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Modified Macdonald polynomials and μ-Mahonian statistics

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Abstract. We establish an equidistribution between the pairs of statistics $(\mathsf{inv}, \mathsf{maj})$ and $(\mathsf{quinv}, \mathsf{maj})$ on any row-equivalency class $[\tau]$ where τ is a filling of a given Young diagram. In particular if τ is a filling of a rectangular Young diagram, the triples $(\mathsf{inv}, \mathsf{quinv}, \mathsf{maj})$ and $(\mathsf{quinv}, \mathsf{inv}, \mathsf{maj})$ have the same distribution over $[\tau]$. Our main result affirms a conjecture proposed by Ayyer, Mandelshtam and Martin, thus presenting the equivalence between two refined formulas for the modified Macdonald polynomials.

Keywords: modified Macdonald polynomial, bijection, inversion, queue inversion, the major index, equidistribution

1 Introduction and main results

Macdonald polynomials $P_\mu(X; q, t)$ indexed by partitions are polynomials in infinitely many variables $X = \{x_1, x_2, \ldots\}$ with coefficients in the field $\mathbb{Q}(q, t)$ of rational functions of two variables q and t. Several important classes of symmetric polynomials are well–studied specializations of Macdonald polynomials such as Schur polynomials (when $q = t$), Hall–Littlewood polynomials (when $q = 0$) and Jack polynomials (when $q = t^\alpha$ and let $t \to 1$).

Macdonald polynomials $P_\mu(X; q, t)$ are defined as the unique basis for the ring of symmetric functions over the field $\mathbb{Q}(q, t)$ satisfying certain orthogonality and lower triangularity properties. The orthogonality is defined with respect to the Hall scalar product, while the triangularity refers to the expansion of $P_\mu(X; q, t)$ in terms of the monomial symmetric functions $m_\lambda(X)$. Since the coefficients in this expansion may have nontrivial denominators, Macdonald introduced the *integral form* of $P_\mu(X; q, t)$, denoted by $J_\mu(X; q, t)$, which is defined as the product of $P_\mu(X; q, t)$ and a polynomial determined by the arm- and leg-lengths of the Young diagram of μ. Specifically,

$$J_\mu(X; q, t) = \sum_\lambda K_{\lambda\mu}(q, t) s_\lambda[X(1-t)], \tag{1.1}$$

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where $f[X]$ denotes the plethystic substitution of X into the symmetric function f, s_λ denotes the Schur function and $K_{\lambda\mu}(q,t)$ is the (q,t)-Kostka number.

Subsequently, another widely studied variant of Macdonald polynomials, called *the modified Macdonald polynomials* $\tilde{H}_\mu(X;q,t)$ was introduced by Garsia and Haiman [4]. Let $\tilde{K}_{\lambda\mu}(q,t) = t^{n(\mu)}K_{\lambda\mu}(q,t^{-1})$ where $n(\mu) = \sum_i (i-1)\mu_i$. Then

$$\tilde{H}_\mu(X;q,t) = \sum_\lambda \tilde{K}_{\lambda\mu}(q,t)s_\lambda(X).$$

Haiman remarkably discovered that $\tilde{H}_\mu(X;q,t)$ equals the Frobenius series of a space spanned by certain polynomials together with all of their partial derivatives [7]. In parallel, the combinatorial investigation of modified Macdonald polynomials has been greatly advanced by the celebrated breakthrough on the surprising connections between $\tilde{H}_\mu(X;q,t)$ and μ-Mahonian statistics on fillings of Young diagrams, due to Haglund, Haiman and Loehr [6].

Before we state the combinatorial formula of $\tilde{H}_\mu(X;q,t)$, let us first review the definitions of μ-Mahonian statistics inv, quinv and maj of fillings. A partition $\mu = (\mu_1, \cdots, \mu_k)$ of n is a sequence of positive integers such that $\mu_i \geq \mu_{i+1}$ for all $1 \leq i < k$ and $|\mu| = \mu_1 + \cdots + \mu_k = n$. The integer μ_i is called the ith part of μ and k is the length of μ, denoted by $\ell(\mu)$. The Young diagram of μ is an array of boxes with μ_i boxes in the ith row from bottom to top, with the first box of each row left-justified. A box has coordinates (i,j) if it is in the ith row from bottom to top and jth column from left to right. Let μ' be the transpose of μ; that is, the Young diagram of μ' is obtained from that of μ by reflecting across the main diagonal (boxes with coordinates (i,i)).

A filling of the Young diagram of μ is a function $\sigma : \mu \to \mathbb{P}$ (where $\mathbb{P}$ is the set of positive integers), which assigns to each box z of the Young diagram of μ (denoted by $z \in \mu$) to a positive integer $\sigma(z)$. We use $\mathsf{South}(z)$ to denote the box immediately below z. Define

$$x^\sigma = \prod_{z\in\mu} x_{\sigma(z)}$$

to be the monomial corresponding to the content of σ. A *descent* (resp. *non-descent*) of a filling σ is a pair of entries $(\sigma(z), \sigma(\mathsf{South}(z)))$ such that $\sigma(z) > \sigma(\mathsf{South}(z))$ (resp. $\sigma(z) \leq \sigma(\mathsf{South}(z))$). We define $\mathsf{Des}(\sigma) = \{z \in \mu : \sigma(z) > \sigma(\mathsf{South}(z))\}$ to be the *descent set* of σ and $\mathsf{des}(\sigma) = |\mathsf{Des}(\sigma)|$. Let $\mathsf{leg}(z)$ be the number of boxes strictly above box z in its column. Then

$$\mathsf{maj}(\sigma) = \sum_{z\in\mathsf{Des}(\sigma)} (\mathsf{leg}(z)+1),$$

which is called *the major index* of σ.

Let $\mathcal{T}(\mu)$ be the set of all fillings of the Young diagram of μ. Given a filling $\sigma \in \mathcal{T}(\mu)$, let $\sigma(\mu'_i+1, i) = 0$ for all $1 \le i \le \mu_1$. In other words, we add a box with entry 0 above the topmost box of each column of σ. Furthermore, let $\sigma(0, i) = \infty$ for all $1 \le i \le \mu_1$. That is, we add a box with entry ∞ below the bottommost box of each column of σ.

Define $\mathcal{Q}(a, b, c) = 1$ if one of the conditions $a < b < c$, $b < c < a$, $c < a < b$ or $a = b \ne c$ holds; otherwise, $\mathcal{Q}(a, b, c) = 0$. A *queue inversion triple* of σ is a triple (a, b, c) of entries in σ such that (as illustrated on the left of the diagram)

1. b and c are in the same row, with c to the right of b;
2. a and b are in the same column, with b immediately below a;
3. $\mathcal{Q}(a, b, c) = 1$ with $c \ne 0$.

An *inversion triple* of σ is a triple (a, b, c) of entries in σ satisfying: 2, 3 and

4. a and c are in the same row, with c to the right of a.

This is illustrated on the right of the diagram.

$$\begin{array}{|c|ccc}\cline{1-1} a & & & \\ \cline{1-1}\cline{4-4} b & \cdots & \multicolumn{1}{|c|}{c} \\ \cline{1-1}\cline{4-4}\end{array} \qquad \begin{array}{|c|c|c|}\cline{1-1}\cline{3-3} a & \cdots & c \\ \cline{1-1}\cline{3-3} b & \multicolumn{2}{l}{} \\ \cline{1-1}\end{array}$$

Let $\mathsf{quinv}(\sigma)$ and $\mathsf{inv}(\sigma)$ be the numbers of queue inversion triples and inversion triples of σ, respectively.

Here the statistics maj, inv and quinv are natural extensions of the major index, the inversion numbers and the queue inversion numbers for permutations, respectively. The reading word $\omega(\sigma)$ of σ is defined as the sequence formed by the entries in the top row, read from left to right, followed by the entries in the next row from left to right, and so on. When the Young diagram consists of a single row (resp. a single column), one has $\mathsf{inv}(\sigma) = \mathsf{inv}(\omega(\sigma))$ and $\mathsf{quinv}(\sigma) = \mathsf{coinv}(\omega(\sigma))$ (resp. $\mathsf{maj}(\sigma) = \mathsf{maj}(\omega(\sigma))$), indicating that the statistics inv, quinv and maj reduce to the classical Mahonian statistic on permutations of a multiset [5, 13]. In general, the statistics inv, quinv and maj defined on the fillings of the Young diagram of μ are called μ-Mahonian statistics.

We are now ready to present the combinatorial formula of $\tilde{H}_\mu(X; q, t)$ by Haglund, Haiman and Loehr [6]:

$$\tilde{H}_\mu(X; q, t) = \sum_{\sigma:\mu\to\mathbb{P}} x^\sigma t^{\mathsf{maj}(\sigma)} q^{\mathsf{inv}(\sigma)}. \tag{1.2}$$

Recently, Corteel, Haglund, Mandelshtam, Mason and Williams [3] proved a compact formula for $\tilde{H}_\mu(X; q, t)$, expressed as a sum over sorted tableaux, and conjectured an equivalent form of Equation (1.2):

$$\tilde{H}_\mu(X; q, t) = \sum_{\sigma:\mu\to\mathbb{P}} x^\sigma t^{\mathsf{maj}(\sigma)} q^{\mathsf{quinv}(\sigma)}. \tag{1.3}$$

This conjecture was confirmed by Ayyer, Mandelshtam and Martin [1] who verified that the right hand side of Equation (1.3) satisfies certain orthogonal and triangular conditions that uniquely determine the modified Macdonald polynomials $\tilde{H}_\mu(X;q,t)$.

Interestingly, a refinement of the equality between Equations (1.2) and (1.3) was conjectured by Ayyer, Mandelshtam and Martin ([1, Conjecture 10.3]). Our main result confirms this conjecture; see Equation (1.4) of Theorem 1.1. As a byproduct of our approach, we establish the equidistribution (1.5) between the triples $(\mathsf{inv}, \mathsf{quinv}, \mathsf{maj})$ and $(\mathsf{quinv}, \mathsf{inv}, \mathsf{maj})$ for all fillings of a rectangular shape $\mu = (n^m)$ (this is a partition with exactly m parts of size n). Two fillings σ, τ of a given diagram are called *row-equivalent*, denoted by $\sigma \sim \tau$, if the multisets of entries in the ith row of σ and τ are identical for all i. The precise statement of our main result is the following.

Theorem 1.1. *Let $[\sigma]$ denote the row-equivalency class of σ, then*

$$\sum_{\tau \in [\sigma]} t^{\mathsf{maj}(\tau)} q^{\mathsf{inv}(\tau)} = \sum_{\tau \in [\sigma]} t^{\mathsf{maj}(\tau)} q^{\mathsf{quinv}(\tau)}. \tag{1.4}$$

If σ is a rectangular filling, then

$$\sum_{\tau \in [\sigma]} t^{\mathsf{maj}(\tau)} q^{\mathsf{inv}(\tau)} u^{\mathsf{quinv}(\tau)} = \sum_{\tau \in [\sigma]} t^{\mathsf{maj}(\tau)} u^{\mathsf{inv}(\tau)} q^{\mathsf{quinv}(\tau)}. \tag{1.5}$$

It should be noted that the symmetric distribution (1.5) does not hold for an arbitrary filling σ; we provide such an example in Remark 3.6. The proof of Theorem 1.1 makes significant use of two operators: the reverse operator and the flip operator (also called the column-switching operator).

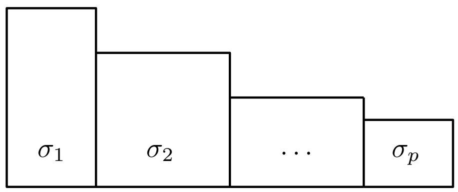
Remark 1.2. Bhattacharya, Ratheesh, and Viswanath [2, 12] proved three special cases of (1.4): when the entries of σ in each column strictly decrease from top to bottom (in the French representation of the Young diagram), when $q = 0$, and when the fillings τ maximize the number of inversions or queue inversions.

The rest of the extended abstract is organized as follows: in Section 2 the reverse operator and flip operator as the starting point of our proof are described. Section 3 is devoted to proving Theorem 1.1, with more details provided in [8].

2 Reverse and flip operators

This section is focused on two operators: the reverse operator and the queue inversion flip operator introduced by Ayyer, Mandelshtam and Martin [1]. The latter was inspired by the inversion flip operator proposed by Loehr and Niese [10].

Both operators are related to the decomposition of the Young diagram of μ into rectangles [3]. Each Young diagram of μ is regarded as a concatenation of maximal rectangles, with heights strictly decreasing from left to right. For any $\sigma \in \mathcal{T}(\mu)$, let σ_i denote the filling of the ith rectangle of the Young diagram of μ from left to right. We write $\sigma = \sigma_1 \sqcup \cdots \sqcup \sigma_p$ where p is the number of distinct parts of μ' and the height of σ_1 is μ'_1; see Figure 2.1 for an example.



3	3	2
1	5	1
2	4	3
1	3	3

2	3	3
1	5	1
3	4	2
3	3	1

Figure 2.1: A decomposition of $\sigma = \sigma_1 \sqcup \sigma_2 \sqcup \cdots \sqcup \sigma_p$ (left), a rectangular filling σ (middle) and its reverse (right).

Definition 2.1 (The reverse operator rev)**.** For a partition μ and a filling $\sigma = \sigma_1 \sqcup \cdots \sqcup \sigma_p \in \mathcal{T}(\mu)$, define $\mathsf{rev}(\sigma) = \mathsf{rev}(\sigma_1) \sqcup \cdots \sqcup \mathsf{rev}(\sigma_p)$ as the reverse of σ where the filling $\mathsf{rev}(\sigma_i)$ is obtained from σ_i by reversing the sequence of entries in each row (see Figure 2.1).

We adopt some notations from [1, 11] to describe the flip operator. We say that the integer i is *μ-compatible* if $\mu'_i = \mu'_{i+1} \geq 1$ and $1 \leq i < \mu_1$. That is, i is μ-compatible if columns i and $i+1$ are of equal height in the Young diagram of μ.

Definition 2.2 (The flip operator ρ_i^r)**.** For a filling $\sigma \in \mathcal{T}(\mu)$, a μ-compatible index i and an integer r such that $1 \leq r \leq \mu'_i$, let $t_i^{(r)}$ be the operator that acts on σ by interchanging the entries $\sigma(r,i)$ and $\sigma(r,i+1)$. For $1 \leq s \leq r \leq \mu'_i$, let

$$t_i^{[s,r]} := t_i^{(s)} \circ t_i^{(s+1)} \cdots \circ t_i^{(r)}$$

denote the flip operator that swaps the entries of boxes (x,i) and $(x,i+1)$ for all x with $s \leq x \leq r$. The flip operator ρ_i^r is defined as follows.

- If $\sigma(x,i) = \sigma(x,i+1)$ for all $1 \leq x \leq r$, then $\rho_i^r(\sigma) = \sigma$.
- Otherwise, let k be the largest integer such that $\sigma(k,i) \neq \sigma(k,i+1)$ and $k \leq r$. Let h be the largest integer satisfying $h \leq k$, $\sigma(h,i) \neq \sigma(h,i+1)$ and

$$\mathcal{Q}(\sigma(h,i), \sigma(h-1,i), \sigma(h-1,i+1)) = \mathcal{Q}(\sigma(h,i+1), \sigma(h-1,i), \sigma(h-1,i+1)),$$

  where we set $\sigma(0,j) = \infty$ for $1 \leq j \leq \mu_1$. Define $\rho_i^r(\sigma) = t_i^{[h,k]}(\sigma)$.

We call row k the starting row, and row h the terminating row of ρ_i^r. For simplicity, we write

$$\rho_i = \rho_i^{\mu_i'} \quad \text{and} \quad t_i = t_i^{(\mu_i')}.$$

By definition $\rho_i^r \circ \rho_i^r(\sigma) = \sigma$; that is, ρ_i^r is an involution on $\mathcal{T}(\mu)$.

Remark 2.3. We point out two differences between the flip operator in Definition 2.2 and the queue inversion flip operator that appears in [1, 11]. First, the latter always starts from the topmost row (that is, ρ_i), whereas our flip operator is allowed to begin at any row. Second, we add the condition $\sigma(h,i) \neq \sigma(h,i+1)$ to the terminating row h. These two modifications are intended to quantify the changes of quinv and inv (see Theorem 3.4), which ultimately contributes to the proof of Theorem 1.1.

Example 2.4. Let $\mu = (2^6)$. For the filling $\sigma \in \mathcal{T}(\mu)$, the μ-compatible index $i = 1$ and $r = \mu_1' = 6$, we show that $\rho_1(\sigma) = \rho_1^6(\sigma) = t_1^{[3,5]}(\sigma)$.

Since $\boxed{9\,|\,3}$ is the topmost row with different entries, the flipping process ρ_1 begins at this row; that is, $k = 5$. We next determine h. Since $\mathcal{Q}(9,2,5) \neq \mathcal{Q}(3,2,5)$ and $\mathcal{Q}(2,3,9) \neq \mathcal{Q}(5,3,9)$, while $\mathcal{Q}(3,5,8) = \mathcal{Q}(9,5,8)$ and $3 \neq 9$, we obtain $h = 3$ by Definition 2.2. Equivalently, the flipping process ρ_1 terminates in row 3. Consequently, $\rho_1(\sigma) = t_1^{[3,5]}(\sigma)$.

$$\sigma = \begin{array}{|c|c|}\hline 3 & 3 \\ \hline 9 & 3 \\ \hline 2 & 5 \\ \hline 3 & 9 \\ \hline 5 & 8 \\ \hline 9 & 6 \\ \hline \end{array} \to \begin{array}{|c|c|}\hline 3 & 3 \\ \hline 3 & 9 \\ \hline 2 & 5 \\ \hline 3 & 9 \\ \hline 5 & 8 \\ \hline 9 & 6 \\ \hline \end{array} \to \begin{array}{|c|c|}\hline 3 & 3 \\ \hline 3 & 9 \\ \hline 5 & 2 \\ \hline 3 & 9 \\ \hline 5 & 8 \\ \hline 9 & 6 \\ \hline \end{array} \to \begin{array}{|c|c|}\hline 3 & 3 \\ \hline 3 & 9 \\ \hline 5 & 2 \\ \hline 9 & 3 \\ \hline 5 & 8 \\ \hline 9 & 6 \\ \hline \end{array} = \rho_1(\sigma)$$

3 Our proof strategy

The proof is bijective: for a partition μ, we construct a bijection $\varphi : \mathcal{T}(\mu) \to \mathcal{T}(\mu)$ satisfying $\varphi(\sigma) \sim \sigma$ and

$$(\mathsf{quinv}, \mathsf{maj})(\varphi(\sigma)) = (\mathsf{inv}, \mathsf{maj})(\sigma). \tag{3.1}$$

In particular, if the Young diagram of μ is a rectangle, we prove that

$$(\mathsf{inv}, \mathsf{quinv}, \mathsf{maj})(\varphi(\sigma)) = (\mathsf{quinv}, \mathsf{inv}, \mathsf{maj})(\sigma).$$

The bijection φ is a composition of two bijections associated with the flip operator ρ_i^r. The first one γ is described in Theorem 3.1 below; it reduces to the reverse operator when the Young diagram of μ is rectangular. Let $\sigma = \sigma_1 \sqcup \cdots \sqcup \sigma_p$, and define

$$\kappa(\sigma) = \sum_{i=1}^{p} (\mathsf{quinv}(\sigma_i) - \mathsf{inv}(\mathsf{rev}(\sigma_i))). \tag{3.2}$$

For any filling τ, τ_1 denotes the leftmost rectangular filling in the decomposition of τ. We define $N\mathsf{des}(\tau) = (a_1, \ldots, a_k)$ where a_i is the number of non-descents in column i of τ and set $\mathsf{ndes}(\tau) = a_1 + \cdots + a_k$ to be the number of non-descents of τ.

Theorem 3.1. *There is a bijection $\gamma : \mathcal{T}(\mu) \to \mathcal{T}(\mu)$ satisfying $\gamma(\sigma) = \tau \sim \sigma$,*

$$\mathsf{quinv}(\tau) = \mathsf{inv}(\sigma) + \kappa(\tau), \tag{3.3}$$

$$\mathsf{maj}(\tau) = \mathsf{maj}(\sigma), \tag{3.4}$$

$$N\mathsf{des}(\sigma_1) = N\mathsf{des}(\mathsf{rev}(\tau_1)), \tag{3.5}$$

and the topmost rows of σ and τ are the reverses of each other.

The second bijection $\theta : \mathcal{T}(\mu) \to \mathcal{T}(\mu)$ acts on each rectangle of any filling independently and decreases the number of queue inversion triples by $\kappa(\gamma(\sigma))$ while preserving the major index. Using this, we find the bijection φ with the property (3.1).

3.1 The proof of Theorem 1.1

In this first step, we discuss the change of statistics quinv, inv and maj by the reverse operator and the flip operator in Lemmas 3.2 and 3.3.

Lemma 3.2. *For a partition $\mu = (n^m)$ and a filling $\sigma \in \mathcal{T}(\mu)$, let x_i be the number of non-descents in column i of σ. Then,*

$$\mathsf{quinv}(\sigma) - \mathsf{inv}(\mathsf{rev}(\sigma)) = \mathsf{inv}(\sigma) - \mathsf{quinv}(\mathsf{rev}(\sigma)) = \sum_{i=1}^{n} x_i(n - 2i + 1). \tag{3.6}$$

Lemma 3.3. *For a partition μ and a μ-compatible index i, let $\sigma \in \mathcal{T}(\mu)$ and suppose that*

$$\rho_i^r(\sigma) = t_i^{[\kappa_1, \kappa_2]}(\sigma).$$

Let $\begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline\end{array}$ and $\begin{array}{|c|c|}\hline s & t \\ \hline u & v \\ \hline\end{array}$ be parts of σ between columns i and $i+1$, such that $\begin{array}{|c|c|}\hline c & d \\ \hline\end{array}$ corresponds to the starting row κ_2 and $\begin{array}{|c|c|}\hline s & t \\ \hline\end{array}$ corresponds to the terminating row κ_1. We set $\sigma(0, j) = \infty$ and $\sigma(\mu_j' + 1, j) = 0$ for $1 \le j \le \mu_1$.

- *If* $\mathcal{Q}(a,c,d) = \mathcal{Q}(b,c,d) = 0$, *then*

$$\mathsf{quinv}(\sigma) + 1 = \mathsf{quinv}(\rho_i^r(\sigma)).$$

Equivalently, if $\mathcal{Q}(a,c,d) = \mathcal{Q}(b,c,d) = 1$, *then*

$$\mathsf{quinv}(\sigma) - 1 = \mathsf{quinv}(\rho_i^r(\sigma)).$$

- *If* $\mathcal{Q}(a,c,d) = \mathcal{Q}(b,c,d)$ *and* $\mathcal{Q}(s,u,t) = \mathcal{Q}(s,v,t) = 0$, *then*

$$\mathsf{inv}(\sigma) + 1 = \mathsf{inv}(\rho_i^r(\sigma)).$$

Equivalently, if $\mathcal{Q}(a,c,d) = \mathcal{Q}(b,c,d)$ *and* $\mathcal{Q}(s,u,t) = \mathcal{Q}(s,v,t) = 1$, *then*

$$\mathsf{inv}(\sigma) - 1 = \mathsf{inv}(\rho_i^r(\sigma)).$$

- *For all cases, that is, whenever* $\mathcal{Q}(a,c,d) = \mathcal{Q}(b,c,d)$, *we have*

$$\mathsf{maj}(\sigma) = \mathsf{maj}(\rho_i^r(\sigma)),$$

and the number of queue inversion triples (resp. inversion triples) induced between columns i *or* $i+1$ *and column* j *of* σ *for any* $j > i+1$ *is invariant under* ρ_i^r.

In the second step, with the help of Lemmas 3.2 and 3.3, we establish the involution ϕ_i in Theorem 3.4, which relates the change of inv and quinv with the number of non-descent pairs.

Theorem 3.4. *For a partition* μ *and a* μ*-compatible index* i, *let* $\sigma \in \mathcal{T}(\mu)$ *and let* x_i *be the number of non-descents in the ith column of* σ. *Then there is an involution* $\phi_i : \mathcal{T}(\mu) \to \mathcal{T}(\mu)$ *such that* $\phi_i(\sigma) \sim \sigma$, *and for* $\nu \in \{\mathsf{inv}, \mathsf{quinv}\}$,

$$\mathsf{maj}(\phi_i(\sigma)) = \mathsf{maj}(\sigma), \tag{3.7}$$
$$\nu(\phi_i(\sigma)) = \nu(\sigma) + x_{i+1} - x_i, \tag{3.8}$$
$$N\mathsf{des}(\phi_i(\sigma)) = s_i N\mathsf{des}(\sigma), \tag{3.9}$$

where $s_i(\ldots x_i, x_{i+1} \ldots) = (\ldots x_{i+1}, x_i \ldots)$.

In the last step, building upon Equation (3.7)–Equation (3.9) of ϕ_i, we construct the bijections θ and γ, thereby completing the proof of Theorem 1.1.

Theorem 3.5 (The second part of Theorem 1.1)**.** *If the Young diagram of* μ *is a rectangle, then there exists a bijection* $\theta : \mathcal{T}(\mu) \to \mathcal{T}(\mu)$ *such that* $\sigma \sim \theta(\sigma)$ *and* $(\mathsf{inv}, \mathsf{quinv}, \mathsf{maj})\sigma = (\mathsf{quinv}, \mathsf{inv}, \mathsf{maj})\theta(\sigma)$.

Proof. For $\mu = (n^m)$ and $\sigma \in \mathcal{T}(\mu)$, let x_i be the number of non-descents in column i of σ. Then $N\mathsf{des}(\mathsf{rev}(\sigma)) = (x_n, \dots, x_1)$. Define a bijection $\theta : \mathcal{T}(\mu) \to \mathcal{T}(\mu)$ as a product of the ϕ_i's as follows. In the first step, we apply the bijections ϕ_i for i from $n-1$ to 1 on $\mathsf{rev}(\sigma)$ and let $\tau_1 = \phi_1 \circ \cdots \circ \phi_{n-1}(\mathsf{rev}(\sigma))$. By Theorem 3.4, the number of queue inversion triples increases by

$$\nu(\tau_1) - \nu(\mathsf{rev}(\sigma)) = \sum_{i=2}^{n}(x_1 - x_i) = (n-1)x_1 - \sum_{i=2}^{n} x_i$$

for $\nu \in \{\mathsf{inv}, \mathsf{quinv}\}$ and $N\mathsf{des}(\tau_1) = (x_1, x_n, \dots, x_2)$. In the next step, we apply the bijections ϕ_i for i from $n-1$ to 2 on τ_1 and let $\tau_2 = \phi_2 \circ \cdots \circ \phi_{n-1}(\tau_1)$, yielding that the number of queue inversion triples increases by $(n-2)x_2 - (x_3 + \cdots + x_n)$ and $N\mathsf{des}(\tau_2) = (x_1, x_2, x_n, \dots, x_3)$. Continue this process until deriving τ_{n-1} satisfying $N\mathsf{des}(\tau_{n-1}) = (x_1, x_2, \dots, x_n)$. Define $\theta(\sigma) = \tau_{n-1}$. From Equation (3.7), we have $\mathsf{maj}(\theta(\sigma)) = \mathsf{maj}(\sigma)$. Furthermore,

$$\begin{aligned}\nu(\theta(\sigma)) &= \nu(\mathsf{rev}(\sigma)) + \sum_{i=1}^{n-1}(n-i)x_i - \sum_{i=2}^{n}(i-1)x_i \\ &= \nu(\mathsf{rev}(\sigma)) + \sum_{i=1}^{n} x_i(n-2i+1).\end{aligned}$$

Comparing with Equation (3.6), we conclude that $(\mathsf{inv}, \mathsf{quinv})(\sigma) = (\mathsf{quinv}, \mathsf{inv})(\theta(\sigma))$, as claimed. ☐

Remark 3.6. In the table below, we provide a row-equivalency class of a non-rectangular diagram to disprove Equation (1.5) for arbitrary fillings.

$[\sigma]$	$\begin{smallmatrix}3&&\\4&1&2\\3&3&3\end{smallmatrix}$	$\begin{smallmatrix}3&&\\4&2&1\\3&3&3\end{smallmatrix}$	$\begin{smallmatrix}3&&\\1&2&4\\3&3&3\end{smallmatrix}$	$\begin{smallmatrix}3&&\\1&4&2\\3&3&3\end{smallmatrix}$	$\begin{smallmatrix}3&&\\2&4&1\\3&3&3\end{smallmatrix}$	$\begin{smallmatrix}3&&\\2&1&4\\3&3&3\end{smallmatrix}$
maj	2	2	2	2	2	2
inv	0	1	2	1	2	3
quinv	3	2	2	1	0	1

We proceed by establishing Theorem 1.1 by Theorem 3.1 and Theorem 3.5.

Proof of Theorem 1.1. For any filling $\sigma \in \mathcal{T}(\mu)$, let $\tau = \gamma(\sigma)$. Consider the rectangular decomposition of $\tau = \tau_1 \sqcup \cdots \sqcup \tau_p$ and define $\pi = \theta(\mathsf{rev}(\tau_1)) \sqcup \cdots \sqcup \theta(\mathsf{rev}(\tau_p))$. We shall see that $\varphi(\sigma) = \pi$ satisfying $\pi \sim \sigma$, $\mathsf{quinv}(\pi) = \mathsf{inv}(\sigma)$ and $\mathsf{maj}(\pi) = \mathsf{maj}(\sigma)$. Since the number of queue inversion triples or inversion triples located in two columns of unequal heights is invariant under ρ_i^r by Lemma 3.3, Theorem 3.5 guarantees that

$$\begin{aligned}\mathsf{quinv}(\pi)-\mathsf{quinv}(\tau)=\sum_{i=1}^{p}(\mathsf{quinv}(\theta(\mathsf{rev}(\tau_i)))-\mathsf{quinv}(\tau_i))&=\sum_{i=1}^{p}(\mathsf{inv}(\mathsf{rev}(\tau_i))-\mathsf{quinv}(\tau_i))\\&=-\kappa(\tau).\end{aligned}$$

Combining with Equation (3.3) of Theorem 3.1, we conclude that $\mathsf{quinv}(\pi)=\mathsf{inv}(\sigma)$. Furthermore, $\pi\sim\sigma$ and $\mathsf{maj}(\pi)=\mathsf{maj}(\sigma)$ are direct consequences of Theorem 3.1 and Theorem 3.5, as desired. □

Example 3.7. Let $\mu=(7,7,5,5,5,2)$. The filling $\sigma\in\mathcal{T}(\mu)$ and the corresponding $\varphi(\sigma)$ are displayed as below:

$$\sigma=\begin{array}{|c|c|c|c|c|c|c|}\hline 5&4\\\cline{1-5} 9&3&6&1&3\\\cline{1-5} 2&5&9&4&8\\\cline{1-5} 3&9&7&3&5\\\hline 5&8&4&6&4&8&7\\\hline 9&3&6&5&2&10&1\\\hline\end{array}\qquad\varphi(\sigma)=\begin{array}{|c|c|c|c|c|c|c|}\hline 4&5\\\cline{1-5} 6&3&1&3&9\\\cline{1-5} 8&4&5&9&2\\\cline{1-5} 3&3&5&7&9\\\hline 7&8&4&4&6&8&5\\\hline 10&1&6&2&5&9&3\\\hline\end{array}$$

One can easily calculate that $(\mathsf{inv},\mathsf{maj})(\sigma)=(\mathsf{quinv},\mathsf{maj})(\varphi(\sigma))=(40,33)$, while $\mathsf{quinv}(\sigma)\neq\mathsf{inv}(\varphi(\sigma))$ as $\mathsf{quinv}(\sigma)=32$ and $\mathsf{inv}(\varphi(\sigma))=34$.

4 Discussion

We propose some interesting open questions. A *column strict filling* (CSF) of a Young diagram is a filling whose entries along each column are strictly decreasing from top to bottom. In other words, the major index of a column strict filling reaches its maximal value.

Bhattacharya, Ratheesh, and Viswanath [2] found the symmetric distribution (1.5) for all the CSFs of an arbitrary Young diagram. That is, let $\mathsf{CSF}(\mu)$ be the set of CSFs of the Young diagram of μ, then

$$\sum_{\tau\in[\sigma]\cap\mathsf{CSF}(\mu)}t^{\mathsf{inv}(\tau)}u^{\mathsf{quinv}(\tau)}=\sum_{\tau\in[\sigma]\cap\mathsf{CSF}(\mu)}u^{\mathsf{inv}(\tau)}t^{\mathsf{quinv}(\tau)}.\tag{4.1}$$

Open problem 4.1. Equation (4.1) is not directly seen from the bijection φ. It would be interesting to find out whether the bijection φ is applicable to prove Equation (4.1).

Loehr and Niese [10, Equations (6.1) and (6.2)] showed that

$$\sum_{\tau\in[\sigma]} q^{\mathsf{maj}(\tau')} = \sum_{\tau\in[\sigma]} q^{\mathsf{inv}(\tau)} = \sum_{\tau\in[\sigma]} q^{\mathsf{quinv}(\tau)} = \prod_{i=1}^{\ell(\mu)} \begin{bmatrix} a_{i,1}+\cdots+a_{i,N} \\ a_{i,1},\ldots,a_{i,N} \end{bmatrix}_q, \tag{4.2}$$

where τ' is obtained by transposing the filling τ. The ith row of σ contains exactly $a_{i,1}$ copies of 1, $a_{i,2}$ copies of 2, and so on. The rightmost term of Equation (4.2) is a product of q-multinomial coefficients (see for instance [9, 13]).

Open problem 4.2. Let $G(t,q,u)$ be the left-hand side of Equation (1.5), can one derive a formula of $G(t,q,u)$ with the symmetric property $G(t,q,u) = G(t,u,q)$? Is there a formula for Equation (1.4) as a natural generalization of Equation (4.2)?

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