# Algebraic Combinatorics at the University of Wales, Aberystwyth

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Over a long period, research at Aberystwyth has been concerned with the representations of the symmetric groups and their generalisations, reflection groups (real and complex) and Weyl groups, but from the combinatorial point of view. This follows the British tradition in algebraic combinatorics established by A. Young and later D. E. Littlewood, who worked at two other constituents of the University of Wales, Swansea (where A. R. Richardson was his mentor) and Bangor.

The work covers the following areas:

- I Projective (spin) representations of symmetric groups
- II Projective representations of reflection groups
- III Representations of Weyl groups
- IV Representations of complex reflection groups
- V Hall-Littlewood symmetric functions

## I Projective (spin) representations of symmetric groups

This work goes back to the author's PhD thesis (1959) under the supervision of D. E. Littlewood on the 'Spin representations of symmetric groups'. An excellent account of the background to this subject area is given in C. Bessenrodt's review article [B].

This problem was first considered by I. Schur who in the classic paper [5] obtained some remarkable results including a complete description of the irreducible spin characters of symmetric groups $\mathcal{S}_n$ which complemented the corresponding results for the ordinary characters of $\mathcal{S}_n$ obtained by him and F. G. Frobenius. He showed that this was equivalent to considering a covering group $\hat{\mathcal{S}}_n$ of $\mathcal{S}_n$ and he introduced in particular the class of symmetric functions now known as Schur $Q$-functions which corresponded to his Schur functions in the ordinary case. However, this work was done prior to the introduction of the combinatorial object of tableau and standard tableau to the subject by A. Young. Thus, the basic idea from the start has been to introduce the corresponding combinatorial objects for spin representations. Indeed, the main object was to generalise in this context all the combinatorial ideas which have added to the literature in such a rich way in the ordinary case over the years.

Thus, over the years one has seen the concepts of partitions, tableaux, standard tableaux, hooks and degree formula in terms of hooks, Schur functions, Nakayama conjecture in the ordinary case replaced by strict partitions, shifted tableaux, shifted standard tableaux, bars and degree formula in terms of bars, Schur $Q$-functions and a Nakayama-type conjecture for spin representations (see [S, M1, M2]).

Since that time, there have been other major developments.

**Modular representations**

Up to recent times, there has been very little progress on the modular (spin) representations of $\mathcal{S}_n$. The author's conjecture [M2] on the $p$-block decomposition in terms of $p$-bars has been proved by J. F. Humphreys and M. Cabanes. Some basic combinatorial ideas which were necessary in this context were presented in [MY1] and the decomposition matrices were calculated for small $n$ [MY2] (see also A. K. Yaseen's PhD thesis (1987) which included more calculations). This led to general results on the decomposition matrices in char 3 [BMO] and char 5 by Andrews, Bessenrodt and Olsson which turned out to be connected with deep combinatorial ideas on partitions due to G. E. Andrews. This work continues and will be the basis for the collaboration of the author with Bessenrodt and Olsson in the Algebraic Combinatorics Project.

**Construction of representations**

Up to very recent times, there was no explicit construction of the irreducible spin representations of $\mathcal{S}_n$; however in 1988, M. L. Nazarov [N] in a remarkable paper showed how to generalise Young's orthogonal form in this context. More recently he has generalised Young's symmetriser in this context, but the formula he gives in this case is not in a form which leads to an easily calculated explicit form. Andrew Jones in his PhD thesis (1995) has made considerable progress in producing such an explicit closed formula.

## II Projective representations of reflection groups

It is clearly of interest to extend all of the above to the irreducible reflection groups in general; basically here also the main motivation has been to develop correct combinatorial concepts in this context. The approach has been mainly an *ad hoc* one dealing with each type separately. However, some progress has been made in producing general results; for example [M4] where ideas from Clifford algebras are exploited to obtain basic spin representations for all reflection groups. Most of the *ad hoc* approach appears in [M3] (spin character tables for exceptional Weyl groups) and in the PhD thesis of the author's students and in their published work, E. W. Read (1973) ($\mathrm{B}_n$, $\mathrm{D}_n$ type and also exceptional Weyl groups), F. Khosraviyani (1981), M. Saleem (1989) (decomposition matrices for the exceptional Weyl groups). This work will continue in the future; there are many general results which are still to be obtained.

## III Representations of Weyl groups

Although a great deal of progress has been made in generalising the representation theory of symmetric groups to Weyl groups, or reflection groups, very little has been done using the combinatorial approach. The first such attempt appeared in [M6], where the basic combinatorial ideas of tableaux, tabloids, etc. which were successful in the case of symmetric groups as exemplified in the work of G. D. James, were interpreted in the context of root systems. This showed under what conditions these basic ideas could be extended to Weyl groups. More recently, we have returned to this problem; see for example S. Halicioglu's PhD thesis (1992) and also [HM] where the ideas have been developed further. Lee Hawkins in his PhD thesis (1995) will show how I. G. Macdonald's approach to the irreducible modules can be extended; this work is very computational in character and focusses on the practical problem of the construction of all the irreducible modules and finding bases for these modules.

## IV Representations of complex reflection groups

This is a further generalisation of the above work. Here the question is far more difficult; the imprimitive reflection groups $G(m,p,n)$ are generalisations of the Weyl groups of type $A_n$ $(G(1,1,n))$, type $B_n$ $(G(2,1,n))$ and type $D_n$ $(G(2,2,n))$ and the hyperoctahedral groups $G(m,1,n)$. Considerable work has been done on the ordinary and projective representations of the hyperoctahedral groups; see the theses of Saeed-al-Islam (1980), M. Munir (1988), M. Saleem (1989) and H. I. Jones (1993) and also the more recent paper of J. R. Stembridge [St]. M. C. Hughes (1981) in his PhD thesis commenced on extending these ideas to complex reflection groups in general. Here, the problem is considerably more difficult as the idea of a root system is not as well developed in this context. His results were published in a series of papers culminating with a paper in Communications in Algebra. H. Can (1995) in his PhD thesis has taken up these ideas leading to a more successful general theory. There is scope for considerably more work of a combinatorial nature in this problem.

## V Hall-Littlewood symmetric functions

As these are generalisations of Schur functions and Schur $Q$-functions, Hall-Littlewood functions have always been central to the work, see for example, the author's 1976 survey article [M5], which describes their connection with representation theory. More recently, these have been considered when the parameter $t$ involved in their definition is a root of unity (Schur functions are the case $t = 0$ and Schur $Q$-functions, the case $t = -1$). N. Sultana considered this special case in her thesis; see also [MS]. As the title of that paper suggests, there is a close connection here with the modular representations of the symmetric group, and indeed, of Hecke algebras and the general linear group. This problem has been taken up by B. Leclerc and hopefully will lead to further collaboration with the group in Marne-la-Valée.

A list of PhD theses written under the author's supervision is appended which gives a good indication of the research interests.

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