T-Theory
An Overview

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Abstract

T-Theory is the name that we adopt for the theory of trees, injective envelopes of metric spaces, and all of the areas that are connected with these topics, which has been developed over the last 10-15 years in Bielefeld. Its motivation was originally – and still is to a large extent – the development of mathematical tools for reconstructing phylogenetic trees. T-theory expanded considerably when its relationships with the theory of affine buildings, valuated matroids, and decompositions of metrics were discovered. In this paper, we give a brief introduction to this theory, which we hope will serve as a useful reference to some of the main results, and also as a guide for further investigations into what T-theory has to offer.

1 Introduction

T-theory originated from a question raised by Manfred Eigen in the late seventies. At that time, he was trying to fit the twenty distinct t-RNA molecules of the *E. coli* bacterium, whose primary sequence structures were then known, into a tree. In doing this, he realized that there was an obstruction to finding such a tree even when only four sequences were to be processed. So he wondered:

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- Does the vanishing of this obstruction for all quartets in a given family of sequences imply the existence of a globally fitting tree?
- What could be used as a substitute for a tree if no globally fitting tree existed?

It soon became clear that the answer to the first question was yes. Also, even though this tree can be constructed recursively, its final shape does not depend upon the order in which the sequences are processed. This suggested the existence of a construction which, for a system of sequences fitting into a tree would produce that tree, and which for an arbitrary system would produce something that could be used as a tree substitute. Trying to find such a construction led to the T-construction described in Section 2. While studying that construction, a surprising amount of additional insights have accumulated over the last ten years – new theorems, unsuspected applications, and unforseen relationships with other subjects studied in theoretical mathematics. So a whole new branch of discrete mathematics, briefly called T-*theory*, has emerged.

In this note, a brief survey of T-theory is presented, with special emphasis on, as yet, unpublished results. Although, for the sake of conciseness, proofs have been omitted, we hope that the definitions and results stated here are clear enough to serve as a guide for further explorations into what T-theory may have to offer.

We are grateful to the editors of this special volume on "Discrete Metric Spaces" for having invited us to present a survey of T-theory in this context.

2 The T-Construction

Let $X = (X, d)$ be a metric space and, in cases where no confusion may arise, denote the distance between two points x, y of X by $xy := d(x, y)$. Let $\mathbb{R}^X$ denote the set of all functions which map X into $\mathbb{R}$, endowed with the L_∞-norm, given by the formula

$$||f|| := \sup_{x \in X} |f(x)| \in \mathbb{R} \cup \{+\infty\},$$

for any element f of $\mathbb{R}^X$. To the pair (X, d), we associate a subset $P_{(X,d)}$ of $\mathbb{R}^X$ defined by

$$P_{(X,d)} := \{f \in \mathbb{R}^X \mid f(x) + f(y) \geq xy \text{ for all } x, y \in X\}.$$

We also denote $P_{(X,d)}$ by P_X or $P(d)$, according to whether we wish to emphasize the dependence of $P_{(X,d)}$ on X or d, respectively. The *tight span* of (X,d), which we denote by $T_{(X,d)}$, T_X, or $T(d)$, is defined to be the set of minimal elements of $P_{(X,d)}$ with respect to the pointwise partial ordering of $\mathbb{R}^X$ (where $f \leq g$ if and only if $f(x) \leq g(x)$ for all $x \in X$). In [11], the tight span was introduced, and it was observed that the space $T_{(X,d)}$ can also be regarded as the set

$$T_{(X,d)} := \{f \in \mathbb{R}^X \mid f(x) = \sup_{y \in X}\{xy - f(y)\} \text{ for all } x \in X\}.$$

There is a canonical map, $h = h_X$, of the space (X,d) into $T_{(X,d)}$, which is given by $x \mapsto h_x$, where the function h_x is defined by the formula

$$h_x(y) := xy \text{ for all } y \in X.$$

Since

$$\begin{aligned} f(y) - xy &= \sup_{z \in X}\{yz - f(z)\} - xy \\ &= \sup_{z \in X}\{yz - f(z) - xy\} \\ &\leq \sup_{z \in X}\{yx + xz - f(z) - xy\} \\ &= \sup_{z \in X}\{xz - f(z)\} \\ &= f(x), \end{aligned}$$

for all $x, y \in X$ and $f \in T_X$, we have

$$\begin{aligned} \|f - h_x\| &= \sup_{y \in X} |f(y) - h_x(y)| \\ &= \sup_{y \in X} |f(y) - xy| \\ &= \max(\sup_{y \in X}\{xy - f(y)\}, \sup_{y \in X}\{f(y) - xy\}) \\ &= f(x) \end{aligned}$$

for all $f \in T_X$ and $x \in X$. In particular, for $f, g \in T_X$,

$$\|f - g\| \leq \|f - h_x\| + \|h_x - g\| = f(x) + g(x) < \infty,$$

so that the pair $(T_X, \|\cdot,\cdot\|)$ is a metric space, where the metric $\|\cdot,\cdot\|$ is defined by the formula $\|f, g\| = \|f - g\|$, for all f, g in T_X. Also, note that

$$\|h_x - h_y\| = \|h_y - h_x\| = h_x(y) = xy,$$

which implies that the map h_X is an isometric embedding of X into T_X.

3 Basic Properties of the T-Construction

In this section, we give a summary of some basic results concering $T_{(X,d)}$ where (X, d) is a metric space. Proofs of these results may be found in [22] and [11].

3.1 J. R. Isbell's description of T_X

In [22], J. R. Isbell studied what he called the *injective envelope* of a metric space. Here, we briefly recall his result and its very close relation to the T-construction. First, a *non-expansive* map between two metric spaces (X, d) and (Y, d') is a map $f : X \to Y$ such that $d'(f(x), f(y))$ is less than or equal to $d(x, y)$, for all x, y in X. Next, we define a metric space Y to be *isojective*[1] if, given any isometric embedding $e : X \to X'$ of metric spaces and any non-expansive map $f : X \to Y$, there exists a non-expansive map $f' : X' \to Y$ with $f = f' \circ e$. A non-expansive map $e : X \to E$ between metric spaces X and E is called an *isojective envelope* of X if E is isojective, e is an isometric embedding, and for every isometric embedding e' of X into another isojective metric space E' there exists a unique isometric embedding $e'' : E \to E'$ with $e' = e'' \circ e$. It follows immediately that any two isojective envelopes $e : X \to E$ and $f : X \to F$ of X are necessarily isomorphic, that is, given e and f, there exists a unique isometric bijection $h : E \to F$ with $f = h \circ e$. Theorem 2.1 of [22] tells us that for each metric space X the map $h_X : X \to T_X$ is an isojective envelope of X, that is,

- for each metric space X there exists an isojective envelope, and
- T_X together with the map h_X can be characterized – up to canonical isomorphism – by the property of providing that isojective envelope.

[1] In [22], such spaces have been called *injective*. We propose the term *isojective* instead, since – due to the fact that there are many monomorphic, non-expansive maps which are *not* isometric – there is a slight difference between the concept of an *injective space* as defined in [22] and the concept of an *injective object* as defined in homological algebra. In fact, injective objects in the latter sense do not exist in the category of metric spaces and non-expansive maps. Consequently, the *injective envelope* of a metric space as defined in [22] also does not coincide with the categorical concept of an injective envelope; so we are going to call such an envelope an *isojective envelope*. This topic is also addressed in more generality in [23, p192-3].

3.2 The space T_X is contractible

First, note that P_X is clearly contractible. In [11, p331], numerous possibilities for defining retractions of P_X onto T_X are described. Obviously, the existence of just one such retraction implies that T_X is contractible. We briefly describe a canonical way to define a retraction p, since this retraction has played a crucial role in proving results in the situation where a group is acting on the space X (for example see [13]).

For any $f \in P_X$, define the map f^* by the formula

$$f^*(x) = \sup_{y \in X}\{xy - f(y)\},$$

and note that $f \in P_X$ implies (and, actually, is equivalent to) $f^* \leq f$. Now, define a new map f' by the formula

$$f'(x) = 1/2(f(x) + f^*(x)).$$

It is not hard to show that $f' \in P_X$ and, therefore,

$$f^* \leq f'^* \leq f' \leq f.$$

We now repeat this process and define a monotonically decreasing sequence of functions by setting $f^{(0)} := f$ and $f^{(n+1)} := f^{n\prime}$, where $n \in \mathbb{N}_0$. The sequence $(f^{(n)})_{n \in \mathbb{N}_0}$ converges to some $\tilde{f}$ which belongs to T_X. The element $p(f)$ is defined to be equal to $\tilde{f}$.

3.3 The combinatorial dimension of a metric space

In [11], the combinatorial dimension of a metric space X is defined and investigated. We denote the combinatorial dimension of X by $\dim_{\text{comb}}(X)$. The following conditions are then equivalent:

- $\dim_{\text{comb}}(X) < n$;
- $\dim T_Y < n$ for all finite $Y \subseteq X$;
- for all $x_1, x_{-1}, \ldots, x_n, x_{-n}$ in X, there exists a permutation α of $I := \{\pm 1, \ldots, \pm n\}$, with $\alpha \neq -\mathrm{Id}_I$ and $\sum_{i \in I} x_i x_{-i} \leq \sum_{i \in I} x_i x_{\alpha(i)}$.

3.4 Consequences of X being finite

Suppose that X is finite. For each pair of points x, y of X, consider

$$H^+_{(x,y)} := \{f \in \mathbb{R}^X \mid f(x) + f(y) \geq xy\}$$

and

$$H_{(x,y)} := \{f \in \mathbb{R}^X \mid f(x) + f(y) = xy\}.$$

Clearly, $H^+_{(x,y)}$ is a closed half space of the finite dimensional space $\mathbb{R}^X$, whose boundary is equal to $H_{(x,y)}$. Moreover, we have the equality

$$P_X = \bigcap_{(x,y) \in X^2} H^+_{(x,y)}.$$

Hence, P_X is a convex (though not a compact) polytope in $\mathbb{R}^X$. For any $f \in P_X$, we define

$$\begin{aligned} K(f) &:= \{(x,y) \in X^2 \mid f(x) + f(y) = xy\} \\ &= \{(x,y) \in X^2 \mid f \in H_{(x,y)}\}, \end{aligned}$$

and

$$\begin{aligned} S(f) &:= P_X \cap \bigcap_{(x,y) \in K_f} H_{(x,y)} \\ &= \{g \in P_X \mid K(f) \subseteq K(g)\}. \end{aligned}$$

As usual, we call $S(f)$ the *facette* of f (relative to P_X). Using this terminology, the space T_X can be characterised as follows (see [12, Lemma 1]): For any function $f \in \mathbb{R}^X$, the following three statements are equivalent:

- f is contained in T_X.
- $S(f)$ is a subset of T_X.
- $S(f)$ is compact.

These statements imply that T_X is compact and, moreover, that T_X inherits a canonical cellular structure from the stratification of the convex polytope P_X, defined by the family of its facettes. In particular, the space T_X has a well-defined dimension, which can be shown to be bounded from above by $\lfloor \#X/2 \rfloor$, and we have the equalities

$$\dim_{\text{comb}}(X) = \dim T_X = \max\{\dim S(f) \mid f \in T_X\}.$$

3.5 The tight extension of T_X is equal to T_X

An extension (Y, d') of a metric space (X, d) is defined to be a *tight extension* if for any pseudo-metric $d'' : Y \times Y \to \mathbb{R}$ – i.e. any map $d'' : Y \times Y \to \mathbb{R}$ such that $d''(x, x) = 0$ and $d''(x, y) \leq d''(x, z) + d''(y, z)$, for all $x, y, z \in Y$ – satisfying the conditions

$$d''(x_1, x_2) = d(x_1, x_2), \text{ for any } x_1, x_2 \in X$$

and

$$d''(y_1, y_2) \leq d'(y_1, y_2), \text{ for any } y_1, y_2 \in Y,$$

one necessarily has $d''(y_1, y_2) = d'(y_1, y_2)$, for all y_1, y_2 in Y. Obviously, a tight extension of a tight extension of X is itself a tight extension of X. It has been shown in [11] that the space T_X is the *universal tight extension* of X, in the sense that it is a tight extension of X, it contains, up to canonical isometries, every other tight extension of X, and it has no proper tight extension itself. In particular, the map $h_{T_X} : T_X \to T_{T_X}$ is a bijection – a fact which, of course, also follows from Isbell's functorial description of T_X.

3.6 When is X equal to T_X?

This question is answered by [11, Theorem 2]. It is shown that the following statements are equivalent:

- The space X is equal to T_X, that is, the embedding $h_X : X \to T_X$ is a bijection or, equivalently, a surjection.
- The space X has no proper tight extension.
- The space X is an isojective metric space.
- For every $f \in P_X$, there exists an $x \in X$ such that for all $y \in X$ we have $xy \leq f(y)$.
- For every $f \in T_X$, there exists an $x \in X$ such that $f(x) = 0$.

4 Trees

4.1 One of many possible definitions of $\mathbb{R}$-trees

An $\mathbb{R}$-*tree* is a complete metric space $X = (X, d)$ satisfying the following conditions:

- for any $x, y \in X$, there exists a unique isometric embedding $\phi = \phi_{x,y}$ of the closed interval $[0, xy] \subseteq \mathbb{R}$ into X such that $\phi(0) = x$, $\phi(xy) = y$, and, therefore, $\phi_{x,y}([0, xy]) = \langle x, y \rangle := \{z \in X \mid xy = xz + zy\}$;

- for any injective continuous map $\phi : [0, 1] \hookrightarrow X : t \longmapsto x_t$ of the unit interval $[0, 1] \subseteq \mathbb{R}$ into X, one has $\phi([0, 1]) \subseteq \langle x_0, x_1 \rangle$, and therefore $\phi([0, 1]) = \langle x_0, x_1 \rangle$.

If X is an $\mathbb{R}$-tree, then $h_X : X \to T_X$ is a bijection (see [11]).

4.2 A striking example: the Real Tree

Let $X_{\mathbb{R}}$ denote the set of all bounded subsets of $\mathbb{R}$ which contain their infimum, and define a map d from $X_{\mathbb{R}} \times X_{\mathbb{R}}$ to $\mathbb{R}$ by

$$d(x, y) := 2 \cdot \max\{\sup(x \Delta y), \inf x, \inf y\} - (\inf x + \inf y)$$

for all such subsets x and y of $\mathbb{R}$ (where $x \Delta y$ denotes the symmetric difference of the subsets x and y). Then d is a metric on $X_{\mathbb{R}}$, and $X_{\mathbb{R}}$ is an $\mathbb{R}$-tree relative to this metric. We call $X_{\mathbb{R}}$ the *Real Tree*. It has many intriguing properties, the most interesting one being that, for every $x \in X_{\mathbb{R}}$, the cardinality of the set of connected components of the set $X_{\mathbb{R}} - \{x\}$ is equal to the cardinality of the *powerset* of $\mathbb{R}$ (see [17]).

4.3 The four-point condition

For a metric space X, the following statements are equivalent (see [11]):

- X satisfies the *four-point condition*, that is,

  $$uv + xy \leq \max\{xu + yv, xv + yu\}$$

  holds for all $u, v, x, y \in X$;

- X can be embedded isometrically into an $\mathbb{R}$-tree;

- T_X is an $\mathbb{R}$-tree.

Moreover, in such a case, T_X is the *smallest* $\mathbb{R}$-tree into which X can be embedded isometrically. Finally, a metric space X is an $\mathbb{R}$-tree if and only if it is complete, (arcwise) connected, and it satisfies the four-point condition.

4.4 δ-Hyperbolic spaces

The concept of a δ-hyperbolic metric space is of interest in the theory of hyperbolic groups (see [21], for example). Let δ be a non-negative real number. A metric space (X, d) is called δ-hyperbolic if and only if it satisfies the following "relaxed" four-point condition;

$$uv + xy \leq \max\{xu + yv, xv + yu\} + \delta \text{ for all } x, y, u, v \in X.$$

It is not hard to show that if X is δ-hyperbolic then the space T_X is δ-hyperbolic as well. Thus, in particular, if a group G acts isometrically on a δ-hyperbolic space then the T-construction provides a δ-hyperbolic *contractible* space for G to act on isometrically. This may be of particular interest when there exists a length function $l : G \to \mathbb{R}_{\geq 0}$ on G such that G is δ-hyperbolic with respect to the induced metric (cf. Section 7).

4.5 Finitely generated $\mathbb{R}$-trees

If X is a *finite* metric space satisfying the four-point condition, then the $\mathbb{R}$-tree T_X can be viewed as a graph theoretical tree (with positively weighted edges, whose weights are represented by their lengths). The vertices are those elements $p \in T_X$ with $T_X - \{p\}$ being either connected (the *leaves* of that tree, which are necessarily of the form h_x for some $x \in X$) or consisting of at least three connected components, while the edges correspond to the subsets of the form $\langle p, q\rangle$ $(p \neq q)$ of T_X with p, q being vertices and with no vertices being contained in the *open edge* $\langle p, q\rangle - \{p, q\}$ (or, equivalently, with the property that there are precisely two connected components in the complement of that open edge in T_X).

In addition, these edges are in one-to-one correspondence to the *d-splits* of X (see Section 5), that is, pairs $\{A, B\}$ of non-empty subsets of X with $A \cap B = \emptyset$ and $A \cup B = X$ such that

$$aa' + bb' < \min\{ab + a'b', ab' + a'b\}$$

holds for all $a, a' \in A$ and $b, b' \in B$ and, therefore, $aa' + bb' < ab + a'b' = ab' + a'b$, since d satisfies the four-point condition. This correspondence is given by associating to each open edge e the split induced on X by the decomposition of $T_X - e$ into its two connected components, that is, if $x \in X$ belongs to either A or B, then A (or B) consists of all $y \in X$ for which h_y and h_x are in the same connected component of $T_X - e$.

4.6 Ends of $\mathbb{R}$-trees

Given an $\mathbb{R}$-tree X, an *end* of X is an equivalence class of isometric embeddings $\varphi : \mathbb{R}_{\geq 0} \hookrightarrow X$ of the non-negative real numbers into X where φ is equivalent to ψ if and only if there exist $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}_{\geq 0}$ with $\alpha + \beta \geq 0$ and $\varphi(t) = \psi(t + \alpha)$ for all $t \geq \beta$.

If one chooses some $x_0 \in X$, then each end φ of X can be represented by an isometric embedding $\varphi_0 : \mathbb{R}_{\geq 0} \hookrightarrow X$ with $\varphi_0(0) = x_0$. Let $E(X) = E_{x_0}(X)$ be the set of all such isometric embeddings φ_0, and assume $E(X) \neq \emptyset$. Define a map $v : E(X) \times E(X) \to \mathbb{R} \cup \{-\infty\}$ by

$$v(\varphi_0, \psi_0) := -2 \cdot \sup\{t \in \mathbb{R}_{\geq 0} \mid \varphi_0(t) = \psi_0(t)\}.$$

This map satisfies the conditions:

(1) $v(\varphi_0, \psi_0) = v(\psi_0, \varphi_0)$,

(2) $v(\varphi_0, \psi_0) = -\infty \Leftrightarrow \varphi_0 = \psi_0$, and

(3) $v(\varphi_0, \varphi_0') + v(\psi_0, \psi_0') \leq \max\{v(\varphi_0, \psi_0) + v(\varphi_0', \psi_0'), v(\varphi_0, \psi_0') + v(\varphi_0', \psi_0)\}$,

for all $\varphi_0, \varphi_0', \psi_0, \psi_0' \in E(X)$. Clearly, v differs from a metric satisfying the four-point condition only by the fact that the diagonal $\{(\varphi_0, \varphi_0) \mid \varphi_0 \in E(X)\} \subseteq E(X) \times E(X)$ is the v-preimage, not of 0, but of $-\infty$ – though, of course, just changing the value of v from $-\infty$ to 0 for all pairs from the diagonal would lead to a violation of (3). Hence, there is a more fundamental difference between metrics satisfying the four-point condition and maps which satisfy the above three conditions.

Even so, one can carry out the T-construction on the pair $(E(X), v)$ as before by defining the space $T = T_{(E(X),v)}$ to be the set

$$\{p : E(X) \to \mathbb{R} \mid p(\varphi_0) = \sup_{\psi_0 \in E(X)} \{v(\varphi_0, \psi_0) - p(\psi_0)\} \text{ for all } \varphi_0 \in E(X)\},$$

on which, as before, a metric can be defined by the map

$$(p, q) \mapsto \sup_{\varphi_0 \in E(X)} |p(\varphi_0) - q(\varphi_0)| \text{ for all } p, q \in T.$$

T can then be identified canonically with a sub-$\mathbb{R}$-tree of (X, d), and one has $T = X$ if and only if for all $x, y \in X$ there exists some $z \in X - \{y\}$ with $y \in \langle x, z \rangle$ or, equivalently, if for every $x \in X$ there exists an isometric embedding $\varphi : \mathbb{R} \hookrightarrow X$ of the real numbers into X with x in its image

$\varphi(\mathbb{R})$ (see [25]). In general, T corresponds to the union of the images of all isometric embeddings of $\mathbb{R}$ into X, which we denote by $X_0 \subseteq X$. The correspondence is given by associating to each $x \in X_0$ the map $p_x : E(X) \to \mathbb{R}$ which associates to each $\varphi_0 \in E(X)$ the real number $\alpha \in \mathbb{R}$ for which $\varphi_0(t) = \varphi(t+\alpha)$ holds for all $t >> 0$, where $\varphi : \mathbb{R}_{\geq 0} \hookrightarrow X$ is the unique isometric embedding with $\varphi(0) = x$ which is equivalent to φ_0, that is, for which such an $\alpha \in \mathbb{R}$ exists.

4.7 Trees from ends

Given a non-empty set E and a map $v : E \times E \to \mathbb{R} \cup \{-\infty\}$ satisfying (4.6, (1)-(3)), $T_{(E,v)}$ is an $\mathbb{R}$-tree, and the set of ends of this $\mathbb{R}$-tree is the *"v-adic" completion* of E (see [16] and [25]).

Important examples for such pairs (E,v) arise as follows: for a prime number p, let $w_p : \mathbb{Q} \to \mathbb{Z} \cup \{-\infty\}$ denote the p-adic valuation of the rational number field $\mathbb{Q}$. If one puts $E := \mathbb{Q}^2 \backslash \{0\}$ and

$$v := w_p \circ \det : E \times E \to \mathbb{Z} \cup \{-\infty\},$$

the composition of the determinant with the p-adic valuation, then the pair (E,v) satisfies (4.6,(1)-(3)). Moreover, the equivalence classes of the ends of the $\mathbb{R}$-tree $T_{(E,v)}$ are in one-to-one correspondence to the points in the projective line over the p-adic completion $\mathbb{Q}_p$ of $\mathbb{Q}$. Of course, corresponding results hold for any pair (F,w) where F is a field and $w : F \to \mathbb{R} \cup \{-\infty\}$ is a valuation of F.

4.8 The ends of the Real tree

The space of ends of the Real Tree as defined in 4.2 is easy to describe: it is isomorphic to the set E of all subsets of $\mathbb{R}$ which are bounded from above, plus some additional element $*$ (represented by the isometry $\phi_* : \mathbb{R}_{\geq 0} \to X_{\mathbb{R}} : t \to \{t\}$) and can be endowed with the map v from $E \times E$ to $\mathbb{R} \cup \{-\infty\}$ defined by $v(e,f) := \sup(e \Delta f)$, if $e, f \neq *$, $v(e,*) = v(*,e) := 0$, if $e \neq *$, and, of course, $v(*,*) := -\infty$. Indeed, the simplest way to construct $X_{\mathbb{R}}$ and to study its properties, is to analyse the pair (E,v) first and then to identify $X_{\mathbb{R}}$ with $T_{(E,v)}$ (see [17]).

4.9 Buildings

One can generalize T-theory to pairs (E,v) of "higher" rank, satisfying appropriate analogues of (4.6, (1)-(3)):

A *simple valuated matroid* is a pair (E, v) consisting of a set E and a map $v : \mathcal{P}_{\text{fin}}(E) \to \mathbb{R} \cup \{-\infty\}$ (where $\mathcal{P}_{\text{fin}}(E) := \{x \subseteq E \mid \#x < \infty\}$) satisfying the following variant of the Steinitz exchange condition:

(SEP) for all $x, y \in \mathcal{P}_{\text{fin}}(E)$ and $a \in x \backslash y$ there exists some $b \in y \backslash x$ with

$$v(x) + v(y) \leq v((x \cup \{b\}) \backslash \{a\}) + v((y \cup \{a\}) \backslash \{b\})$$

(see [18] and [19]). Then one can define the space $T_{(E,v)}$ to be the set

$$\{p : E \to \mathbb{R} \mid p(e) = \sup_{x \subseteq \mathcal{P}_{\text{fin}}(E \backslash \{e\})} \{v(x \cup \{e\}) - \sum_{a \in x} p(a)\} \text{ for all } e \in E\},$$

where, again, the map

$$(p, q) \mapsto \sup_{e \in E} |p(e) - q(e)| \text{ for all } p, q \in T_{(E,v)}$$

defines a metric on $T_{(E,v)}$. For these "higher-dimensional" analogues of $\mathbb{R}$-trees, it is possible to define *ends* in such a way that, as above, the set of ends of $T_{(E,v)}$ is the completion of E with respect to v, as defined in [16] (see [25]).

Again, p-adic numbers give rise to important examples: if $E := \mathbb{Q}_p^m \backslash \{0\}$ and

$$\begin{aligned} v := w_p \circ \det : \mathcal{P}_{\text{fin}}(E) &\to \mathbb{R} \cup \{-\infty\} \\ x &\mapsto \begin{cases} w_p \circ \det(e_1, \ldots, e_m), & \text{if } x = \{e_1, \ldots, e_m\}, \\ -\infty, & \text{else,} \end{cases} \end{aligned}$$

then the pair (E, v) is a (well-defined!) simple valuated matroid (see [19]), and the associated space $T_{(E,v)}$ is the euclidean building for the general linear group $GL_m(\mathbb{Q}_p)$, the simplicial structure of $T_{(E,v)}$, as introduced in Section 3, reflecting the chamber complex structure of the building (see [25]). Of course, this also holds more generally for any valuation $w : F \to \mathbb{R} \cup \{-\infty\}$ of some field F.

Finally, let us note that it might be of interest to study the δ-relaxation of these concepts, too, in particular, if one wanted to capture, in the context of combinatorial group theory, properties characteristic to the arithmetic groups of higher rank.

5 Coherent Decompositions

5.1 Split decompositions

We begin this section with a question. Suppose that d is a metric defined on a finite set X, which additively decomposes into two metrics (or pseudo-metrics) d_1 and d_2, that is, the equality $d(x,y) = d_1(x,y) + d_2(x,y)$ holds for all x, y in X. Then, what can we say about the relationship between $T(d)$ and $T(d_1) + T(d_2) := \{f_1 + f_2 \mid f_1 \in T(d_1), f_2 \in T(d_2)\}$? Here, of course, we set

$$P(d') := \{f \in \mathbb{R}^X \mid f(x) + f(y) \geq d'(x,y) \text{ for all } x, y \in X\},$$

and

$$T(d') := \{f \in \mathbb{R}^X \mid f(x) = \sup_{y \in X}\{d'(x,y) - f(y)\} \text{ for all } x \in X\},$$

for every map $d' : X \times X \to \mathbb{R}$ (or even $\mathbb{R} \cup \{-\infty\}$), whether it is a metric, a pseudo-metric, or any other map.

In [6], some progress is made in answering this question, which we summarize here. We start with a little background on the subject of *split decompositions*. A *split*, S, of a set X is simply a bipartition of X into two non-empty sets, say A and B. The *split (pseudo) metric*, δ_S, associated to this split is defined by the formula

$$\delta_S(x,y) = \begin{cases} 0 & \text{if } x, y \in A \text{ or } x, y \in B, \\ 1 & \text{otherwise.} \end{cases}$$

For every pair A, B of non-empty subsets of X, we can associate the *isolation index*, $\alpha^d_{A,B}$, with respect to any pseudo-metric d, which is defined as

$$\alpha^d_{A,B} := 1/2 \cdot \min_{a,a' \in A, b,b' \in B}\{\max\{ab + a'b', a'b + ab', aa' + bb'\} - aa' - bb'\}.$$

If the pair A, B forms a split, S, of X and the isolation index of S, $\alpha^d_S := \alpha^d_{A,B}$, is positive, then we call S a *d-split*. Let $\mathcal{S} = \mathcal{S}(d)$ denote the set of all d-splits of X. The main theorem of [6] states that

$$d_0 := d - \sum_{S \in \mathcal{S}} \alpha^d_S \cdot \delta_S$$

is a *split-prime* pseudo-metric, which, by definition, is a pseudo-metric d_0 such that $\alpha^{d_0}_S = 0$ for all splits S of X. We call d_0 the *split-prime residue* of d.

Let us now return to the original question. Clearly, if a metric d decomposes as $d = d_1 + d_2$, then $P(d_1) + P(d_2)$ is a subset of $P(d)$. Further, $P(d)$ is equal to $P(d_1) + P(d_2)$ if and only if the minimal members of $P(d_1 + d_2)$ decompose, that is, if $T(d)$ is contained in $P(d_1) + P(d_2)$ in which case, for every decomposition $f = f_1 + f_2$ of some map $f \in T(d)$ with $f_1 \in P(d_1)$ and $f_2 \in P(d_2)$, one must have $f_1 \in T(d_1)$ and $f_2 \in T(d_2)$. If this holds, we call d_1 and d_2 *coherent*. More generally, we define k pseudo-metrics $d_1, \ldots, d_k$ to be *coherent* if $P(d_1 + \ldots + d_k)$ is equal to $P(d_1) + \ldots + P(d_k)$, and in this case we say that the pseudo-metrics $d_1, \ldots, d_k$ constitute a *coherent decomposition* of the pseudo-metric $d := d_1 + \ldots + d_k$. Theorem 8 of [6] links split decompositions with coherent decompositions. We state this theorem here for completeness:

Let d be a metric on X. Assume that

$$d = d_1 + \sum_{S \in \mathcal{S}_1} \lambda_S \cdot \delta_S$$

is a decomposition of d such that d_1 is a pseudo-metric, $\mathcal{S}_1$ is a collection of splits of X, and $\lambda_S > 0$ for all members of $\mathcal{S}_1$. Then this constitutes a coherent decomposition of d, that is,

$$P(d) = P(d_1) + \sum_{S \in \mathcal{S}_1} P(\lambda_S \cdot \delta_S) = P(d_1) + \sum_{S \in \mathcal{S}_1} \lambda_S \cdot P(\delta_S)$$

holds, if and only if $\lambda_S \leq \alpha_S^d$ for all $S \in \mathcal{S}_1$, in which case one has $\alpha_S^{d_1} = \alpha_S^d$, for all splits S not in $\mathcal{S}_1$, and $\alpha_S^{d_1} = \alpha_S^d - \lambda_S$, for all splits S in $\mathcal{S}_1$ – in particular, the split-prime residues of d and d_1 must coincide.

5.2 Split decomposition, trees, and phylogenetic analysis

It has been shown in [6] that, if d satifies the four-point condition, then

$$d = \sum_{S \in \mathcal{S}} \alpha_S^d \cdot \delta_S.$$

In other words, if d is a tree-like metric, then the split-prime residue of d vanishes, while – as mentioned in Section 4.5 – the d-splits are precisely the splits induced by the edges of the associated tree, their weights corresponding to the length of these edges. Split decomposition was designed, in particular, to analyse phylogenetic distance data, which are in general not too far from satisfying the four-point condition as they somehow reflect the *phylogenetic tree*, but, of course, they rarely satisfy it precisely. Applications of split decomposition to biology are discussed in [14], [10], [5], and [1].

5.3 Cyclic split systems

According to [6], there are at most $\binom{n}{2}$ d-splits for any (pseudo) metric d, defined on a set X of cardinality n. Moreover, this upper bound is attained if and only if the metric is *cyclic*, that is, there exists a bijection $\varphi : X \rightarrow \{1, \ldots, n\}$ such that the d-splits are precisely all splits of the form

$$\{\varphi^{-1}(\{a, a+1, \ldots, b-1\}) \, , \, X - \varphi^{-1}(\{a, a+1, \ldots, b-1\})\}$$

(with $1 \leq a < b \leq n$), in which case the split-prime residue of d necessarily vanishes. In [26], it has been shown that an arbitrary system of splits is contained in a cyclic split system if and only if the smallest system of splits containing the given system and containing, for any two of its splits $S_1 = \{A_1, B_1\}$ and $S_2 = \{A_2, B_2\}$ with $A_1 \cap A_2 \neq \emptyset \neq B_1 \cap B_2$, the split $\{A_1 \cap A_2, B_1 \cup B_2\}$ is *weakly compatible* (that is, it does not contain three splits, $S_1 = \{A_1, B_1\}, S_2 = \{A_2, B_2\}$, and $S_3 = \{A_3, B_3\}$ such that there exist elements $a, a_1, a_2, a_3 \in X$ with $a \in A_1 \cap A_2 \cap A_3$ and "$a_i \in A_j$ if and only if $i = j$ for all $i, j \in \{1, 2, 3\}$"). In turn, this is (essentially) equivalent to the fact that there exists a "nice" *planar representation* of the given split system (for details, see [26]).

5.4 Split decompositions and overlapping clustering

From the point of view of cluster theory, split decomposition can be viewed as a particular instance of overlapping clustering procedures. This point of view has been worked out in detail in [4], [7], and [9], where different aspects have been stressed. In [4], *weak hierarchies* have been introduced which are set systems (or *hypergraphs*) related to similarity measures in the same way that split systems are related to metrics. In [7], lattice theoretic aspects of the weak-hierarchy concept are worked out in detail, leading to a deeper understanding of weak hierarchies and to far-reaching generalizations of *local* and *global* similarity data. In [9], the connection between various kinds of split systems on the one hand and corresponding quaternary relations on the other is analysed, leading to axiomatic characterizations of various important classes of split systems.

5.5 Split decompositions and the Travelling Salesman's Problem

For any metric space X of cardinality n and any (pseudo) metric $d : X \times X \to \mathbb{R}$, define

$$TSP(d) := \min\{\sum_{i=1}^{n} d(\varphi(i), \varphi(i-1)) \mid \varphi : \{0, 1, \ldots, n\} \twoheadrightarrow X, \varphi(0) = \varphi(n)\}.$$

It is easy to see that for any decomposition of d of the form

$$d = d_1 + \sum_{S \in \mathcal{S}} \lambda_S \cdot \delta_S$$

with d_1 a pseudo-metric, $\mathcal{S}$ a system of splits and $\lambda_S > 0$ for all $S \in \mathcal{S}$, one has

$$TSP(d) \geq 2 \cdot \sum_{S \in \mathcal{S}} \lambda_S,$$

so that, in particular,

$$TSP(d) \geq 2 \cdot \sum_{S \in \mathcal{S}(d)} \alpha_S^d.$$

It can be shown (see [20]) that the following three statements are equivalent:

- The split-prime residue of $d|_{Y \times Y}$ vanishes for all $Y \subseteq X$ with $\#Y = 5$ and the system $\mathcal{S}(d)$ of d-splits can be embedded into a cyclic split system.
- $TSP(d) = 2 \cdot \sum_{S \in \mathcal{S}(d)} \alpha_S^d$.
- $TSP(d) = \sup\{2 \cdot \sum_{S \in \mathcal{S}} \lambda_S\}$, where the supremum is taken over all decompositions of the form $d = d_1 + \sum_{S \in \mathcal{S}} \lambda_S \cdot \delta_S$ as discussed above.

5.6 Embedding metric spaces into the rectilinear plane: a six-point criterion

The main result obtained by H.-J. Bandelt and V. Chepoi in [2] states that a metric space embeds into the rectilinear plane (i.e. is L^1-embeddable in $\mathbb{R}^2$) if and only if every subspace with five or six points does. The proof of this result takes advantage of split decomposition theory, of which we have discussed some of the basic concepts above. In particular, the proof uses the concept of a *totally decomposable* metric space, (X, d), which means that the metric d can be written in the form

$$d = \sum_{S \in \mathcal{S}} \alpha_S^d \cdot \delta_S,$$

where $\mathcal{S}$ is the set of all d-splits, i.e. the split-prime residue of d vanishes. As shown in [6], this holds for (X, d) if (and only if) it holds for every five-point subspace of X.

6 The Block Decomposition

In [15], a unique additive decomposition of a metric d defined on a finite set X is introduced which is called its *block decomposition*. This decomposition is a particular instance of a coherent decomposition, though it differs in many ways from the split decomposition described in the previous section. The block decomposition arises from particular properties of the topology of T_X when X is finite, and we briefly describe the ideas giving rise to it here.

If R is an equivalence relation on X, then we denote its set of equivalence classes by X/R. We say that two equivalence relations R_1 and R_2 on X are *compatible* if they satisfy the following conditions:

- neither R_1 nor R_2 is equal to $X \times X$;
- there exist sets $A_i \in X/R_i$, $i = 1, 2$, such that $A_1 \cup A_2 = X$.

A set of equivalence relations is *compatible* if any two relations contained within that set are compatible. Note that each pseudo-metric d defined on X induces an equivalence relation on X by setting x equivalent to y, for x, y in X, if and only if the distance between x and y relative to d is equal to zero. A *t-decomposition* of the pseudo-metric d is a finite set of pseudo-metrics $\mathcal{D}$ defined on X such that;

- $d = \sum_{d' \in \mathcal{D}} d'$;
- the set of equivalence relations induced on X by the elements of $\mathcal{D}$ is compatible.

In [15], we define the concept of a *d-tree*. A d-tree is a (graph theoretical) tree, T, with vertex set $W \dot{\cup} V$ and edge set $E \subseteq \{\{w, v\} \mid w \in W,\, v \in V\}$, which satifies the following conditions:

- the set W contains the set X;
- the set X, considered as a subset of T, contains every vertex of degree one;
- each $v \in V$ is labelled by a metric d_v which is defined on the neighborhood of v in T, $N(v) := \{w \in W \mid \{w, v\} \in E\}$;
- for every two vertices x, y in X, one has

$$d(x, y) = \sum_{i=1}^{m} d_{v_i}(w_{i-1}, w_i),$$

  where $x = w_0, v_1, w_1, v_2, \ldots, v_m, w_m = y$ are the consecutive vertices in the (unique) shortest path from x to y in T.

In [15], it is shown that there is a one-to-one correspondence between (isomorphism classes of) d-trees and t-decompositions of d.

We now give an example of a block decomposition. Let d denote the graph metric defined on the six-point set $\{1, \ldots, 6\}$ pictured in Figure 1, where the metric assigns length 1 to each simple edge and length 2 to each double edge. The d-tree associated to d is pictured in Figure 2 and consists of two "blocks", labelled by v_1 and v_2, which are connected at the vertex w_1. The set W contains the vertices $\{1, \ldots, 6\}$, together with the vertex w_1, and the set V is equal to the union of the two vertices v_1 and v_2. The metrics d_{v_1} and d_{v_2} are pictured in Figure 3. The block decomposition of d is given by $d = d_1 + d_2$, where the metrics d_1 and d_2 are defined as follows. Let A equal the set $\{1, 2, 3\}$ and B equal the set $\{4, 5, 6\}$. Then we have

$$d_1(i, j) = \begin{cases} 0 & \text{if } i = j \text{ or } \{i, j\} \in B \times B, \\ 1 & \text{if } \{i, j\} \in \{\{1, 2\}, \{1, 3\}\}, \{2\} \times B, \text{ or } \{3\} \times B, \\ 2 & \text{if } \{i, j\} = \{2, 3\} \text{ or } \{i, j\} \in \{1\} \times B, \end{cases}$$

and

$$d_2(i, j) = \begin{cases} 0 & \text{if } i = j \text{ or } \{i, j\} \in A \times A, \\ 1 & \text{if } \{i, j\} \in \{\{4, 5\}, \{5, 6\}\}, A \times \{4\}, \text{ or } A \times \{6\}, \\ 2 & \text{if } \{i, j\} = \{4, 6\} \text{ or } \{i, j\} \in A \times \{5\}. \end{cases}$$

.

We close this section with an explanation of how the topology of T_X enters into the block decompositions. In [11, Theorem 6] it is shown that if

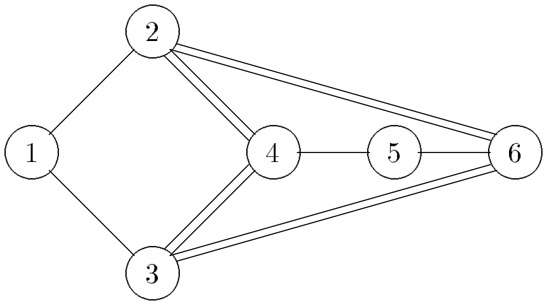

Figure 1: The graph representing metric d

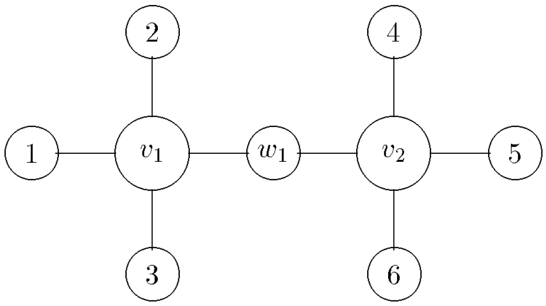

Figure 2: The d-tree associated to the metric d

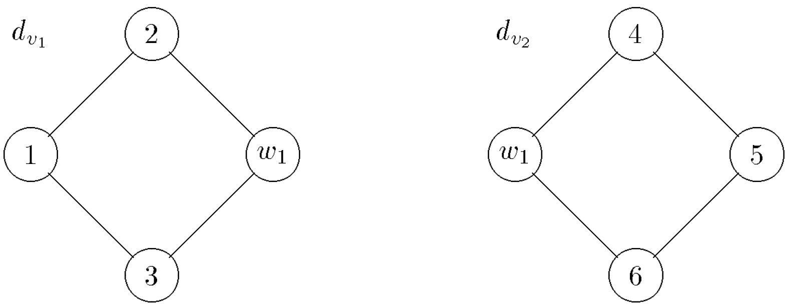

Figure 3: The metrics d_{v_1} and d_{v_2}

there exists a split $A \cup B$ of X and a map $f \in P_X$ satisfying $f(x)+f(y) = xy$, for all $x \in A$ and $y \in B$, then f is an element of T_X. Moreover, a map $f \in T_X$ is of this form if and only if it is either of the form h_x for some x in X, or it is a *cut point* of T_X, that is, the set $T_X - \{f\}$ consists of at least two connected components. Given a finite set $\mathcal{F}$ of such points contained in T_X which contains all of the points h_x for $x \in X$, we construct a d-tree, whose vertices in W correspond to the elements of $\mathcal{F}$, and whose vertices in V correspond to the connected components of $T_X - \mathcal{F}$, while an edge connects a component with an element of $\mathcal{F}$ if and only if that point is contained in the boundary of the component. Employing the fact that T_X is compact, we then show that every d-tree can be obtained in this way and that there exists – up to a splitting of intervals – a unique *finest* such d-tree. The t-decomposition to which this finest d-tree corresponds is precisely the block decomposition.

7 T-Theory and Groups

Let G be an arbitrary group, endowed with a length function $l : G \to \mathbb{R}_{\geq 0}$, i.e. a map, l, satisfying the following conditions:

- $l(g) = l(g^{-1}) \geq 0$;
- $l(g) = 0 \Leftrightarrow g = 1$;
- $l(gh) \leq l(g) + l(h)$;

for all g, h contained in G. Then the group G can be considered as a metric space, where we define the distance, D, between any two elements g, h in G to be $D(g, h) := l(gh^{-1})$. By employing this idea, one can use the T-construction on the metric space (G, D) to investigate relationships between properties of the group G and the length function l. For example, in [11, Theorem 10] it is shown that if G is a group, endowed with an interger valued length function, which satisfies the condition

$$\sup\{l(g^k) \mid k \in \mathbb{Z}\} = \infty$$

for all elements $g \in G$ not equal to the identity, and if $\dim_{\text{comb}} T_{(G,D)}$ is less than or equal to n, then the cohomological dimension of G is also less than or equal to n. In the case when n is equal to one, this recovers a result of R. Lyndon (see [24] and also [9]) as, by a famous result of Stallings, it implies that G must be free.

One can also use the T-construction to investigate group actions on finite metric spaces. In [12], the case where X is a finite metric space whose group of isometries acts transitively on X, and where one has the equality $\dim_{\text{comb}}(X) = \lfloor \#X/2 \rfloor$, is studied. For example, it is shown that the Feit-Thompson Theorem can be recovered, using T-theory, from it's simple consequence that any finite simple group acts transitively as a group of isometries on some finite metric space X satisfying $\dim_{\text{comb}}(X) = \lfloor \#X/2 \rfloor$.

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