Constructing Irreducible Representations of Weyl Groups

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We describe a construction of irreducible representations of Weyl groups based on a remarkably simple procedure given by I. G. Macdonald. For a given Weyl group W with root system R, each subsystem S of R gives rise to an irreducible representation of W. In general, however, not all the irreducible representations can be realised in this way. We show that other special subsets of R lead to representations unobtainable via the subsystem approach. The focus of this work is to determine explicitly Macdonald's representations and various computational techniques are given for finding generating sets and bases for the irreducible W-modules produced by the construction. To illustrate the success of these techniques, we enumerate examples in the Weyl group of type E_6.

1 Introduction

Although the characters of Weyl groups of all types have been known for some time, the problem of describing the corresponding irreducible representations has proved to be a not inconsiderable task. In view of this, the 1972 paper of I. G. Macdonald (see [Mac72]) is remarkable – he describes a simple construction providing irreducible representations which relies on nothing more complicated than subsystems of the root system of the Weyl group. However, a complete set of irreducible representations is not generally given by the approach.

We introduce certain special subsets of root systems (including subsystems) in section 2, and show how these may be used in Macdonald's approach in section 3. It will turn out that we can describe some of the "missing" representations using these subsets. The remaining sections are devoted to the computational aspects of the construction. In section 4, we determine generating sets for the W-modules corresponding to the representations of section 3, and consider the problems associated with finding bases for these modules. The fruits of our labours are presented as examples in section 5.

2 Special Subsets of Root Systems

In this section we will look at certain subsets of the root system R of a Weyl group W – the subsets of interest are parabolic subsets, invertible subsets and subsystems. We begin by defining a notion of closure for subsets of a root system.

Definition 2.1 Let R be a root system. A subset $P \subseteq R$ is *closed* if $\alpha,\ \beta \in P,\ \alpha+\beta \in R$ imply that $\alpha + \beta \in P$. ■

Now follows our principal definition of the subsets of interest.

Definition 2.2 A subset $P \subseteq R$ is said to be *invertible* if both P and $R \backslash P$ are closed (in the sense of Definition 2.1). A *parabolic* subset is a closed set $P \subseteq R$ such that $P \cup (-P) = R$. A *subsystem* S of R is a closed subset $S \subseteq R$ which is itself a root system in the space which it spans. ■

It should be noted that subsystems S of R may also exist which are not closed. For our discussions, however, we shall assume the closure of subsystems as defined in Definition 2.2 above.

Let π be a simple system in R, and let R^+ (respectively R^-) denote the corresponding set of positive (respectively negative) roots. For each subset $\Delta \subseteq \pi$ we shall denote by R_Δ the root system consisting of all $\alpha \in R$ which are linear combinations of roots of Δ. Then immediately Δ is a simple system in R_Δ. Write R_Δ^+ (respectively R_Δ^-) for the set of positive (respectively negative) roots of R_Δ with respect to Δ. Note that $R_\Delta^+ = R^+ \cap R_\Delta$. We have the following result.

Proposition 2.3 *For $\Delta \subseteq \pi$, let $P_\Delta = R^+ \cup R_\Delta$. Then the P_Δ, $\Delta \subseteq \pi$, are representatives of W-orbits of parabolic subsets of R.*

Proof See [Bou68]. ■

For invertible subsets, we can establish a similar result with a little more effort. Let $\mathcal{P}$ be the set of all ordered pairs (Δ, Δ') where $\Delta' \subseteq \Delta \subseteq \pi$ and Δ' is orthogonal to $\Delta \backslash \Delta'$. To such a pair, associate the set

$$P(\Delta, \Delta') = R_{\Delta'} \cup (R^+ \backslash R_\Delta^+).$$

Then $P(\Delta, \Delta')$ is an invertible subset of R and, moreover,

Theorem 2.4 *The sets $P(\Delta, \Delta')$, $(\Delta, \Delta') \in \mathcal{P}$, are representatives of W-orbits of invertible subsets in R.*

Proof See [DCH94]. ■

Note that the set $P(\Delta, \Delta')$ is parabolic if and only if $\Delta = \Delta'$. Also we have $P(\Delta, \Delta) = P_\Delta$.

Remark 2.5

1. Consider again a representative P_Δ of a W-orbit of a parabolic subset of R. We have that
$$P_\Delta = R^+ \cup R_\Delta,$$
where both R and R_Δ have simple system Δ, and $R_\Delta \subseteq R$. Then $R_\Delta^+ \subseteq R^+$, and
$$\begin{aligned} P_\Delta &= R^+ \cup R_\Delta \\ &= R^+ \cup (R_\Delta^+ \cup R_\Delta^-) \\ &= (R^+ \cup R_\Delta^+) \cup R_\Delta^- \\ P_\Delta &= R^+ \,\dot\cup\, R_\Delta^- \end{aligned}$$
This decomposition will prove useful later when determining representations using Macdonald's method.

2. There is a relationship between the representatives P_Δ and $P(\Delta, \Delta')$ as we now show. Recall that we define the representatives $P(\Delta, \Delta')$ by
$$\begin{aligned} P(\Delta, \Delta') &= R_{\Delta'} \cup (R^+ \backslash R_\Delta^+) \\ &= (R_{\Delta'}^+ \cup R_{\Delta'}^-) \cup (R^+ \backslash R_\Delta^+) \\ &= R_{\Delta'}^- \cup (R_{\Delta'}^+ \cup (R^+ \backslash R_\Delta^+)) \\ &= R_{\Delta'}^- \,\dot\cup\, T, \quad \text{where } T = R_{\Delta'}^+ \cup (R^+ \backslash R_\Delta^+) \subseteq R^+ \end{aligned}$$
We have that $\Delta' \subseteq \Delta \subseteq \pi$ so that $R_{\Delta'} \subseteq R_\Delta \subseteq R$ which means $R_{\Delta'}^- \subseteq R_\Delta^- \subseteq R^-$ since all have base Δ. Thus,
$$P_\Delta = P(\Delta, \Delta') \cup V \cup X,$$
where $V = R_\Delta^- \backslash R_{\Delta'}^-$ and $X = R^+ \backslash T$.

3. Note that any representative P_Δ *always* contains a root system of R, namely R_Δ. There may exist other root systems S such that $S \subseteq P_\Delta$ but R_Δ is the largest such root system. Any representative $P(\Delta, \Delta')$ also necessarily contains a root system, namely $R_{\Delta'}$. ■

We now look at how we may define subgroups of the Weyl group $W(R)$ using subsets of the root system.

Definition 2.6 For any subset $A \subseteq R$, let $W(A)$ be the subgroup of $W(R)$ generated by the $\tau_a,\ a \in A$. If A is a subsystem of R, we call $W(A)$ a *Weyl subgroup* of $W(R)$. ■

It should be clear that $W(P_\Delta) = W(R)$, and thus $W(P) = W(R)$ for any parabolic subset P. It is less easy to describe $W(P(\Delta, \Delta'))$.

We describe an equivalence relation on subsets of the root system.

Definition 2.7 Let S, T be closed subsets of R. Then S is *W-conjugate* to T if there exists $w \in W(R)$ such that $S = wT$. ■

This notion of conjugacy will be important when we study the representation theory in later sections.

Example 2.8 Let W be of type A_3 with simple system $\pi = \{\epsilon_1 - \epsilon_2,\ \epsilon_2 - \epsilon_3,\ \epsilon_3 - \epsilon_4\}$, where $\pi \subset \mathbf{Q}^4$ with standard basis $\{\epsilon_1,\ \epsilon_2,\ \epsilon_3,\ \epsilon_4\}$, then the non-conjugate subsystems are

$$\begin{array}{ll} S_1 = \emptyset & \text{with simple system } J_1 = \emptyset \\ S_2 = \mathrm{A}_1 & \text{with simple system } J_2 = \{\epsilon_1 - \epsilon_2\} \\ S_3 = \mathrm{A}_2 & \text{with simple system } J_3 = \{\epsilon_1 - \epsilon_2,\ \epsilon_2 - \epsilon_3\} \\ S_4 = 2\mathrm{A}_1 & \text{with simple system } J_4 = \{\epsilon_1 - \epsilon_2,\ \epsilon_3 - \epsilon_4\} \\ S_5 = \mathrm{A}_3 & \text{with simple system } J_5 = \{\epsilon_1 - \epsilon_2,\ \epsilon_2 - \epsilon_3,\ \epsilon_3 - \epsilon_4\} \quad \blacksquare \end{array}$$

The graphs which are Dynkin diagrams of Weyl subgroups may be obtained by a standard algorithm, due independently to Dynkin [Dyn57] and Borel and de Siebenthal [BS49]. This algorithm depends upon the extended Dynkin diagram.

Definition 2.9 The unique element $\tilde{\alpha} \in R$ with maximal height is called the *longest root* (or *highest root*) of R. The *extended Dynkin diagram* of R is formed by adding one further node to the Dynkin diagram of R corresponding to $-\tilde{\alpha}$. $\blacksquare$

The Dynkin diagrams of all possible Weyl subgroups are obtained as follows (this description appears in [Car72]):

Algorithm 2.10

- *Form the extended Dynkin diagram of R*
- *Delete one or more nodes in all possible ways from the extended Dynkin diagram*
- *Take the duals of the diagrams obtained in the same way from the root system $\tilde{R}$ dual to R (the dual system being that system obtained by interchanging long and short roots)*
- *Repeat the process with the Dynkin diagram obtained and continue any number of times.* $\blacksquare$

3 Macdonald's Construction

Let V be a finite dimensional vector space over the rationals $\mathbf{Q}$ with basis $\mathcal{B}$. Equip V with a positive definite inner product $(\cdot,\cdot)$ and let V^* be the dual of V, i.e. V^* is the space $\mathrm{Hom}_{\mathbf{Q}}(V,\mathbf{Q})$ of all linear maps $\psi:\ V \to \mathbf{Q}$. The inner product $(\cdot,\cdot)$ on V gives rise to one on V^*, denoted by $<\cdot,\cdot>$.

Let R be a reduced root system in V^*, in the sense of Bourbaki (see [Bou68]), and let S be a subsystem of R. Choose an ordering on R and let R^+ and S^+ denote the sets of positive roots with respect to this ordering. Let

$$\pi_S = \prod_{\alpha \in S^+} \alpha.$$

Then π_S is a product of functions and, more precisely, is a homogeneous rational-valued polynomial function on V. We call π_S the *Macdonald polynomial* corresponding to the subsystem S.

The space of all rational-valued polynomial functions on V is the symmetric algebra Σ=Sym(V^*), and the Weyl group $W = W(R)$ of R acts on Σ as follows :

$$\text{if } w \in W,\ \phi \in \Sigma,\ x \in V \text{ then } (w\phi)(x) = \phi(w^{-1}x).$$

Let P_S be the subspace of Σ spanned by the polynomial functions $w\pi_S$ for all $w \in W$ (with the action as defined above). Note that P_S is finite dimensional whereas Σ is infinite dimensional. We may consider any vector space V as an abelian group just by neglecting the scalar multiplication within V. View P_S in this way then P_S is a W-module. (We may also consider P_S as being a $\mathbf{Q}W$-module.)

The representation of W afforded by the W-module P_S is automatically a rational representation. We call P_S the *Macdonald module* corresponding to the subsystem S.

The main result of Macdonald's paper is the following theorem (which is proved therein).

Theorem 3.1

- *P_S is an absolutely irreducible W-module ;*
- *if S' is another subsystem of R and $|S| \neq |S'|$ then the modules P_S and $P_{S'}$ are* **not** *isomorphic.* ■

Remark 3.2

1. This construction applies in the more general situation of a finite Coxeter group W and a reflection subgroup of W, the only difference in this case being that the root system R has to be replaced by a root system in the sense of Steinberg [Ste67] (he does not insist on the crystallographic condition) and the rational vector space V by a real vector space. This does not affect the statement or proof of Theorem 3.1, but the representations so obtained will not generally be rational.

2. This construction supplies "most" but not in general all of the irreducible representations of a Weyl group W. For W of type A_n or B_n it goes back to Specht [Spe35, Spe37], who obtained all the irreducible representations in this way in these two cases. For a Weyl group of type D_n, however, the only subsystems of the root system are disjoint unions of root systems of types A and D, and by a counting argument all the irreducible representations of D_n are not obtained via this method.

3. If $S,\ S'$ are W-conjugate subsystems of R, then it is clear that P_S and $P_{S'}$ are isomorphic W-modules. However, the converse does not hold. The situation here has analogies with that encountered by Carter [Car72] in the classification of the conjugacy classes of Weyl groups; most but not all conjugacy classes are represented by Coxeter elements of Weyl subgroups, and a class can have essentially distinct representations of this sort.

4. For any root system R, there are two trivial subsystems S, viz. $S = \emptyset$ and $S = R$. We can describe immediately the representations arising in these cases.
   - $S = \emptyset$
     We clearly have $w\pi_S = \pi_S$ for all $w \in W$, so $w \mapsto 1$ for all $w \in W$ and the identity representation of W is given.
   - $S = R$
     Here $S^+ = R^+$ so π_S is the product of all the positive roots of R. It is well known that W acts by sign on R^+, i.e. $w\pi_S = (sgn(w))\pi_S$. Hence, $w \mapsto sgn(w)$ for all $w \in W$, and the corresponding representation leads to the sign character of W.
5. There is another easy situation to describe, namely for a subsystem of type A_1. Let R be of rank n and let S be a subsystem of type A_1 in R. Then the dimension of P_S is n.

   To see this, let $S = \{r, -r\}$ be the subsystem of type A_1, where $r \in R^+$. Then $\pi_S = r$. Now $W(R)$ permutes R so the generating set for P_S will be simply R itself. A basis for R is given by a simple system for R, and so has order n. Thus, the module P_S is of dimension n. ■

3.1 Using Parabolic Subsets

Recall from Proposition 2.3 that representatives P_Δ of W-orbits of parabolic subsets of R are given by

$$P_\Delta = R^+ \cup R_\Delta$$

for each subset $\Delta \subseteq \pi$. We now describe how Macdonald's construction may be applied to parabolic subsets in place of subsystems. The setup is essentially the same as before but now define the Macdonald polynomial to be

$$\pi_P = \prod_{\alpha \in P_\Delta} \alpha$$

for a representative P_Δ. The action of $W = W(R)$ remains the same and we let M_P be the subspace of Σ spanned by $w\pi_P$ for all $w \in W$. Again M_P is a W-module and we seek to prove its absolute irreducibility just as it was proved that the module P_S corresponding to a subsystem S was absolutely irreducible as a W-module.

We make use of the decomposition of P_Δ into disjoint sets given in Remark 2.5 (1), that is,

$$P_\Delta = R^+ \,\dot\cup\, R_\Delta^-.$$

Hence, we may write our Macdonald polynomial π_P as follows

$$\begin{aligned} \pi_P &= \prod_{\alpha \in P_\Delta} \alpha \\ &= \prod_{\alpha \in R^+} \alpha \prod_{\beta \in R_\Delta^-} \beta \\ \pi_P &\stackrel{\text{def}}{=} \pi_{R^+} \cdot \pi_{R_\Delta^-}. \end{aligned}$$

This enables us to prove the main result, essentially by using the fact that W acts by sign on R^+ again.

Theorem 3.3 *M_P is an absolutely irreducible W-module.* ■

We have used the parabolic subsets P_Δ thus far since we have an explicit form for their construction. However, the construction applies to any parabolic subset P as we now show.

Take any parabolic subset P and construct its W-orbit $\{wP \ : \ w \in W\}$. For some $\Delta \subseteq \pi$ we then have P_Δ belonging to this W-orbit. In particular, there exists an element $w_0 \in W(R)$ such that $P_\Delta = w_0 P$, in which case $\pi_{P_\Delta} = w_0 \pi_P$. Now, the module M_P has presentation

$$M_P = \langle w\pi_P \ : \ w \in W(R) \rangle$$

and the module M_{P_Δ} has presentation

$$\begin{aligned} M_{P_\Delta} &= \langle w\pi_{P_\Delta} \ : \ w \in W(R) \rangle \\ M_{P_\Delta} &= \langle w(w_0\pi_P) \ : \ w \in W(R) \rangle \\ M_{P_\Delta} &= \langle (ww_0)\pi_P \ : \ w \in W(R) \rangle \\ \Rightarrow M_{P_\Delta} &= \langle w'\pi_P \ : \ w' \in W(R) \rangle, \end{aligned}$$

i.e. the modules M_P and M_{P_Δ} are isomorphic W-modules and so the irreducible representations of $W(R)$ afforded by these modules will be identical.

Remark 3.4

1. There are again two trivial situations, namely when $\Delta = \emptyset$ or $\Delta = \pi$. If $\Delta = \emptyset$ then $P_\Delta = R^+$ and $\pi_P = \pi_S$ where $S = R$, so the corresponding character is the sign character of W. When $\Delta = \pi$, $P_\Delta = R^+ \cup R^-$ so that π_P is the product of all the roots of R. Since W acts transitively on R, $w\pi_P = \pi_P$ for all $w \in W$, i.e. $w \mapsto 1$ for all $w \in W$ and we have the identity character.

2. If R_Δ is a subsystem S of R for $\Delta \subseteq \pi$ then it should be clear from our decomposition of P_Δ that the character χ afforded by M_P will be $\chi = (sgn) \times \rho$, where ρ is the character afforded by P_S. ■

3.2 Using Invertible Subsets

Recall from Theorem 2.4 that representatives $P(\Delta, \Delta')$ of W-orbits of invertible subsets of R are given by

$$P(\Delta, \Delta') = R_{\Delta'} \cup (R^+ \backslash R^+_\Delta)$$

for each pair (Δ, Δ') satisfying $\Delta' \subseteq \Delta \subseteq \pi$ with Δ' orthogonal to $\Delta \backslash \Delta'$. We modify the original Macdonald construction here in exactly the same way as in §3.1, namely replace subsystems by the invertible subsets $P(\Delta, \Delta')$ throughout. Our Macdonald polynomial is now

$$\pi_{P'} = \prod_{\alpha \in P(\Delta, \Delta')} \alpha$$

for a representative $P(\Delta, \Delta')$. The action of $W = W(R)$ is as in the original setup and we let $N_{P'}$ be the subspace of Σ spanned by $w\pi_{P'}$ for all $w \in W$. $N_{P'}$ is a W-module and it turns out that $N_{P'}$ is absolutely irreducible only when $\Delta = \Delta'$ or $(\Delta, \Delta') = (\pi, \emptyset)$.

Generally the module $N_{P'}$ is reducible, but all is not lost since we may easily prove that the module M_P appears as a component in the direct sum decomposition of $N_{P'}$ into irreducibles. The author has been unable, however, to describe the other components in this decomposition.

Remark 3.5 We have used special examples of invertible subsets, namely the representatives $P(\Delta, \Delta')$. Just as in the parabolic case, however, we can argue that any invertible subset may be used in the construction, by consideration of its W-orbit and taking one of the representatives $P(\Delta, \Delta')$ which corresponds to this orbit. ∎

4 Computational Techniques

Having introduced the required theory in sections 2 and 3, we now consider how to determine explicitly the representations we have described. The computational problem arises in two parts – firstly, we need to efficiently determine generating sets for our Macdonald modules then, secondly, find bases for them. We present an algorithm (which lends itself to straightforward computer implementation) for determining generating sets for the modules arising from subsystems and parabolic subsets, and describe three methods of dealing with basis determination.

4.1 Generating Sets for Macdonald Modules

For a given subsystem S of R, we have the following generating set for the module P_S,

$$\Gamma = \{w\pi_S \ : \ w \in W(R)\}.$$

A naive attempt to calculate this set would require $|W|$ action calculations, one for each element of W. We can, however, reduce the number of calculations we need to carry out using the following approach.

Let $w\pi_S = \prod_{\alpha \in S^+} w(\alpha)$. Each $w(\alpha) \in R$ by the transitivity of the W-action on R so we may write $w(\alpha) = \epsilon_{\alpha,w} r_{\alpha,w}$, where $\epsilon_{\alpha,w} = \pm 1$ and $r_{\alpha,w} \in R^+$, for each $\alpha \in S^+$. Then

$$w\pi_S = \prod_{\alpha \in S^+} \epsilon_{\alpha,w} r_{\alpha,w} = \prod_{\alpha \in S^+} \epsilon_{\alpha,w} \prod_{\alpha \in S^+} r_{\alpha,w}.$$

Since each $\epsilon_{\alpha,w} = \pm 1$, $\prod_{\alpha \in S^+} \epsilon_{\alpha,w} = \pm 1$ and so

$$w\pi_S = \pm \prod_{\alpha \in S^+} r_{\alpha,w}.$$

Let $l \ : W \to \mathbf{N} \cup \{0\}$ denote the length function on W. Then we have the following result.

Theorem 4.1 *Given $w \in W$ and a subsystem $S \subseteq R$, there exists $z_{w,S} \in W$ (with $l(z_{w,S}) \leq l(w)$) and $\sigma_{w,S} \in Sym(S^+)$ such that*

$$r_{\alpha,w} = z_{w,S}(\sigma_{w,S}(\alpha)) \quad \text{for all } \alpha \in S^+ \quad ∎$$

This leads in a natural way to a consideration of the set

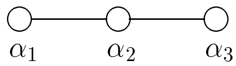
$$D_S = \{w \in W \ : \ w(\alpha) \in R^+ \text{ for all } \alpha \in S^+\},$$

since we then have (by virtue of Theorem 4.1) that

Theorem 4.2 $P_S = \langle w\pi_S \ : \ w \in W\rangle \cong \langle w\pi_S \ : \ w \in D_S\rangle$ ■

There is a well-known algorithmic construction of the set D_S for any subsystem S (see, for example [GP93]). Denote by D_k the set of elements of D_S of length k. We begin by setting $D_0 = \{e\}$. Now assume that D_k has already been computed, for $k \geq 0$. Then D_{k+1} is the set of elements $w\tau_i$, where $\tau_i \in \{\tau_{\alpha_r} \ : \ \alpha_r \in \pi\}$ (a reflection for each i), $w \in D_k$, and $w\tau_i(\alpha) \in R^+$ for all $\alpha \in S^+$. If we keep track of the simple reflections τ_i by which we had to multiply the elements in D_k to get those in D_{k+1}, we obtain reduced expressions for all the elements of D_S as words in the simple reflections.

Example 4.3 Consider W of type A_3 with Dynkin diagram



and simple system $\pi = \{\alpha_1, \ \alpha_2, \ \alpha_3\}$ in $(\mathbf{Q}^4)^*$. Consider the subsystem S whose positive system comprises $\{\alpha_1, \ \alpha_3\}$ then $\pi_S = \alpha_1\alpha_3$. Denote by τ_i the reflection along the simple root α_i. Starting the algorithm off with $D_0 = \{e\}$, we get the following run:

$$D_0 = \{e\}, \ D_1 = \{\tau_2\}, \ D_2 = \{\tau_2\tau_1, \ \tau_2\tau_3\}, \ D_3 = \{\tau_2\tau_3\tau_1\}, \ D_4 = \{\tau_2\tau_3\tau_1\tau_2\}$$

and hence $D_S = \{e, \ \tau_2, \ \tau_2\tau_1, \ \tau_2\tau_3, \ \tau_2\tau_3\tau_1, \ \tau_2\tau_3\tau_1\tau_2\}$. We can illustrate this recursive construction conveniently in a diagram of the following form

$$\begin{tikzcd}
 & \alpha_1\alpha_3 \arrow[d, "\tau_2"] & \\
 & (\alpha_1+\alpha_2)(\alpha_2+\alpha_3) \arrow[dl, "\tau_1"'] \arrow[dr, "\tau_3"] & \\
\alpha_2(\alpha_1+\alpha_2+\alpha_3) \arrow[dr, "\tau_3"'] & & (\alpha_1+\alpha_2+\alpha_3)\alpha_2 \arrow[dl, "\tau_1"] \\
 & (\alpha_2+\alpha_3)(\alpha_1+\alpha_2) \arrow[d, "\tau_2"] & \\
 & \alpha_3\alpha_1 &
\end{tikzcd}$$

The sets D_k can be read off the diagram level by level. ■

The above example illustrates an important point. The polynomials of $\{w\pi_S \;:\; w \in D_S\}$ are distinct if we consider the order of factors as being important but are not distinct in the usual sense as polynomials in commuting indeterminants α_i. As far as generation of the module P_S is concerned, of course, the order of factors is irrelevant. This motivates the following definition.

Definition 4.4 Denote by G_S a subset of D_S such that

$$\langle w\pi_S \;:\; w \in G_S\rangle = \langle w\pi_S \;:\; w \in D_S\rangle$$

and if $w\pi_S = w'\pi_S$ for $w,\ w' \in G_S$ then $w = w'$. ■

It should be clear that the algorithm for constructing D_S needs little modification in order to obtain a corresponding G_S – all we need do, in fact, is keep a record of the polynomials arising at each step and ensure that we do not copy any of them in later steps.

Example 4.5 Refer again to Example 4.3. With the modified algorithm described above, we obtain

$$G_0 = \{e\},\ G_1 = \{\tau_2\},\ G_2 = \{\tau_2\tau_1\}$$

or, diagrammatically,

$$\begin{array}{c} \alpha_1\alpha_3 \\ \downarrow \tau_2 \\ (\alpha_1+\alpha_2)(\alpha_2+\alpha_3) \\ \downarrow \tau_1 \\ \alpha_2(\alpha_1+\alpha_2+\alpha_3) \end{array}$$ ■

The algorithm presented for construction of G_S will be referred to as GENSET throughout and the corresponding set of polynomials is $P = \{w\pi_S \;:\; w \in G_S\} = \{p_1,\ \ldots,\ p_k\}$. GENSET is straightforward to implement on a computer and runs very efficiently. Unfortunately, although $|P| \ll |W|$, P is not generally $\mathbf{Q}$-linearly independent and so we still have the problem of determining a basis for the Macdonald module P_S – this is addressed in the next section.

Remark 4.6 In view of Remark 3.4, it should be apparent that GENSET is equally valid (subject to small modifications) when subsystems are replaced by parabolic subsets. We have been unable, however, to describe an algorithm for constructing a generating set for the module $N_{P'}$ arising from an invertible subset of the root system. ■

4.2 Bases for Macdonald Modules

Three methods have been used in attempting to find bases of Macdonald modules. The first of these, the all-monomial method, was the initial method of attack and is the natural approach. Unfortunately, it is very inefficient for large examples which led to the consideration of other approaches. The second approach, the leading monomial method, was motivated by a study of Gröbner basis techniques and utilises monomial orderings. The drawback here is that generally only a 'near' basis is given (that is, a large linearly independent subset of P which may not be sufficient to give a generating set). The final approach we present here, the rational vector method, is our latest approach to the problem and, as such, is still in early stages of development. We expect to able to refine this approach in the future and the author anticipates using this method for large examples.

1. The all-monomial method

   For each polynomial of P, decompose into its constituent monomials

   $$p_i = \sum_{j=1}^{l_i} a_{ij} m_{ij} \qquad (a_{ij} \in \mathbf{Q};\ i = 1,\ \ldots, k)$$

   and put $M_i = \bigcup_{j=1}^{l_i} \{m_{ij}\}$ to be the set of all monomials involved in p_i. Let $M = \bigcup_{i=1}^{k} M_i$ be the set of all monomials involved in all the polynomials of P, and let $t = |M|$. For convenience of representation, map the monomials of M onto indeterminates $Y_1,\ \ldots,\ Y_t$. Express each polynomial p_i as a linear combination of the Y_j. We begin our basis $\mathcal{B}$ with $p_1 = \pi_S$, $\mathcal{B} = \{\pi_S\}$. Now work through $p_2,\ \ldots,\ p_k$ checking whether each p_i may be expressed as a linear combination of elements of $\mathcal{B}$, adding to $\mathcal{B}$ any p_i for which such a combination does not exist. This checking essentially involves solving systems of linear equations in the indeterminates Y_j. After testing all of $p_2,\ \ldots,\ p_k$, we have that $\mathcal{B}$ is a basis for P_S.

   In practice, this approach is hopelessly inefficient. As $|S|$ increases, t increases rapidly and the corresponding systems of equations in the above procedure become impractical to attempt to solve. However, even with the problems, the method has been a good workhorse and the author has computed such bases when working with certain subsystems in the Weyl group of type E_6.

2. The leading monomial method

   The backbone of the approach we describe is monomial orderings. The order we need to describe is an *admissible total ordering* $\geq$ on M, this being an ordering such that

   - $m \geq 1$ for all $m \in M$
   - $m_1 \geq m_2 \Rightarrow m_1 \cdot n \geq m_2 \cdot n$ for all $n,\ m_1,\ m_2 \in M$

   An example of such an ordering is the lexicographical order on M, which orders monomials according to their exponent tuples.

Definition 4.7 The *leading monomial* of a polynomial p with respect to $\geq$ is the monomial appearing in p which is maximal among those in p. We denote it by $LM(p)$. Similarly, the monomial appearing in p which is minimal among those in p will be called the *trailing monomial*, denoted by $TM(p)$. ■

We order polynomials in the natural way, i.e. by ordering their constituent monomials. We give here three simple results on polynomials which will prove useful in our approach to determining a near basis for the Macdonald module.

Proposition 4.8 *Let p, q be polynomials. Then $LM(pq) = LM(p) \cdot LM(q)$ and $TM(pq) = TM(p) \cdot TM(q)$.* ■

Proposition 4.9 *Let $P = \{p_1, \ldots, p_r\}$ be a set of polynomials such that*

$$|\{LM(p_i) \ : \ i = 1, \ldots, r\}| = r.$$

Then P is $\mathbf{Q}$-linearly independent. ■

Remark 4.10 Note that Proposition 4.9 can be rephrased in terms of trailing monomials also, merely by reversing the order. ■

Proposition 4.11 *Let $P = \{p_1, \ldots, p_r\}$ be a set of $\mathbf{Q}$-linearly independent polynomials. Suppose the polynomial p is such that $TM(p) < TM(p_i)$ for $i = 1, \ldots, r$. Then $P' = \{p_1, \ldots, p_r, p\}$ is also $\mathbf{Q}$-linearly independent.* ■

Having introduced monomial orders and the above properties of polynomials, we are now in a position to describe how to determine a 'near' basis for the Macdonald module P_S. Construct the list of leading monomials of the $p_i \in P$,

$$LM_P = (LM(p_1), \ LM(p_2), \ \ldots, \ LM(p_k)).$$

This is straightforward to construct thanks to Proposition 4.8 – to find the leading monomial of p_i, all we need do is multiply together the leading monomials of its factors. This has the advantage of not needing to expand the polynomials p_i.

Proposition 4.9 says that we may select a $\mathbf{Q}$-linearly independent subset of P immediately by choosing polynomials with distinct leading monomials. Carry out this choice in such a way that the trailing monomials of the polynomials we choose are maximal amongst all such choices. Let the resulting set be P'.

We know that P' is $\mathbf{Q}$-linearly independent but we cannot say that it forms a basis in general. The next step will sometimes allow us to add more elements to P' while maintaining $\mathbf{Q}$-linear independence.

Construct the list of trailing monomials of the polynomials of P', $TM_{P'}$ (similarly to LM_P). Proposition 4.11 says that we can add to P' any polynomial $p_i \in P$ such that $TM(p_i) < TM(p')$ for all $p' \in P'$. Carry out this step as often as possible and let the resulting set be P''.

It should now be clear why we chose the elements of P' to have maximal trailing monomials – by doing so, we improve the chances of being able to add to P' in the above way.

We now have $P'' \subseteq P$ being $\mathbf{Q}$-linearly independent. In favourable cases, P'' is in fact a basis for the module P_S as required. When it is not, of course, there are polynomials of P which we can add to P'' to form a basis – the choice of these polynomials is at present an open question. Note that it is *not* the case that if $|P'| = |P''|$ then P' is a basis.

We have described a construction which generally determines a large $\mathbf{Q}$-linearly independent subset of P. Note that this subset is determined without expanding any polynomials at any stage which is an important point when considering computer implementations where $|S|$ is large. Often the set we obtain by the above procedure is a basis for the Macdonald module but, unfortunately, this is not always the case.

3. The rational vector method

Our latest approach to the basis determination problem will be referred to as the rational vector method. Suppose the polynomials $p_i \in \mathbf{Q}[\alpha_1, \ldots, \alpha_d]$ (i.e. the polynomials of P are in at most d variables). We have that π_S is homogeneous so that each $w\pi_S$ $(w \in W)$ is homogeneous and is of the same degree as π_S. Thus, P is a set of homogeneous polynomials of degree N say.

Definition 4.12 Define

$$V_0 = \{v = (v_1, \ldots, v_d) \in \mathbf{Q}^d \;:\; v_i \in \{1, \ldots, N\},\ i = 1, \ldots, d\}$$

and let $M = |V_0| = N^d$. ■

Definition 4.13 For $i = 1, \ldots, k$ define

$$w_i = (p_i(v_1), \ldots, p_i(v_M)), \quad v_j \in V_0 \ (j = 1, \ldots, M) \quad ■$$

We then have the following result

Theorem 4.14 *The set P is $\mathbf{Q}$-linearly dependent if and only if the k vectors w_i, $1 \leq i \leq k$, are $\mathbf{Q}$-linearly dependent in $\mathbf{Q}^M$.* ■

Note that $M = N^d$ becomes large rapidly as N, d increase. Also we will generally have M large compared to k.

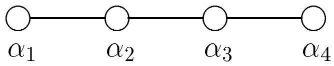
Let W_0 be the $k \times M$ matrix whose rows are the vectors w_i of Definition 4.13. In order to check the $\mathbf{Q}$-linear dependence of the w_i, it suffices to calculate the $k \times k$ determinants of portions of the matrix W_0 looking for a non-zero determinant, stopping either if

- some determinant is non-zero $\Rightarrow$ P is $\mathbf{Q}$-linearly independent
- all $k \times k$ determinants are zero $\Rightarrow$ P is $\mathbf{Q}$-linearly dependent
- the first r $k \times k$ determinants are zero $\Rightarrow$ P is probably $\mathbf{Q}$-linearly dependent (Exactly how large r has to be chosen so that we can be reasonably certain of the linear dependence is unclear)

It seems likely that special properties of the polynomials of the sets P (for example, homogeneity) may enable us to restrict $|V_0|$ considerably and hence make this approach more attractive. Currently, restrictions of V_0 have been found by a trial-and-error approach.

It should be noted that all of the methods described above apply to general 'polynomial spaces', we have not used any special properties of the set P which arises from GENSET.

Example 4.15 Consider the Weyl group of type A_4 with Dynkin diagram



and simple system $\pi = \{\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4\}$ in $(\mathbf{Q}^5)^*$. Consider the subsystem S of type A_3 leading to $\pi_S = \alpha_1\alpha_2\alpha_3(\alpha_1+\alpha_2)(\alpha_2+\alpha_3)(\alpha_1+\alpha_2+\alpha_3)$. Then GENSET gives

$$
\begin{array}{ll}
p_1 & \alpha_1\alpha_2\alpha_3(\alpha_1+\alpha_2)(\alpha_2+\alpha_3)(\alpha_1+\alpha_2+\alpha_3) \\
 & \Big\downarrow \tau_4 \\
p_2 & \alpha_1\alpha_2(\alpha_3+\alpha_4)(\alpha_1+\alpha_2)(\alpha_2+\alpha_3+\alpha_4)(\alpha_1+\alpha_2+\alpha_3+\alpha_4) \\
 & \Big\downarrow \tau_3 \\
p_3 & \alpha_1(\alpha_2+\alpha_3)\alpha_4(\alpha_1+\alpha_2+\alpha_3)(\alpha_2+\alpha_3+\alpha_4)(\alpha_1+\alpha_2+\alpha_3+\alpha_4) \\
 & \Big\downarrow \tau_2 \\
p_4 & (\alpha_1+\alpha_2)\alpha_3\alpha_4(\alpha_1+\alpha_2+\alpha_3)(\alpha_3+\alpha_4)(\alpha_1+\alpha_2+\alpha_3+\alpha_4) \\
 & \Big\downarrow \tau_1 \\
p_5 & \alpha_2\alpha_3\alpha_4(\alpha_2+\alpha_3)(\alpha_3+\alpha_4)(\alpha_2+\alpha_3+\alpha_4)
\end{array}
$$

In the all-monomial approach, there are 42 monomials in all so the p_i will be expressed as polynomials in 42 variables. It turns out that the basis we obtain is $\mathcal{B} = \{p_1,\ p_2,\ p_3,\ p_4\}$.

The leading monomial approach is more sensible. We use lexicographical order, with $\alpha_1 > \alpha_2 > \alpha_3 > \alpha_4$. We then have

$$LM_P = (\alpha_1^3\alpha_2^2\alpha_3,\ \alpha_1^3\alpha_2^2\alpha_3,\ \alpha_1^3\alpha_2^2\alpha_4,\ \alpha_1^3\alpha_3^2\alpha_4,\ \alpha_2^3\alpha_3^2\alpha_4),$$

so there are four distinct leading monomials. We have to choose between p_1 and p_2 – the corresponding trailing monomials are $\alpha_1\alpha_2^2\alpha_3^3$ and $\alpha_1\alpha_2^2\alpha_4^3$ respectively so select p_1. Hence, $P' = \{p_1,\ p_3,\ p_4,\ p_5\}$. The only non-$P'$ polynomial is thus p_2 but its trailing monomial is not less than all of those in P' ($TM(p_2) > TM(p_5)$ for example), so we do not add it to P'.

This completes the construction. In this case, the set $P' = \{p_1,\ p_3,\ p_4,\ p_5\}$ is a basis for P_S ($p_2 = p_1 + p_3 - p_4 + p_5$).

Finally, in the rational vector method we use $d = 4$ and $N = 6$ so that the vectors w_i lie in $\mathbf{Q}^{1296}$. Clearly it would be impractical to test these vectors for $\mathbf{Q}$-linear dependence. It turns out that it is sufficient to let V_0 consist of just

$$(1,1,1,1),\ (2,1,1,1),\ (1,2,1,1),\ (1,1,2,1),\ (1,1,1,2)$$

to obtain the basis $\mathcal{B} = \{p_1,\ p_2,\ p_3,\ p_4\}$.

The all-monomial method has been used to compute in this example but it should be clear that it is not the best method here. The leading monomial method is quick and in fact produces a basis but the rational vector method is also useful with the restricted set V_0. ∎

5 Examples

We now look at the results of applying Macdonald's method using subsystems, parabolic subsets and invertible subsets. We will consider the Weyl groups of types A_3, G_2, D_4 and E_6. In the first two cases, a complete set of irreducible characters is obtained by the 'classical' Macdonald approach (i.e. just by using subsystems) whereas the final two cases illustrate that our other special subsets of root systems are useful in this construction. It should be noted that the algorithm GENSET and the various basis determination methods of section 4 have all been implemented in the symbolic computation system MAPLE (see [Maple91]).

Example 5.1 Consider W of type A_3 as in Example 4.3. Then let e, τ_1, $(\tau_1\tau_3)$, $(\tau_1\tau_2)$, $(\tau_1\tau_2\tau_3)$ be representatives of conjugacy classes C_1, C_2, C_3, C_4, C_5 respectively of $W(\mathrm{A}_3)$. The character table of $W(\mathrm{A}_3)$ is given by

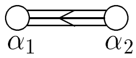
	C_1	C_2	C_3	C_4	C_5
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	2	0	2	-1	0
χ_4	3	1	-1	0	-1
χ_5	3	-1	-1	0	1

Our results are as follows

type of subsystem	simple system	corresponding character
A_3	$\{\alpha_1,\ \alpha_2,\ \alpha_3\}$	χ_2
A_2	$\{\alpha_1,\ \alpha_2\}$	χ_5
$2\mathrm{A}_1$	$\{\alpha_1,\ \alpha_3\}$	χ_3
A_1	$\{\alpha_1\}$	χ_4
$\emptyset$	$\emptyset$	χ_1

Note that a full set of irreducible representations is given (in accordance with Remark 3.2 (2)). ∎

Example 5.2 Consider the Weyl group of type G_2, which is the dihedral group of order 12. The Dynkin diagram of type G_2 is



Let $e,\ \tau_2,\ \tau_1\tau_2,\ (\tau_1\tau_2)^2,\ (\tau_1\tau_2)^3,\ \tau_1$ be representatives of conjugacy classes $C_1,\ C_2,\ C_3,\ C_4,\ C_5,\ C_6$ respectively of $W(\mathrm{G}_2)$. Then the character table of $W(\mathrm{G}_2)$ is

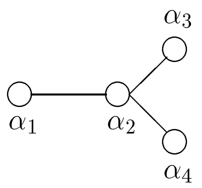
	C_1	C_2	C_3	C_4	C_5	C_6
χ_1	1	1	1	1	1	1
χ_2	1	-1	1	1	1	-1
χ_3	1	-1	-1	1	-1	1
χ_4	1	1	-1	1	-1	-1
χ_5	2	0	-1	-1	2	0
χ_6	2	0	1	-1	-2	0

Our results are as follows

type of subsystem	simple system	corresponding character
G_2	$\{\alpha_1,\ \alpha_2\}$	χ_2
$\tilde{\mathrm{A}}_2$	$\{\alpha_2,\ 3\alpha_1+\alpha_2\}$	χ_3
A_2	$\{\alpha_1,\ \alpha_1+\alpha_2\}$	χ_4
$2\mathrm{A}_1$	$\{\alpha_1,\ 3\alpha_1+2\alpha_2\}$	χ_5
A_1	$\{\alpha_1\}$	χ_6
$\emptyset$	$\emptyset$	χ_1

We have thus constructed a complete set of irreducible $W(\mathrm{G}_2)$-modules. ■

Example 5.3 Consider the Weyl group of type D_4 with Dynkin diagram



and simple system $\pi = \{\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4\}$ in $(\mathbf{Q}^4)^*$. The character table of $W(\mathrm{D}_4)$, with representatives of conjugacy classes $C_1,\ \ldots,\ C_{13}$ taken as $e,\ \tau_2,\ (\tau_2\tau_1),\ (\tau_2\tau_3\tau_1),\ (\tau_1\tau_2\tau_3\tau_1\tau_2\tau_1),$ $(\tau_2\tau_4\tau_1),\ (\tau_1\tau_2\tau_4\tau_1\tau_2\tau_1),\ (\tau_2\tau_4\tau_3),\ (\tau_3\tau_2\tau_4\tau_2\tau_3\tau_2),\ (\tau_2\tau_4\tau_3\tau_1),\ (\tau_4\tau_2\tau_3\tau_1\tau_2\tau_4\tau_1\tau_2\tau_3\tau_1\tau_2\tau_1),$ $(\tau_2\tau_4\tau_3\tau_1\tau_2),\ (\tau_3\tau_1\tau_2\tau_4\tau_1\tau_2)$, is

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}	C_{12}	C_{13}
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	-1	1	-1	1	-1	1	-1	1	1	1	-1	1
χ_3	2	0	-1	0	2	0	2	0	2	-1	2	0	2
χ_4	3	1	0	1	3	-1	-1	-1	-1	0	3	1	-1
χ_5	3	1	0	-1	-1	-1	-1	1	3	0	3	1	-1
χ_6	3	-1	0	-1	3	1	-1	1	-1	0	3	-1	-1
χ_7	3	-1	0	1	-1	1	-1	-1	3	0	3	-1	-1
χ_8	3	-1	0	1	-1	-1	3	1	-1	0	3	-1	-1
χ_9	3	1	0	-1	-1	1	3	-1	-1	0	3	1	-1
χ_{10}	4	-2	1	0	0	0	0	0	0	-1	-4	2	0
χ_{11}	4	2	1	0	0	0	0	0	0	-1	-4	-2	0
χ_{12}	6	0	0	0	-2	0	-2	0	-2	0	6	0	2
χ_{13}	8	0	-1	0	0	0	0	0	0	1	-8	0	0

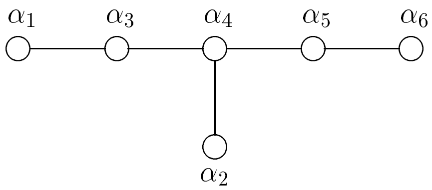
Firstly, we apply the classical Macdonald approach via subsystems to yield the following results.

type of subsystem	simple system	corresponding character
D_4	$\{\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4\}$	χ_2
A_3	$\{\alpha_1,\ \alpha_2,\ \alpha_3\}$	χ_6
A_3	$\{\alpha_1,\ \alpha_2,\ \alpha_4\}$	χ_8
A_3	$\{\alpha_2,\ \alpha_3,\ \alpha_4\}$	χ_7
A_2	$\{\alpha_1,\ \alpha_2\}$	χ_{13}
$4A_1$	$\{\alpha_1,\ \alpha_3,\ \alpha_4,\ \alpha_1+2\alpha_2+\alpha_3+\alpha_4\}$	χ_3
$3A_1$	$\{\alpha_1,\ \alpha_3,\ \alpha_4\}$	χ_{13}
$2A_1$	$\{\alpha_1,\ \alpha_3\}$	χ_4
$2A_1$	$\{\alpha_1,\ \alpha_4\}$	χ_9
$2A_1$	$\{\alpha_3,\ \alpha_4\}$	χ_5
A_1	$\{\alpha_1\}$	χ_{11}
$\emptyset$	$\emptyset$	χ_1

We see that 11 of the 13 irreducible representations are obtained, missing are those corresponding to $\chi_{10},\ \chi_{12}$. To complete the story, we turn out attention to the parabolic and invertible subsets.

Selecting $\Delta = \{\alpha_1\} \subset \pi$, the character yielded by use of the parabolic subset P_Δ is in fact χ_{10}. A complete enumeration using the P_Δ for all $\Delta \subseteq \pi$ does not give the only remaining character, the self-conjugate degree 6 character χ_{12}. Our last resort is the use of invertible subsets – in this case, it turns out that using $(\Delta, \Delta') = (\{\alpha_1,\ \alpha_3\}, \{\alpha_1\})$ gives rise to a reducible representation with character $\chi_4 + \chi_6 + \chi_{12}$ so we may obtain our missing representation of degree 6 by consideration of the appropriate factor module. ■

Example 5.4 Consider now the Weyl group of type E_6 with simple system in $(\mathbf{Q}^8)^*$ given by $\pi = \{\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4,\ \alpha_5,\ \alpha_6\}$ and Dynkin diagram



There are 17 non-conjugate subsystems whose simple system is contained in π, viz.

$$\mathrm{E}_6,\ \mathrm{D}_5,\ \mathrm{A}_5,\ \mathrm{A}_4+\mathrm{A}_1,\ 2\mathrm{A}_2+\mathrm{A}_1,\ 2\mathrm{A}_2,\ \mathrm{A}_4,\ \mathrm{D}_4,\ \mathrm{A}_2+2\mathrm{A}_1,$$
$$\mathrm{A}_3+\mathrm{A}_1,\ \mathrm{A}_3,\ \mathrm{A}_2+\mathrm{A}_1,\ 3\mathrm{A}_1,\ \mathrm{A}_2,\ 2\mathrm{A}_1,\ \mathrm{A}_1,\ \emptyset$$

and three other non-conjugate subsystems (see [IM87]), viz. $3\mathrm{A}_2$, $\mathrm{A}_3+2\mathrm{A}_1$, $4\mathrm{A}_1$, giving 20 subsystems in all to consider. In view of the fact that there are 25 irreducible representations of $W(\mathrm{E}_6)$, we cannot expect to obtain a full set of irreducibles via the classical Macdonald approach. We now present our results for subsystems.

type of subsystem	simple system	degree of corresponding character
E_6	$\{\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4,\ \alpha_5,\ \alpha_6\}$	1
D_5	$\{\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4,\ \alpha_5\}$	20
A_5	$\{\alpha_1,\ \alpha_3,\ \alpha_4,\ \alpha_5,\ \alpha_6\}$	30
$\mathrm{A}_4+\mathrm{A}_1$	$\{\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4,\ \alpha_6\}$	60
$2\mathrm{A}_2+\mathrm{A}_1$	$\{\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_5,\ \alpha_6\}$	80
$2\mathrm{A}_2$	$\{\alpha_1,\ \alpha_3,\ \alpha_5,\ \alpha_6\}$	24
A_4	$\{\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4\}$	81
D_4	$\{\alpha_2,\ \alpha_3,\ \alpha_4,\ \alpha_5\}$	24
$\mathrm{A}_2+2\mathrm{A}_1$	$\{\alpha_2,\ \alpha_3,\ \alpha_5,\ \alpha_6\}$	60
$\mathrm{A}_3+\mathrm{A}_1$	$\{\alpha_1,\ \alpha_2,\ \alpha_4,\ \alpha_5\}$	80
A_3	$\{\alpha_1,\ \alpha_3,\ \alpha_4\}$	81
$\mathrm{A}_2+\mathrm{A}_1$	$\{\alpha_1,\ \alpha_2,\ \alpha_3\}$	64
$3\mathrm{A}_1$	$\{\alpha_1,\ \alpha_2,\ \alpha_5\}$	30
A_2	$\{\alpha_1,\ \alpha_3\}$	30
$2\mathrm{A}_1$	$\{\alpha_1,\ \alpha_2\}$	20
A_1	$\{\alpha_1\}$	6
$\emptyset$	$\emptyset$	1
$3\mathrm{A}_2$	$\{\alpha_1,\ \alpha_3,\ \alpha_5,\ \alpha_6,\ \alpha_2,\ \tilde{\alpha}\}$	10
$\mathrm{A}_3+2\mathrm{A}_1$	$\{\alpha_1,\ \alpha_3,\ \alpha_4,\ \alpha_6,\ \tilde{\alpha}\}$	60
$4\mathrm{A}_1$	$\{\alpha_1,\ \alpha_4,\ \alpha_6,\ \tilde{\alpha}\}$	15

In fact, we yield 17 distinct irreducible representations (for example, the characters of degree 30 arising from subsystems of type $3\mathrm{A}_1$ and A_2 are the same).

Using the parabolic subsets P_Δ for all $\Delta \subseteq \pi$, we obtain a collection of 13 distinct irreducibles, but only three of these are different from those obtained using subsystems – they have degrees 6, 20 and 64. Hence, we have constructed 20 of the 25 representations. The five characters which remain to be determined have degrees 15, 15, 15, 20 and 90.

The remaining irreducible representations have not yet been obtained using the invertible subsets. The problem here is essentially that the degrees of the reducible representations so obtained are too large for our present computational methods to deal with. It is hoped that improvements of the rational vector method will lead to a complete enumeration using the invertible subsets. ■

Acknowledgements

This work forms part of the author's PhD thesis (which will contain the full details) and I would like to thank Professor Alun Morris for his supervision of my postgraduate study. Also Dr. Tom McDonough has provided invaluable guidance with the computing aspects of my work. I am indebted to Dave Rusin of the Department of Mathematical Sciences at Northern Illinois University for useful discussions with regard to the rational vector method and for his initial suggestion of the ideas. I gratefully acknowledge the financial support of the EPSRC.

References

[Bou68] N. Bourbaki. *Groups et Algèbres de Lie*, chapter 4, 5, and 6. Hermann, Paris, 1968.

[BS49] A. Borel and J. De Siebenthal. Les Sous-groupes Fermés Connexes de Rang Maximum des Groupes de Lie Clos. *Commentarii Mathematici Helvetici*, **23**:200–221, 1949.

[Car72] R. W. Carter. Conjugacy Classes in the Weyl Group. *Compositio Mathematica*, **25**:1–59, 1972.

[DCH94] D. Z. Doković, P. Check, and J.-Y. Hée. On closed subsets of root systems. *Canadian Mathematical Bulletin*, **37**(3):338–345, 1994.

[Dyn57] E. B. Dynkin. Semisimple Subalgebras of Semisimple Lie Algebras. *American Mathematical Society Translations*, **6**:111–244, 1957.

[Maple91] B. W. Char et. al. *The Maple V Language Reference Manual.* Springer-Verlag, New York, 1991.

[GP93] M. Geck and G. Pfeiffer. On The Irreducible Characters of Hecke Algebras. *Advances in Mathematics*, **102**:79–94, 1993.

[IM87] A. J. Idowu and A. O. Morris. Some Combinatorial Results for Weyl Groups. *Proceedings of the Cambridge Philosophical Society*, **101**:405–420, 1987.

[Mac72] I. G. Macdonald. Some Irreducible Representations of Weyl Groups. *Bulletin of the London Mathematical Society*, **4**:148–150, 1972.

[Spe35] W. Specht. Die Irreduziblen Darstellungen der Symmetrischen Gruppe. *Mathematische Zeitschriet*, **39**:696–711, 1935.

[Spe37] W. Specht. Darstellungstheorie der Hyperoktaedergruppe. *Mathematische Zeitschriet*, **42**:629–640, 1937.

[Ste67] R. Steinberg. *Lectures on Chevalley Groups*, chapter Appendix. Yale University, New Haven, 1967.