On certain formulas of Karlin and Szegö

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Abstract

Some identities due to Karlin and Szegö which provide a relationship between determinants of classical orthogonal polynomials of Wronskian and Hankel type are shown to be specializations of a general algebraic identity between minors of a matrix.

Résumé

On montre que des familles d'identités découvertes par Karlin et Szegö, qui relient des Wronskiens et des déterminants de Hankel de polynômes orthogonaux classiques, résultent par spécialisation d'une identité algébrique générale entre mineurs d'une matrice.

1 Introduction

Let

$$P_n(x) = \frac{1}{2^n} \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^m \binom{2n-2m}{n} \binom{n}{m} x^{n-2m} \tag{1}$$

denote the nth Legendre polynomial. It was found some fifty years ago by Turán [29, 27] that for $x \in]-1, 1[$ and all $n \geq 0$, there holds

$$\begin{vmatrix} P_n(x) & P_{n+1}(x) \\ P_{n+1}(x) & P_{n+2}(x) \end{vmatrix} < 0\,. \tag{2}$$

Turán's inequality was soon generalized in several ways and there is today a huge literature on Turán type inequalities (see [1, 2, 3, 5, 6, 7, 8, 9, 10, 12, 20, 23, 25, 26, 31] and references therein).

A major contribution to this topic was made by Karlin and Szegö [12], who showed that for even l, the Hankel determinant

$$T(l,n;x) = \begin{vmatrix} Q_n(x) & Q_{n+1}(x) & \cdots & Q_{n+l-1}(x) \\ Q_{n+1}(x) & Q_{n+2}(x) & \cdots & Q_{n+l}(x) \\ \vdots & \vdots & & \vdots \\ Q_{n+l-1}(x) & Q_{n+l}(x) & \cdots & Q_{n+2l-2}(x) \end{vmatrix} \tag{3}$$

has a constant sign for $x \in I$ in each of the following cases:

(i) $Q_n(x) = P_n^{(\lambda)}(x)$ (ultraspherical polynomials, which contain Legendre polynomials for $\lambda = 1/2$) and $I =]-1, 1[$,

(ii) $Q_n(x) = L_n^{(\alpha)}(x)$ (Laguerre polynomials) and $I =]0, +\infty[$,

(iii) $Q_n(x) = H_n(x)$ (Hermite polynomials) and $I =]-\infty, +\infty[$.

Their strategy was to express the determinant $T(l,n;x)$ in terms of the Wronskian of certain orthogonal polynomials of another class. For instance, in the case of Legendre polynomials they proved that

$$\begin{vmatrix} P_n(x) & P_{n+1}(x) & \cdots & P_{n+l-1}(x) \\ P_{n+1}(x) & P_{n+2}(x) & \cdots & P_{n+l}(x) \\ \vdots & \vdots & & \vdots \\ P_{n+l-1}(x) & P_{n+l}(x) & \cdots & P_{n+2l-2}(x) \end{vmatrix} = A_{l,n}(x^2-1)^{l(n+l-1)/2}$$

$$\times \begin{vmatrix} T_l(u) & T_{l+1}(u) & \cdots & T_{l+n-1}(u) \\ T'_l(u) & T'_{l+1}(u) & \cdots & T'_{l+n-1}(u) \\ \vdots & \vdots & & \vdots \\ T_l^{(n-1)}(u) & T_{l+1}^{(n-1)}(u) & \cdots & T_{l+n-1}^{(n-1)}(u) \end{vmatrix}, \tag{4}$$

where $T_l(u)$ is the lth Tchebichev polynomial of the second kind,

$$u = -x(x^2-1)^{-1/2}, \quad T_m^{(k)}(u) = \frac{d^k T_m(u)}{du^k} \tag{5}$$

and

$$A_{l,n} = \frac{(-1)^l}{2} \prod_{p=0}^{l-1} 2^{1-n-2p} \prod_{q=0}^{n-1} \frac{2^{1-q}}{q!}. \tag{6}$$

In each of the three cases (i) (ii) (iii), Karlin and Szegö managed to find and prove a similar formula, which allowed them to reduce their analysis to that of the sign of a Wronskian determinant, which is easier.

Thus a kind of duality emerged between Hankel and Wronskian determinants of classical orthogonal polynomials. However the proofs of Karlin and Szegö did not clearly show what in their formulas resulted from a general algebraic transformation, and what in contrast was due to some particular properties of the orthogonal polynomials under consideration. In trying to clarify this, we obtained a different derivation of the identities of Karlin and Szegö, which consists of two steps:

Step 1. A completely general algebraic identity stating that a Wronskian of orthogonal polynomials is proportional to a Hankel determinant whose elements form a new sequence of polynomials.

Step 2. The verification that in each case considered by Karlin and Szegö, this new sequence is, after change of variable and normalization, another class of classical orthogonal polynomials.

To be more explicit, let us consider a sequence of arbitrary numbers a_n, $n \geq 0$. With this sequence are associated two classes of polynomials in the indeterminate u, defined by:

$$p_n(u) = \begin{vmatrix} a_0 & a_1 & \cdots & a_{n-1} & 1 \\ a_1 & a_2 & \cdots & a_n & u \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n-1} & a_n & \cdots & a_{2n-2} & u^{n-1} \\ a_n & a_{n+1} & \cdots & a_{2n-1} & u^n \end{vmatrix}, \qquad (n \geq 0), \tag{7}$$

$$q_n(u) = \sum_{m=0}^{n} a_m \binom{n}{m} (-u)^{n-m}, \qquad (n \geq 0). \tag{8}$$

Theorem 1 *The following identity holds for all integer values of l and n, $l \geq 1, n \geq 1$:*

$$\begin{vmatrix} p_l(u) & p_{l+1}(u) & \cdots & p_{l+n-1}(u) \\ p_l'(u) & p_{l+1}'(u) & \cdots & p_{l+n-1}'(u) \\ \vdots & \vdots & & \vdots \\ p_l^{(n-1)}(u) & p_{l+1}^{(n-1)}(u) & \cdots & p_{l+n-1}^{(n-1)}(u) \end{vmatrix} = C_{l,n} \begin{vmatrix} q_n(u) & q_{n+1}(u) & \cdots & q_{n+l-1}(u) \\ q_{n+1}(u) & q_{n+2}(u) & \cdots & q_{n+l}(u) \\ \vdots & \vdots & & \vdots \\ q_{n+l-1}(u) & q_{n+l}(u) & \cdots & q_{n+2l-2}(u) \end{vmatrix},$$

where $C_{l,n}$ is independent of u:

$$C_{l,n} = (-1)^{nl} \prod_{k=1}^{n-1} k! \begin{vmatrix} a_0 & a_1 & \cdots & a_{k+l-1} \\ a_1 & a_2 & \cdots & a_{k+l} \\ \vdots & \vdots & & \vdots \\ a_{k+l-1} & a_{k+l} & \cdots & a_{2k+2l-2} \end{vmatrix}.$$

Now, as is well known (see [28], p.27), if

$$a_k = \int_a^b u^k w(u)\, du \tag{9}$$

is the kth moment of the weight function w on $]a, b[$, then p_n is the nth orthogonal polynomial associated with w (up to normalization). Thus Theorem 1 states that the Wronskian of n consecutive orthogonal polynomials

$p_l, \ldots, p_{l+n-1}$ is proportional to a $l \times l$ Hankel determinant of polynomials $q_n, \ldots, q_{n+2l-2}$ defined in a simple way from the moments of the p_k's. In fact, it is easy to see that the q_k's are given by the generating series

$$\sum_{k \geq 0} q_k(u) \frac{t^k}{k!} = e^{-tu} \int_a^b e^{tx} w(x) dx \,. \tag{10}$$

The proof of Theorem 1 will be obtained by application of a 'master identity' of Turnbull on minors of a matrix [30, 16].

Theorem 1 should be regarded as a limiting case of the well-known result of Christoffel for the orthogonal polynomials associated with the weight function $(u - x_1) \cdots (u - x_n) w(u)$ (see [28], p. 30). And indeed, one could also obtain this Theorem by taking an appropriate limit in the formula given by Lascoux and Shi He [14] for the Christoffel polynomials (see below, end of Section 3).

Assuming Theorem 1, the verification of the identities of Karlin and Szegö is therefore essentially reduced to the following algebraic property of the classical polynomials, discovered by Burchnall [4].

Theorem 2 [4] *Let $Q_n(x)$ denote one of the following classes of polynomials:*

$$\text{(i) } Q_n(x) = P_n^{(\lambda)}(x)\,; \quad \text{(ii) } Q_n(x) = L_n^{(\alpha)}(x)\,; \quad \text{(iii) } Q_n(x) = H_n(x)\,.$$

Then we have

$$Q_n(x) = \lambda_n \, \phi(x)^n \, q_n(u) \tag{11}$$

where $q_n(u)$ is of the form (8)*, and in case* (i)*:*

$$\lambda_n = P_n^{(\lambda)}(1), \;\; \phi(x) = (x^2 - 1)^{1/2}, \;\; u = \frac{-x}{(x^2-1)^{1/2}}, \;\; a_{2p} = \frac{(\frac{1}{2})_p}{(\lambda + \frac{1}{2})_p}, \;\; a_{2p+1} = 0,$$

in case (ii)*:*

$$\lambda_n = L_n^{(\alpha)}(0), \quad \phi(x) = -x, \quad u = 1/x, \quad a_m = \frac{1}{(\alpha+1)_m},$$

and in case (iii)*:*

$$\lambda_n = (-2)^n, \quad \phi(x) = 1, \quad u = x, \quad a_{2p} = \frac{(-1)^p \, (2p)!}{2^{2p} \, p!}, \quad a_{2p+1} = 0.$$

Theorem 2 is rather straightforward to check, using the differential relation satisfied by each class of polynomials. On the other hand, the differential relation for general Jacobi polynomials does not allow to write them in the form (11), which explains why there is no identity of the Karlin-Szegö type for these polynomials. One may note that Burchnall had already observed that Turán's inequality (2) follows directly from Theorem 2, as well as its generalization to polynomials of the classes (i), (ii), (iii). He had also given formula (12.1) of [12] for $n = 0, 1$.

Using the results of Karlin and Szegö on Wronskians of orthogonal polynomials ([12], p.6) one deduces immediately from Theorem 1 the following

Corollary 3 *Let w be an arbitrary non-negative weight function on a real interval $]a,b[$, and let $q_n(u)$ be defined by (10). For $l,n \geq 1$, set*

$$T(w;l,n;u) = \begin{vmatrix} q_n(u) & q_{n+1}(u) & \cdots & q_{n+l-1}(u) \\ q_{n+1}(u) & q_{n+2}(u) & \cdots & q_{n+l}(u) \\ \vdots & \vdots & & \vdots \\ q_{n+l-1}(u) & q_{n+l}(u) & \cdots & q_{n+2l-2}(u) \end{vmatrix}.$$

Then, if n is even, $T(w;l,n;u)$ keeps a constant sign for all real u, and if n is odd, $T(w;l,n;u)$ has exactly l real simple zeros strictly interlaced between the $l+1$ zeros of $T(w;l+1,n;u)$.

By Theorem 1, when $n = 1$ the polynomial $T(w;l,1;u)$ is equal up to a numerical factor to the lth orthogonal polynomial $p_l(u)$ associated with w. Thus Corollary 3 is a generalization of the well-known fact that $p_l(u)$ has l real simple zeros interlaced between the zeros of $p_{l+1}(u)$.

The paper is organized as follows. Section 2 provides some background on determinantal identities. It also introduces the Schur function notation for orthogonal polynomials, which is extremely convenient for handling formulas such as Theorem 1. The proof of Theorem 1 is given in Section 3, and its specialization to the Karlin-Szegö identities is considered in Section 4. To make the paper self-contained, we have included a proof of Theorem 2. Section 5 gives examples of specializations of Theorem 1 to other classes of polynomials, like Euler polynomials or Bernoulli polynomials. Finally, Section 6 discusses another family of identities of Karlin and Szegö also contained in [12].

2 Determinantal identities and Schur functions

We begin by reviewing briefly the notation of [16] for minor identities, which is a variant of Turnbull's dot notation (see [30], p.27). Let M be a $n \times p$ matrix with $p > n$, and $a, b, \dots, c$ be n column vectors of M. The maximal minor of M taken on these n columns is denoted by either a bracket or a one line tableau:

$$[ab\dots c] = \boxed{a \quad b \quad \dots \quad c}\,.$$

A product of k minors of M is designated by a $k \times n$ tableau. Thus for $k = 3$,

$$[ab\dots c].[de\dots f].[gh\dots i] = \begin{array}{|cccc|} \hline a & b & \dots & c \\ \hline d & e & \dots & f \\ \hline g & h & \dots & i \\ \hline \end{array}\,.$$

To denote alternating sums of products of minors, we use tableaux with boxes enclosing certain vectors. Let T be a $k \times n$ tableau and A a subset of elements of T. Write i_j for the number of elements of A lying in the jth row of T. The tableau τ obtained from T by boxing all elements of A will be used to denote the sum

$$\tau = \frac{1}{i_1! \cdots i_k!} \sum_{\sigma \in \mathfrak{S}(A)} \operatorname{sgn}(\sigma)\, \sigma(T)\,, \tag{12}$$

where $\sigma(T)$ is the tableau in which the elements of A are permuted by σ. Due to the skew-symmetry of $(a, b, \ldots, c) \mapsto [ab \cdots c]$, if σ permutes between themselves the elements of A lying in the jth row for all j, then clearly $\operatorname{sgn}(\sigma)\, \sigma(T) = T$. Such permutations σ form a Young subgroup $\mathfrak{S}_T$ of $\mathfrak{S}(A)$ of cardinality $i_1! \cdots i_k!$. Hence (12) may be rewritten as

$$\tau = \sum_{\sigma} \operatorname{sgn}(\sigma)\, \sigma(T)\,, \tag{13}$$

where σ ranges now over a set of representatives of the left cosets of $\mathfrak{S}_T$ in $\mathfrak{S}(A)$. For example,

$$\tau \;=\; \begin{array}{|ccc|}\hline \boxed{a} & \boxed{b} & c \\ \hline d & e & \boxed{f} \\ \hline\end{array} \;:=\; \begin{array}{|ccc|}\hline a & b & c \\ \hline d & e & f \\ \hline\end{array} \;-\; \begin{array}{|ccc|}\hline f & b & c \\ \hline d & e & a \\ \hline\end{array} \;-\; \begin{array}{|ccc|}\hline a & f & c \\ \hline d & e & b \\ \hline\end{array}\,.$$

We can now state Turnbull's identity (see [30], p. 48 and [16]).

Theorem 4 *Let τ be a $p \times n$ tableau with set of enclosed elements A of cardinality $\leq n$. Let R be one of the rows of T and denote by B the set of elements of R which are not enclosed. Form a new tableau υ by* (i) *exchanging the elements of A which do not belong to R with elements of B;* (ii) *removing the boxes of the elements of A;* (iii) *boxing the elements of A; then,*

$$\tau = \upsilon\,.$$

Taking for instance

$$\tau = \begin{array}{|cccccc|}\hline \boxed{\mathbf{a}} & \alpha & \beta & \gamma & \delta & \varepsilon \\ \hline \boxed{\mathbf{b}} & \boxed{\mathbf{c}} & f & g & h & i \\ \hline \boxed{\mathbf{d}} & j & k & l & m & o \\ \hline\end{array}$$

and choosing for R the first row, we have $A = \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}\}$, $B = \{\alpha, \beta, \gamma, \delta, \varepsilon\}$, and we obtain

$$\tau = \begin{array}{|cccccc|}\hline \boxed{\mathbf{a}} & \alpha & \beta & \gamma & \delta & \varepsilon \\ \hline \boxed{\mathbf{b}} & \boxed{\mathbf{c}} & f & g & h & i \\ \hline \boxed{\mathbf{d}} & j & k & l & m & o \\ \hline\end{array} = \begin{array}{|cccccc|}\hline \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} & \boxed{\delta} & \boxed{\varepsilon} \\ \hline \boxed{\alpha} & \boxed{\beta} & f & g & h & i \\ \hline \boxed{\gamma} & j & k & l & m & o \\ \hline\end{array} = \upsilon\,.$$

Note that here τ represents a sum of 12 products of minors, whereas υ stands for a sum of 30 products. It has been shown in [16] that many classical determinantal identities are easily obtained as simple specializations of Theorem 4. Such identities would become quite cumbersome to state if one did not use a convenient system of notation showing clearly what are the transformations being performed.

Similarly, as explained by Lascoux in [13], the Schur function approach to orthogonal polynomials simplifies greatly the algebraic aspects of this subject. Recall that the complete homogeneous symmetric functions $S_i(E)$ of a set E of indeterminates are defined via the generating series

$$\sigma(E,t) := \sum_{i\geq 0} S_i(E)\, t^i = \prod_{e\in E} \frac{1}{1-te}\,. \tag{14}$$

In particular $S_0(E) = 1$. We decide that for $i < 0$, $S_i(E) = 0$. If F is a second set of indeterminates, we define the formal sum and difference $E + F$ and $E - F$ by

$$\begin{aligned} \sigma(E+F,t) &:= \sigma(E,t)\,\sigma(F,t)\,, & (15)\\ \sigma(E-F,t) &:= \frac{\sigma(E,t)}{\sigma(F,t)}\,. & (16)\end{aligned}$$

As an example, consider the case when E consists of only one variable $E = \{x\}$, and F of n variables. We have

$$\sigma(x-F,t) = \frac{\prod_{f\in F}(1-tf)}{1-tx} = \prod_{f\in F}(1-tf)\,\sum_{k\geq 0} t^k x^k$$

so that

$$S_n(x-F) = \prod_{f\in F}(x-f)$$

is the monic polynomial with set of roots F.

An important idea, extensively used by Littlewood (see [15], chap. 7), is that any sequence a_k, $k \geq 1$ of elements of a commutative ring R can be regarded as the sequence of complete symmetric functions of a fictitious set of variables E:

$$S_k(E) = a_k\,.$$

Indeed it is well known that the S_k form a set of algebraically independent generators of the ring of symmetric functions (see [19]), and thus one can always define a homomorphism from this ring to R by assigning $S_k \mapsto S_k(E) := a_k$. Of course this is very formal, but it allows to understand that certain identities, between orthogonal polynomials for instance, arise naturally as specializations of identities at the level of symmetric functions. An example of this phenomenon was given in [17] where a conjecture of Favreau for the computation of the linearization coefficients of Bessel polynomials was shown to result from a known formula for multiplying two staircase Schur functions.

In the sequel the notation $S_k(E)$ will therefore indicate nothing but a certain specialization of the ring of symmetric functions. In this context, it is customary to call E an alphabet. A set of indeterminates is regarded as a particular case of alphabet by means of (14). The symmetric functions of a sum $E + F$ or a difference $E - F$ of alphabets are defined via (15) and (16).

Given n alphabets $E_1, \ldots, E_n$ and a sequence $I = (i_1, \ldots, i_n)$ of integers, define the (multi) Schur function

$$S_I(E_1, \ldots, E_n) = \det[S_{i_l+l-k}(E_l)]_{1 \le k,l \le n} \,. \tag{17}$$

We shall often use the exponential notation for sequences I with repeated parts, and write $I = (i_1^{m_1} i_2^{m_2} \cdots i_r^{m_r})$ to indicate the sequence with m_1 terms equal to i_1, m_2 terms equal to i_2 and so on.

As recalled in Section 1, orthogonal polynomials have a determinantal expression which can be seen as a particular instance of Schur function. Indeed, putting

$$S_k(E) = \frac{a_k}{a_0}, \qquad a_k = \int_a^b u^k w(u)\, du \,,$$

we see that

$$S_{(n,\ldots,n,0)}(E, \ldots, E, u) = \begin{vmatrix} S_n(E) & S_{n+1}(E) & \cdots & S_{2n-1}(E) & u^n \\ S_{n-1}(E) & S_n(E) & \cdots & S_{2n-2}(E) & u^{n-1} \\ \vdots & \vdots & & \vdots & \vdots \\ S_1(E) & S_2(E) & \cdots & S_n(E) & u \\ S_0(E) & S_1(E) & \cdots & S_{n-1}(E) & 1 \end{vmatrix} \tag{18}$$

is up to a scalar the nth orthogonal polynomial associated with the weight function w. Here, we consider the single variable u as a particular alphabet by defining

$$\sigma(u,t) = \sum_{i \ge 0} S_i(u)\, t^i := \frac{1}{1-tu} = \sum_{i \ge 0} u^i t^i \,.$$

By subtraction of rows in this determinant we arrive at the following more symmetric expression:

$$\begin{aligned} S_{(n^n\,0)}(E, \ldots, E, u) \;&=\; \begin{vmatrix} S_n(E-u) & S_{n+1}(E-u) & \cdots & S_{2n-1}(E-u) \\ S_{n-1}(E-u) & S_n(E-u) & \cdots & S_{2n-2}(E-u) \\ \vdots & \vdots & & \vdots \\ S_1(E-u) & S_2(E-u) & \cdots & S_n(E-u) \end{vmatrix} \\ &=\; S_{(n^n)}(E-u, \ldots, E-u) \,. \end{aligned}$$

This is now an ordinary Schur function (*i.e.* depending on a single alphabet $E-u$) that we denote more concisely by $S_{(n^n)}(E-u)$.

The formula $S_{(n^n\,0)}(E, \ldots, E, u) = S_{(n^n)}(E-u)$ is in fact a special case of a very useful lemma going back to Jacobi (see [11], p.371).

Lemma 5 *Let F be an alphabet such that $S_k(-F) = 0$ for $k > m$. Then*

$$S_I(E_1, \ldots, E_n) = \begin{vmatrix} S_{i_1}(E_1 - F) & \ldots & S_{i_n+n-1}(E_n - F) \\ \vdots & & \vdots \\ S_{i_1-n+m+1}(E_1 - F) & \ldots & S_{i_n+m}(E_n - F) \\ S_{i_1-n+m}(E_1) & \ldots & S_{i_n+m-1}(E_n) \\ \vdots & & \vdots \\ S_{i_1-n+1}(E_1) & \ldots & S_{i_n}(E_n) \end{vmatrix} \,.$$

Indeed, we have $S_k(E_j-F) = S_k(E_j)+S_{k-1}(E_j)S_1(-F)+\ldots+S_1(E_j)S_{k-1}(-F)+S_k(-F)$. Thus, since $S_k(-F)=0$ for $k>m$, the determinant of the right-hand side is obtained from that of the left-hand side by adding to each of the first $n-m$ rows a linear combination of the next m rows.

Note that if F is a set of m indeterminates, then it satisfies the condition $S_k(-F)=0$ for $k>m$, since by definition

$$\sum_{i\geq 0} S_i(-F)\, t^i = \prod_{f\in F}(1-tf)\,.$$

Finally, as a further example of this alphabet notation, let us consider again the alphabet u consisting of the single variable u. Iterating (15), we define a new alphabet $nu := u+\cdots+u$ by

$$\sum_{i\geq 0} S_i(nu)\, t^i := \sigma(u,t)^n = \frac{1}{(1-tu)^n}$$

so that $S_k(nu) = \binom{n+k-1}{k} u^k$. Similarly, we set

$$\sum_{i\geq 0} S_i(-nu)\, t^i := \sigma(u,t)^{-n} = (1-tu)^n\,,$$

which gives $S_k(-nu) = \binom{n}{k}(-u)^k$. Now one can write for an arbitrary alphabet E:

$$S_n(E-nu) = \sum_{k=0}^{n} S_k(E)\binom{n}{k}(-u)^{n-k}, \tag{19}$$

in which we recognize the polynomial $q_n(u)$ of (8) with $S_k(E) = a_k/a_0$.

3 Proof of Theorem 1

Let E be an arbitrary alphabet. We put

$$\begin{aligned} p_m(u) &= (-1)^{\binom{m+1}{2}} S_{(m^m)}(E-u) = (-1)^{\binom{m+1}{2}} S_{(m^m0)}(E,\ldots,E,u)\,,\\ q_m(u) &= S_m(E-mu)\,. \end{aligned}$$

We first remark that

$$\begin{vmatrix} q_{n+l-1}(u) & q_{n+l}(u) & \cdots & q_{n+2l-2}(u)\\ q_{n+l-2}(u) & q_{n+l-1}(u) & \cdots & q_{n+2l-3}(u)\\ \vdots & \vdots & & \vdots\\ q_n(u) & q_{n+1}(u) & \cdots & q_{n+l-1}(u) \end{vmatrix} = S_{((n+l-1)^l)}(E-nu)\,. \tag{20}$$

Indeed, since all the terms of the sequence $I = ((n+l-1)^l)$ are equal, we can use repeatedly Lemma 5 both in the rows and in the columns of the Schur function $S_{((n+l-1)^l)}(E-nu)$ and subtract the alphabet ku in the $(k+1)$th column and the $(l-k)$th row to get the determinant in the left-hand side of (20).

Denoting by $\mathrm{Wr}\,(f_1(u),\dots,f_k(u))$ the Wronskian of the functions $f_j(u)$, $1\le j\le k$, we deduce from Section 2 that the formula to be proved can be rewritten as

$$\mathrm{Wr}\,(S_{(l^l)}(E-u),\dots,S_{((l+n-1)^{l+n-1})}(E-u)) = C_{l,n}(E)\, S_{((l+n-1)^l)}(E-nu)\,, \quad (21)$$

where

$$C_{l,n}(E) = (-1)^{\binom{n}{2}} \prod_{k=1}^{n-1} k!\, S_{((l+k-1)^{l+k})}(E)\,. \quad (22)$$

Let us introduce the $(l+n)\times(l+3n-2)$ matrix

$$\begin{bmatrix}
1 & 0 & \dots & 0 & S_{l+n-1} & S_{l+n} & \dots & S_{2l+2n-3} & u^{l+n-1} & \frac{du^{l+n-1}}{du} & \dots & \frac{d^{n-1}u^{l+n-1}}{du^{n-1}} \\
0 & 1 & \dots & 0 & S_{l+n-2} & S_{l+n-1} & \dots & S_{2l+2n-4} & u^{l+n-2} & \frac{du^{l+n-2}}{du} & \dots & \frac{d^{n-1}u^{l+n-2}}{du^{n-1}} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
0 & 0 & \dots & 1 & S_{l+1} & S_{l+2} & \dots & S_{2l+n-1} & u^{l+1} & \frac{du^{l+1}}{du} & \dots & \frac{d^{n-1}u^{l+1}}{du^{n-1}} \\
0 & 0 & \dots & 0 & S_l & S_{l+1} & \dots & S_{2l+n-2} & u^l & \frac{du^l}{du} & \dots & \frac{d^{n-1}u^l}{du^{n-1}} \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
0 & 0 & \dots & 0 & S_1 & S_2 & \dots & S_{l+n-1} & u & 1 & \dots & 0 \\
0 & 0 & \dots & 0 & S_0 & S_2 & \dots & S_{l+n-2} & 1 & 0 & \dots & 0
\end{bmatrix}$$

where S_k is short for $S_k(E)$. The column vectors of this matrix will be denoted from left to right by:

$$\mathbf{1},\mathbf{2},\dots,\mathbf{n-1},\sigma_0,\sigma_1,\dots,\sigma_{l+n-2},\mathbf{u},\mathbf{u}',\dots,\mathbf{u}^{(\mathbf{n-1})}\,.$$

Using the notation of Section 2, the left-hand side of (21) is written as

$$\begin{vmatrix}
[\mathbf{12}\dots\mathbf{n-1}\sigma_0\sigma_1\dots\sigma_{l-1}\mathbf{u}] & [\mathbf{12}\dots\mathbf{n-2}\sigma_0\sigma_1\dots\sigma_l\mathbf{u}] & \dots & [\sigma_0\dots\sigma_{l+n-2}\mathbf{u}] \\
[\mathbf{12}\dots\mathbf{n-1}\sigma_0\sigma_1\dots\sigma_{l-1}\mathbf{u}'] & [\mathbf{12}\dots\mathbf{n-2}\sigma_0\sigma_1\dots\sigma_l\mathbf{u}'] & \dots & [\sigma_0\dots\sigma_{l+n-2}\mathbf{u}'] \\
\vdots & \vdots & & \vdots \\
[\mathbf{1}\dots\mathbf{n-1}\sigma_0\dots\sigma_{l-1}\mathbf{u}^{(\mathbf{n-1})}] & [\mathbf{1}\dots\mathbf{n-2}\sigma_0\dots\sigma_l\mathbf{u}^{(\mathbf{n-1})}] & \dots & [\sigma_0\dots\sigma_{l+n-2}\mathbf{u}^{(\mathbf{n-1})}]
\end{vmatrix}$$

$$= \begin{array}{|cccccccccc|}
\hline
\mathbf{1} & \mathbf{2} & \dots & \mathbf{n-2} & \mathbf{n-1} & \sigma_0 & \sigma_1 & \dots & \sigma_{l-1} & \boxed{\mathbf{u}} \\
\hline
\mathbf{1} & \mathbf{2} & \dots & \mathbf{n-2} & \sigma_0 & \sigma_1 & \sigma_2 & \dots & \sigma_l & \boxed{\mathbf{u}'} \\
\hline
\vdots & \vdots & & & & & & & & \vdots \\
\hline
\mathbf{1} & \sigma_0 & \dots & \dots & \dots & \dots & \dots & \dots & \sigma_{l+n-1} & \boxed{\mathbf{u}^{(\mathbf{n-2})}} \\
\hline
\sigma_0 & \sigma_1 & \dots & \dots & \dots & \dots & \dots & \dots & \sigma_{l+n-2} & \boxed{\mathbf{u}^{(\mathbf{n-1})}} \\
\hline
\end{array}$$

Now using the transformation of Theorem 4 with R being the last row, this tableau is equal to

$$
\begin{array}{|cccccccccc|}
\hline
\mathbf{1} & \mathbf{2} & \ldots & \mathbf{n-1} & \mathbf{n-1} & \sigma_0 & \sigma_1 & \ldots & \sigma_{l-1} & \boxed{\sigma_l} \\
\hline
\mathbf{1} & \mathbf{2} & \ldots & \mathbf{n-2} & \sigma_0 & \sigma_1 & \sigma_2 & \ldots & \sigma_l & \boxed{\sigma_{l+1}} \\
\hline
\vdots & \vdots & & & & & & & & \vdots \\
\hline
\mathbf{1} & \sigma_0 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \sigma_{l+n-1} & \boxed{\sigma_{l+n-2}} \\
\hline
\boxed{\sigma_0} & \boxed{\sigma_1} & \ldots & \ldots & \boxed{\sigma_{l-1}} & \mathbf{u} & \mathbf{u}' & \ldots & \mathbf{u}^{(\mathbf{n-2})} & \mathbf{u}^{(\mathbf{n-1})} \\
\hline
\end{array}
$$

$$
= \begin{array}{|cccccccccc|}
\hline
\mathbf{1} & \mathbf{2} & \ldots & \mathbf{n-1} & \mathbf{n-1} & \sigma_0 & \sigma_1 & \ldots & \sigma_{l-1} & \sigma_l \\
\hline
\mathbf{1} & \mathbf{2} & \ldots & \mathbf{n-2} & \sigma_0 & \sigma_1 & \sigma_2 & \ldots & \sigma_l & \sigma_{l+1} \\
\hline
\vdots & \vdots & & & & & & & & \vdots \\
\hline
\mathbf{1} & \sigma_0 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \sigma_{l+n-1} & \sigma_{l+n-2} \\
\hline
\sigma_0 & \sigma_1 & \ldots & \ldots & \sigma_{l-1} & \mathbf{u} & \mathbf{u}' & \ldots & \mathbf{u}^{(\mathbf{n-2})} & \mathbf{u}^{(\mathbf{n-1})} \\
\hline
\end{array}
$$

Note that all boxes have been deleted in the second tableau because all the other permutations of the letters enclosed give rise to tableaux with two equal letters on some row, which are therefore equal to zero in view of the skew-symmetry of the determinant. Thus, we have rewritten the Wronskian of (21) as a product of n determinants, where only the last one depends on the variable u. Explicitly, we have obtained that the left-hand side of (21) is equal to

$$S_{(l^{l+1})}(E)\, S_{((l+1)^{l+2})}(E) \cdots S_{((l+n-2)^{l+n-1})}(E)\, \Delta(u)\,,$$

where

$$
\Delta(u) = \begin{vmatrix}
S_{l+n-1}(E) & S_{l+n}(E) & \ldots & S_{2l+n-2}(E) & u^{l+n-1} & \frac{du^{l+n-1}}{du} & \ldots & \frac{d^{n-1}u^{l+n-1}}{du^{n-1}} \\
S_{l+n-2}(E) & S_{l+n-1}(E) & \ldots & S_{2l+n-3}(E) & u^{l+n-2} & \frac{du^{l+n-2}}{du} & \ldots & \frac{d^{n-1}u^{l+n-2}}{du^{n-1}} \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
S_1(E) & S_2(E) & \ldots & S_l(E) & u & 1 & \ldots & 0 \\
S_0(E) & S_1(E) & \ldots & S_{l-1}(E) & 1 & 0 & \ldots & 0
\end{vmatrix}.
$$

Now, noting that

$$\frac{d^k u^m}{du^k} = k! \binom{m}{k} u^{m-k} = k!\, S_{m-k}\left((k+1)u\right)\,,$$

we have

$$
\Delta(u) = 1!\,2! \cdots (n-1)! \begin{vmatrix}
S_{l+n-1}(E) & \ldots & S_{2l+n-2}(E) & S_{l+n-1}(u) & \ldots & S_l(nu) \\
\vdots & & \vdots & \vdots & & \vdots \\
S_0(E) & \ldots & S_{l-1}(E) & S_0(u) & \ldots & S_{-n+1}(nu)
\end{vmatrix}.
$$

Finally, using Lemma 5 to subtract the alphabet nu in the l first rows of this last determinant, we see that it reduces to

$$\begin{vmatrix} S_{l+n-1}(E-nu) & \dots & S_{2l+n-2}(E-nu) & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & & \vdots & \vdots \\ S_n(E-nu) & \dots & S_{l+n-1}(E-nu) & 0 & \dots & 0 & 0 \\ S_{n-1}(E) & \dots & S_{l+n-2}(E) & S_{n-1}(u) & \dots & S_1((n-1)u) & 1 \\ S_{n-2}(E) & \dots & S_{l+n-3}(E) & S_{n-2}(u) & \dots & 1 & 0 \\ \vdots & & \vdots & \vdots & \iddots & \vdots & \vdots \\ S_0(E) & \dots & S_{l-1}(E) & 1 & \dots & 0 & 0 \end{vmatrix}$$

$$= \quad (-1)^{\binom{n}{2}} \begin{vmatrix} S_{l+n-1}(E-nu) & \dots & S_{2l+n-2}(E-nu) \\ \vdots & & \vdots \\ S_n(E-nu) & \dots & S_{l+n-1}(E-nu) \end{vmatrix}$$

so that

$$\Delta(u) = (-1)^{\binom{n}{2}}\, 1!\,2!\,\cdots\,(n-1)!\, S_{((l+n-1)^l)}(E-nu)\,,$$

as required. This finishes the proof of Theorem 1.

We end this section by noting the closely related expression given by Lascoux and Shi He [14] of the Christoffel polynomials

$$\psi_{l,n} := \det[S_{((l+i-1)^{l+i-1})}(E-f_j)]_{1\le i,j\le n}\,,$$

where $F = \{f_1, \dots, f_n\}$ is a set of n indeterminates (see [28], p.30). Lascoux and Shi He found that

$$\psi_{l,n} = \prod_{1\le i,j\le n} (f_i - f_j) \prod_{1\le j\le n-1} S_{((l+j-1)^{l+j})}(E)\ S_{((l+n-1)^l)}(E-F)\,. \tag{23}$$

Theorem 1 may be regarded as the limiting case $f_i \longrightarrow u$ of (23). Indeed, $\psi_{l,n}$ is clearly skew-symmetric in the f_i, and therefore is divisible by the Vandermonde determinant $\prod_{1\le i,j\le n}(f_i - f_j)$. The quotient may be expressed as a discrete Wronskian, that is, a Wronskian determinant where derivatives are replaced by divided differences. Once this division is performed, one can actually pass to the limit $f_i \longrightarrow u$ and recover the usual Wronskian of Theorem 1.

4 Specialization to classical orthogonal polynomials

In the derivation of the Karlin-Szegö identities, the following lemma, which follows immediately from (19), will be used.

Lemma 6 *A sequence of functions $f_k(u)$ is of the form*

$$f_k(u) = S_k(E - ku)\,, \qquad (k \geq 0)$$

for a certain alphabet E, if and only if $f_0 \equiv 1$ and

$$\frac{df_k(u)}{du} = -k f_{k-1}(u)\,, \qquad (k \geq 1)\,.$$

In this case, the alphabet E is specified by $S_k(E) = f_k(0)$.

4.1 Ultraspherical polynomials

Let

$$P_k^{(\lambda)}(x) = \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} (-1)^m \binom{k-m+\lambda-1}{k-m} \binom{k-m}{m} (2x)^{k-2m} \tag{24}$$

denote the kth ultraspherical polynomial with parameter $\lambda > -1/2$ (see [28], (4.7.31)). (For $\lambda = 0$, the right-hand side of (24) is 0, but $\lim_{\lambda \to 0} \frac{1}{\lambda} P_k^{(\lambda)}(x) = \frac{2}{k}\, T_k(x)$ and the subsequent formulas remain valid provided $P_k^{(\lambda)}(x)$ is replaced by this limit [28], (4.7.8).) We put

$$f_k(u) = (x^2-1)^{-k/2}\, \frac{P_k^{(\lambda)}(x)}{P_k^{(\lambda)}(1)}\,,$$

where $u = -x(x^2-1)^{-1/2}$. Then,

$$\begin{aligned} \frac{df_k(u)}{du} &= \frac{d}{dx}\left\{ (x^2-1)^{-k/2} \frac{P_k^{(\lambda)}(x)}{P_k^{(\lambda)}(1)} \right\} \frac{dx}{du} \\ &= (x^2-1)^{-(k-1)/2} \left\{ (x^2-1) \frac{d}{dx}\left(\frac{P_k^{(\lambda)}(x)}{P_k^{(\lambda)}(1)} \right) - kx \frac{P_k^{(\lambda)}(x)}{P_k^{(\lambda)}(1)} \right\}\,. \end{aligned}$$

Now, using the classical differential relation ([28], (4.7.27))

$$(1-x^2) \frac{d}{dx}\left(\frac{P_k^{(\lambda)}(x)}{P_k^{(\lambda)}(1)} \right) = -kx \frac{P_k^{(\lambda)}(x)}{P_k^{(\lambda)}(1)} + k \frac{P_{k-1}^{(\lambda)}(x)}{P_{k-1}^{(\lambda)}(1)}\,,$$

we obtain

$$\frac{df_k(u)}{du} = -k(x^2-1)^{-(k-1)/2} \frac{P_{k-1}^{(\lambda)}(x)}{P_{k-1}^{(\lambda)}(1)} = -k\, f_{k-1}(u)\,.$$

Thus, by Lemma 6, $f_k(u) = S_k(\mathcal{U}_\lambda - ku)$ for the alphabet $\mathcal{U}_\lambda$ specified by

$$S_k(\mathcal{U}_\lambda) = f_k(0) = (-1)^{-k/2} \frac{P_k^{(\lambda)}(0)}{P_k^{(\lambda)}(1)} = \begin{cases} \dfrac{(\frac{1}{2})_l}{(\lambda+\frac{1}{2})_l} & \text{if } k = 2l, \\ 0 & \text{if } k = 2l+1, \end{cases} \tag{25}$$

where we have used Pochammer's symbol $(a)_n := a(a+1)\cdots(a+n-1)$. Thus, Theorem 2 is verified in case (i). It follows that

$$
\begin{aligned}
\det\left[\frac{P^{(\lambda)}_{n+i+j}(x)}{P^{(\lambda)}_{n+i+j}(1)}\right]_{0\le i,j\le l-1} &= \det\left[(x^2-1)^{(n+i+j)/2}\, S_{n+i+j}(\mathcal{U}_\lambda-(n+i+j)u)\right]_{0\le i,j\le l-1} \\
&= (x^2-1)^{l(n+l-1)/2}\det\left[S_{n+i+j}(\mathcal{U}_\lambda-(n+i+j)u)\right]_{0\le i,j\le l-1} \\
&= (-1)^{\binom{l}{2}}(x^2-1)^{l(n+l-1)/2}S_{((n+l-1)^l)}(\mathcal{U}_\lambda-nu)\,.
\end{aligned}
$$

On the other hand, the moments of the weight function

$$
w_\lambda(u)=(1-u^2)^{\lambda-1/2}\,,\qquad (-1<u<1)
$$

associated with the polynomials $P^{(\lambda)}_n(u)$ are readily computed, and one finds

$$
\frac{\int_{-1}^1 u^k\, w_\lambda(u)du}{\int_{-1}^1 w_\lambda(u)du}=\begin{cases}\dfrac{(\frac{1}{2})_l}{(\lambda+1)_l} & \text{if } k=2l,\\ 0 & \text{if } k=2l+1.\end{cases}
$$

Thus, comparing with (25), we have

$$
\frac{\int_{-1}^1 u^k\, w_\lambda(u)du}{\int_{-1}^1 w_\lambda(u)du}=S_k(\mathcal{U}_{\lambda+1/2})\,. \tag{26}
$$

Note the shift $\lambda \longrightarrow \lambda+1/2$ which explains in particular the relationship between Legendre and Tchebichev polynomials expressed by (4). It follows from (26) that the Schur function $S_{(n^n)}(\mathcal{U}_\lambda-u)$ is equal up to a numerical factor to $P^{(\lambda-1/2)}_n(u)$. Therefore, applying Theorem 1 we find that

$$
\det\left[\frac{P^{(\lambda)}_{n+i+j}(x)}{P^{(\lambda)}_{n+i+j}(1)}\right]_{0\le i,j\le l-1} = A^{(\lambda)}_{l,n}\,(x^2-1)^{l(n+l-1)/2}\,\mathrm{Wr}\left(P^{(\lambda-1/2)}_l(u),\dots,P^{(\lambda-1/2)}_{l+n-1}(u)\right),
$$

where $A^{(\lambda)}_{l,n}$ is a constant depending only on l,n and λ. By (22), to evaluate this constant it remains to compute the specialized Schur functions $S_{(m^{m+1})}(\mathcal{U}_\lambda)$. We omit this calculation and only mention that it may be done using the following result of Saalschütz [22]:

Lemma 7 *Write* $(2m+1)!!=(2m+1)(2m-1)\dots 3.1$. *Then*

$$
\begin{vmatrix}
(2k+1)!! & (a)_1\,(2k-1)!! & \dots & (a)_{n-1}\,(2(k-n)+3)!! \\
(2k+3)!! & (a+1)_1\,(2k+1)!! & \dots & (a+1)_{n-1}\,(2(k-n)+5)!! \\
\vdots & \vdots & & \vdots \\
(2(k+n)-1)!! & (a+n-1)_1\,(2(k+n)-3)!! & \dots & (a+n-1)_{n-1}\,(2k+3)!!
\end{vmatrix}
$$

$$
=(2(k-a)+1)^{n-1}(2(k-a)-1)^{n-2}\cdots(2(k-a-n)+5)(2k+1)!!\cdots(2(k-n)+3)!!
$$

This completes our derivation of the identity of Karlin and Szegö for ultraspherical polynomials ([12], (14.1)).

4.2 Laguerre polynomials

The (generalized) Laguerre polynomials are given by ([28], (5.1.6))

$$L_k^{(\alpha)}(x) = \sum_{m=0}^{k} (-1)^m \binom{k+\alpha}{k-m} \frac{1}{m!}\, x^m \,. \tag{27}$$

Defining $\mathcal{L}_\alpha$ by

$$S_m(\mathcal{L}_\alpha) = \frac{1}{(1+\alpha)_m}\,, \qquad (m \geq 0) \tag{28}$$

we obtain easily that

$$\frac{L_k^{(\alpha)}(x)}{L_k^{(\alpha)}(0)} = (-x)^k\, S_k(\mathcal{L}_\alpha - ku), \tag{29}$$

where $u = 1/x$.

On the other hand, the moments of the weight function

$$w_\alpha(u) = x^\alpha e^{-x}\,, \qquad (x > 0)$$

are given by

$$\frac{\int_0^{+\infty} u^k\, w_\alpha(u) du}{\int_0^{+\infty} w_\alpha(u) du} = (1+\alpha)_k\,.$$

It follows by a simple calculation that the polynomial $S_{(m^m)}(\mathcal{L}_\alpha - u)$ is equal up to a numerical factor to $x^{-m}\, L_m^{(-\alpha-2m)}(-x)$, and thus, by application of Theorem 1, we obtain

$$\det \left[\frac{L_{n+i+j}^{(\alpha)}(x)}{L_{n+i+j}^{(\alpha)}(0)} \right]_{0 \leq i,j \leq l-1} \;=\; B_{l,n}^{(\alpha)}\, x^{l(n+l-1)}\, \mathrm{Wr}\left(\lambda_l(u), \ldots, \lambda_{l+n-1}(u)\right),$$

where

$$\lambda_p(u) := u^p\, L_p^{(-\alpha-2p)}(-u^{-1})\,.$$

The value of $B_{l,n}^{(\alpha)}$ may be calculated from (22) and the easily checked formula:

$$S_{(m^{m+1})}(\mathcal{L}_\alpha) = \prod_{k=0}^{m} \frac{(-1)^{\binom{m+1}{2}}\, k!}{(1+\alpha)_{m+k}}\,,$$

and this completes our derivation of the identity of Karlin and Szegö for Laguerre polynomials ([12], (16.1)).

4.3 Hermite polynomials

The Hermite polynomials are given by ([28], (5.5.4))

$$H_k(x) = k! \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} (-1)^m \frac{1}{m!(k-2m)!}\, (2x)^{k-2m}\,. \tag{30}$$

Setting

$$S_m(\mathcal{H}) = \begin{cases} \dfrac{(-1)^l\,(2l)!}{2^{2l}\,l!} & \text{if } m = 2l, \\ 0 & \text{if } m = 2l+1, \end{cases} \tag{31}$$

we obtain immediately that

$$\frac{H_k(x)}{(-2)^k} = S_k(\mathcal{H} - kx). \tag{32}$$

On the other hand, the moments of the weight function

$$w(x) = e^{-x^2}, \qquad (x \in \mathbf{R})$$

are easily found to be

$$\frac{\int_{-\infty}^{+\infty} x^k\, w(x)dx}{\int_{-\infty}^{+\infty} w(x)dx} = \begin{cases} \dfrac{(2l)!}{2^{2l}\,l!} & \text{if } k = 2l, \\ 0 & \text{if } k = 2l+1, \end{cases} .$$

This implies, by application of Theorem 1, that

$$\det\left[\frac{H_{n+i+j}(x)}{(-2)^{n+i+j}}\right]_{0\le i,j\le l-1} = E_{l,n}\,\mathrm{Wr}\,(H_l(\sqrt{-1}x), \ldots, H_{l+n-1}(\sqrt{-1}x)), \tag{33}$$

where $E_{l,n}$ is given by

$$E_{l,n} = i^{\lfloor \frac{l+n}{2} \rfloor - \lfloor \frac{l}{2} \rfloor}\, 2^{(n+l-1)(l-n)/2}\, \frac{\prod_{k=n}^{l+n-1} k!}{\prod_{k=l}^{l+n-1} k!}.$$

Note that (33) is not the formula (18.1) of [12]. Indeed, the formula of Karlin and Szegö, which is simpler, involves a Wronskian and a Hankel determinant of the same order n, and is thus of a different type.

5 Miscellaneous examples

Example 1. Let us consider the polynomial

$$d_k(x) = \sum_{i=0}^{k} (-1)^i \frac{k!}{i!} x^{k-i},$$

which for $x = 1$ gives the number of permutations in $\mathfrak{S}_k$ without fixed point. We have

$$d_k(x) = x^k\, S_k(\mathcal{D} - ku),$$

where $u = 1/x$ and $S_m(\mathcal{D}) = m!$. The Hankel determinant $S_{(m^{m+1})}(\mathcal{D})$ is easily found to be

$$S_{(m^{m+1})}(\mathcal{D}) = (-1)^{m(m+1)/2} \left(\prod_{k=0}^{m} k!\right)^2,$$

and by Theorem 1, we get

$$\det\left[d_{n+i+j}(x)\right]_{0\le i,j\le l-1}$$
$$= (-1)^{n(n+1)/2}\prod_{k=1}^{l} k! \prod_{k=n}^{l+n-1} k!\; x^{(l+1)(l+n-1)}\,\mathrm{Wr}\left(L_l(u),\ldots,L_{l+n-1}(u)\right),$$

where $u = 1/x$ and $L_m(u) = L_m^{(0)}(u)$ is the (ordinary) Laguerre polynomial.

Example 2. Let $E_n(x)$ denote the nth Euler polynomial defined via the generating series

$$\sum_{n\ge 0} E_n(x)\,\frac{t^n}{n!} = \frac{2e^{tx}}{e^t+1}\,.$$

Then, we have

$$E_n(x) = 2^{-n}\sum_{k=0}^{n} E_k\binom{n}{k}(2x-1)^{n-k}\,,$$

where $E_k = 2^k E_k(1/2)$ is the kth Euler number. (Note that $E_{2l+1}=0$). Thus, putting

$$S_k(\mathcal{E}) = E_k\,,$$

we have $E_n(x) = 2^{-n}\,S_n(\mathcal{E}-nu)$, with $u = 1-2x$.

The calculation of the orthogonal polynomials $S_{(n^n)}(\mathcal{E}-u)$ is known to be equivalent to the determination of the (formal) continued fraction expansion of the associated power series $\sum_{n\ge 0} S_n(\mathcal{E})z^n$. But this continued fraction has been computed by Stieltjes [24], who found that

$$\sum_{n\ge 0} E_n u^{-n-1} = \int_0^\infty \frac{e^{-ut}}{\operatorname{ch} t}dt = \cfrac{1}{u+\cfrac{1}{u+\cfrac{4}{u+\cfrac{9}{u+\cfrac{16}{u+\cdots}}}}} \tag{34}$$

It follows that

$$S_{(n^n)}(\mathcal{E}-u) = (-1)^n\left(\prod_{k=1}^{n-1} k!\right)^2 \sum_{l=0}^{\lfloor\frac{n}{2}\rfloor}\sigma_{l,n}\,u^{n-2l}\,, \tag{35}$$

where

$$\sigma_{l,n} = \sum k_1^2\,k_2^2\cdots k_l^2\,, \tag{36}$$

the sum running over all integer sequences $(k_i)_{1\le i\le l}$ satisfying

$$1\le k_i < n\,,\qquad k_{i+1}-k_i\ge 2\,. \tag{37}$$

Therefore the specialization of Theorem 1 to the sequence of Euler polynomials reads

$$\det\left[E_{n+i+j}(x)\right]_{0\le i,j\le l-1} = F_{l,n}\,\mathrm{Wr}\left(\pi_l(u),\ldots,\pi_{l+n-1}(u)\right), \tag{38}$$

where

$$F_{l,n} = \frac{(-1)^{nl+l(l-1)/2}(1!\,2!\,\cdots(l-1)!)^2}{2^{l(l+n-1)}\,1!\,2!\,\cdots(n-1)!}, \tag{39}$$

and

$$\pi_m(u) = \sum_{k=0}^{\lfloor\frac{m}{2}\rfloor} \sigma_{k,m}\,u^{m-2k} \tag{40}$$

is the denominator of the mth partial fraction of (34).

Example 3. Let $B_n(x)$ denote the nth Bernoulli polynomial defined via the generating series

$$\sum_{n\ge0} B_n(x)\,\frac{t^n}{n!} = \frac{te^{tx}}{e^t-1}\,.$$

Then, we have

$$B_n(x) = \sum_{k=0}^{n} B_k \binom{n}{k} x^{n-k}\,,$$

where $B_k = B_k(0)$ is the kth Bernoulli number. (Note that $B_{2l+1} = 0$ for $l\ge1$). Thus, putting

$$S_k(\mathcal{B}) = B_k\,,$$

we have $B_n(x) = S_n(\mathcal{B}-nu)$, with $u=-x$.

The calculation of the orthogonal polynomials $S_{(n^n)}(\mathcal{B}-u)$ is equivalent to the continued fraction expansion of $\sum_{n\ge0} S_n(\mathcal{B})z^n$ which has been obtained by Rogers [21]:

$$\sum_{n\ge0} B_n u^{-n-1} = \int_0^\infty \frac{te^{-ut}}{e^t-1}dt = 2\int_0^\infty \frac{ye^{-(1+2u)y}}{\operatorname{sh} y}dy$$

$$= \cfrac{2}{1+2u+\cfrac{1^4}{3(1+2u)+\cfrac{2^4}{5(1+2u)+\cfrac{3^4}{7(1+2u)+\cfrac{4^4}{9(1+2u)+\cdots}}}}} \tag{41}$$

It follows that

$$S_{(n^n)}(\mathcal{B}-u) = (-1)^n\,S_{((n-1)^n)}(\mathcal{B})\,\varphi_n(u+1/2) \tag{42}$$

where

$$S_{((n-1)^n)}(\mathcal{B}) = \frac{1}{4^{\binom{n}{2}}\,(2n-1)!!} \prod_{k=2}^{n-1} \frac{k!^4}{(2k-1)!!^2}\,, \qquad \varphi_n(w) = \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor} \tau_{l,n}\, w^{n-2l}\,, \tag{43}$$

and

$$\tau_{l,n} = \frac{1}{4^l} \sum \frac{k_1^4}{(2k_1-1)(2k_1+1)} \cdots \frac{k_l^4}{(2k_l-1)(2k_l+1)}\,, \tag{44}$$

the sum running over all integer sequences $(k_i)_{1\le i\le l}$ satisfying

$$1 \le k_i < n\,, \qquad k_{i+1} - k_i \ge 2\,. \tag{45}$$

Therefore the specialization of Theorem 1 to the sequence of Bernoulli polynomials reads

$$\det\,[B_{n+i+j}(x)]_{0\le i,j\le l-1} = G_{l,n}\,\mathrm{Wr}\left(\varphi_l\left(\frac{1}{2}-x\right),\,\ldots,\,\varphi_{l+n-1}\left(\frac{1}{2}-x\right)\right)\,, \tag{46}$$

where

$$G_{l,n} = \frac{(-1)^{nl+l(l-1)/2}}{4^{\binom{l}{2}}\,1!\,2!\,\cdots(n-1)!\,(2l-1)!!} \prod_{j=2}^{l-1} \frac{j!^4}{(2j-1)!!^2}\,. \tag{47}$$

6 A transformation of alphabets

There are other formulas in [12], Section 28 which suggest an algebraic approach. They can all be deduced from the following general

Proposition 8 *Let E be an arbitrary alphabet. Define for $r \ge l$*

$$\Delta_{l,r}(u) = \begin{vmatrix} S_l(E-lu) & \cdots & S_{2l-1}(E-(2l-1)u) & S_{r+l}(E-(r+l)u) \\ S_{l-1}(E-(l-1)u) & \cdots & S_{2l-2}(E-(2l-2)u) & S_{r+l-1}(E-(r+l-1)u) \\ \vdots & \ddots & \vdots & \vdots \\ S_0(E) & \cdots & S_{l-1}(E-(l-1)u) & S_r(E-ru) \end{vmatrix}\,.$$

Then, we have

$$\Delta_{l,r}(u) = \binom{r}{l}\, S_{(l^{l+1})}(E)\, S_{r-l}(d_l E - (r-l)u)\,, \tag{48}$$

where the alphabet $d_l E$ is defined from the alphabet E by

$$S_m(d_l E) = \frac{S_{(l^l(l+m))}(E)}{\binom{l+m}{l} S_{(l^{l+1})}(E)}\,, \qquad (m \ge 0).$$

The proof of Proposition 8 is elementary. First, by several applications of Lemma 5, one can write

$$\Delta_{l,r}(u) = S_{(l^r)}(E, \ldots, E, E - ru)\,.$$

Then, one uses (19) to rewrite this determinant as a sum and obtain the right-hand side of (48).

Taking this into account, the formulas of Section 28 of [12] for determinants of the type $\Delta_{l,r}(u)$ whose elements are orthogonal polynomials of the classes (i) (ii) (iii) of Theorem 2, are equivalent to the following properties of the alphabets $\mathcal{U}_\lambda$, $\mathcal{L}_\lambda$ and $\mathcal{H}$ associated with these polynomials (see Section 4):

$$d_l\mathcal{U}_\lambda = \mathcal{U}_{\lambda+l}\,, \quad d_l\mathcal{L}_\alpha = \mathcal{L}_{\alpha+2l}\,, \quad d_l\mathcal{H} = \mathcal{H}\,. \tag{49}$$

We believe that these properties, which represent nontrivial evaluations of certain determinants, are to be added to the remarkable algebraic properties of the classical polynomials discovered by Burchnall and recalled in Theorem 2.

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