"Combinatorial Geometry" Research Group, TU Berlin

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Combinatorics is the branch of Mathematics dealing with finite structures, while Geometry deals with spatial structures. Thus Combinatorial Geometry deals with the close interaction of discrete geometric objects and the combinatorial objects and data that determine them.

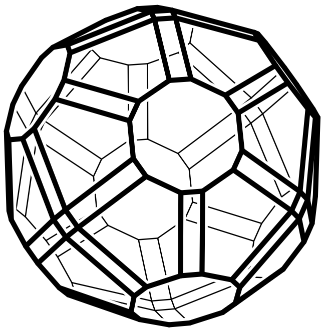
In this framework, essential aspects of a great variety of geometric structures can be studied in terms of combinatorial data (such as the number and incidence structure of the points, lines, planes etc. involved). At the same time, many combinatorial objects can be represented by geometric models (e.g. graphs, complexes, and polytopes), which leads to additional insight and new methods for their analysis.

The new methods of combinatorial geometry rely on a systematic development of the combinatorial models for geometric structures. Moreover, one uses that the structure, completeness and complexity of combinatorial objects can be measured in terms of topological and algebraic invariants [1]. The evolving systematic theory of combinatorial structures can be applied in many mathematical situations. For this the reduction to combinatorial problems is often not new (see e.g. the method of cell decomposition and stratifications of topological spaces) — the new element in the game is the (algebraic and topological) theory that is available to deal with the combinatorial data generated this way.

Our research program, supported by "Algebraic Combinatorics" network in the EU Human Capital and Mobility Programme, deals with selected questions that seem to be in the focus of current interest. In the following we describe five ranges of topics studied within our research group, with very brief sketches of the structures and problems studied, of current research work and some recent progress and success. We are happy to provide more information, details and explanations.

1. Polytope Theory.

Polytope theory has made rapid, substantial progress in the last few years (see Ziegler [19]), and has experienced more and more interest from "pure mathematics", due to its close connections to parts of algebraic geometry (e.g. toric varieties), optimization (linear programming), commutative algebra, and so on. Thus, for example, the construction of the "permuto-associahedra" as symmetric, convex polytopes (Reiner & Ziegler [9]) was motivated by a problem by Kapranov from Homotopy Theory.

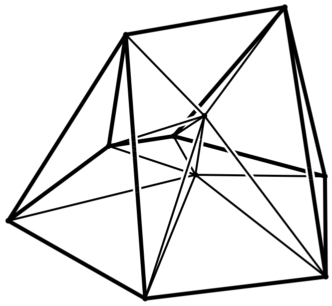


A comprehensive goal in our research on polytopes asks to understand more about the algebraic structure of polytopes, including diameter questions, rationality, decomposition theory, as well as the universal constructions within the category of polytopes.

The *realization spaces* of polytopes are objects of particular importance and interest. Here one studies the configuration space given by all the realizations of some discrete object, and asks for the (topological and arithmetic) complexity of this space. A class of structures is called *universal* if (essentially) all semialgebraic sets can appear as configuration spaces of objects in the class. Mnëv's celebrated "Universality Theorem" for oriented states that planar line configurations are universal in this sense. By now, even a simple proof of a much more powerful "Universal Partition Theorem" is available, see Richter-Gebert [12].

Quite spectacular progress was recently achieved by Richter-Gebert [11, 14]: a strong Universality Theorem (and a Universal Partition Theorem) for 4-dimensional polytopes, which has many interesting corollaries, among them

- realization spaces of 4-polytopes can have complicated homotopy types,
- for 4-polytopes one cannot prescribe the shape of a 2-face, (a Schlegel diagram for the simplest example, a 4-polytope **X** with 10 vertices, is indicated below),



- not every combinatorial type can be realized with rational coordinates,
- for integral realizations of rational 4-polytopes, one needs at least doubly exponentially large coordinates.

After these complete answers for 4-polytopes, some of the open problems about 3-polytopes need to be reevaluated, and studied anew.

2. Homotopy Theory of Finite Structures.

Homotopy methods provide essential new tools for problems of Algebraic Combinatorics. Combinatorial methods here led to elementary proofs for the famous homology formulas of Goresky & MacPherson, and even to homotopy formulas (Ziegler & Živaljević [16]), with applications in complexity theory (Björner, Lovász & Yao). Surveys of this line of research, which has led to a quite complete picture of the topological structure of arrangements, are given in Björner [2] and in Ziegler [18, Chap. 1]. The "homotopy theory techniques" (in particular, the diagram method of Ziegler & Živaljević [16]) still have great potential. We are working both on the tools and on extending the range of applicability.

For this one needs a systematic theory of *diagrams of spaces* and their *homotopy limits*. Substantial progress in this direction is given in Welker, Ziegler & Živaljević [15], providing a collection of versatile tools for the manipulation and comparison of homotopy limits of different posets, including analogs of the main homotopy comparison results for order complexes, in particular of the Quillen Fibration Theorem and the Björner-Walker Complementation Formula.

This makes the diagram construction available in much more general contexts. Applications include the topology of toric varieties, as well as the homotopy types of posets, with a new proof of Björner's [3] Generalized Homotopy Complementation Formula. In the future we will test and work out applications to combinatorial, geometric and topological situations (esp. configurations, tilings, knots, arrangements) of special interest. A particularly challenging problem here is the classification of line configurations in real or projective 3-space.

This research is done in close collaboration with the Stockholm Group (Prof. Anders Björner) of the EU network.

3. Matroid Theory, and the Grassmannians.

Oriented Matroids [4] are a versatile and widely applicable model for geometric objects. In this model, one can, in particular, profitably study realizability and embeddability problems.

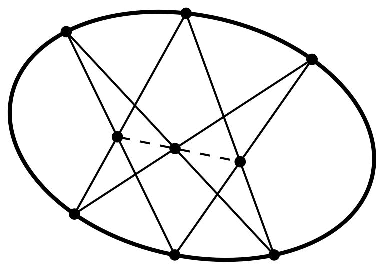
The oriented matroid models associated with (stratifications of) the finite-dimensional real Grassmannians are important structures that link several topics mentioned above. Here we investigate the local (inductive resp. shelling) structure of *spaces of oriented matroids* resp. *spaces of strings* in polytope theory, which are discrete models for finite-dimensional Grassmannians. This is also closely related to the investigations of the Paris Group (Prof. Alain Lascoux).

Here our recent research is connected to two most important problems:

- the "Generalized Baues Conjecture" of Billera, Kapranov & Sturmfels, for which we provided *counterexamples* in Rambau & Ziegler [8], and
- the cohomology structure of the "OM-Grassmannian," see Mnëv & Ziegler [7]. Here research is directed towards the key problem of MacPherson's theory of "combinatorial differential manifolds." This is closely related to our work on the geometric realization of extension spaces via zonotopal tilings, via the Bohne-Dress Theorem (see Richter-Gebert & Ziegler [13]).
- The structure of the *complex matroids* introduced and studied in Ziegler [17]. Spaces of such complex matroids provide models for complex Grassmannians, and should eventually lead to the analogs of Chern classes for complex combinatorial differential manifolds.

4. Automatic Theorem Proving.

This is a large, important field of research, connecting geometry with combinatorics and algebra. Here the aim is to design algorithms that can use the structural data of hypotheses and conclusions of a geometric theorem (incidence properties, angles, etc.) in order to automatically generate a proof. New approaches and tools [10] are based on methods of invariant theory (going back to Felix Klein's work!), which can be translated into effective algorithms.



In collaboration with H. Crapo (INRIA, Paris) we are developing a software package [5, 6] that provides an interactive, graphic interface, linked to a powerful prover, for dealing with geometric theorems. Here work is still in progress — but the performance of the current prototype version is already impressively strong!

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