TWO-ROWED A-TYPE HECKE ALGEBRA REPRESENTATIONS AT ROOTS OF UNITY[†]

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In this paper, we describe a study into the explicit construction of irreducible representations of the Hecke algebra $H_n(q)$ of type A_{n-1} in the non-generic case where q is a root of unity. The approach is via the Specht modules of $H_n(q)$ which are irreducible in the generic case, and possess a natural basis indexed by Young tableaux. The general framework in which the irreducible non-generic $H_n(q)$-modules are to be constructed is set up and exploited in the case of two-part partitions. For such partitions, we obtain the composition series of the Specht modules, describe a basis for each irreducible module in terms of a subset of the set of standard tableaux, and detail an algorithm by which their explicit matrix representations may be calculated. Plentiful examples are given. Full proofs will be given elsewhere.

1 Introduction and notation

The Hecke algebra $H_n(q)$ (of type A_{n-1}) is the unital associative algebra over $\mathbb{C}$, generated by h_i, $i = 1, 2, \ldots, n-1$, subject to the relations:

$$\begin{aligned} &h_i h_{i+1} h_i = h_{i+1} h_i h_{i+1}; \\ &h_i h_j = h_j h_i \qquad \text{for } |i-j| > 1; \\ &h_i^2 = (q-1) h_i + q. \end{aligned} \tag{1}$$

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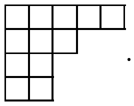
The parameter $q \in \mathbb{C}$ will be permitted to take any non-zero value. It is said to be generic if $q = 1$ or $q^p \neq 1$ for $p = 2, 3, 4, \ldots$. Otherwise, if q is a primitive pth root of unity for $p \geq 2$, it is said to be non-generic. In this latter case, $[p]_q = 0$ where we define $[x]_q = 1 + q + q^2 + \cdots + q^{x-1}$. In addition, define $[x]_q! = [x]_q[x-1]_q \cdots [2]_q$.

When $q = 1$, $H_n(q)$ may be identified with the group algebra $\mathbb{C}S_n$ of the symmetric group on n symbols, through identifying each h_i with the simple transposition $s_i = (i, i+1) \in S_n$.

If $w = s_{i_1} s_{i_2} \cdots s_{i_k}$ and $w \in S_n$ cannot be expressed as a shorter product of the generators s_i, then $s_{i_1} s_{i_2} \cdots s_{i_k}$ is said to be a reduced expression for w and the value of k is the length $l(w)$ of w. Thereupon, the relations (1) imply that the map $h : \mathbb{C}S_n \to H_n(q)$ for which $h(s_i) = h_i$ and $h(ww') = h(w)h(w')$ for $w, w' \in S_n$ satisfying $l(ww') = l(w) + l(w')$, and extended linearly, is well defined. It follows that if $l(w) = k$ and $w = s_{i_1} s_{i_2} \cdots s_{i_k}$, then $h(w) = h_{i_1} h_{i_2} \cdots h_{i_k}$. Furthermore, a basis of $H_n(q)$ is provided by $\{h(w) : w \in S_n\}$.

It may be shown that if q is generic then $H_n(q)$ is isomorphic to $\mathbb{C}S_n$ [DJ86, Wn88] and the representation theory of $H_n(q)$ is much the same as that of S_n. In particular, the inequivalent irreducible representations of $H_n(q)$ are indexed by partitions λ of n. That is, by finite integer sequences $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ for which $\lambda_1 + \lambda_2 + \cdots + \lambda_r = n$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0$. A partition for which no part λ_i is repeated more than $p-1$ times is said to be p-regular. In Section 2, an explicit construction of the irreducible modules of $H_n(q)$ with q generic will be described. This generalisation of the well known Specht module construction (see [JK81]) was first described in [KWy92], and is based on the use of Young diagrams, Young tableaux and q-analogues of Young symmetrisers. The Young diagram F^λ associated with the partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ is a left-adjusted, top-adjusted array of square boxes such that the ith row (counting from the top) contains λ_i boxes. For instance, if $\lambda = (5, 3, 2, 2)$, then

$F^\lambda =$  . (2)

Filling (or replacing) each of the n boxes of F^λ with elements of $\{1, 2, \ldots, n\}$ so that no element appears more than once, yields what is known as a Young tableau. Of the possible $n!$ tableaux of a given shape, those for which the entries are increasing across each row and down each column are known as *standard* tableaux. Examples may be found at (16), (30) and (31). That particular standard tableau of shape λ for which the entries increase down first the leftmost column and then down successive columns taken left to right is denoted t_-^λ. For example,

$$t_-^{(5,3,2,2)} = \begin{array}{lllll} 1 & 5 & 9 & 11 & 12 \\ 2 & 6 & 10 & & \\ 3 & 7 & & & \\ 4 & 8 & & & \end{array} . \tag{3}$$

The total number of standard tableaux of shape λ is equal to f^λ, the dimension of the irreducible representation of S_n (and $H_n(q)$ with q generic) labelled by λ (see [JK81]). In fact, the Specht module construction enables a basis to be identified naturally with the set of standard tableaux.

2 The Specht modules

If λ is a partition of n, the Specht module S^λ of $H_n(q)$ is defined to be the linear span of the vectors v_{t^λ}, indexed by Young tableaux t^λ and subject to certain relations (which will be defined below). The *natural* action of $H_n(q)$ on these vectors is defined in the following way. We say that the entry i precedes j in t^λ if i occurs before j on reading the entries of t^λ down the first and then successive columns. If x^λ is identical to t^λ apart from the transposition of i and $i+1$, then h_i acts on v_{t^λ} as follows:

$$h_i v_{t^\lambda} = \begin{cases} v_{x^\lambda} & \text{if } i \text{ precedes } i+1 \text{ in } t^\lambda; \\ q v_{x^\lambda} + (q-1) v_{t^\lambda} & \text{if } i+1 \text{ precedes } i \text{ in } t^\lambda. \end{cases} \tag{4}$$

It is possible to express every v_{z^λ} in terms of standard tableaux, by means of the following two types of relation:

1. **Column relations.** Entries within a column may be transposed, if the corresponding vector is multiplied by -1. Thus if x^λ differs from z^λ only in that a single pair of entries within a column are transposed then:

$$v_{z^\lambda} = -v_{x^\lambda}. \tag{5}$$

For example (denoting v_{t^λ} by t^λ for typographical reasons),

$$\begin{array}{llllll} 1 & 8 & 5 & 10 & 4 & 12 \\ 6 & 11 & 3 & & & \\ 9 & 2 & 7 & & & \\ 13 & & & & & \end{array} = - \begin{array}{llllll} 1 & 8 & 5 & 10 & 4 & 12 \\ 6 & 2 & 3 & & & \\ 9 & 11 & 7 & & & \\ 13 & & & & & \end{array} = - \begin{array}{llllll} 1 & 2 & 3 & 10 & 4 & 12 \\ 6 & 8 & 5 & & & \\ 9 & 11 & 7 & & & \\ 13 & & & & & \end{array} . \tag{6}$$

2. **Garnir relations.** Assume that z^λ is such that its entries increase down each column. If z^λ is not standard then an adjacent pair of entries exists with that on the left greater than that on the right. Consider these two entries together with all those below the left one and all those above the right one. For example, we could consider the highlighted entries in:

$$\begin{array}{cccccc} 1 & 2 & \mathbf{3} & 10 & 4 & 12 \\ 6 & \mathbf{8} & \mathbf{5} \\ 9 & \mathbf{11} & 7 \\ 13 \end{array} . \tag{7}$$

Now form all possible tableaux t^λ by permuting these entries in all ways such that the permuted entries are increasing down the portions of each of the two columns being considered. The Garnir relation is then the following expression in which the sum is over all such tableaux:

$$(-q)^{l(w_{z^\lambda})} \sum_{t^\lambda} (-q)^{-l(w_{t^\lambda})} v_{t^\lambda} = 0, \tag{8}$$

where $w_{t^\lambda} \in S_n$ maps t^λ_- to t^λ. The above example gives the Garnir relation:

$$\begin{array}{cccccc} 1 & 2 & \mathbf{3} & 10 & 4 & 12 \\ 6 & \mathbf{8} & \mathbf{5} \\ 9 & \mathbf{11} & 7 \\ 13 \end{array} - q \begin{array}{cccccc} 1 & 2 & \mathbf{3} & 10 & 4 & 12 \\ 6 & \mathbf{5} & \mathbf{8} \\ 9 & \mathbf{11} & 7 \\ 13 \end{array} + q^2 \begin{array}{cccccc} 1 & 2 & \mathbf{5} & 10 & 4 & 12 \\ 6 & \mathbf{3} & \mathbf{8} \\ 9 & \mathbf{11} & 7 \\ 13 \end{array}$$
$$+q^2 \begin{array}{cccccc} 1 & 2 & \mathbf{3} & 10 & 4 & 12 \\ 6 & \mathbf{5} & \mathbf{11} \\ 9 & \mathbf{8} & 7 \\ 13 \end{array} - q^3 \begin{array}{cccccc} 1 & 2 & \mathbf{5} & 10 & 4 & 12 \\ 6 & \mathbf{3} & \mathbf{11} \\ 9 & \mathbf{8} & 7 \\ 13 \end{array} + q^4 \begin{array}{cccccc} 1 & 2 & \mathbf{8} & 10 & 4 & 12 \\ 6 & \mathbf{3} & \mathbf{11} \\ 9 & \mathbf{5} & 7 \\ 13 \end{array} = 0. \tag{9}$$

As in the example above, these relations do not necessarily immediately express an arbitrary v_{t^λ} in terms of standard tableaux. However, it may be shown through employing a suitable order on the set of tableaux [JK81], that repeated application of the column and Garnir relations enables any term to be rendered in terms of standard tableaux in a finite number of steps. This completes the construction of the irreducible Specht module S^λ of $H_n(q)$ since the number of standard tableaux is equal to the dimension of the representation of $H_n(q)$ indexed by λ and consequently,

$$\{v_{t^\lambda} : t^\lambda \text{ is standard}\} \tag{10}$$

is a basis for S^λ.

As an example, consider representing $h_1 \in H_5(q)$ in the Specht module $S^{(3,2)}$, by acting on each $v_{t^{(3,2)}}$ for which $t^{(3,2)}$ is standard (once more v_{t^λ} is written as t^λ):

$$
\begin{aligned}
h_1 \begin{matrix}1&3&5\\2&4&\end{matrix} &= \begin{matrix}2&3&5\\1&4&\end{matrix} = -\begin{matrix}1&3&5\\2&4&\end{matrix}, \\
h_1 \begin{matrix}1&2&5\\3&4&\end{matrix} &= \begin{matrix}2&1&5\\3&4&\end{matrix} = q\begin{matrix}1&2&5\\3&4&\end{matrix} - q^2\begin{matrix}1&3&5\\2&4&\end{matrix}, \\
h_1 \begin{matrix}1&3&4\\2&5&\end{matrix} &= \begin{matrix}2&3&4\\1&5&\end{matrix} = -\begin{matrix}1&3&4\\2&5&\end{matrix}, \\
h_1 \begin{matrix}1&2&4\\3&5&\end{matrix} &= \begin{matrix}2&1&4\\3&5&\end{matrix} = q\begin{matrix}1&2&4\\3&5&\end{matrix} - q^2\begin{matrix}1&3&4\\2&5&\end{matrix}, \\
h_1 \begin{matrix}1&2&3\\4&5&\end{matrix} &= \begin{matrix}2&1&3\\4&5&\end{matrix} = q\begin{matrix}1&2&3\\4&5&\end{matrix} - q^2\begin{matrix}1&4&3\\2&5&\end{matrix} = q\begin{matrix}1&2&3\\4&5&\end{matrix} - q^3\begin{matrix}1&3&4\\2&5&\end{matrix} + q^4\begin{matrix}1&3&5\\2&4&\end{matrix}.
\end{aligned}
$$

Here, column relations have been used in the first and third calculations, and Garnir relations have been used in the second, fourth and last (twice), to express each result in terms of the standard tableaux. Consequently, in the representation labelled by the partition $(3,2)$, h_1 is mapped to the matrix (where zeros are denoted by dots):

$$
\begin{pmatrix}
-1 & -q^2 & \cdot & \cdot & q^4 \\
\cdot & q & \cdot & \cdot & \cdot \\
\cdot & \cdot & -1 & -q^2 & -q^3 \\
\cdot & \cdot & \cdot & q & \cdot \\
\cdot & \cdot & \cdot & \cdot & q
\end{pmatrix}. \tag{11}
$$

The matrices representing the generators h_i of $H_n(q)$ in each irreducible representation for $n \leq 5$ given in [KWy92] have been produced in a similar way.

3 The Young symmetriser and its annihilators

For each entry a of t^{λ}_{-} which is not at the bottom of a column, define the *column element*:

$$C^\lambda_a = 1 + h_a. \tag{12}$$

Its action on $v_{t^{\lambda}_{-}}$ gives rise to a Column relation (cf. (5)):

$$C^\lambda_a v_{t^{\lambda}_{-}} = 0. \tag{13}$$

The *Garnir element* G^λ_a is defined for each a which is not at the end of a row of t^{λ}_{-}, through first letting d be the entry to the right of a, b be the entry at the bottom of the column containing a, and c ($= b+1$) the entry at the top of the column containing d in t^λ_-. With W_{ij} the subgroup of S_n permuting only $\{i, i+1, \ldots, j\}$, let $\mathcal{G}^\lambda_a$ be a set of left coset representatives for $W_{ab} \times W_{cd}$ in W_{ad} chosen so that each representative is of minimal length in its coset (it is unique). Then let [KWy92]:

$$G^\lambda_a = q^l \sum_{d \in \mathcal{G}^\lambda_a} (-q)^{-l(d)} h(d), \tag{14}$$

where l is the length of the longest element in $\mathcal{G}^\lambda_a$. Its action on $v_{t^\lambda_-}$ gives rise to a Garnir relation (cf. (8)):

$$G^\lambda_a v_{t^\lambda_-} = 0. \tag{15}$$

It is easily shown that the general column and Garnir relations of Section 2 are a consequence of (13) and (15). These properties themselves arise by identifying $v_{t^\lambda_-}$ with the q-analogue $Y^\lambda(q)$ of the Young symmetriser. $Y^\lambda(q)$ was originally defined in [DJ86, Gy86] and cast in a form suitable for the current purposes in [KWy92, BKW93]. However, as is seen, only its $2n - r - \lambda_1$ annihilators C_a and G_a are required in the construction of the Specht module S^λ. Thus S^λ may be defined as the free module generated by a non-zero vector (say $v_{t^\lambda_-}$) subject to (13) and (15). This viewpoint of S^λ will be utilised in what follows.

To illustrate it, consider $\lambda = (6, 3, 3, 1)$, for which:

$$t^\lambda_- = \begin{array}{llllll} 1 & 5 & 8 & 11 & 12 & 13 \\ 2 & 6 & 9 & & & \\ 3 & 7 & 10 & & & \\ 4 & & & & & \end{array} . \tag{16}$$

Here we have the seven column elements $1 + h_1$, $1 + h_2$, $1 + h_3$, $1 + h_5$, $1 + h_6$, $1 + h_8$ and $1 + h_9$, each of which annihilates v_{t^λ}. There are nine Garnir elements G^λ_a for $a = 1, 2, 3, 5, 6, 7, 8, 11, 12$, each of which annihilates v_{t^λ}. Typically:

$$\begin{aligned} G^\lambda_6 &= q^4 - q^3 h_7 + q^2 h_6 h_7 + q^2 h_8 h_7 - q h_6 h_8 h_7 + h_7 h_6 h_8 h_7; \\ G^\lambda_8 &= q^3 - q^2 h_{10} + q h_9 h_{10} - h_8 h_9 h_{10}; \\ G^\lambda_{11} &= q - h_{11}. \end{aligned} \tag{17}$$

In fact $G^\lambda_6 v_{t^\lambda_-} = 0$ gives rise to (9).

4 Decomposing S^λ at roots of unity

In the generic case when q is not a root of unity, each Specht module S^λ of $H_n(q)$ is irreducible. However, this is no longer so if q is a root of unity, although S^λ remains well-defined. For q a primitive pth root of unity, let D_p^λ be the irreducible $H_n(q)$-module obtained by factoring out the maximal proper submodule from S^λ. It is shown in [DJ86] that in this case,

$$\{D_p^\lambda : \lambda \text{ is } p\text{−regular}\} \tag{18}$$

is a complete and irredundant set of irreducible $H_n(q)$-modules. Very little is known about the D_p^λ or the composition series of S^λ in terms of the D_p^λ except in a few specific cases (see [Jm90] for $n \leq 10$, [CK92] for $n \leq 5$, and [JM95] for various results when $p = 2$).

The viewpoint developed in the previous section provides a means of tackling these questions in a quite general way. It relies on the fact that within the set (18), the module D_p^μ is characterised by the presence of a non-zero vector $v_{\underline{t}^\mu}$ which is annihilated by the set of column and Garnir elements, C_a^μ and G_a^μ defined above. This follows because, via (4), $v_{\underline{t}^\mu}$ generates the whole of S^μ, and hence $v_{\underline{t}^\mu}$ cannot be present in any proper submodule of S^μ. Thus, to determine whether S^λ is reducible, it is sufficient to prove the existence of a non-zero $v^\mu \in S^\lambda$ having the same set of annihilators as $v_{\underline{t}^\mu} \in S^\mu$ for some p-regular partition $\mu \neq \lambda$ of n. Conversely, the absence of all such $v_{\underline{t}^\mu}$ would prove S^λ to be irreducible. (In fact, the results of [DJ86] and [DJ87] considerably restrict the set of μ for which D_p^μ may occur as a composition factor of S^λ.)

As an example, consider $\lambda = (3, 2)$. We will show that if $p = 3$ then

$$v^\mu = (1 + h_4)v_{\underline{t}^{(3,2)}} = \begin{matrix} 1 & 3 & 5 \\ 2 & 4 & \end{matrix} + \begin{matrix} 1 & 3 & 4 \\ 2 & 5 & \end{matrix} \tag{19}$$

is annihilated by the column and Garnir elements of $\mu = (4, 1)$, and hence that $S^{(3,2)}$ has a submodule $D_3^{(4,1)}$. Since $\underline{t}^{(4,1)} = {\scriptsize\begin{matrix} 1 & 3 & 4 & 5 \\ 2 & & & \end{matrix}}$, the column and Garnir elements of $\mu = (4, 1)$, are:

$$\begin{array}{ll} i) & (1 + h_1); \\ ii) & (q^2 - qh_2 + h_1h_2); \\ iii) & (q - h_3); \\ iv) & (q - h_4). \end{array} \tag{20}$$

Acting on (19) with each of these, using (4) gives:

$$
\begin{aligned}
&i) \quad (1+h_1)v^\mu = \begin{smallmatrix}1&3&5\\2&4\end{smallmatrix} + \begin{smallmatrix}2&3&5\\1&4\end{smallmatrix} + \begin{smallmatrix}1&3&4\\2&5\end{smallmatrix} + \begin{smallmatrix}2&3&4\\1&5\end{smallmatrix} = 0;\\
&ii) \quad (q^2 - qh_2 + h_1h_2)v^\mu = q^2\begin{smallmatrix}1&3&5\\2&4\end{smallmatrix} - q\begin{smallmatrix}1&2&5\\3&4\end{smallmatrix} + \begin{smallmatrix}2&1&5\\3&4\end{smallmatrix}\\
&\qquad\qquad + q^2\begin{smallmatrix}1&3&4\\2&5\end{smallmatrix} - q\begin{smallmatrix}1&2&4\\3&5\end{smallmatrix} + \begin{smallmatrix}2&1&4\\3&5\end{smallmatrix} = 0;\\
&iv) \quad (q-h_4)v^\mu = q\begin{smallmatrix}1&3&5\\2&4\end{smallmatrix} - \begin{smallmatrix}1&3&4\\2&5\end{smallmatrix} + q\begin{smallmatrix}1&3&4\\2&5\end{smallmatrix} - q\begin{smallmatrix}1&3&5\\2&4\end{smallmatrix} - (q-1)\begin{smallmatrix}1&3&4\\2&5\end{smallmatrix} = 0;\\
&iii) \quad (q-h_3)v^\mu = q\begin{smallmatrix}1&3&5\\2&4\end{smallmatrix} - \begin{smallmatrix}1&4&5\\2&3\end{smallmatrix} + q\begin{smallmatrix}1&3&4\\2&5\end{smallmatrix} - \begin{smallmatrix}1&4&3\\2&5\end{smallmatrix}\\
&\qquad = (1+q)\begin{smallmatrix}1&3&5\\2&4\end{smallmatrix} + q\begin{smallmatrix}1&3&4\\2&5\end{smallmatrix} - q\begin{smallmatrix}1&3&4\\2&5\end{smallmatrix} + q^2\begin{smallmatrix}1&3&5\\2&4\end{smallmatrix} = (1+q+q^2)\begin{smallmatrix}1&3&5\\2&4\end{smallmatrix} = 0,
\end{aligned}
$$

since if $p=3$, then $1+q+q^2=0$. Therefore, $D_3^{(4,1)}$ is a submodule of $S^{(3,2)}$. It may be shown that the 4-dimensional $S^{(4,1)}$ is irreducible when $p=3$, so that $D_3^{(4,1)} \equiv S^{(4,1)}$. Hence $D_3^{(3,2)}$ is of dimension $5-4=1$. It is spanned by $v_{t_-^{(3,2)}}$.

In order to generalise the v^μ of the previous example, let $\lambda = (\lambda_1, \lambda_2)$ and for $1 \le x \le \lambda_2$, define the standard tableau t_{x-}^λ as follows. After ignoring the rightmost x boxes of the bottom row and the rightmost $\lambda_1 - \lambda_2$ boxes of the top row, fill the diagram as for $t_-^{(\lambda_2,\lambda_2-x)}$. Then put the entries $\{2\lambda_2 - x + 1, \ldots, n\}$ in increasing order across first the remaining x boxes of the bottom row and then the remaining $\lambda_1 - \lambda_2$ boxes of the top row. For example:

$$
t_{2-}^{(4,2)} = \begin{matrix}1&2&\mathbf{5}&\mathbf{6}\\\mathbf{3}&\mathbf{4}\end{matrix}, \quad t_{3-}^{(7,4)} = \begin{matrix}1&3&4&5&\mathbf{9}&\mathbf{10}&\mathbf{11}\\2&\mathbf{6}&\mathbf{7}&\mathbf{8}\end{matrix}, \quad t_{1-}^{(7,4)} = \begin{matrix}1&3&5&7&\mathbf{9}&\mathbf{10}&\mathbf{11}\\2&4&6&\mathbf{8}\end{matrix}, \tag{21}
$$

where the latter set of entries have been highlighted. In addition, define the standard tableau t_{x+}^λ which also has the entries $\{1, 2, \ldots, 2\lambda_2 - x\}$ placed exactly as for $t_-^{(\lambda_2,\lambda_2-x)}$. The entries $\{2\lambda - x + 1, \ldots, n\}$ are then placed in increasing order across first the remaining $\lambda_1 - \lambda_2$ boxes of the top row and then the remaining x boxes of the bottom row. For example:

$$
t_{2+}^{(4,2)} = \begin{matrix}1&2&\mathbf{3}&\mathbf{4}\\\mathbf{5}&\mathbf{6}\end{matrix}, \quad t_{3+}^{(7,4)} = \begin{matrix}1&3&4&5&\mathbf{6}&\mathbf{7}&\mathbf{8}\\2&\mathbf{9}&\mathbf{10}&\mathbf{11}\end{matrix}, \quad t_{1+}^{(7,4)} = \begin{matrix}1&3&5&7&\mathbf{8}&\mathbf{9}&\mathbf{10}\\2&4&6&\mathbf{11}\end{matrix}. \tag{22}
$$

For $a \le b \le m$, let $\mathcal{D}_{a,m}^b$ be the set of left coset representatives of $W_{b,m}$ in $W_{a,m}$ chosen so that each representative is of minimal length in its coset (once more, it is unique). Then define:

$$v^{(\lambda_1+x,\lambda_2-x)} = \sum_{d\in\mathcal{D}^{2\lambda_2+1}_{\lambda_2+i,n}} v_{dt^\lambda_{x-}}. \tag{23}$$

(For each of the tableaux in (21), this sum is over all tableaux in which the highlighted entries have been permuted such that those in the top row are in increasing order.)

Theorem 1. *For $\lambda = (\lambda_1, \lambda_2)$ and q a primitive pth root of unity, let $x = p - 1 - (\lambda_1 - \lambda_2) \bmod p$ and $\mu = (\lambda_1 + x, \lambda_2 - x)$. If $0 < x \leq \lambda_2$ then $v^\mu \in S^\lambda$ satisfies:*

1. *$C^\mu_a v^\mu = 0$ for all a not at the bottom of a column of t^μ_-;*
2. *$G^\mu_a v^\mu = 0$ for all a not at the end of a row of t^μ_-.*

Furthermore, if v^μ is written in terms of standard tableaux then, subject to $[p]_q = 0$, each polynomial coefficient has a polynomial factor $[x]_q!$. Moreover the coefficient of $v_{t^\lambda_{x+}}$ in $v^\mu/[x]_q!$ is 1.

This theorem therefore has the consequence that $S^{(\lambda_1,\lambda_2)}$ has a submodule D^μ_p where μ is determined by p as in the statement of Theorem 1. It may be further shown that all submodules of $S^{(\lambda_1,\lambda_2)}$ arise in this way, and this enables the full composition of S^λ in the case of two-part partitions λ to be determined. The result is best expressed using the notion of a boundary strip (sometimes called a rim hook [JK81]) of a Young diagram F^λ. It is a continuous strip of boxes obtained by starting at the rightmost end of a row of F^λ and, for a number of steps, recursively passing to the box below if one exists, otherwise passing to the box to the left. The strip ends at the bottom of any column to the left of, or in, the column in which it started. The length of the boundary strip is the number of boxes that it comprises.

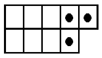
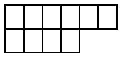
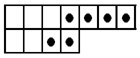
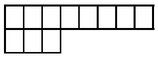
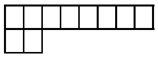
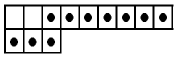
Theorem 2. *If $\lambda = (\lambda_1, \lambda_2)$ and q is a primitive pth root of unity then S^λ is reducible if and only if for some integer $k > 0$, F^λ has a boundary strip of length kp having at least one, but not more than $p-1$ boxes in the second row (or equivalently, if there exists an integer $k > 0$ such that $\lambda_1 - \lambda_2 + 2 \leq kp \leq \min\{\lambda_1 + 1, \lambda_1 - \lambda_2 + p\}$). If so, S^λ has an irreducible submodule corresponding to the diagram obtained by moving all the boxes of the boundary strip into the top row. The corresponding quotient module is irreducible. That is:*

$$S^{(\lambda_1,\lambda_2)} = D^{(\mu_1,\mu_2)}_p \ni\!\!\!\!- D^{(\lambda_1,\lambda_2)}_p, \tag{24}$$

where

$$(\mu_1, \mu_2) = (\lambda_1 + p - 1 - (\lambda_1 - \lambda_2) \bmod p, \lambda_2 - p + 1 + (\lambda_1 - \lambda_2) \bmod p) \quad (25)$$

This theorem is illustrated by the following table, which for various $\lambda = (\lambda_1, \lambda_2)$ and p, displays the Young diagram F^λ with the appropriate boundary strip indicated, says whether S^λ is reducible, and shows its composition series.

λ	p	F^λ	S^λ	Composition
$(5,4)$	3		reducible	$S^{(5,4)} = D_3^{(6,3)} \mathbin{+\!\!\!\!\supset} D_3^{(5,4)}$
$(6,4)$	3		irreducible	$S^{(6,4)} = D_3^{(6,4)}$
$(7,4)$	3		reducible	$S^{(7,4)} = D_3^{(9,2)} \mathbin{+\!\!\!\!\supset} D_3^{(7,4)}$
$(8,3)$	3		irreducible	$S^{(8,3)} = D_3^{(8,3)}$
$(8,2)$	5		irreducible	$S^{(8,2)} = D_5^{(8,2)}$
$(9,3)$	5		reducible	$S^{(9,3)} = D_5^{(12)} \mathbin{+\!\!\!\!\supset} D_5^{(9,3)}$

Theorem 2 has the consequence that the character $\tilde{\chi}_p^\lambda$ of the irreducible representation corresponding to D_p^λ may be expressed as a finite sum over the characters $\chi^\lambda(q)$ of the generic representations of $H_n(q)$ (which themselves may be calculated using the methods and formulae of [KWy90, KWy92, Rm91, Vj91]).

Theorem 3. *If $S^{(\lambda_1,\lambda_2)}$ is reducible then, using the notation of Theorem 2,*

$$\tilde{\chi}_p^\lambda = \sum_{j=0}^{[\lambda_2/p]} \chi^{(\lambda_1+jp,\lambda_2-jp)}(q) - \sum_{j=0}^{[\mu_2/p]} \chi^{(\mu_1+jp,\mu_2-jp)}(q), \quad (26)$$

where $[x]$ is the largest integer less than or equal to x.

Of course, this Theorem may be used to give the dimension of D_p^λ in terms of the dimensions f^ν of the irreducible representations of S_n. For example,

$$\begin{aligned}
\dim D_4^{(4,2)} &= f^{(4,2)} - f^{(5,1)} = 9 - 5 = 4; \\
\dim D_2^{(4,2)} &= f^{(4,2)} + f^{(6,0)} - f^{(5,1)} = 9 + 1 - 5 = 5; \\
\dim D_3^{(6,5)} &= f^{(6,5)} + f^{(9,2)} - f^{(7,4)} - f^{(10,1)} = 132 + 44 - 165 - 10 = 1.
\end{aligned} \quad (27)$$

In fact, $D_3^{(6,5)} \cong S^{(1^{11})}$ (note that (1^{11}) is not a 3-regular partition).

5 The root-standard basis

When q is a primitive pth root of unity, the irreducible $H_n(q)$-module $D_p^{(\lambda_1,\lambda_2)}$ may be constructed along lines similar to the construction of the Specht modules. A basis for $D_p^{(\lambda_1,\lambda_2)}$ may be defined in terms of a certain subset of the set of standard tableaux. In order to specify this set, let

$$T^{(\lambda_1,\lambda_2)} = \begin{matrix} a_1\, a_2\, a_3 \,\cdot\, \cdot\, \cdot\, \cdot\, \cdot\, a_{\lambda_1} \\ b_1\, b_2\, b_3 \,\cdot\, \cdot\, b_{\lambda_2} \end{matrix}, \tag{28}$$

and say that T^λ is s-strip standard at the ith position if:

$$b_i < a_{i+s-2} \tag{29}$$

(or if $i > \lambda_1 - s + 2$, when of course a_{i+s-2} is undefined).

Definition 1. *If $\lambda = (\lambda_1, \lambda_2)$ and the positive integers p and k are such that $\lambda_1 - \lambda_2 + 2 \leq kp \leq \min\{\lambda_1 + 1, \lambda_1 - \lambda_2 + p\}$, then T^λ is said to be p-*root standard *if T^λ is standard and either:*

1. *T^λ is kp-strip standard at positions $1, 2, \ldots, \lambda_2$;*

or 2. *to the right of the rightmost position of a non-standard kp-strip, there is a position at which T^λ is $((k-1)p+2)$-strip standard.*

Note that in the important case of $k = 1$, the second condition here can never be satisfied because standardness denies 2-strip standardness. In this case, the tableaux are identical to those defined in [Wn88] for the corresponding representations.

As an example, consider $\lambda = (7, 4)$ and $p = 3$ (so that $k = 2$). In this case, the following are 3-root standard:

$$\begin{matrix} 1 & 2 & 3 & 4 & 6 & 8 & 10 \\ 5 & 7 & 9 & 11 \end{matrix}, \quad \begin{matrix} 1 & 3 & 4 & 5 & 6 & 7 & 11 \\ 2 & 8 & 9 & 10 \end{matrix}, \quad \begin{matrix} 1 & 2 & 3 & 4 & 5 & 9 & 10 \\ 6 & 7 & 8 & 11 \end{matrix}; \tag{30}$$

whereas the following are not 3-root standard:

$$\begin{matrix} 1 & 3 & 5 & 6 & 7 & 8 & \mathbf{9} \\ 2 & 4 & \mathbf{10} & 11 \end{matrix}, \quad \begin{matrix} 1 & 3 & 4 & 5 & 6 & \mathbf{7} & 10 \\ 2 & \mathbf{8} & 9 & 11 \end{matrix}, \quad \begin{matrix} 1 & 2 & 3 & 4 & \mathbf{5} & 8 & 10 \\ \mathbf{6} & 7 & 9 & 11 \end{matrix}; \tag{31}$$

with the highlighted entries indicating the offending 6-strip.

Theorem 4. *If $\lambda = (\lambda_1, \lambda_2)$, the dimension of D_p^λ is equal to the number of p-root standard tableaux of shape λ.*

Let d_p^λ denote the dimension of D_p^λ. In the case $p = 2$, we immediately deduce from Theorem 1 that if $n = \lambda_1 + \lambda_2$ is odd then $d_2^\lambda = f^\lambda$. Furthermore, if n is even (so that $\lambda_1 - \lambda_2$ is even and positive), then Definition 1 forces the entry n to be in the final box of the top row. The other entries are then restricted only to be in a standard configuration. Thus if λ is 2-regular:

$$d_2^{(\lambda_1,\lambda_2)} = \begin{cases} f^{(\lambda_1,\lambda_2)} & \text{if } n \text{ is odd;} \\ f^{(\lambda_1-1,\lambda_2)} & \text{if } n \text{ is even.} \end{cases} \tag{32}$$

This result was obtained in [JM95] by considering the restriction of the $H_n(-1)$-module D_2^λ to $H_{n-1}(-1)$. In fact, either of these arguments may be generalised to yield:

$$d_p^{(\lambda_1,\lambda_2)} = d_p^{(\lambda_1-1,\lambda_2)} \quad \text{if } (\lambda_1 - \lambda_2 + 2) \bmod p \equiv 0. \tag{33}$$

6 Explicit $D_p^{(\lambda_1,\lambda_2)}$

As indicated above, the non-generic $H_n(q)$-module D_p^λ may be explicitly constructed with basis:

$$\{v_{t^\lambda} : t^\lambda \text{ is } p\text{−root standard}\}. \tag{34}$$

As for the Specht module, the explicit construction process relies on being able to express terms indexed by arbitrary tableaux in terms of those that are in the basis. For the explicit construction of D_p^λ, the column and Garnir relations are retained and are supplemented by additional relations. These relations are described here.

By using the column and Garnir relations, any term may be expressed in terms of standard tableaux. The rewriting of v_{t^λ} with t^λ standard in terms of p-root standard tableaux involves v^μ (as defined in (23)) and similar expressions. So assume that t^λ is standard and that i is the largest number such that t^λ is not kp-strip standard at the ith position. Three cases need to be considered.

Case 1. $i > \lambda_2 - p$. Let $x = \lambda_2 - i + 1$ which is the number of boxes in the second row to the right of, and including, the non strip-standard position. Now let:

$$v_0 = \frac{1}{[x]_q!} \sum_{d \in \mathcal{D}^{2\lambda_2+1}_{\lambda_2+i,\lambda_2+i+kp-2}} v_{dt^\lambda_{x-}} = \sum_{\text{standard } z^\lambda} c(z^\lambda) v_{z^\lambda}, \tag{35}$$

where the column and Garnir relations have been used to obtain the sum over standard tableaux, and where each $c(t^\lambda)$ is a polynomial in q. From Theorem 1, it can be seen that $c(t^\lambda_{*+}) = 1$ here where t^λ_{*+} has $\{\lambda_2 + i + kp - 1, \ldots, n\}$ in the last $\lambda_1 - i - kp + 2$ boxes of the top row but is otherwise identical to $t^{(i+kp-2,\lambda_2)}_{x+}$. Then, as may be shown, the quotienting out of the submodule implies that $v_0 = 0$. Thereby, an expression for $v_{t^\lambda_{*+}}$ is obtained in terms of other tableaux. Acting on the tableaux that index the vectors in this expression with $w \in S_n$ defined such that $t^\lambda = wt^\lambda_{*+}$, may be shown to yield an expression for $v_{t^\lambda} \in D^\lambda_p$:

$$v_{t^\lambda} = - \sum_{\text{standard } z^\lambda \neq t^\lambda_{*+}} c(z^\lambda) v_{wz^\lambda}. \tag{36}$$

For $\lambda = (4,2)$ and $p = 4$ (so that $k = 1$), we will consider a number of examples. First let

$$t^\lambda_1 = \begin{matrix} 1 & 3 & 4 & 5 \\ 2 & 6 & & \end{matrix}, \tag{37}$$

for which $i = 2$ and $x = 1$, whereupon

$$t^\lambda_{x-} = \begin{matrix} 1 & 3 & \mathbf{5} & \mathbf{6} \\ 2 & \mathbf{4} & & \end{matrix}, \tag{38}$$

$t^\lambda_{*+} = t^\lambda_1$, $w = 1$ and we require $\mathcal{D}^5_{4,6} = \{1, s_4, s_5 s_4\}$ which permutes the highlighted entries. Then on setting this particular case of (35) to zero yields:

$$\begin{matrix} 1 & 3 & 4 & 5 \\ 2 & 6 & & \end{matrix} = - \begin{matrix} 1 & 3 & 4 & 6 \\ 2 & 5 & & \end{matrix} - \begin{matrix} 1 & 3 & 5 & 6 \\ 2 & 4 & & \end{matrix}, \tag{39}$$

an expression which immediately gives $v_{t^\lambda_1}$ in terms of 4-root standard tableaux.

For the tableau

$$t^\lambda_2 = \begin{matrix} 1 & 2 & 3 & 4 \\ 5 & 6 & & \end{matrix}, \tag{40}$$

again $i = 2$, and $x = 1$, so that t^λ_{x-} and t^λ_{*+} are as above, but now $w = s_4 s_3 s_2$. Thus, it is required to act on the tableaux in the expression (39) with $s_4 s_3 s_2$, which gives:

$$\begin{matrix} 1 & 2 & 3 & 4 \\ 5 & 6 & & \end{matrix} = - \begin{matrix} 1 & 2 & 3 & 6 \\ 5 & 4 & & \end{matrix} - \begin{matrix} 1 & 2 & 4 & 6 \\ 5 & 3 & & \end{matrix}. \tag{41}$$

Here the resulting terms are not 4-root standard and further processing would be required to render $v_{t^\lambda_2}$ in terms of 4-root standard tableaux.

For the tableau

$$t_3^\lambda = \begin{array}{llll} 1 & 2 & 3 & 6 \\ 4 & 5 & & \end{array}, \tag{42}$$

$i = 1$ and $x = 2$, whereupon

$$t_{x-}^\lambda = \begin{array}{llll} 1 & 2 & \mathbf{5} & 6 \\ \mathbf{3} & \mathbf{4} & & \end{array}, \tag{43}$$

$t_{*+}^\lambda = t_3^\lambda$, $w = 1$ and we require $\mathcal{D}_{3,5}^5 = \{1, s_3, s_4, s_4s_3, s_3s_4, s_3s_4s_3\}$, Thus, in this case, (35) yields:

$$v_0 = \begin{array}{llll} 1 & 2 & 5 & 6 \\ 3 & 4 & & \end{array} + \begin{array}{llll} 1 & 2 & 5 & 6 \\ 4 & 3 & & \end{array} + \begin{array}{llll} 1 & 2 & 4 & 6 \\ 3 & 5 & & \end{array} + \begin{array}{llll} 1 & 2 & 4 & 6 \\ 5 & 3 & & \end{array} + \begin{array}{llll} 1 & 2 & 3 & 6 \\ 4 & 5 & & \end{array} + \begin{array}{llll} 1 & 2 & 3 & 6 \\ 5 & 4 & & \end{array}, \tag{44}$$

which, on standardisation using the column and Garnir relations, results in:

$$v_0 = (1+q)\left((q^3-q^2)\begin{array}{llll} 1 & 3 & 5 & 6 \\ 2 & 4 & & \end{array} - q^2\begin{array}{llll} 1 & 3 & 4 & 6 \\ 2 & 5 & & \end{array} + \begin{array}{llll} 1 & 2 & 5 & 6 \\ 3 & 4 & & \end{array} + \begin{array}{llll} 1 & 2 & 4 & 6 \\ 3 & 5 & & \end{array} + \begin{array}{llll} 1 & 2 & 3 & 6 \\ 4 & 5 & & \end{array}\right), \tag{45}$$

where, in this particular case, it has not been necessary to use $[4]_q = 0$ to enable the factor $[x]_q = (1+q)$ to be extracted. Setting this expression to zero yields the requisite expression for $v_{t_3^\lambda}$:

$$\begin{array}{llll} 1 & 2 & 3 & 6 \\ 4 & 5 & & \end{array} = (q-1)\begin{array}{llll} 1 & 3 & 5 & 6 \\ 2 & 4 & & \end{array} - \begin{array}{llll} 1 & 3 & 4 & 6 \\ 2 & 5 & & \end{array} - \begin{array}{llll} 1 & 2 & 5 & 6 \\ 3 & 4 & & \end{array} - \begin{array}{llll} 1 & 2 & 4 & 6 \\ 3 & 5 & & \end{array}, \tag{46}$$

where use has been made of $q^2 = -1$.

Case 2. $i \le \lambda_2 - p$ **and** $k = 1$**.** The relations obtained in this case are similar to those obtained in Case 1. They may be viewed as those in that case having been moved leftward through the tableaux. Here let $x = p-1$ which is again the number of boxes in the second row over which a symmetrisation process takes place. Now define t_*^λ to be identical to t_-^λ in the first $i-1$ columns and also from columns $i+p-1$ to the last. The remaining entries $\{2i-1, \ldots, 2(i+p)-4\}$ are placed in the remaining positions, across first the top row and then across the bottom row (so that in its first $i+p-2$ columns, t_*^λ is identical to $t_{x-}^{(i+p-2,i+p-2)}$). For example if $\lambda = (12, 9)$, $p = 5$ and $i = 4$ then

$$t_*^{(12,9)} = \begin{array}{|ccc|cccc|ccccc|} \hline 1 & 3 & 5 & 7 & 8 & 9 & 10 & 15 & 17 & 19 & 20 & 21 \\ \hline 2 & 4 & 6 & \mathbf{11} & \mathbf{12} & \mathbf{13} & \mathbf{14} & 16 & 18 & & & \\ \hline \end{array}, \tag{47}$$

where the partitioning gives an indication as to how the entries have been entered. Now, in analogy with (35), let:

$$v_0 = \frac{1}{[x]_q!} \sum_{d \in W_{2i+p-2,2(i+p-2)}} v_{dt_*^\lambda} = \sum_{\text{standard } z^\lambda} c(z^\lambda) v_{z^\lambda}, \tag{48}$$

where again the column and Garnir relations have been used to obtain the sum over standard tableaux. As before, $c(t_*^\lambda) = 1$. This v_0 also lies in the submodule and thus setting $v_0 = 0$ results in an expression for $v_{t_*^\lambda} \in D_p^\lambda$. To obtain an expression for v_{t^λ}, it is again valid to act on each standard tableau with $w \in S_n$ for which $wt_*^\lambda = t^\lambda$:

$$v_{t^\lambda} = -\sum_{\text{standard } z^\lambda \neq t_*^\lambda} c(z^\lambda) v_{wz^\lambda}. \tag{49}$$

To illustrate this case, let $\lambda = (4,4)$, $p = 4$ (so that $k = 1$) and

$$t_4^\lambda = \begin{array}{cccc} 1 & 2 & 3 & 6 \\ 4 & 5 & 7 & 8 \end{array}, \tag{50}$$

for which $i = 1$ and $x = 3$, whereupon

$$t_*^\lambda = \begin{array}{cccc} 1 & 2 & 3 & 7 \\ \mathbf{4} & \mathbf{5} & \mathbf{6} & 8 \end{array}, \tag{51}$$

and $w = s_6$. Then using $W_{4,6} = \{1, s_4, s_5, s_5s_4, s_4s_5, s_4s_5s_4\}$, in accordance with (48),

$$\begin{aligned} [3]_q! v_0 = {} & \begin{array}{cccc} 1 & 2 & 3 & 7 \\ 4 & 5 & 6 & 8 \end{array} + \begin{array}{cccc} 1 & 2 & 3 & 7 \\ 5 & 4 & 6 & 8 \end{array} + \begin{array}{cccc} 1 & 2 & 3 & 7 \\ 4 & 6 & 5 & 8 \end{array} + \begin{array}{cccc} 1 & 2 & 3 & 7 \\ 6 & 4 & 5 & 8 \end{array} + \begin{array}{cccc} 1 & 2 & 3 & 7 \\ 5 & 6 & 4 & 8 \end{array} + \begin{array}{cccc} 1 & 2 & 3 & 7 \\ 6 & 5 & 4 & 8 \end{array} \\ = {} & q^4(1+q)^2 \begin{array}{cccc} 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \end{array} - q^3(1+q) \left(\begin{array}{cccc} 1 & 3 & 4 & 7 \\ 2 & 5 & 6 & 8 \end{array} + \begin{array}{cccc} 1 & 2 & 5 & 7 \\ 3 & 4 & 6 & 8 \end{array} + \begin{array}{cccc} 1 & 2 & 4 & 7 \\ 3 & 5 & 6 & 8 \end{array} \right) \\ & + (1+q)(1+q+q^2) \begin{array}{cccc} 1 & 2 & 3 & 7 \\ 4 & 5 & 6 & 8 \end{array}, \end{aligned} \tag{52}$$

for which the $[3]_q!$ factor on the right is made manifest on using $[4]_q = 0$ in the form $q^3 = -1 - q - q^2$. This results in:

$$v_0 = -q(1+q) \begin{array}{cccc} 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \end{array} + \begin{array}{cccc} 1 & 3 & 4 & 7 \\ 2 & 5 & 6 & 8 \end{array} + \begin{array}{cccc} 1 & 2 & 5 & 7 \\ 3 & 4 & 6 & 8 \end{array} + \begin{array}{cccc} 1 & 2 & 4 & 7 \\ 3 & 5 & 6 & 8 \end{array} + \begin{array}{cccc} 1 & 2 & 3 & 7 \\ 4 & 5 & 6 & 8 \end{array}, \tag{53}$$

whence, on setting $v_0 = 0$, $q^2 = -1$, and acting on each tableau with $w = s_6$, we get the requisite expression for $v_{t_4^\lambda}$:

$$\begin{array}{cccc} 1 & 2 & 3 & 6 \\ 4 & 5 & 7 & 8 \end{array} = (q-1) \begin{array}{cccc} 1 & 3 & 5 & 6 \\ 2 & 4 & 7 & 8 \end{array} - \begin{array}{cccc} 1 & 3 & 4 & 6 \\ 2 & 5 & 7 & 8 \end{array} - \begin{array}{cccc} 1 & 2 & 5 & 6 \\ 3 & 4 & 7 & 8 \end{array} - \begin{array}{cccc} 1 & 2 & 4 & 6 \\ 3 & 5 & 7 & 8 \end{array}. \tag{54}$$

Note the similarity to (46). This is because essentially the same symmetrisation process has taken place.

Case 3. $i \le \lambda_2 - p$ **and** $k > 1$. The final case may also be viewed as the symmetrisation process moved leftward. However, the situation here is not so straightforward in that the entries to the right of the symmetrised section are not constant across the analogue of (35) and (48). For the first part of the following algorithm, these entries are ignored. Again let $x = p-1$ and consider just t^ν_{x-} where $\nu = (kp+i-2, p+i-2)$. This tableau is used exactly as in (35) to give:

$$v_0^{\text{ig}} = \frac{1}{[x]_q!} \sum_{d \in \mathcal{D}^{2\nu_2+1}_{\nu_2+i,\nu_2+i+kp-2}} v_{dt^\nu_{x-}} = \sum_{\text{standard } z^\nu} c(z^\nu) v_{z^\nu}, \tag{55}$$

the Garnir and column relations having been used as for tableaux of shape ν. Now for each standard tableau z^ν in the sum here, form the tableau z^λ_{aug} by first appending the entries $\{2i+(k+1)p-3, \ldots, 2\lambda_2+(k-1)p\}$ one at a time, alternately onto the bottom row and onto the top row. The remaining entries $\{2\lambda_2+(k-1)p+1, \ldots, n\}$ (if any), are used to complete the top row. For example, if $\lambda = (17,9)$, $p = 5$, (so that $k = 2$) and $i = 3$, then each z^λ_{aug} will be a standard tableau of the form:

$$z^{(17,9)}_{\text{aug}} = \begin{array}{|cc|ccccccccc|ccc|ccc|} \hline 1 & 3 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & 19 & 21 & 23 & 24 & 25 & 26 \\ \hline 2 & 4 & \bullet & \bullet & \bullet & \bullet & 18 & 20 & 22 \\ \cline{1-9} \end{array}, \tag{56}$$

where each $\bullet$ represents an entry from $\{5,6,\ldots,17\}$. Now from v_0^{ig}, form the sum:

$$v_0^{\text{aug}} = \frac{1}{[x]_q!} \prod_{i \in \mathcal{E}^{2\lambda_2+(k-1)p-1}_{2i+(k+1)p-3}} (h_i - q) \sum_{\text{standard } z^\nu} c(z^\nu) v_{z^\lambda_{\text{aug}}}, \tag{57}$$

where $\mathcal{E}^b_a = \{a, a+2, a+4, \ldots, b\}$. In fact, the action of each h_i here may be accomplished (as may be seen from (4)) directly through the action of s_i on the tableau: $h_i v_{z^\lambda_{\text{aug}}} = v_{s_i z^\lambda_{\text{aug}}}$. In this sum, the term $v_{t^\lambda_*}$ has coefficient 1 where t^λ_* is defined as t^ν_{x+} augmented with the entries $\{2i+(k+1)p-3, \ldots, 2\lambda_2+(k-1)p\}$ placed one at a time, alternately onto the *top* row and onto the *bottom* row. Again, the remaining entries $\{2\lambda_2+(k-1)p+1, \ldots, n\}$ (if any), are used to complete the top row. For the example $\lambda = (17,9)$, $p = 5$, $i = 3$ considered above:

$$t^{(17,9)}_* = \begin{array}{|cc|ccccccccc|ccc|ccc|} \hline 1 & 3 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 18 & 20 & 22 & 24 & 25 & 26 \\ \hline 2 & 4 & 14 & 15 & 16 & 17 & 19 & 21 & 23 \\ \cline{1-9} \end{array}. \tag{58}$$

Now let w be such that $t^\lambda = wt^\lambda_*$. Unfortunately, in this case, the action of w on v_0^{aug}, given by (57), must take place through the Hecke algebra action of $h(w)$. Nonetheless, in the standardised expression for $h(w)v_0^{\mathrm{aug}}$, the term v_{t^λ} appears with a coefficient of 1, whereupon, on setting $h(w)v_0^{\mathrm{aug}}$ to zero (it also lies in the submodule), an expression for $v_{t^\lambda} \in D_p^\lambda$ is obtained.

Due to the large number of terms involved in instances of this case, working a full example is impractical. So we will outline the example $\lambda = (9,5)$, $p = 4$ (so that $k = 2$) and

$$t_5^\lambda = \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 10 & 13 \\ 8 & 9 & 11 & 12 & 14 \end{matrix}, \tag{59}$$

which is not 4-root standard with $i = 1$. So in this case we consider $\nu = (7,3)$ and

$$t_{x-}^\nu = \begin{matrix} 1 & 2 & 3 & \mathbf{7} & \mathbf{8} & \mathbf{9} & \mathbf{10} \\ \mathbf{4} & \mathbf{5} & \mathbf{6} \end{matrix}. \tag{60}$$

Then (55) produces a sum over all 75 standard tableaux of shape ν. Augmenting each of these with the entries $\{11, 12, 13, 14\}$ as indicated, results in a sum over 75 tableaux each of the form:

$$\begin{matrix} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & 12 & 14 \\ \bullet & \bullet & \bullet & 11 & 13 \end{matrix}. \tag{61}$$

The action of $(h_{11} - q)(h_{13} - q)$ then results in a sum over 300 terms. Amongst them appears

$$t_*^\lambda = \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 11 & 13 \\ 8 & 9 & 10 & 12 & 14 \end{matrix} \tag{62}$$

with a coefficient 1. Acting on this sum with $h(w)$ where $w = s_{10}$, and setting the result to zero, then yields the requisite expression for $v_{t_5^\lambda}$ in terms of 299 other tableaux.

In each of the three cases considered above, the resulting expression for v_{t^λ} may include terms that are not p-root standard (indeed this was the situation in the previous example, as it was also in (41)). If an expression solely in terms of p-root standard tableaux is required then the above relations together with the column and Garnir relations would have to recursively applied in order to obtain such an expression. This is guaranteed eventually since, as in Section 3, the ordering on the tableaux shows that an improvement takes place with each invocation.

7 Final example

For $\lambda = (4,2)$, the Specht module S^λ is 9-dimensional. When $p = 4$, the corresponding irreducible module D_4^λ is 4-dimensional, as was given in (27). The methods of Section 2 enable the action of each h_i on each of the four vectors indexed by the 4-root standard tableaux

$$\begin{matrix}1\,3\,5\,6\\2\,4\end{matrix}\;,\qquad \begin{matrix}1\,2\,5\,6\\3\,4\end{matrix}\;,\qquad \begin{matrix}1\,3\,4\,6\\2\,5\end{matrix}\;,\qquad \begin{matrix}1\,2\,4\,6\\3\,5\end{matrix}\;, \tag{63}$$

to be written in terms of standard tableaux. Those terms which are then not 4-root standard may be expressed in terms of such using (39), (46), and the immediate result of the action of s_2 on the tableaux of (39).

Consider the action of h_3 on the terms indexed by the tableaux (63) above:

$$\begin{aligned}
h_3 \begin{matrix}1\,3\,5\,6\\2\,4\end{matrix} &= \begin{matrix}1\,4\,5\,6\\2\,3\end{matrix} = -\begin{matrix}1\,3\,5\,6\\2\,4\end{matrix}\;,\\
h_3 \begin{matrix}1\,2\,5\,6\\3\,4\end{matrix} &= \begin{matrix}1\,2\,5\,6\\4\,3\end{matrix} = q\begin{matrix}1\,2\,5\,6\\3\,4\end{matrix} - q^2\begin{matrix}1\,3\,5\,6\\2\,4\end{matrix}\;,\\
h_3 \begin{matrix}1\,3\,4\,6\\2\,5\end{matrix} &= \begin{matrix}1\,4\,3\,6\\2\,5\end{matrix} = q\begin{matrix}1\,3\,4\,6\\2\,5\end{matrix} - q^2\begin{matrix}1\,3\,5\,6\\2\,4\end{matrix}\;,\\
h_3 \begin{matrix}1\,2\,4\,6\\3\,5\end{matrix} &= \begin{matrix}1\,2\,3\,6\\4\,5\end{matrix} = (q-1)\begin{matrix}1\,3\,5\,6\\2\,4\end{matrix} - \begin{matrix}1\,3\,4\,6\\2\,5\end{matrix} - \begin{matrix}1\,2\,5\,6\\3\,4\end{matrix} - \begin{matrix}1\,2\,4\,6\\3\,5\end{matrix}\;,
\end{aligned}$$

where the methods of Section 2 were sufficient in all but the last calculation, where (46) was used. Thus D_4^λ gives rise to the following representation matrix for h_3:

$$\begin{pmatrix} -1 & 1 & 1 & q-1\\ . & q & . & -1\\ . & . & q & -1\\ . & . & . & -1 \end{pmatrix}, \tag{64}$$

where $q^2 = -1$ has been used. The representation matrices for the other generators of $H_6(q)$ may be calculated in a similar manner.

It is interesting to note that for $\lambda = (4,2)$ and $p = 2$, the calculation proceeds in an almost identical manner. However, an extra basis vector is present since, by (27), D_2^λ is 5-dimensional. This vector is indexed by the tableau on the left of (46). That (46) cannot be used in this $p = 2$ case may be traced to the appearance of the $[2]_q$ factor in its derivation.

8 Footnote

The algorithms which have been described in this paper have been implemented in the computer algebra package SYMMETRICA. In the generic and the two-rowed non-generic cases, they enable representation matrices to be readily obtained for any element of $H_n(q)$. Further routines can check that the matrices so produced indeed provide a representation in that they respect (1). These have been used to check the algorithms presented in this paper in a large number of cases. In addition, in either the generic or two-rowed non-generic case, routines are available to generate and enumerate the appropriate standard tableaux, and to render an arbitrary tableau in terms of the appropriate standard tableaux. Calculations in $H_n(q)$ itself may also be undertaken.

SYMMETRICA has been developed at Bayreuth University, and is an extensive package of routines concerned with the symmetric and related groups, together with their representations and combinatorics. For more information about SYMMETRICA, access the WorldWideWeb page *http://btm2xd.mat.uni-bayreuth.de* or email *sym@btm2x2.mat.uni-bayreuth.de*. The full SYMMETRICA package, with documentation, is available from the above WorldWideWeb site or via FTP from *ftp://btm2x7.mat.uni-bayreuth.de/dist/SYM.tar.Z*.

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