta_4(q^2)$. However, it appears that there are two quadratic relations:

$$\theta_3(q^2) = \left(\frac{\theta_3^2(q) + \theta_4^2(q)}{2}\right)^{1/2}$$

and

$$\theta_4(q^2) = \left(\theta_3^2(q)\theta_4^2(q)\right)^{1/2}.$$

This is Gauss' parametrization of the arithmetic-geometric mean iteration. See [13, Ch. 2] for details.

> **Exercise 6.** Define $a(q)$, $b(q)$ and $c(q)$ as in **Exercise 2**. Find homogeneous relations between the functions $a(q)$, $b(q)$, $c(q)$, $a(q^3)$, $b(q^3)$, $c(q^3)$. In particular, try to get $a(q^3)$ and $b(q^3)$ in terms of $a(q)$, $b(q)$. See [12] for more details. These results lead to a cubic analog of the AGM due to Jon and Peter Borwein [10], [11].

4.2. `findhomcombo`. The function `findhomcombo` is a variant of `findhom`. Suppose f is a q-series and L is a list of q-series. `findhomcombo(f,L,q,n,topshift,etaoption)` tries to express f as a homogeneous polynomial in the members of L. If `etaoption=yes` then each monomial in the combination is "converted" into an eta-product using `etamake`.

We illustrate this function with certain Eisenstein series. For p an odd prime define

$$\chi(m) = \left(\frac{m}{p}\right) \qquad \text{(the Legendre symbol)}.$$

Suppose k is an integer, $k \geq 2$, and $(p-1)/2 \equiv k \pmod 2$. Define the Eisenstein series

$$U_{p,k}(q) := \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \chi(m) n^{k-1} q^{mn}.$$

Then $U_{p,k}$ is a modular form of weight k and character χ for the congruence subgroup $\Gamma_0(p)$. See [28], [20] for more details. The classical result is the following identity found by Ramanujan [32, Eq. (1.52), p. 354]:

$$U_{5,2} = \frac{\eta(5\tau)^5}{\eta(\tau)}.$$

Kolberg [28] has found many relations between such Eisenstein series and certain eta products. The eta function $\eta(\tau)$ is a modular form of weight $\frac{1}{2}$ [27, p. 121]. Hence the modular forms

$$B_1 := \frac{\eta(5\tau)^5}{\eta(\tau)}, \qquad B_2 := \frac{\eta(\tau)^5}{\eta(5\tau)}$$

are modular forms of weight $\frac{(5-1)}{2} = 2$. In fact, it can be shown that they are modular forms on $\Gamma_0(5)$ with character $\left(\frac{\cdot}{5}\right)$. We might therefore expect that $U_{5,6}$ can be written as a homogeneous cubic polynomial in B_1, B_2. We write a short MAPLE program to compute the Eisenstein series $U_{p,k}$.

```
> with(numtheory):
> UE:=proc(q,k,p,trunk)
>  local x,m,n:
>  x:=0:
>  for m from 1 to trunk do
>   for n from 1 to trunk/m do
>    x:=x + L(m,p)*n^(k-1)*q^(m*n):
>   od:
>  od:
> end:
```

The function UE(q,k,p,trunk) returns the q-expansion of $U_{p,k}$ up through q^{trunk}. We note that L(m,p) returns the Legendre symbol $\left(\dfrac{m}{p}\right)$. We are now ready to study $U_{5,6}$.

```
> with(qseries):
> f := UE(q,6,5,50):
> B1 := etaq(q,1,50)^5/etaq(q,5,50):
> B2 := q*etaq(q,5,50)^5/etaq(q,1,50):
> findhomcombo(f,[B1,B2],q,3,0,yes);
                    # of terms , 25
  -----possible linear combinations of degree------, 3
```

$$\left\{\eta(5\,\tau)^3\eta(\tau)^9 + 40\,\eta(5\,\tau)^9\eta(\tau)^3 + 335\,\frac{\eta(5\,\tau)^{15}}{\eta(\tau)^3}\right\}$$

$$\left\{{X_1}^2X_2 + 40\,X_1{X_2}^2 + 335\,{X_2}^3\right\}$$

It would appear that

$$U_{5,6} = \eta(5\,\tau)^3\eta(\tau)^9 + 40\,\eta(5\,\tau)^9\eta(\tau)^3 + 335\,\frac{\eta(5\,\tau)^{15}}{\eta(\tau)^3}.$$

The proof is a straightforward exercise using the theory of modular forms.

Exercise 7. Define the following eta products:

$$C_1 := \frac{\eta(7\tau)^7}{\eta(\tau)}, \qquad C_2 := \eta(\tau)^3\,\eta(7\tau)^3, \qquad C_3 := \frac{\eta(\tau)^7}{\eta(7\tau)}.$$

What is the weight of these modular forms?

Write $U_{7,3}$ in terms of C_1, C_2, C_3.

The identity that you should find was originally due to Ramanujan. Also see Fine [15, p. 159] and [19, Eq. (5.4)].

If you are ambitious find $U_{7,9}$ in terms of C_1, C_2, C_3.

4.3. **findnonhom.** In section 4.1 we introduced the function `findhom` to find homogeneous relations between q-series. The nonhomogeneous analog is `findnonhom`.

The syntax of `findnonhom` is the same as `findhom`. Typically (but not necessarily) `findhom` is used to find relations between modular forms of a certain weight. To find relations between modular functions we would use `findnonhom`. We consider an example involving theta functions.

```
> with(qseries):
> F := q -> theta3(q,500)/theta3(q^5,100):
> U := 2*q*theta(q^10,q^25,5)/theta3(q^25,20);
```

$$U := \frac{2\,q\,(q^{575}+q^{360}+q^{195}+q^{80}+q^{15}+1+q^{35}+q^{120}+q^{255}+q^{440}+q^{675})}{2\,q^{625}+2\,q^{400}+2\,q^{225}+2\,q^{100}+2\,q^{25}+1}$$

```
> EQNS := findnonhom([F(q),F(q^5),U],q,3,20);
                    # of terms , 61
          -----RELATIONS-----of order---, 3
```

$$\mathtt{EQNS} := \left\{-1 - X_1X_2X_3 + {X_2}^2 + {X_3}^2 + X_3\right\}$$

```
> ANS:=EQNS[1];
```

$$\mathtt{ANS} := -1 - X_1X_2X_3 + {X_2}^2 + {X_3}^2 + X_3$$

```
> CHECK := subs(X[1]=F(q),X[2]=F(q^5),X[3]=U,ANS):
> series(CHECK,q,500);
```

$$O\left(q^{500}\right)$$

We define
$$F(q) := \frac{\theta_3(q)}{\theta_3(q^5)},$$
and
$$U(q) := 2\frac{\sum\limits_{n=-\infty}^{\infty} q^{25n^2+10n+1}}{\theta_3(q^{25})}.$$
We note that $U(q)$ and $F(q)$ are modular functions since they are ratios of theta series. From the session above we see that it appears that
$$1 + F(q)F(q^5)U(q) = F(q^5)^2 + U(q)^2 + U(q).$$
Observe how we were able to verify this equation to high order. When `findnonhom` returns a set of relations the variable X has been declared *global*. This is so we can manipulate the relations. It this way we were able to assign `ANS` to the relation found and then use `subs` and `series` to check it to order $O\left(q^{500}\right)$.

4.4. `findnonhomcombo`. The syntax of `findnonhomcombo` is the same as `findhomcombo`. We consider an example involving eta functions. First we define
$$\xi := \frac{\eta(49\,\tau)}{\eta(\tau)},$$
and
$$T := \left(\frac{\eta(7\,\tau)}{\eta(\tau)}\right)^4,$$
Using the theory of modular functions it can be shown that one must be able to write T^2 in terms of T and powers of ξ. We now use `findnonhomcombo` to get T^2 in terms of ξ and T.

```
>  xi:=q^2*etaq(q,49,100)/etaq(q,1,100):
>  T:=q*(etaq(q,7,100)/etaq(q,1,100))^4:
>  findnonhomcombo(T^2,[T,xi],q,7,-15);
                    # of terms , 42
                 matrix is , 37, x, 42
   -----possible linear combinations of degree------, 7
```

$$\left\{147\,{X_2}^5 + 343\,{X_2}^7 + 343\,{X_2}^6 + X_2 + 49\,{X_2}^4 + 49\,X_1{X_2}^3 + 7\,X_1X_2 \right.$$
$$\left. +21\,{X_2}^3 + 7\,{X_2}^2 + 35\,X_1{X_2}^2\right\}$$

Then it seems that
$$T^2 = \left(35\,\xi^2 + 49\,\xi^3 + 7\,\xi\right)T$$
$$+ 343\,\xi^7 + 343\,\xi^6 + 147\,\xi^5 + 49\,\xi^4 + 21\,\xi^3 + 7\,\xi^2 + \xi.$$

This is the modular equation used by Watson[41] to prove Ramanujan's partition congruences for powers of 7. Also see [5] and [26], and see [16] for an elementary treatment.

Exercise 8. Define

$$\xi := \frac{\eta(25\,\tau)}{\eta(\tau)},$$

and

$$T := \left(\frac{\eta(5\,\tau)}{\eta(\tau)}\right)^6.$$

Use `findnonhomcombo` to express T as a polynomial in ξ of degree 5. The modular equation you find was used by Watson to prove Ramanujan's partition congruences for powers of 5. See [23] for an elementary treatment.

Exercise 9. Define $a(q)$ and $c(q)$ as in **Exercise 2**. Define

$$x(q) := \frac{c(q)^3}{a(q)^3},$$

and the classical Eisenstein series (usually called E_3; see [35, p. 93])

$$N(q) := 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1-q^n}.$$

Use `findnonhomcombo` to express $N(q)$ in terms of $a(q)$ and $x(q)$.

HINT: $N(q)$ is a modular form of weight 6 and $a(q)$ and $c(q)$ are modular forms of weight 1. See [8] for this result and many others.

4.5. `findpoly`. The function `findpoly` is used to find a polynomial relation between two given q-series with degrees specified.

`findpoly(x,y,q,deg1,deg2,check)` returns a possible polynomial in X, Y (with corresponding degrees deg1, deg2) which is satisfied by the q-series x and y.

If check is assigned then the relation is checked to $O\left(q^{\text{check}}\right)$.

We illustrate this function with an example involving theta functions and the function $a(q)$ and $c(q)$ encountered in **Exercises 2** and **7**. It can be shown that

$$a(q) = \theta_3(q)\theta_3(q^3) + \theta_2(q)\theta_2(q^3).$$

See [12] for details. This equation provides a better way of computing the q-series expansion of $a(q)$ than the definition. In **Exercise 2** you would have found that

$$c(q) = 3\ \frac{\eta^3(3\,\tau)}{\eta(\tau)}.$$

See [12] for a proof. Define

$$y := \frac{c^3}{a^3},$$

and

$$x := \left(\frac{\theta_2(q)}{\theta_2(q^3)}\right)^2 + \left(\frac{\theta_3(q)}{\theta_3(q^3)}\right)^2.$$

We use `findpoly` to find a polynomial relation between x and y.

```
>  with(qseries):
>  x1 := radsimp(theta2(q,100)^2/theta2(q^3,40)^2):
>  x2 := theta3(q,100)^2/theta3(q^3,40)^2:
>  x := x1+x2:
>  c := q*etaq(q,3,100)^9/etaq(q,1,100)^3:
>  a := radsimp(theta3(q,100)*theta3(q^3,40)+theta2(q,100)
  *theta2(q^3,40)):
>  c := 3*q^(1/3)*etaq(q,3,100)^3/etaq(q,1,100):
>  y := radsimp(c^3/a^3):
>  P1:=findpoly(x,y,q,3,1,60);
WARNING: X,Y are global.
```

`dims , 8, 18`

`The polynomial is`

$$(X+6)^3\,Y - 27\,(X+2)^2$$

`Checking to order, 60`

$$\mathrm{O}\left(q^{59}\right)$$

$$\mathtt{P1} := (X+6)^3\,Y - 27\,(X+2)^2$$

It seems that x and y satisfy the equation

$$p(x,y) = (x+6)^3\,y - 27(x+2)^2 = 0.$$

Therefore it would seem that

$$\frac{c^3}{a^3} = 27\,\frac{(x+2)^2}{(x+6)^3}.$$

See [8, pp. 4237–4240] for more details.

Exercise 10. Define

$$m := \left(\frac{\theta_3(q)}{\theta_3(q^3)}\right)^2.$$

Use `polyfind` to find $y = \frac{c^3}{a^3}$ as a rational function in m. The answer is Eq.(12.8) in [8].

5. Sifting coefficients

Suppose we are given a q-series

$$A(q) = \sum_{n=0}^{\infty} a_n q^n.$$

Occasionally it will turn out the generating function

$$\sum_{n=0}^{\infty} a_{mn+r} q^n$$

will have a very nice form. A famous example for $p(n)$ is due to Ramanujan:

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\prod_{n=1}^{\infty}\frac{(1-q^{5n})^6}{(1-q^n)^5}.$$

See [1, Cor. 10.6]. In fact, G.H. Hardy and Major MacMahon [31, p. xxxv] both agreed that this is Ramanujan's most beautiful identity.

Suppose s is the q-series

$$\sum a_i q^i + \mathrm{O}(q^T)$$

then `sift(s,q,n,k,T)` returns the q-series

$$\sum a_{ni+k} q^i + \mathrm{O}(q^{T/n}).$$

We illustrate this function with another example from the theory of partitions. Let $pd(n)$ denote the number of partitions of n into distinct parts. Then it is well known that

$$\sum_{n=0}^{\infty} pd(n)q^n = \prod_{n=1}^{\infty}(1+q^n) = \prod_{n=1}^{\infty}\frac{(1-q^{2n})}{(1-q^n)}.$$

We now examine the generating function of $pd(5n+1)$ in MAPLE.

```
> PD:=series(etaq(q,2,200)/etaq(q,1,200),q,200):
```

```
> sift(PD,q,5,1,199);
```

$$1+4\,q+5010688\,q^{26}+53250\,q^{15}+668\,q^{7}+12\,q^{2}+165\,q^{5}
+12076\,q^{12}+1087744\,q^{22}+109420549\,q^{35}+76\,q^{4}+32\,q^{3}
+340\,q^{6}+1260\,q^{8}+2304\,q^{9}+4097\,q^{10}+7108\,q^{11}+20132\,q^{13}
+32992\,q^{14}+84756\,q^{16}+133184\,q^{17}+206848\,q^{18}+317788\,q^{19}
+728260\,q^{21}+20792120\,q^{30}+2368800\,q^{24}+483330\,q^{20}
+1611388\,q^{23}+3457027\,q^{25}+7215644\,q^{27}+10327156\,q^{28}
+14694244\,q^{29}+29264960\,q^{31}+40982540\,q^{32}+57114844\,q^{33}
+79229676\,q^{34}+150473568\,q^{36}+206084096\,q^{37}
+281138048\,q^{38}+382075868\,q^{39}$$

```
> PD1:=":
> etamake(PD1,q,38);
```

$$\frac{\eta(5\,\tau)^3\,\eta(2\,\tau)^2}{q^{5/24}\,\eta(10\,\tau)\,\eta(\tau)^4}$$

So it would seem that

$$\sum_{n=0}^{\infty} pd(5n+1)q^n = \prod_{n=1}^{\infty} \frac{(1-q^{5n})^3(1-q^{2n})^2}{(1-q^{10n})(1-q^n)^4}.$$

This result was found originally by Rødseth [33].

Exercise 11. Rødseth also found the generating functions for $pd(5n+r)$ for $r = 0, 1, 2, 3$ and 4. For each r use `sift` and `jacprodmake` to identify these generating functions as infinite products.

6. Product Identities

At present, the package contains the Triple Product identity, the Quintuple Product identity and Winquist's identity. These are the most commonly used of the Macdonald identities [30], [37], [38]. The Macdonald identities are the analogs of the Weyl denominator for affine roots systems. Hopefully, a later version of this package will include these more general identities.

6.1. The Triple Product Identity.

The triple product identity is

$$\sum_{n=-\infty}^{\infty} (-1)^n z^n q^{n(n-1)/2} = \prod_{n=1}^{\infty}(1-zq^{n-1})(1-q^n/z)(1-q^n), \tag{6.7}$$

where $z \neq 0$ and $|q| < 1$. The Triple Product Identity is originally due to Jacobi [24, Vol I]. The first combinatorial proof of the triple product identity is due to Sylvester [39]. Recently, Andrews [3] and Lewis [29] have found nice combinatorial proofs. The triple product occurs frequently in the theory of partitions. For instance, most proofs of the Rogers-Ramanujan identity crucially depend on the triple product identity.

`tripleprod(z,q,T)` returns the q-series expansion to order $O(q^T)$ of Jacobi's triple product (6.7). This expansion is found by simply truncating the right side of (6.7).

```
>    tripleprod(z,q,10);
```

$$\frac{q^{21}}{z^6}-\frac{q^{15}}{z^5}+\frac{q^{10}}{z^4}-\frac{q^6}{z^3}+\frac{q^3}{z^2}-\frac{q}{z}+1-z+z^2q-z^3q^3+z^4q^6-z^5q^{10}+z^6q^{15}$$

```
>    tripleprod(q,q^3,10);
```

$$q^{57}-q^{40}+q^{26}-q^{15}+q^7-q^2+1-q+q^5-q^{12}+q^{22}-q^{35}+q^{51}$$

The last calculation is an illustration of Euler's Pentagonal Number Theorem [1, Cor. 1.7 p.11]:

(6.8)
$$\prod_{n=1}^{\infty}(1-q^n)=\prod_{n=1}^{\infty}(1-q^{3n-1})(1-q^{3n-2})(1-q^{3n})=\sum_{n=-\infty}^{\infty}(-1)^nq^{n(3n-1)/2}.$$

6.2. **The Quintuple Product Identity.** The following identity is the Quintuple Product Identity:

(6.9)
$$\begin{aligned}&(-z,q)_\infty(-\frac{q}{z},q)_\infty(z^2q,q^2)_\infty(\frac{q}{z^2},q^2)_\infty(q,q)_\infty\\&=\sum_{m=-\infty}^{\infty}\left((-z)^{-3\,m}-(-z)^{3\,m-1}\right)q^{\frac{m(3\,m+1)}{2}}.\end{aligned}$$

Here $|q|<1$ and $z\neq 0$. This identity is the BC_1 case of the Macdonald identities [30]. The quintuple product identity is usually attributed to Watson [40]. However it can be found in Ramanujan's lost notebook [32, p. 207]. Also see [7] for more history and some proofs.

The function `quinprod(z,q,T)` returns the quintuple product identity in different forms:

(i) If T is a positive integer it returns the q-expansion of the right side of (6.9) to order $O(q^T)$.
(ii) If $T=$ *prodid* then `quinprod(z,q,prodid)` returns the quintuple product identity in product form.
(iii) If $T=$ *seriesid* then `quinprod(z,q,seriesid)` returns the quintuple product identity in series form.

```
> quinprod(z,q,prodid);
```

$$(-z,\, q)_\infty\, (-\frac{q}{z},\, q)_\infty\, (z^2\, q,\, q^2)_\infty\, (\frac{q}{z^2},\, q^2)_\infty\, (q,\, q)_\infty =$$
$$(\frac{q^2}{z^3},\, q^3)_\infty\, (q\, z^3,\, q^3)_\infty\, (q^3,\, q^3)_\infty + z\, (\frac{q}{z^3},\, q^3)_\infty\, (q^2\, z^3,\, q^3)_\infty\, (q^3,\, q^3)_\infty$$

```
> quinprod(z,q,seriesid);
```

$$(-z,\, q)_\infty \quad (-\frac{q}{z},\, q)_\infty \quad (z^2\, q,\, q^2)_\infty \quad (\frac{q}{z^2},\, q^2)_\infty \quad (q,\, q)_\infty =$$
$$\sum_{m=-\infty}^{\infty} ((-z)^{(-3\,m)} - (-z)^{(3\,m-1)})\, q^{(1/2\,m\,(3\,m+1))}$$

```
> quinprod(z,q,3);
```

$$(z^{12} + \frac{1}{z^{11}})\, q^{22} + (-z^9 - \frac{1}{z^8})\, q^{12} + (z^6 + \frac{1}{z^5})\, q^5 + (-z^3 - \frac{1}{z^2})\, q + 1 + z$$
$$+ (-\frac{1}{z^3} - z^4)\, q^2 + (\frac{1}{z^6} + z^7)\, q^7 + (-\frac{1}{z^9} - z^{10})\, q^{15} + (\frac{1}{z^{12}} + z^{13})\, q^{26}$$

Let's examine a more interesting application. Euler's infinite product may be dissected according to the residue of the exponent of q mod 5:

$$\prod_{n=1}^{\infty}(1-q^n) = E_0(q) + qE_1(q^5) + q^2E_2(q^5) + q^3E_3(q^5) + q^4E_2(q^5).$$

By (6.8) we see that $E_3 = E_4 = 0$ since $n(3n-1)/2 \equiv 0$, 1 or 2 mod 5. Let's see if we can identify E_0.

```
> with(qseries):
> EULER:=etaq(q,1,500):
> E0:=sift(EULER,q,5,0,499);
```

$$E0 := 1 + q - q^3 - q^7 - q^8 - q^{14} + q^{20} + q^{29} + q^{31} + q^{42} - q^{52} - q^{66}$$
$$- q^{69} - q^{85} + q^{99}$$

```
> jacprodmake(E0,q,50);
```

$$\frac{\mathrm{JAC}(\,2, 5, \infty\,)\, \mathrm{JAC}(\,0, 5, \infty\,)}{\mathrm{JAC}(\,1, 5, \infty\,)}$$

```
> jac2prod(");
```

$$\frac{(\,q^5, q^5\,)_\infty\ (\,q^2, q^5\,)_\infty\ (\,q^3, q^5\,)_\infty}{(\,q, q^5\,)_\infty\ (\,q^4, q^5\,)_\infty}$$

```
> quinprod(q,q^5,20):
> series(",q,100);
```

$$1+q-q^3-q^7-q^8-q^{14}+q^{20}+q^{29}+q^{31}+q^{42}-q^{52}-q^{66}-q^{69}-q^{85}+O\left(q^{99}\right)$$

From our MAPLE session it appears that

$$E_0 = \frac{(q^5;q^5)_\infty (q^2;q^5)_\infty (q^3;q^5)_\infty}{(q;q^5)_\infty (q^4;q^5)_\infty}, \tag{6.10}$$

and that this product can be gotten by replacing q by q^5 and z by q in the product side of the quintuple product identity (6.9).

Exercise 12. (i) Use the quintuple product identity (6.9) and Euler's pentagonal number theorem to prove (6.10) above.
(ii) Use MAPLE to identify and prove product expressions for E_1 and E_2.
(iii) This time see if you can repeat (i), (ii) but split the exponent mod 7.
(iv) Can you generalize these results to arbitrary modulus?
Atkin and Swinnerton-Dyer found a generalization. See Lemma 6 in [6].

6.3. **Winquist's Identity.** Back in 1969, Lasse Winquist [43] discovered a remarkable identity

(6.11)

$$\begin{aligned}
&(a;q)_\infty(q/a;q)_\infty(b;q)_\infty(q/b;q)_\infty(ab;q)_\infty(q/(ab);q)_\infty(a/b;q)_\infty \\
&\quad (b/(aq);q)_\infty(q;q)_\infty^2 \\
&= \sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty}(-1)^{n+j}((a^{-3n}-a^{3n+3})(b^{-3m}-b^{3m+1}) \\
&\qquad\qquad + (a^{-3m+1}-a^{3m+2})(b^{3n+2}-b^{-3n-1}))q^{3n(n+1)/2+m(3m+1)/2}.
\end{aligned}$$

By dividing both sides by $(1-a)(1-b)$ and letting $a,\, b \to 1$ he was able to express the product $\prod_{n=1}^{\infty}(1-q^n)^{10}$ as a double series and prove Ramanujan's partition congruence

$$p(11n+6) \equiv 0 \pmod{11}.$$

This was probably the first truly elementary proof of Ramanujan's congruence modulo 11. The interested reader should see Dyson's article [14] for some fascinating history on identities for powers of the Dedekind eta function and how they led to the Macdonald identities. A new proof of Winquist's identity has been found recently by S.-Y. Kang [25]. Mike Hirschhorn [22] has found a four-parameter generalization of Winquist's identity.

The function `winquist(a,b,q,T)` returns the q-expansion of the right side of (6.11) to order $O(q^T)$.

We close with an example. For $1 < k < 33$ define

$$Q(k) = \prod_{n=1}^{\infty} (1-q^k)(1-q^{33-k})(1-q^{33}).$$

Now define the following functions:

$$A_0 = Q(15), \quad A_3 = Q(12), \quad A_7 = Q(6), \quad A_8 = Q(3), \quad A_9 = Q(9);$$

$$\begin{aligned} B_0 &= Q(16) - q^2 Q(5), \\ B_1 &= Q(14) - qQ(8), \\ B_2 &= Q(13) - q^3 Q(2), \\ B_4 &= Q(7) + qQ(4), \\ B_7 &= Q(10) + q^3 Q(1). \end{aligned}$$

These functions occur in Theorem 6.7 of [17] as well as the function $A_0 B_2 - q^2 A_9 B_4$.

```
> with(qseries):
> Q:=n->tripleprod(q^n,q^33,10):
> A0:=Q(15):   A3:=Q(12):   A7:=Q(6):
> A8:=Q(3):   A9:=Q(9):
> B2:=Q(13)-q^3*Q(2):   B4:=Q(7)+q*Q(4):
> IDG:=series( ( A0*B2-q^2*A9*B4),q,200):
> series(IDG,q,10);
```

$$1 - q^2 - 2\,q^3 + q^5 + q^7 + q^9 + O\left(q^{11}\right)$$

```
> jacprodmake(IDG,q,50);
```

$$\frac{\mathrm{JAC}(\,2, 11, \infty\,)\,\mathrm{JAC}(\,3, 11, \infty\,)^2\,\mathrm{JAC}(\,5, 11, \infty\,)}{\mathrm{JAC}(\,0, 11, \infty\,)^3}$$

```
> jac2prod(");
```

$$(\,q^2, q^{11}\,)_\infty\,(\,q^9, q^{11}\,)_\infty\,(\,q^{11}, q^{11}\,)_\infty\,(\,q^3, q^{11}\,)_\infty{}^2\,(\,q^8, q^{11}\,)_\infty{}^2\,(\,q^5, q^{11}\,)_\infty\,(\,q^6, q^{11}\,)_\infty$$

```
> series(winquist(q^5,q^3,q^11,10),q,20);
```

$$1 - q^2 - 2\,q^3 + q^5 + q^7 + \mathrm{O}(\,q^9\,)$$

```
> series(IDG-winquist(q^5,q^3,q^11,10),q,60);
```

$$\mathrm{O}(\,q^{49})$$

From our MAPLE session it seems that

(6.12)
$$A_0B_2 - q^2A_9B_4 = (q^2;q^{11})_\infty(q^9;q^{11})_\infty(q^{11};q^{11})_\infty(q^3;q^{11})^2_\infty(q^8;q^{11})^2_\infty (q^5;q^{11})_\infty(q^6;q^{11})_\infty,$$

and that this product appears in Winquist's identity on replacing q by q^{11} and letting $a = q^5$ and $b = q^3$.

Exercise 13. (i) Prove (6.12) by using the triple product identity (6.7) to write the right side of Winquist's identity (6.11) as a sum of two products.

(ii) In a similar manner find and prove a product form for
$$A_0B_0 - q^3A_7B_4.$$

References

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