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GBG-rank generating functions for ordinary, self-conjugate, and doubled distinct partitions

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Abstract. BG-ranks and GBG-ranks, introduced by Berkovich and Garvan, are combinatorial statistics that have been used to prove some partition congruences and refinements. In this paper, we give a thorough study on these statistics and obtain formulas for the GBG-rank generating functions for ordinary partitions, self-conjugate partitions, and doubled distinct partitions.

Keywords: partition ranks, cranks, BG-ranks, GBG-ranks, self-conjugate partitions, doubled distinct partitions

1 Introduction

In the theory of partitions, the most well-known combinatorial statistics are Dyson's rank and crank. In 1944, Dyson [10] defined the rank of a partition claiming that this statistic combinatorially explains Ramanujan's mod 5 and 7 partition congruences. Dyson's claim was confirmed by Atkin and Swinnerton-Dyer [3] in 1955. In the same paper, Dyson [10] also conjectured the existence of another statistic for Ramanujan's mod 11 congruence, namely crank, and this conjecture was settled by Andrews and Garvan [2] in 1988.

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In 1990, Garvan, Kim and Stanton [12] found another crank, which splits the set of partitions into t equinumerous classes for $t = 5, 7, 11$, and thus gives a combinatorial account for the three congruences of Ramanujan. In their proof, they constructed two bijections, in which t-core partitions are an essential component. Since then, their crank along with Dyson's rank and crank has been adopted to prove other partition congruences and refinements.

In [5, 6], utilizing the idea of Garvan, Kim and Stanton, Berkovich and Garvan introduced another partition statistic, namely BG-rank, and used it to prove some generalizations and refinements of Ramanujan's mod 5 partition congruence. The BG-rank involves 2-core partitions. Later, Berkovich and Garvan [7] further generalized this statistic by considering t-core partitions.

Recently, the BG-rank has played an important role in the recent works of Chern, Li, Stanton, Xue and Yee [9] and Li, Seo, Stanton and Yee [14] on the enumeration of Kleshchev bipartitions. Motivated by their works, in this paper, we study the BG-rank and its generalization GBG-rank more in depth.

An integer partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ is a weakly decreasing sequence of positive integers. These positive integers are called the parts of λ. The sum of all parts of λ and the number of parts are denoted by $|\lambda|$ and $\ell(\lambda)$, respectively. If $|\lambda| = n$, then λ is called a partition of n and denoted by $\lambda \vdash n$. It is a convention that the empty sequence $\varnothing$ is considered a partition of 0. We denote by $\mathcal{P}$ the set of all partitions.

For a partition λ, its *Young diagram* Y_λ, also known as the *Ferrers diagram*, is a graphical representation of λ, which consists of n boxes placed left justified in rows with λ_i boxes in the i-th row. In Figure 1, the Young diagram of $\lambda = (5, 4, 2, 1)$ is illustrated.

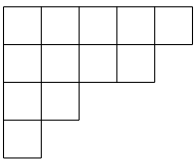

Figure 1: The Young diagram $Y_{(5,4,2,1)}$ of the partition $(5, 4, 2, 1)$

We label the box in the i-th row and the j-th column of the Young diagram Y_λ by (i, j). For a positive integer t, the *t-residue of the box* (i, j) is defined as follows [13]:

$$Res(i, j) := j - i \mod t.$$

The Young diagram with each box filled with its t-residue is called the *t-residue diagram of* λ. The 3-residue diagram of the partition $(5, 4, 2, 1)$ is shown in Figure 2.

For a positive integer t, the *GBG-rank of λ mod t* is the sum of ζ_t^{j-i} for $(i, j) \in Y_\lambda$, where ζ_t is a t-th root of unity. In this paper, we denote the GBG-rank mod t by $\omega_t(\lambda)$,

0	1	2	0	1
2	0	1	2	
1	2			
0				

Figure 2: The 3-residue diagram of the partition $(5,4,2,1)$

namely,

$$\omega_t(\lambda) := \sum_{(i,j)\in Y_\lambda} \zeta_t^{j-i}.$$

We also define the *t-GBG diagram of* λ by assigning ζ_t^{j-i} to each box (i,j) in the Young diagram. The 3-GBG diagram of the partition $(5,4,2,1)$ is given in Figure 3, where $\zeta := \zeta_3$. Since $1+\zeta_3+\zeta_3^2 = 0$, the GBG-rank of $(5,4,2,1)$ mod 3 is 0.

1	ζ	ζ^2	1	ζ
ζ^2	1	ζ	ζ^2	
ζ	ζ^2			
1				

Figure 3: The 3-GBG diagram of the partition $(5,4,2,1)$

For $t = 2$, the GBG-rank is called the BG-rank [5]. Berkovich and Garvan used the BG-rank to prove some generalizations and refinements of Ramanujan's mod 5 partition congruence [5, 6]. They [7] also studied the number of distinct values of GBG-ranks of s-core partitions mod t. In [8], Berkovich and Uncu gave the generating functions for ordinary and strict partitions with the largest part bounded and a fixed integral BG-rank. These results were generalized by Berkovich and Dhar [4] to integral GBG-ranks mod t for any prime t for ordinary partitions and partitions into parts repeating no more than $t-1$ times. The proofs of these results rely on the characterization of t-core partitions in the work of Garvan, Kim and Stanton [12].

Our main objective of this paper is to give a more direct way to compute the generating function for partitions with a fixed GBG-rank mod t, which enables us to get the generating function for non-integral values of GBG-ranks mod t. This approach can also be adopted for other types of partitions such as self-conjugate partitions and doubled distinct partitions. We only state our results on ordinary partitions in this section. More results on the other partitions will be discussed in later sections.

We fix integers t and ℓ with $t \geq 2$. For $\omega \in \zeta_t^\ell \mathbb{Z}$, let

$$G_t(\omega, q) := \sum_{\substack{\lambda\in\mathcal{P} \\ \omega_t(\lambda)=\omega}} q^{|\lambda|}.$$

Theorem 1.1. *For any prime t and $k, \ell \in \mathbb{Z}$, we have*

$$G_t(k\zeta_t^\ell, q) = \begin{cases} \dfrac{q^{tk^2-(t-1)k}}{(q^t;q^t)_\infty^t} & \text{if} \quad \ell = 0, \\ \dfrac{q^{tk^2+k}}{(q^t;q^t)_\infty^t} & \text{if} \quad 1 \le \ell \le t-1. \end{cases}$$

Here, we use the following standard q-series notation: For $a, q \in \mathbb{C}$ and a positive integer n,

$$(a;q)_0 := 1, \quad (a;q)_n := \prod_{j=0}^{n-1}(1-aq^j), \quad (a;q)_\infty := \lim_{n\to\infty}(a;q)_n, \quad |q| < 1.$$

Remark 1.2. *Letting $N \to \infty$ in [4, Theorem 1.1] and [4, Theorem 4.1] gives Theorem 1.1 for $\ell = 0$ and $1 \le \ell \le t-1$, respectively.*

By restricting the number of parts and the size of parts of partitions, we can also obtain the finite form of the generating function $G_t(\omega, q)$. Let

$$G_{M,N,t}(\omega, q) := \sum_{\substack{\lambda \in \mathcal{P}, \lambda_1 \le N, \ell(\lambda) \le M \\ \omega_t(\lambda) = \omega}} q^{|\lambda|}.$$

Theorem 1.3. *For any prime t, any $k, \ell \in \mathbb{Z}$, and nonnegative integers M and N, we have*

$$G_{M,N,t}(k\zeta_t^\ell, q) = \begin{cases} q^{tk^2-(t-1)k} \begin{bmatrix} N_{t,1}+M_{t,1} \\ N_{t,1}-k \end{bmatrix}_{q^t} \begin{bmatrix} N_{t,t}+M_{t,t} \\ N_{t,t}+k \end{bmatrix}_{q^t} \prod_{j=2}^{t-1} \begin{bmatrix} N_{t,j}+M_{t,j} \\ N_{t,j} \end{bmatrix}_{q^t} & \text{if} \quad \ell = 0, \\ q^{tk^2+k} \begin{bmatrix} N_{t,\ell+1}+M_{t,\ell+1} \\ N_{t,\ell+1}-k \end{bmatrix}_{q^t} \begin{bmatrix} N_{t,\ell}+M_{t,\ell} \\ N_{t,\ell}+k \end{bmatrix}_{q^t} \prod_{\substack{1 \le j \le t \\ j \ne \ell, \ell+1}} \begin{bmatrix} N_{t,j}+M_{t,j} \\ N_{t,j} \end{bmatrix}_{q^t} & \text{if} \quad 1 \le \ell \le t-1, \end{cases}$$

where $M_{t,j} := \left\lfloor \frac{M+j-1}{t} \right\rfloor$ and $N_{t,j} := \left\lfloor \frac{N-j}{t} \right\rfloor + 1$.

Here, $\begin{bmatrix} m \\ n \end{bmatrix}_q$ is the q-binomial coefficient defined by

$$\begin{bmatrix} m \\ n \end{bmatrix}_q = \begin{cases} \dfrac{(q;q)_m}{(q;q)_n(q;q)_{m-n}} & \text{for } m \ge n \ge 0, \\ 0 & \text{otherwise.} \end{cases}$$

Since

$$\lim_{m\to\infty} \begin{bmatrix} m \\ n \end{bmatrix}_q = \frac{1}{(q;q)_n},$$

letting $M \to \infty$ for $\ell = 0$ in Theorem 1.3 yields the following identity, which was given by Berkovich and Dhar [4, Theorem 1.1].

Corollary 1.4. *For $\nu \in \{0, 1 \ldots, t-1\}$,*

$$G_{\infty,tN+\nu,t}(k,q) = \frac{q^{tk^2-(t-1)k}}{(q^t;q^t)_{N+k}(q^t;q^t)_{N+\lceil \frac{\nu}{t} \rceil -k}(q^t;q^t)_{N+\lceil \frac{\nu-1}{t} \rceil} \cdots (q^t;q^t)_{N+\lceil \frac{\nu-(t-2)}{t} \rceil}}.$$

Similarly, letting $M \to \infty$ in Theorem 1.3 for $1 \leq \ell \leq t-1$ gives [4, Theorem 4.1].

Corollary 1.5. *For $\nu \in \{0, 1, \ldots, t-1\}$ and $\ell \in \{1, 2, \ldots, t-1\}$,*

$$G_{\infty,tN+\nu,t}(k\zeta_t^\ell, q) = \frac{q^{tk^2+k}}{(q^t;q^t)_{N+\lceil \frac{\nu-\ell+1}{t} \rceil +k}(q^t;q^t)_{N+\lceil \frac{\nu-\ell}{t} \rceil -k} \prod\limits_{\substack{1 \leq j \leq t \\ j \neq \ell, \ell+1}} (q^t;q^t)_{N+\lceil \frac{\nu-j+1}{t} \rceil}}.$$

The rest of this paper is organized as follows. In Section 2, we recall some necessary definitions. In Section 3, we give a general formula for the GBG-rank generating function for ordinary partitions, from which Theorems 1.1 and 1.3 can be derived. We also give an asymptotic formula for the number of partitions with integral GBG-ranks mod t. Sections 4 and 5 are devoted to GBG-ranks for self-conjugate partitions and doubled distinct partitions, respectively. Adapting the method used for ordinary partitions, we give the GBG-rank generating functions and refinements for self-conjugate partitions and doubled distinct partitions.

2 Preliminaries

The hook of the box (i,j) in the Young diagram of a partition λ is the following set of boxes:

$$H_{(i,j)} := \{(i,m) \in Y_\lambda \mid m \geq j\} \cup \{(k,j) \in Y_\lambda \mid k \geq i\}.$$

The size of $H_{(i,j)}$ is called the hook length of the box (i,j) and denoted by $h_{(i,j)}$. For a positive integer t, λ is called a t-core partition if no hooks have length divisible by t.

A Frobenius symbol for a positive integer n is a two-rowed array

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_s \\ b_1 & b_2 & \cdots & b_s \end{pmatrix}$$

such that $\sum_{j=1}^{s}(a_j + b_j + 1) = n$, $a_1 > a_2 > \cdots > a_s \geq 0$, and $b_1 > b_2 > \cdots > b_s \geq 0$. There is a natural one-to-one correspondence between partitions and Frobenius symbols. For a partition λ, let s be the largest integer such that $\lambda_s - s \geq 0$, i.e., s is the side of the largest square of λ that fits inside the Young Diagram of λ. This largest square is unique and called the *Durfee square* of λ. We now consider the following two-rowed array:

$$\begin{pmatrix} \lambda_1 - 1 & \lambda_2 - 2 & \cdots & \lambda_s - s \\ \lambda'_1 - 1 & \lambda'_2 - 2 & \cdots & \lambda'_s - s \end{pmatrix},$$

where λ'_j is the number of boxes in the j-th column of the Young diagram of λ. Clearly, this satisfies the conditions for Frobenius symbols, and this process is reversible. Thus, there is a unique Frobenius symbol associated with λ. In this paper, however, we will denote the Frobenius symbol of λ by

$$\mathfrak{F}(\lambda) := (\lambda_1, \lambda_2 - 1 \ldots, \lambda_s - s + 1 \mid \lambda'_1 - 1, \lambda'_2 - 2, \ldots, \lambda'_s - s).$$

Note that each number on the left of the bar in $\mathfrak{F}(\lambda)$ is one larger than the corresponding entry in the above two-rowed array notation. In Figure 3, the Frobenius symbol of the partition $(5, 4, 2, 1)$ is

$$(5, 3 \mid 3, 1).$$

The *conjugate of* λ is the partition resulting from reflecting the Young diagram of λ about the main diagonal, and we denote the conjugate partition by λ'. We call λ a *self-conjugate partition* if $\lambda = \lambda'$.

Let $MD(\lambda)$ be the set of hook lengths in the main diagonal of λ. Then, for

$$\mathfrak{F}(\lambda) = (a_1, \ldots, a_s \mid b_1, \ldots, b_s),$$

we can easily see that

$$MD(\lambda) = \{a_1 + b_1, a_2 + b_2, \ldots, a_s + b_s\}.$$

If λ is self-conjugate, then

$$\mathfrak{F}(\lambda) = (\lambda_1, \lambda_2 - 1 \ldots, \lambda_s - s + 1 \mid \lambda_1 - 1, \lambda_2 - 2, \ldots, \lambda_s - s), \tag{2.1}$$

so

$$MD(\lambda) = \{2\lambda_1 - 1, 2\lambda_2 - 3, \ldots, 2\lambda_s - (2s - 1)\},$$

i.e., all elements of $MD(\lambda)$ are distinct and odd, and $MD(\lambda)$ determines λ. Thus,

$$\sum_{\lambda \in \mathcal{SC}} q^{|\lambda|} = (-q; q^2)_\infty,$$

where $\mathcal{SC}$ denotes the set of self-conjugate partitions.

For a partition λ into distinct parts, the shifted Young diagram of λ is the diagram resulting from shifting the i-th row to the right by $i - 1$ boxes for each i in its Young diagram. The doubled distinct partition of λ, denoted by $\lambda\lambda$, is the partition whose Young diagram is defined by adding λ_i boxes to the $(i-1)$st column of the shifted Young diagram of λ. That is, for a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ with $\lambda_1 > \lambda_2 > \cdots > \lambda_\ell$,

$$\mathfrak{F}(\lambda\lambda) = (\lambda_1 + 1, \lambda_2 + 1, \ldots, \lambda_\ell + 1 \mid \lambda_1 - 1, \lambda_2 - 1, \ldots, \lambda_\ell - 1) \tag{2.2}$$

and

$$MD(\lambda\lambda) = \{2\lambda_1, 2\lambda_2, \ldots, 2\lambda_\ell\}.$$

Thus,

$$\sum_{\lambda\lambda \in \mathcal{DD}} q^{|\lambda\lambda|} = (-q^2; q^2)_\infty,$$

where $\mathcal{DD}$ denotes the set of doubled distinct partitions.

3 Results on ordinary partitions

In this section and throughout the rest of this paper, we assume that t is a positive integer with $t \geq 2$ unless it is stated otherwise. We also assume that ζ is a t-th root ζ_t of unity.

We first note that it follows from Frobenius symbols that the generating function for partitions is

$$\sum_{\lambda \in \mathcal{P}} q^{|\lambda|} = [z^0](-zq;q)_\infty(-z^{-1};q)_\infty,$$

where $[z^0]G(z)$ means the constant term of $G(z)$. Next, we will consider the following GBG-rank generating function:

$$\sum_{\lambda \in \mathcal{P}} x^{\omega_t(\lambda)} q^{|\lambda|}.$$

For a nonnegative integer j, let $r_j := r_j(\zeta) := \sum_{i=0}^{j-1} \zeta^i$. Note that $r_0 = 0$ because the sum is empty. Also, for $0 \leq j \leq t-1$,

$$\sum_{i=1}^{t-j} \zeta^{-i} = \sum_{i=1}^{t-j} \zeta^{t-i} = \sum_{i=j}^{t-1} \zeta^i = -r_j \tag{3.1}$$

since $r_t = 1 + \zeta + \cdots + \zeta^{t-1} = 0$.

Let λ be a partition with the following Frobenius symbol:

$$\mathfrak{F}(\lambda) = (a_1, \ldots, a_s \mid b_1, \ldots, b_s)$$

for some positive integer s. Then the GBG-rank of λ mod t is

$$\omega_t(\lambda) = \sum_{k=1}^{s} \left(\sum_{i=0}^{a_k-1} \zeta^i + \sum_{i=1}^{b_k} \zeta^{-i} \right) = \sum_{k=1}^{s} \left(\sum_{i=0}^{\bar{a}_k-1} \zeta^i + \sum_{i=1}^{\bar{b}_k} \zeta^{-i} \right) = \sum_{k=1}^{s} (r_{\bar{a}_k} - r_{t-\bar{b}_k}),$$

where $\bar{a}_k$ and $\bar{b}_k$ are the least nonnegative residues of a_k and b_k mod t, respectively, and the last equality follows from (3.1). Thus,

$$\begin{aligned}
\sum_{\lambda \in \mathcal{P}} x^{\omega_t(\lambda)} q^{|\lambda|} &= [z^0] \prod_{j=1}^{t} \left(-x^{r_j} z q^j; q^t \right)_\infty \left(-x^{-r_j} z^{-1} q^{t-j}; q^t \right)_\infty \\
&= \frac{1}{(q^t;q^t)_\infty^t} [z^0] \prod_{j=1}^{t} \sum_{n_j=-\infty}^{\infty} z^{n_j} x^{n_j r_j} q^{\frac{t n_j (n_j-1)}{2} + j n_j} \\
&= \frac{1}{(q^t;q^t)_\infty^t} \sum_{\substack{n_1, \ldots, n_t = -\infty \\ n_1 + \cdots + n_t = 0}}^{\infty} x^{n_1 r_1 + \cdots + n_t r_t} q^{\sum_{j=1}^{t} \left(\frac{t n_j (n_j-1)}{2} + j n_j \right)},
\end{aligned} \tag{3.2}$$

where the second equality follows from the Jacobi triple product identity [1]:

$$(-x;q)_\infty(-q/x;q)_\infty(q;q)_\infty = \sum_{n=-\infty}^{\infty} x^n q^{n(n-1)/2}. \tag{3.3}$$

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. For $\ell = 0$, we have $\omega_t(\lambda) = k \in \mathbb{Z}$. When t is a prime, the minimal polynomial of ζ over $\mathbb{Z}$ is $1 + x + \cdots + x^{t-1}$, so $1, \zeta, \ldots, \zeta^{t-2}$ are linearly independent over $\mathbb{Z}$. Hence, the power of x in (3.2) is an integer if and only if $n_1 + n_t = 0$ and $n_2 = \cdots = n_{t-1} = 0$. Thus,

$$\sum_{\substack{\lambda\in\mathcal{P}\\ \omega_t(\lambda)\in\mathbb{Z}}} x^{\omega_t(\lambda)} q^{|\lambda|} = \frac{1}{(q^t;q^t)_\infty^t} \sum_{n_1=-\infty}^{\infty} x^{n_1} q^{\frac{tn_1(n_1-1)}{2}+\frac{tn_1(n_1+1)}{2}+n_1-tn_1}. \tag{3.4}$$

The theorem immediately follows from comparing the coefficients of x^{n_1} on both sides.

Next we consider the case when $1 \le \ell \le t-1$. We first note that $\zeta^\ell = r_{\ell+1}(\zeta) - r_\ell(\zeta)$. Again, due to the minimal polynomial of ζ and the linear independence of $1, \zeta, \ldots, \zeta^{t-2}$ over $\mathbb{Z}$, we see that the power of x in (3.2) is in $\zeta^\ell\mathbb{Z}$ if and only if $n_\ell + n_{\ell+1} = 0$ and $n_1 = \ldots = n_{\ell-1} = n_{\ell+2} = \cdots = n_t = 0$. Thus,

$$\sum_{\substack{\lambda\in\mathcal{P}\\ \omega_t(\lambda)\in\zeta^\ell\mathbb{Z}}} x^{\omega_t(\lambda)} q^{|\lambda|} = \frac{1}{(q^t;q^t)_\infty^t} \sum_{n_{\ell+1}=-\infty}^{\infty} x^{n_{\ell+1}\zeta^\ell} q^{tn_{\ell+1}^2+n_{\ell+1}},$$

which completes the proof. □

The finite form of $G_t(k\zeta^\ell, q)$ in Theorem 1.3 can be proved in a similar way by applying the following finite form of the Jacobi triple product identity [11]:

$$(-1/x;q)_M(-xq;q)_N = \sum_{n=-M}^{N} \begin{bmatrix} N+M \\ M+n \end{bmatrix}_q x^n q^{n(n+1)/2}. \tag{3.5}$$

Next, we will give the asymptotic ratio for partitions with integral GBG-ranks mod t among all ordinary partitions by applying Wright's circle method. Let

$$F_t(q) := \sum_{n\ge 0} a_t(n) q^n = \frac{1}{(q^t;q^t)_\infty^t} \sum_{k=-\infty}^{\infty} q^{tk^2-(t-1)k}.$$

Theorem 3.1. *For a fixed $t \ge 2$,*

$$a_t(n) = \frac{t^{\frac{t-1}{2}}}{2(24)^{t/4}} n^{-(t+2)/4} e^{\pi\sqrt{2n/3}} + O\left(n^{-\frac{t+3}{4}} e^{\pi\sqrt{2n/3}}\right) \text{ as } n \to \infty.$$

Moreover, the asymptotic ratio for ordinary partitions to have integral GBG-ranks mod t is

$$\frac{a_t(n)}{p(n)} = \frac{t^{(t-1)/2}}{\sqrt{2}(24)^{(t-2)/4}} n^{-(t-2)/4}\left(1 + O\left(n^{-1/4}\right)\right) \text{ as } n \to \infty,$$

where $p(n)$ is the partition function of n.

4 Results on self-conjugate partitions

For a nonnegative integer j, let $r_j^* := r_j^*(\zeta) := \sum_{i=-j}^{j} \zeta^i$.

As seen in Section 2, self-conjugate partitions are completely determined by their hook lengths in the main diagonal, which are distinct odd numbers.

Let λ be a self-conjugate partition with $MD(\lambda) = \{2d_1+1, 2d_2+1, \ldots, 2d_s+1\}$. Then, $\omega_t(\lambda) = \sum_{j=1}^{s} r_{d_j}^*$ and $r_j^* = r_{j+2t}^*$. Thus,

$$\sum_{\lambda \in \mathcal{SC}} x^{\omega_t(\lambda)} q^{|\lambda|} = \prod_{j=0}^{t-1} (-x^{r_j^*} q^{2j+1}; q^{2t})_\infty.$$

Utilizing this formula, we can obtain the following theorem.

Theorem 4.1. *We have*

$$\sum_{\lambda \in \mathcal{SC}} x^{\omega_t(\lambda)} q^{|\lambda|} = \begin{cases} \dfrac{(-q^t; q^{2t})_\infty}{(q^{2t}; q^{2t})_\infty^{(t-1)/2}} \displaystyle\prod_{j=0}^{(t-3)/2} \sum_{n=-\infty}^{\infty} x^{nr_j^*} q^{tn^2-(t-2j-1)n} & \text{if } t \text{ is odd,} \\ \dfrac{1}{(q^{2t}; q^{2t})_\infty^{t/2}} \displaystyle\prod_{j=0}^{t/2-1} \sum_{n=-\infty}^{\infty} x^{nr_j^*} q^{tn^2-(t-2j-1)n} & \text{if } t \text{ is even.} \end{cases}$$

For a nonnegative integer N, let $\mathcal{SC}_N$ be the set of self-conjugate partitions into parts less than or equal to N. Clearly, each partition in $\mathcal{SC}_N$ has at most N parts.

Theorem 4.2. *For a nonnegative integer N, we have*

$$\sum_{\lambda \in \mathcal{SC}_N} x^{\omega_t(\lambda)} q^{|\lambda|}$$
$$= \begin{cases} (-q^t; q^{2t})_{N_{t,\frac{t-1}{2}}} \displaystyle\prod_{j=0}^{(t-3)/2} \sum_{n=-N_{t,t-j-1}}^{N_{t,j}} \begin{bmatrix} N_{t,j}+N_{t,t-j-1} \\ N_{t,j}-n \end{bmatrix}_{q^{2t}} x^{nr_j^*} q^{tn^2-(t-2j-1)n} & \text{if } t \text{ is odd,} \\ \displaystyle\prod_{j=0}^{t/2-1} \sum_{n=-N_{t,t-j-1}}^{N_{t,j}} \begin{bmatrix} N_{t,j}+N_{t,t-j-1} \\ N_{t,j}-n \end{bmatrix}_{q^{2t}} x^{nr_j^*} q^{tn^2-(t-2j-1)n} & \text{if } t \text{ is even,} \end{cases}$$

where $N_{t,j} := \lfloor \frac{N-j-1}{t} \rfloor + 1$.

Note that when $t = 2, 3, 4, 6$, we have $r_j^* \in \mathbb{Z}$ for all positive integers j, which yields that $\omega_t(\lambda) \in \mathbb{Z}$.

Next, we consider integral GBG-ranks mod t for an odd prime t. Let $GSC_t(k, q)$ be the generating function for the number of self-conjugate partitions with the GBG-rank mod t equal to k.

Theorem 4.3. *For an odd prime t and an integer k,*

$$GSC_t(k, q) = \frac{(-q^t; q^{2t})_\infty q^{tk^2-(t-1)k}}{(q^{2t}; q^{2t})_\infty^{(t-1)/2}}.$$

For a nonnegative integer N, let $GSC_{N,t}(k, q)$ be the generating function for the number of self-conjugate partitions into parts less than or equal to N with the GBG-rank mod t equal to k.

Theorem 4.4. *For an odd prime t and integers k, N with $N \geq 0$,*

$$GSC_{N,t}(k, q) = q^{tk^2-(t-1)k}(-q^t; q^{2t})_{N_{t,\frac{t-1}{2}}} \begin{bmatrix} N_{t,0} + N_{t,t-1} \\ N_{t,0} - k \end{bmatrix}_{q^{2t}} \prod_{j=1}^{(t-3)/2} \begin{bmatrix} N_{t,j} + N_{t,t-j-1} \\ N_{t,t-j-1} \end{bmatrix}_{q^{2t}},$$

where $N_{t,j} = \lfloor \frac{N-j-1}{t} \rfloor + 1$.

Remark 4.5. *Theorem 4.4 is the same as [4, Theorem 1.3] when N is replaced by $tN + \nu$.*

5 Results on doubled distinct partitions

For a nonnegative integer j, let $r'_j := r'_j(\zeta) := \sum_{i=-j+1}^{j} \zeta^i$.

As seen in Section 2, doubled distinct partitions are completely determined by their hook lengths in the main diagonal, which are distinct even numbers. Thus, for a doubled distinct partition $\lambda\lambda$ with $MD(\lambda\lambda) = \{2d_1, 2d_2, \ldots, 2d_s\}$, we see that $\omega_t(\lambda\lambda) = \sum_{j=1}^{s} r'_{d_j}$.

In a similar way to the self-conjugate partition case, we get the following generating function for doubled distinct partitions with its GBG-rank mod t.

Theorem 5.1. *We have*

$$\sum_{\lambda\lambda \in \mathcal{DD}} x^{\omega_t(\lambda\lambda)} q^{|\lambda\lambda|} = \begin{cases} \dfrac{(-q^{2t}; q^{2t})_\infty}{(q^{2t}; q^{2t})_\infty^{(t-1)/2}} \displaystyle\prod_{j=1}^{(t-1)/2} \sum_{n=-\infty}^{\infty} x^{nr'_j} q^{tn^2-(t-2j)n} & \text{if } t \text{ is odd,} \\ \dfrac{(-q^t; q^{2t})_\infty (-q^{2t}; q^{2t})_\infty}{(q^{2t}; q^{2t})_\infty^{t/2-1}} \displaystyle\prod_{j=1}^{t/2-1} \sum_{n=-\infty}^{\infty} x^{nr'_j} q^{tn^2-(t-2j)n} & \text{if } t \text{ is even.} \end{cases}$$

We also get the finite form of the results in Theorem 5.1. For a nonnegative integer N, let $\mathcal{DD}_N$ be the set of doubled distinct partitions into parts less than or equal to N. By the definition, each partitions in $\mathcal{DD}_N$ has at most $N-1$ parts.

Theorem 5.2. *For a nonnegative integer N, we have*

$$\sum_{\lambda\lambda\in\mathcal{DD}_N} x^{\omega_t(\lambda\lambda)}q^{|\lambda\lambda|} = \begin{cases} (-q^{2t};q^{2t})_{N_{t,t}} \displaystyle\prod_{j=1}^{(t-1)/2} \sum_{n=-N_{t,t-j}}^{N_{t,j}} \begin{bmatrix} N_{t,j}+N_{t,t-j} \\ N_{t,j}-n \end{bmatrix}_{q^{2t}} x^{nr'_j} q^{tn^2-(t-2j)n} & \text{if } t \text{ is odd,} \\ (-q^t;q^{2t})_{N_{t,\frac{t}{2}}} (-q^{2t};q^{2t})_{N_{t,t}} \displaystyle\prod_{j=1}^{t/2-1} \sum_{n=-N_{t,t-j}}^{N_{t,j}} \begin{bmatrix} N_{t,j}+N_{t,t-j} \\ N_{t,j}-n \end{bmatrix}_{q^{2t}} x^{nr'_j} q^{tn^2-(t-2j)n} & \text{if } t \text{ is even,} \end{cases}$$

where $N_{t,j} = \lfloor \frac{N-j-1}{t} \rfloor + 1$.

Let $GDD_t(k,q)$ be the generating function for the number of doubled distinct partitions with the GBG-rank mod t equal to k. We have the following results.

Theorem 5.3. *For a nonzero integer k, we have $GDD_t(k,q) = 0$. Also, if t is an odd prime, then*

$$GDD_t(0,q) = \frac{(-q^{2t};q^{2t})_\infty}{(q^{2t};q^{2t})_\infty^{(t-1)/2}},$$

and if $t = 2$, then

$$GDD_2(0,q) = (-q^2;q^2)_\infty.$$

Moreover, for a doubled distinct partition $\lambda\lambda$, if $\omega_t(\lambda\lambda) \in \mathbb{Z}$, then $\omega_t(\lambda\lambda) = 0$.

Theorem 5.4. *For an odd prime t and an integer k, we have*

$$GDD_t(k+k\zeta,q) = \frac{(-q^{2t};q^{2t})_\infty q^{tk^2-(t-2)k}}{(q^{2t};q^{2t})_\infty^{(t-1)/2}}.$$

For a nonnegative integer N, let $GDD_{N,t}(k,q)$ be the generating function for the number of doubled distinct partitions into parts less than or equal to N with the GBG-rank mod t equal to k.

Theorem 5.5. *For integers t,k,N with $t \geq 2$ and $N \geq 0$, $GDD_{N,t}(k,q) = 0$ when $k \neq 0$, and for odd prime t,*

$$GDD_{N,t}(0,q) = (-q^{2t};q^{2t})_{N_{t,t}} \prod_{j=1}^{(t-1)/2} \begin{bmatrix} N_{t,j}+N_{t,t-j} \\ N_{t,j} \end{bmatrix}_{q^{2t}},$$

where $N_{t,j} = \left\lfloor \frac{N-j-1}{t} \right\rfloor + 1$.

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