HALL-LITTLEWOOD FUNCTIONS AND KOSTKA-FOULKES POLYNOMIALS IN REPRESENTATION THEORY

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Abstract

This paper presents a survey of recent applications of Hall-Littlewood functions and Kostka-Foulkes polynomials to the representation theory of the general linear group $GL(n, \mathbf{C})$ and of the symmetric group $\mathbf{S}_n$. The reviewed topics include the q-analogue of Kostant's partition function, vertex operators, generalized exponents of $GL(n, \mathbf{C})$ and $\mathbf{S}_n$-harmonic polynomials. We also give a detailed description of the various combinatorial interpretations of Kostka-Foulkes polynomials. We conclude with the study of Hall-Littlewood functions at roots of unity, which provide a combinatorial description of certain plethysms.

Résumé

Cet article donne un aperçu de plusieurs applications récentes des fonctions de Hall-Littlewood et des polynômes de Kostka-Foulkes aux représentations du groupe linéaire $GL(n, \mathbf{C})$ et du groupe symétrique $\mathbf{S}_n$. Parmi les sujets abordés figurent notamment le q-analogue de la fonction de partition de Kostant, les opérateurs vertex, les exposants généralisés de $GL(n, \mathbf{C})$ et les polynômes harmoniques de $\mathbf{S}_n$. Nous donnons aussi une description détaillée des diverses interprétations combinatoires des polynômes de Kostka-Foulkes. Nous concluons par l'étude des fonctions de Hall-Littlewood aux racines de l'unité, qui fournit une interprétation combinatoire de certains pléthysmes.

1 Introduction

The original motivation for defining Hall polynomials, Hall algebra and Hall-Littlewood symmetric functions comes from group theory. It is well known that any finite Abelian p-group G is a direct sum of cyclic subgroups, of orders $p^{\lambda_1}, p^{\lambda_2}, \ldots, p^{\lambda_r}$, say, where we may suppose that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r > 0$. The partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ determines G to within isomorphism, and is called the *type* of G. The investigation of the lattice structure of finite Abelian p-groups leads to the problem of enumerating the chains of subgroups

$$1 = H_0 \leq H_1 \leq \ldots \leq H_s = G$$

in G of type λ, such that H_i/H_{i+1} has type μ^i for $i = 1, 2, \ldots, s$. Let $g^{\lambda}_{\mu^1 \ldots \mu^s}(p)$ denote the number of such chains. Hall showed [**Ha**] that

(i) $g^{\lambda}_{\mu^1\ldots\mu^s}(p)$ is a polynomial function of p;

(ii) the numbers $g^{\nu}_{\lambda\mu}(p)$ can be taken as the structure constants of an associative and commutative algebra $H(p)$, isomorphic to the algebra of symmetric functions;

(iii) the polynomial function $g^{\nu}_{\lambda,\mu}(p)$ vanishes identically in p if and only if the corresponding multiplication constant for the Schur functions $c^{\nu}_{\lambda\mu} = \langle s_\lambda\, s_\mu\, ,\, s_\nu\rangle$ is zero;

(iv) the leading term of $g^{\nu}_{\lambda\mu}(p)$ is equal to $c^{\nu}_{\lambda\mu}\, p^{n(\lambda)-n(\mu)-n(\nu)}$, where $n(\lambda) = \sum_i (i-1)\lambda_i$.

In fact, it has been brought to light recently by Johnsen [**Jo**] that as early as 1900, Steinitz had arrived at the very same results in a forgotten note on the theory of Abelian groups [**St**].

The Hall algebra $H(p)$ with generators $u_\lambda(p)$ may be identified with the algebra of symmetric functions by sending the complete homogeneous symmetric function h_n onto the sum $\sum_{|\lambda|=n} u_\lambda(p)$, thus introducing a new family of symmetric functions $u_\lambda(p)$. Hall was able to show that the functions $u_{1^n}(p)$ coincide up to a power of p with the elementary symmetric functions:

$$u_{1^n}(p) = p^{-n(n-1)/2} e_n\ ,$$

but it remained to Littlewood to obtain an explicit expression for the symmetric functions $u_\lambda(p)$ for any partition λ. Defining $P_\lambda(q)$ by formulas (15) and (16) below, Littlewood [**Li1**] found that

$$u_\lambda(p) = p^{-n(\lambda)} P_\lambda(p^{-1})\ . \tag{1}$$

It turns out that the $P_\lambda(q)$, called now Hall-Littlewood functions, interpolate between two fundamental bases of the algebra of symmetric functions, namely, the basis of Schur functions s_λ when $q=0$, and the basis of monomial functions m_λ when $q=1$. The entries $K_{\lambda\mu}$ of the transition matrix from m_μ to s_λ are well known in representation theory: $K_{\lambda\mu}$ is the multiplicity of the weight μ in an irreducible $\mathfrak{gl}\,(n,\mathbf{C})$-module with highest weight λ. These $K_{\lambda\mu}$, or Kostka numbers, have several combinatorial expressions, the most popular being that $K_{\lambda\mu}$ is the number of Young tableaux of shape λ and weight μ. They can also be evaluated in terms of Kostant's partition function $\mathcal{P}(\alpha)$ (see Section 2 below). $\mathcal{P}(\alpha)$ is defined as the number of ways of writing the vector $\alpha = (\alpha_1,\ldots,\alpha_n)$ as a sum

$$\alpha = \sum_{1\le i<j\le n} \nu_{ij}\,\epsilon_{ij}\ ,\qquad m_{ij}\ge 0$$

of positive roots $\epsilon_{ij} = (0,\ldots,1,\ldots,-1,\ldots,0)$. A natural q-analogue $\mathcal{P}_q(\alpha)$ is obtained by taking into account the 'length' $\sum \nu_{ij}$ of each partition, which gives back by means of Kostant's formula a q-analogue $K_{\lambda\mu}(q)$ of the Kostka numbers $K_{\lambda\mu}$. These polynomials $K_{\lambda\mu}(q)$ happen to be none other than the entries of the transition matrix from the basis $P_\mu(q)$ to the basis s_λ.

Apart from the Hall algebra, the best known applications of Hall-Littlewood functions include the character theory of finite linear groups [**Gr**], projective and modular representations of symmetric groups [**S**], [**Mo2**] and various topics in algebraic geometry [**HS**] [**He**] [**Lu1**] [**Lu2**]. The aim of these notes is to present some applications to the ordinary representation theory of $\mathbf{S}_n$ and of $GL(n,\mathbf{C})$, with emphasis on the combinatorial aspects.

In particular, the interpretation of Kostka-Foulkes polynomials in terms of the charge of Young tableaux [**LS2**] [**Sc**], and in terms of the quantum numbers of rigged configurations [**KKR**] [**KR**] are presented and illustrated by many examples. The notations for symmetric functions are those of [**Mcd**].

2 Raising operators and Kostant's partition function

The Jacobi-Trudi determinant for Schur functions can be compactly expressed by means of *raising operators*

$$s_\lambda = \prod_{i<j}(1 - R_{ij})\, h_\lambda \tag{2}$$

where $R_{ij}\, h_\lambda := h_{R_{ij}(\lambda)}$ and $R_{ij}(\lambda) := (\ldots, \lambda_i + 1, \ldots, \lambda_j - 1, \ldots)$.

One would like to be able to invert formula (2), *i.e.* to write

$$h_\lambda = \prod_{i<j}(1 - R_{ij})^{-1} s_\lambda \ . \tag{3}$$

The simplest way to justify such a formal manipulation seems to adopt the following point of view about raising operators. Rather than acting on elements of $\mathbf{Z}^n$, we prefer to use *linear maps* from the group algebra $\mathbf{Z}[\mathbf{Z}^n]$ to some commutative ring R.

It is customary to use as a basis of $\mathbf{Z}[\mathbf{Z}^n]$ the *formal exponentials* $(e^\alpha)_{\alpha\in\mathbf{Z}^n}$, satisfying the relations $e^\alpha e^\beta = e^{\alpha+\beta}$. We shall follow this practice, and furthermore introduce n independent indeterminates $x_1, \ldots, x_n$ in order to identify $\mathbf{Z}[\mathbf{Z}^n]$ with the ring of polynomials $\mathbf{Z}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$, by setting $e^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. We set $X = \{x_1, \ldots x_n\}$ and $X^\vee = \{x_1^{-1}, \ldots x_n^{-1}\}$.

Now, let f be any $\mathbf{Z}$-module homomorphism from $\mathbf{Z}[\mathbf{Z}^n]$ to a commutative ring R. We define $g = R_{ij} f$ by

$$g(e^\lambda) := f\left(\frac{x_i}{x_j} e^\lambda\right) \ . \tag{4}$$

In the applications we have in mind, R will be the ring *Sym* of symmetric functions, or of symmetric polynomials in n variables. Introducing in both cases the two linear maps

$$s : \ e^\lambda \longmapsto s(\lambda) = s_\lambda \quad \text{and} \quad h : \ e^\lambda \longmapsto h(\lambda) = h_\lambda$$

(with $h_i = 0$ for $i < 0$) and the element

$$\theta_n = \prod_{1\le i<j\le n} \left(1 - \frac{x_i}{x_j}\right)$$

we see that (2) can be written

$$s(e^\lambda) = h(\theta_n e^\lambda) \ ,$$

from which it follows immediately by linearity that

$$h(e^\lambda) = s(\theta_n^{-1} e^\lambda)$$

which is exactly formula (3). We thus see that this formula has to be interpreted as an identity between linear maps, $h = \prod_{i<j}(1-R_{ij})^{-1}s$, evaluated at e^λ.

Consider now the function $\mathcal{P} :\ \mathbf{Z}^n \longrightarrow \mathbf{N}$ defined by

$$\theta_n^{-1} = \prod_{i<j}\left(1-\frac{x_i}{x_j}\right)^{-1} = \sum_{\alpha\in\mathbf{Z}^n}\mathcal{P}(\alpha)e^\alpha \ . \tag{5}$$

This is an example of a *vector partition function.* It counts the number of ways a vector α can be expressed as a sum of the vectors

$$\epsilon_{ij} := \epsilon_i - \epsilon_j \quad (i<j)$$

where $\epsilon_i = (0,\dots,1,\dots,0)$ is the canonical basis of $\mathbf{Z}^n$.

The vectors ϵ_{ij} $(i \neq j)$ are nothing but the root system A_{n-1}, and those ϵ_{ij} with $i<j$ are generally taken as the positive roots. The sublattice $P = \{\alpha | \sum_i \alpha_i = 0\}$ of $\mathbf{Z}^n$ can be identified with the weight lattice of $\mathfrak{sl}(n,\mathbf{C})$, and for $\alpha \in P$, $\mathcal{P}(\alpha)$ is the number of ways in which the weight α can be expressed as a sum of positive roots. Thus, $\mathcal{P}$ is called *Kostant's partition function* for the root system A_{n-1}. The partition function can of course be defined for any root system, and much of the theory presented in this paper can be extended to this general setting.

Let us return to (3). Using (5), we can rewrite it as

$$h_\mu = \sum_{\alpha\in\mathbf{Z}^n}\mathcal{P}(\alpha)s_{\mu+\alpha} \tag{6}$$

where the Schur functions $s_{\mu+\alpha}$ have to be standardized, *i.e.* expressed in terms of Schur functions indexed by partitions. Defining $\rho = (n-1, n-2, \dots, 1, 0)$, we see that $s_{\mu+\alpha}$ is equal to $\varepsilon(w)s_\lambda$ if there exist a permutation $w \in \mathbf{S}_n$ and a partition λ such that $\mu+\alpha+\rho = w(\lambda+\rho)$, and is 0 otherwise. Thus,

$$h_\mu = \sum_\lambda\left(\sum_{w\in\mathbf{S}_n}\varepsilon(w)\mathcal{P}\left(w(\lambda+\rho)-(\mu+\rho)\right)\right)s_\lambda \ . \tag{7}$$

On the other hand, the expansion of h_μ on the basis s_λ is well known to be given by the *Kostka numbers*

$$K_{\lambda\mu} = \langle h_\mu\, ,\, s_\lambda\rangle = |\mathrm{Tab}\,(\lambda,\mu)|$$

where $\mathrm{Tab}\,(\lambda,\mu)$ is the set of Young tableaux with shape λ and weight μ. Thus,

$$K_{\lambda\mu} = \sum_{w\in\mathbf{S}_n}\varepsilon(w)\mathcal{P}\left(w(\lambda+\rho)-(\mu+\rho)\right) \ . \tag{8}$$

This formula is *Kostant's weight multiplicity formula* (for $GL(n,\mathbf{C})$). Since one has as well

$$s_\lambda = \sum_\mu K_{\lambda\mu}m_\mu \tag{9}$$

it expresses the multiplicity $K_{\lambda\mu}$ of the weight μ in the irreducible polynomial representation V_λ of $GL(n,\mathbf{C})$ with highest weight λ.

Example 2.1 We put $n = 4$, $\lambda = (3,3,2,0)$, $\mu = (2,2,2,2)$. Then $\lambda + \rho = (6,5,3,0)$ and $\mu + \rho = (5,4,3,2)$. There are only three nonzero summands in (8), as shown in the following table.

$w(\lambda+\rho)$	$w(\lambda+\rho)-(\mu+\rho)$	$\mathcal{P}(w(\lambda+\rho)-(\mu+\rho))$	$\varepsilon(w)$
$(6,5,3,0)$	$(1,1,0,-2)$	7	$+$
$(5,6,3,0)$	$(0,2,0,-2)$	3	$-$
$(6,3,5,0)$	$(1,-1,2,-2)$	1	$-$

Hence, $K_{\lambda\mu} = 7 - 3 - 1 = 3$. The corresponding three tableaux are

$$
\begin{array}{|c|c|}\hline 3 & 4 \\ \hline \end{array}\quad
\begin{array}{|c|c|}\hline 4 & 4 \\ \hline \end{array}\quad
\begin{array}{|c|c|}\hline 3 & 4 \\ \hline \end{array}
$$
$$
\begin{array}{|c|c|c|}\hline 2 & 3 & 4 \\ \hline 1 & 1 & 2 \\ \hline \end{array}\quad
\begin{array}{|c|c|c|}\hline 2 & 3 & 3 \\ \hline 1 & 1 & 2 \\ \hline \end{array}\quad
\begin{array}{|c|c|c|}\hline 2 & 2 & 4 \\ \hline 1 & 1 & 3 \\ \hline \end{array}
$$

3 Hall-Littlewood functions and the q-analogue of the partition function

A natural statistics on ordinary partitions λ of a number n is the *length* (number of parts). That is, the generating function

$$\sum_{n\geq 0} p(n)t^n = \prod_{k\geq 1}(1-t^k)^{-1}$$

can be refined in

$$\sum_{m,n} p(n;m)q^m t^n = \prod_{k\geq 1}(1-qt^k)^{-1}$$

where $p(n;m)$ is the number of partitions of n in m parts. The same can be done with vector partition functions, and the length q-analog of Kostant's partition function has indeed been considered by several authors [**Ka**][**Mcd**][**Lu1**][**Gu1**]. For the root system A_{n-1}, it is given by

$$\theta_n(q)^{-1} := \prod_{1\leq i<j\leq n}\left(1 - q\,\frac{x_i}{x_j}\right)^{-1} = \sum_{\alpha\in\mathbf{Z}^n}\mathcal{P}_q(\alpha)e^{\alpha} \tag{10}$$

the coefficient of q^k in $\mathcal{P}_q(\alpha)$ being the number of ways the weight α can be expressed as a sum of k positive roots.

With this at hand, it is then tempting to define a q-analog of the symmetric function h_λ by

$$Q'_\lambda(X;q) := \prod_{i<j}(1-qR_{ij})^{-1}s_\lambda(X) \tag{11}$$

(the choice of this notation is explained below). One has then

$$Q'_\mu = \sum_\lambda K_{\lambda\mu}(q)s_\lambda \tag{12}$$

where

$$K_{\lambda\mu}(q) = \sum_{w\in\mathbf{S}_n}\varepsilon(w)\mathcal{P}_q\left(w(\lambda+\rho)-(\mu+\rho)\right)\ . \tag{13}$$

Example 3.1 Take $\lambda = (3,3,2,0)$, $\mu = (2,2,2,2)$, as in Example 2.1. The computation of $K_{\lambda\mu}(q)$ takes the form

$w(\lambda+\rho)$	$w(\lambda+\rho)-(\mu+\rho)$	$\mathcal{P}_q(w(\lambda+\rho)-(\mu+\rho))$	$\varepsilon(w)$
$(6,5,3,0)$	$(1,1,0,-2)$	$q^2+3q^3+2q^4+q^5$	$+$
$(5,6,3,0)$	$(0,2,0,-2)$	$q^2+q^3+q^4$	$-$
$(6,3,5,0)$	$(1,-1,2,-2)$	q^3	$-$

Hence, $K_{\lambda\mu}(q) = q^3+q^4+q^5$.

It turns out, although this is by no means obvious, that the polynomials $K_{\lambda\mu}(q)$ have nonnegative coefficients [**LS2**]. They are the *Kostka-Foulkes* polynomials, and the symmetric functions Q'_λ are, up to a simple transformation, the Hall-Littlewood functions.

To be more precise, recall that in the λ-ring formalism, the symmetric functions of the argument $(1-q)X$ are defined by

$$p_k((1-q)X) = (1-q^k)p_k(X)$$

and similarly

$$p_k\left(\frac{X}{1-q}\right) = (1-q^k)^{-1}p_k(X)\ .$$

Then, the usual Hall-Littlewood functions are

$$Q_\mu(X;q) = Q'_\mu((1-q)X;q)\ . \tag{14}$$

Littlewood's original definition [**Li1**] of the Q-functions was, for $X=\{x_1,\ldots,x_n\}$

$$Q_\lambda(X;q) = \frac{(1-q)^{\ell(\lambda)}}{[m_0]_q!}\sum_{w\in\mathbf{S}_n} w\left(e^\lambda\,\frac{\theta_n(q)}{\theta_n(1)}\right) \tag{15}$$

where $m_0 = n-\ell(\lambda)$ and

$$[m_0]_q! = [m_0]_q\,[m_0-1]_q\cdots[1]_q = \prod_{1\le i\le m_0}\frac{1-q^i}{1-q}\ .$$

The normalization factor $[m_0]_q!^{-1}$ is introduced in order to have stability with respect to adjunction of variables, *i.e.*

$$Q_\lambda(x_1,\ldots,x_n\,;\,q) = Q_\lambda(x_1,\ldots,x_{m+n}\,;\,q)|_{x_{m+1}=\ldots=x_{m+n}=0}\ .$$

The coefficients of $Q_\lambda(X;q)$ on the basis of Schur functions are polynomials in q having common factors of the type $1-q^i$. Hence one defines also the symmetric function

$$P_\lambda(X;q) = \frac{1}{(1-q)^{\ell(\lambda)}\,[m_1]_q!\cdots[m_n]_q!}\,Q_\lambda(X;q)\ , \tag{16}$$

which still has polynomial coefficients on the basis (s_λ). Littlewood showed that the symmetric functions obtained by Schur as generating functions for the projective characters of $\mathbf{S}_n$ were the $Q_\lambda(X;-1)$. He also derived the formula

$$Q_\lambda(X;q) = \prod_{i<j}(1-qR_{ij})^{-1}s_\lambda((1-q)X) \tag{17}$$

which is easily seen, in view of definition (11), to be equivalent to (14). Finally, he pointed out that Hall's functions u_λ were given by (1). This relation is established by Morris in [**Mo3**]. From our point of view, $(P_\lambda(X;q))$ is simply the adjoint basis of $(Q'_\lambda(X;q))$ for the usual scalar product of Sym.

4 Morris' recurrence formula, vertex operators and Green polynomials

An important consequence of (17) is the following formula of Morris [**Mo1**], which allows to express a Kostka-Foulkes polynomial $K_{\lambda\mu}(q)$ in terms of the lower weight ones $K_{\alpha\nu}(q)$, where $\nu = (\mu_2, \mu_3, \ldots, \mu_s)$ is obtained from μ by removing its largest part:

$$Q_\mu(X;q) = \sum_{r\geq 0} q^r \sum_\alpha K_{\alpha\nu}(q) \sum_\beta \langle s_\beta h_r \, , \, s_\alpha \rangle s_{(\mu_1+r,\beta)}((1-q)X) \ . \tag{18}$$

This formula looks simpler when stated in terms of *vertex operators*. Recall that for any $f \in Sym$, there is a differential operator D_f on Sym (the *Foulkes derivative*) defined as the adjoint of the multiplication operator $g \mapsto fg$, that is

$$\langle fg \, , \, h \rangle = \langle g \, , \, D_f h \rangle \ .$$

Introducing the generating series

$$\sigma_z(X) := \sum_{r\geq 0} z^r h_r(X) \quad \text{and} \quad \lambda_z(X) := \sum_{r\geq 0} z^r e_r(X) \ , \tag{19}$$

the generating series for the Schur functions indexed by vectors of the form (r, λ) (where λ is a fixed partition) can then be expressed as

$$\sum_{r\in\mathbf{Z}} z^r s_{(r,\lambda)} = \sigma_z D_{\lambda_{-1/z}} s_\lambda \ . \tag{20}$$

This identity is established by expanding the Jacobi-Trudi determinant by its first row, which causes the appearance of the skew Schur functions $s_{\lambda/(1^k)} = D_{e_k} s_\lambda$. The operator

$$\Gamma_z = \sigma_z \, D_{\lambda_{-1/z}} \tag{21}$$

is a typical example of the so-called *vertex operators* of Mathematical Physics. It has been applied to the determination of the rational solutions of various hierarchies of soliton equations and to the construction of representations of certain infinite dimensional Lie algebras (see *e.g.* Kac's book [**Kac**] Chap. 14 for examples and references). Some applications to symmetric functions and group characters can be found in [**T**][**CT**][**ST**]. Remark that in λ-ring notation,

$$D_{\sigma_z} f(X) = f(X+z) \quad \text{and} \quad D_{\lambda_{-z}} f(X) = f(X-z)$$

when z is an element of rank one (which means that $e_r(z) = 0$ for $r > 1$), so that

$$\Gamma_z f(X) = \sigma_z(X) \, f\left(X - \frac{1}{z}\right) \ .$$

Morris' formula can now be rewritten as a generating function identity similar to (20) for the Q'-functions:

$$\sum_{p\in\mathbf{Z}} Q'_{(p,\nu)}(X;q) = \Gamma_1\, Q'_\nu(X+q;q)\ . \tag{22}$$

Indeed, replacing Q by Q' in (18) and summing over $p=\mu_1$, one gets

$$\sum_{p\in\mathbf{Z}} Q'_{(p,\nu)} = \sum_{r\geq 0} q^r \sum_\alpha K_{\alpha\nu}(q) \sum_\beta \langle s_\beta h_r\,,\ s_\alpha\rangle \sum_{m\in\mathbf{Z}} s_{(m,\beta)}$$

$$= \Gamma_1 \left\{ \sum_\alpha K_{\alpha\nu}(q) \sum_\beta \left\langle \sum_{r\geq 0} q^r s_{\alpha/r}\,,\ s_\beta \right\rangle s_\beta \right\}$$

$$= \Gamma_1 \left\{ \sum_\alpha K_{\alpha\nu}(q) s_\alpha(X+q) \right\} = \Gamma_1\, Q'_\nu(X+q) = \Gamma_1 D_{\sigma_q} Q'_\nu\ .$$

By homogeneity, one has as well

$$\sum_{p\in\mathbf{Z}} z^p Q'_{(p,\nu)} = \Gamma_z Q'_\nu(X+q)\ . \tag{23}$$

Example 4.1 Let us compute the stable Schur expansion of the functions $Q'_{(p,2,1,1)}$. First, we have

$$\sum_{p\in\mathbf{Z}} Q'_{p11} = \Gamma_1\, Q'_{11}(X+q) = \Gamma_1\, [q\, s_2(X+q) + s_{11}(X+q)]$$

$$= \Gamma_1 \left[q\,(s_2 + q\, s_1 + q^2) + s_{11} + q\, s_1 \right] = \Gamma_1 \left[q\, s_2 + (q+q^2)\, s_1 + s_{11} + q^3 s_0 \right]$$

so that $Q'_{211} = q\, s_{22} + (q+q^2)\, s_{31} + s_{211} + q^3 s_4$. Now,

$$\sum_{p\in\mathbf{Z}} Q'_{p211} = \Gamma_1 \left[q\, s_{22}(X+q) + (q+q^2)\, s_{31}(X+q) + s_{211}(X+q) + q^3 s_4(X+q) \right]$$

$$= \Gamma_1 \left[q^7 s_0 + (q^4+q^5+q^6)\, s_1 + (2q^3+q^4+q^5)\, s_2 + (q^2+q^3+q^4)\, s_{11} + (q^2+q^3+q^4)\, s_3 \right.$$
$$\left. + q\, s_{111} + (q+2q^2+q^3)\, s_{21} + q^3\, s_4 + (q+q^2)\, s_{31} + q\, s_{22} + s_{211} \right]\ .$$

Equation (22) provides a q-analogue of the *character polynomials* of the symmetric groups (see *e.g.* [**Ke**]) for the *Green polynomials*. The (renormalized) Green polynomials $X^\mu_\lambda(q)$ are q-analogues (at $q=0$) of the characters χ^μ_λ of the symmetric group, defined by

$$X^\mu_\lambda(q) = \langle Q'_\mu\,,\ p_\lambda\rangle\ . \tag{24}$$

If $\mu=(p,\nu)$, it follows from (22) that

$$\begin{aligned} X^{(p,\nu)}_\lambda(q) &= \langle \sigma_1 D_{\lambda_{-1}} Q'_\nu(X+q)\,,\ p_\lambda\rangle \\ &= \langle Q'_\nu(X+q-1)\,,\ p_\lambda(X+1)\rangle \\ &= \langle Q'_\nu(X+q-1)\,,\ \prod_i (1+p_i)^{m_i(\lambda)}\rangle \end{aligned} \tag{25}$$

which yields an expression of $X^\mu_\lambda(q)$ as a polynomial $\Xi^\nu(q\,;\, m_1, m_2, \ldots, m_n)$ in the multiplicities m_i of the parts of λ, which is independent of the first part μ_1 of μ. The polynomials $\Xi^\nu(0\,;\, m_1, m_2, \ldots, m_n)$ are the usual character polynomials of the symmetric group.

Example 4.2 Let $\lambda = (1^{m_1} 2^{m_2} \cdots n^{m_n})$. Then,

$$X_\lambda^{(p,1)}(q) = \langle s_1(X+q-1)\,,\, \prod_i (1+p_i)^{m_i} \rangle$$

$$= \langle p_1 + (q-1)\,,\, 1 + m_1 p_1 + \cdots \rangle = (q-1) + m_1$$

and

$$X_\lambda^{(p,2)} = \langle Q_2'(X+q-1)\,,\, \prod_i (1+p_i)^{m_i} \rangle$$

$$= \langle s_2(X+q-1)\,,\, (1 + m_1 p_1 + \binom{m_1}{2} p_1^2 + \cdots)(1 + m_2 p_2 + \cdots)\cdots \rangle$$

$$= q(q-1) + (q-1)\binom{m_1}{1} + \binom{m_1}{2} + \binom{m_2}{1} .$$

5 Generalized exponents for $GL(n, \mathbf{C})$

Consider the adjoint representation of $G := GL(n, \mathbf{C})$, *i.e.* its action by conjugation on its Lie algebra $\mathfrak{g} = \mathfrak{gl}\,(n, \mathbf{C}) = M_n(\mathbf{C})$

$$\mathrm{Ad}\,(g)A = gAg^{-1} .$$

Since as a G-module $\mathfrak{g} = V \otimes V^*$, where $V = \mathbf{C}^n$ is the defining representation of $GL(n, \mathbf{C})$, with character $\mathrm{ch}_V = \sum_i x_i$ (the formal character ch_U of a polynomial representation can be viewed as the trace of the image of the diagonal matrix $\mathrm{diag}\,(x_1, \ldots, x_n)$ in this representation), the character of the adjoint representation is

$$\mathrm{ch}\,\mathfrak{g} = \left(\sum_{i=1}^n x_i\right)\left(\sum_{j=1}^n x_j^{-1}\right) = \sum_{i,j} \frac{x_i}{x_j} .$$

Consider now the symmetric algebra

$$S^\bullet(\mathfrak{g}) = \bigoplus_{k \geq 0} S^k(\mathfrak{g})$$

as a graded G-module. The character of $S^k(\mathfrak{g})$ is then given by the generating series

$$\sum_{k \geq 0} q^k \mathrm{ch}_{S^k(\mathfrak{g})} = \sigma_q(XX^\vee) = \prod_{i,j} \left(1 - q\,\frac{x_i}{x_j}\right)^{-1} . \tag{26}$$

Let $\mathcal{I} = S^\bullet(\mathfrak{g})^G$ be the algebra of G-invariants in $S^\bullet(\mathfrak{g})$. By a theorem of Kostant [**Ko**], $S^\bullet(\mathfrak{g})$ is a free module over $\mathcal{I}$, generated by the G-harmonic polynomials, which are those elements of $S^\bullet(\mathfrak{g})$ annihilated by all G-invariant differential operators without constant term. One has $S^\bullet(\mathfrak{g}) = \mathcal{I} \otimes \mathcal{H}$, where $\mathcal{H} = \bigoplus_{k \geq 0} \mathcal{H}^k$ is the graded module of harmonic polynomials. Kostant defined the *generalized exponents* of a finite-dimensional G-module E to be the exponents $e_1, \ldots, e_s$ of the nonzero terms in the polynomial

$$F_q(E) := \sum_{k \geq 0} \langle E\,,\, \mathcal{H}^k \rangle q^k = \sum_{i=1}^s c_i q^{e_i} \tag{27}$$

where $\langle E\,,\,\mathcal{H}^k\rangle = \dim_{\mathbf{C}}\mathrm{Hom}_G(E\,,\,\mathcal{H}^k)$ is the intertwining number.

The problem is to compute $F_q(V_\lambda)$ for the irreducible representations V_λ of G. Here, we have to consider *rational* irreducible representations of $GL(n,\mathbf{C})$, whose characters are of the form

$$e_n(X)^m s_\mu(X)$$

where $e_n(X) = x_1\cdots x_n$, $m\in\mathbf{Z}$, and μ is a partition which can be taken of length $\le n-1$ since with n variables $e_n^m(X)s_\mu(X) = s_{\mu+(m^n)}(X)$. For $GL(n,\mathbf{C})$, the result is

Theorem 5.1 *The generating series of the characters of harmonic polynomials is equal to*

$$\sum_{k\ge 0} q^k \mathrm{ch}_{\mathcal{H}^k} = (q)_n \prod_{i,j}\left(1-q\,\frac{x_i}{x_j}\right)^{-1} = \sum_{k\ge 0} e_n(X)^{-k} Q'_{(k^n)}(X) \qquad (28)$$

so that for a decreasing vector α with $|\alpha| = 0$, such that $\lambda = \alpha + (k^n)$ is a partition, one has

$$F_q(V_\alpha) = K_{\lambda,(k^n)}(q)\ . \qquad (29)$$

Several sophisticated proofs of this result appear in the literature. An elementary one, due to Gupta [**Gu1**], can be presented as follows.

Let Ω_n be the operator which sends the monomial e^λ onto the Schur function $s_\lambda(X)$, *i.e.*

$$\Omega_n(f) = \sum_{w\in\mathbf{S}_n} w\left(f\prod_{i<j}\left(1-\frac{x_j}{x_i}\right)^{-1}\right)\ . \qquad (30)$$

It is clear that when f is symmetric, $\Omega_n(fg) = f\Omega_n(g)$. Now, equation (26) can be written

$$(1-q)^n\prod_{i<j}\left(1-q\,\frac{x_j}{x_i}\right)\cdot\sigma_q(XX^\vee) = \prod_{i<j}\left(1-q\,\frac{x_i}{x_j}\right)^{-1}\ ,$$

and we can apply Ω_n to both sides. An easy induction shows that

$$\Omega_n\prod_{i<j}\left(1-q\,\frac{x_j}{x_i}\right)^{-1} = [n]_q!\ ,$$

and the r.h.s. becomes

$$\Omega_n\left(\sum_{\alpha\in\mathbf{Z}^n}\mathcal{P}_q(\alpha)e^\alpha\right) = \sum_{\alpha\in\mathbf{Z}^n}\mathcal{P}_q(\alpha)s_\alpha(X) = \sum_{\gamma\downarrow} K_{\gamma,(0^n)}(q)s_\gamma(X)\ ,$$

(sum over decreasing vectors of integers γ) so that

$$\sum_{k\ge 0} q^k\mathrm{ch}_{\mathcal{H}^k} = (q)_n\sigma_q(XX^\vee) = \sum_{k\ge 0}(x_1\cdots x_n)^{-k}Q'_{(k^n)}(X)\ ,$$

as required.

6 Harmonic polynomials for the symmetric group

The modified Hall-Littlewood functions Q'_λ also appear as graded characteristics of the representations of the symmetric group in certain spaces of polynomials. Recall that the *Frobenius characteristic* is the linear map $\mathcal{F}$ from the representation ring $R(\mathbf{S}_n)$ to the ring of symmetric functions which sends the irreducible representation indexed by a partition λ to the Schur function s_λ. If χ is the character of a representation V, one has (*cf.* [**Mcd**])

$$\mathcal{F}(V) = \sum_{|\alpha|=n} \chi(\alpha)\frac{p_\alpha}{z_\alpha} \ .$$

For a graded module $V = \bigoplus_k V_k$, one defines the *graded characteristic* by

$$\mathcal{F}_q(V) = \sum_k q^k \mathcal{F}(V_k) \ .$$

For example, if $V = \mathbf{C}[x_1, \ldots, x_n]$ is the ring of polynomials endowed with its usual graduation, the representation of $\mathbf{S}_n$ in V_1 is the representation by permutation matrices $w \mapsto g_w$, and the representation in V_k is its k-th symmetric power $w \mapsto S^k(g_w)$. Let w be a permutation of type $\alpha = (1^{m_1} 2^{m_2} \cdots n^{m_n})$. Then, by a classical identity

$$\sum_{k\geq 0} q^k \operatorname{tr} S^k(g_w) = \frac{1}{\det(I - q\, g_w)} = \prod_{i\geq 1} \frac{1}{(1-q^i)^{m_i}} = p_\alpha\left(\frac{1}{1-q}\right)$$

(in λ-ring notation), so that

$$\mathcal{F}_q(V) = \sum_{|\alpha|=n} p_\alpha\left(\frac{1}{1-q}\right)\frac{p_\alpha(X)}{z_\alpha} = h_n\left(\frac{X}{1-q}\right) \ . \tag{31}$$

The harmonic polynomials for the symmetric group $\mathbf{S}_n$ are those polynomials which are annihilated by all differential operators of the form

$$\mathcal{D} = f\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right)$$

where f is a symmetric polynomial without constant term. For example the Vandermonde determinant

$$\Delta(X) = \prod_{r<s}(x_r - x_s)$$

is harmonic, since for $w \in S_n$,

$$w\mathcal{D}\Delta = \mathcal{D}w\Delta = \varepsilon(w)\mathcal{D}w$$

so that $\mathcal{D}\Delta$ is an alternating polynomial of degree $<$ degΔ. This implies that $\mathcal{D}\Delta = 0$ since any alternating polynomial is divisible by Δ. As a consequence, any partial derivative

$$\partial^\alpha\Delta = \frac{\partial^{\alpha_1}}{\partial x_1}\cdots\frac{\partial^{\alpha_n}}{\partial x_n}\Delta$$

of Δ is harmonic since

$$\mathcal{D}\partial^\alpha \Delta = \partial^\alpha \mathcal{D}\Delta = 0 \ .$$

A result of Chevalley (valid for any finite reflection group) shows that as an $\mathbf{S}_n$-module,

$$\mathbf{C}[x_1, \ldots, x_n] = \mathcal{I} \otimes \mathcal{H} \tag{32}$$

where $\mathcal{I} = \mathbf{C}[x_1, \ldots, x_n]^{\mathbf{S}_n}$ is the ring of invariants, and $\mathcal{H}$ the module of harmonic polynomials. Since one clearly has

$$\mathcal{F}_q(\mathcal{I}) = \sum_{k\geq 0} q^k \dim \mathcal{I}_k \ h_n(X) = \frac{1}{(q\,;\,q)_n}\, h_n(X) \ ,$$

it follows that

$$\mathcal{F}_q(\mathcal{H}) = (q\,;\,q)_n h_n \left(\frac{X}{1-q}\right) = \omega\left(Q'_{(1^n)}(X;q)\right) \tag{33}$$

where ω is the involution $h_n \leftrightarrow e_n$.

It can be shown that the other Q' functions also arise as graded characteristics of natural submodules of $\mathcal{H}$ (up to a simple transformation) . However, this is far more complicated. Let us set

$$\tilde{Q}'_\mu(X;q) = \sum_\lambda \tilde{K}_{\lambda\mu}(q) s_\lambda(X) = q^{n(\mu)} Q'_\mu(X;q^{-1})$$

and

$$F_\mu(X;q) = \omega(\tilde{Q}'_\mu(X;q)) \ .$$

(In the special case when $\mu = (1^n)$ we have $\omega(Q'_{(1^n)}) = \tilde{Q}'_{(1^n)}$, so that $F_{(1^n)} = Q'_{(1^n)}$). These symmetric functions first appeared as graded S_n-characteristics in algebraic geometry ([**HS**], see also [**Mcd**] ex. 9 p. 136). To be precise, if we denote by X_μ the submanifold of the flag manifold $F(\mathbf{C}^n)$ formed by the flags which are fixed by a unipotent u such that the nilpotent $u - I$ has its Jordan canonical form described by the partition μ, there is an action of $\mathbf{S}_n$ on the cohomology ring $H^*(X_\mu, \mathbf{Q})$ such that

$$\mathcal{F}_{\sqrt{q}}(H^*(X_\mu)) = \sum_{i\geq 0} q^i \mathcal{F}(H^{2i}(X_\mu)) = F_\mu(X;q) \ .$$

It is well known that the cohomology ring of the full flag manifold $F(\mathbf{C}^n) = X_{(1^n)}$ is isomorphic as an $\mathbf{S}_n$-module to the ring of *coinvariants* (for the action $\sigma f = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$ which is conjugate to the above one)

$$\mathbf{Q}[x_1, \ldots, x_n]/\mathcal{I}^+ \tag{34}$$

where $\mathcal{I}^+$ is the ideal generated by the symmetric polynomials without constant term, and that this ring is itself isomorphic to the module $\mathcal{H}_{\mathbf{Q}}$ of harmonic polynomials (over $\mathbf{Q}$ here).

A description of $H^*(X_\mu)$ as the quotient of the ring of polynomials by a certain ideal is given in [**DCP**], and an isomorphic module of harmonic polynomials is constructed in

[**La**]. Let λ be a partition of n and $\mu = \lambda' = (\mu_1, \mu_2, \ldots, \mu_k)$ be its conjugate. The *Specht polynomial* Δ_λ is by definition the product of Vandermonde determinants

$$\Delta_\lambda = \prod_{i=1}^{k} \Delta(X^{(i)})$$

where $X^{(i)} = \{x_r | \mu_0 + \cdots + \mu_{r-1} < r \leq \mu_0 + \cdots + \mu_r\}$ $(\mu_0 := 0)$ and for $A \subseteq X$

$$\Delta(A) = \prod_{x_r, x_s \in A,\ r<s} (x_r - x_s)\ .$$

Specht's result states that the linear span of the orbit of Δ_λ under $\mathbf{S}_n$ is a model of the irreducible representation of $\mathbf{S}_n$ indexed by the partition λ. Now, one can also consider the linear span L_λ of the orbit of Δ_λ under the *affine symmetric group*, which is generated by the permutations and the translations $T_i : x_i \mapsto x_i + 1$. The elements of this orbit are harmonic, since the Specht polynomials are easily seen to be (up to a scalar factor) partial derivatives of the full Vandermonde $\Delta(X)$. By Taylor's formula, the image of a Specht polynomial under an element of the affine symmetric group is itself a linear combination of partial derivatives of the Vandermonde $\Delta(X)$ (which in fact span $\mathcal{H}$). One can then state [**La**]:

Theorem 6.1 *The linear span L_λ of the orbit of a Specht polynomial Δ_λ under the affine symmetric group is a submodule of $\mathcal{H}$ having as graded characteristic the modified Hall-Littlewood function $\tilde{Q}'_\lambda(X;q)$.*

Example 6.2 We choose $\lambda = (2,1,1)$. The Specht polynomial Δ_{211} may be expressed as a derivative of the full Vandermonde $\Delta = \Delta(x_1, \ldots, x_4)$:

$$\Delta_{211} = -\frac{1}{6}\, \partial^{(0,0,0,3)} \Delta\ .$$

The homogeneous component of degree 3 of L_{211} is the Specht module W_{211}. A basis is for example

$$\{\partial^{(0,0,0,3)}\Delta,\ \partial^{(0,0,3,0)}\Delta,\ \partial^{(0,3,0,0)}\Delta\}\ .$$

The degree 2 component has the basis

$$\{\partial^{(0,1,0,3)}\Delta,\ \partial^{(0,1,3,0)}\Delta,\ \partial^{(0,3,0,1)}\Delta,\ \partial^{(0,0,1,3)}\Delta,\ \partial^{(0,0,3,1)}\Delta\}\ ,$$

and is isomorphic to the direct sum $W_{22} \oplus W_{31}$. The degree 1 component has the basis

$$\{\partial^{(0,1,1,3)}\Delta,\ \partial^{(1,0,1,3)}\Delta,\ \partial^{(1,1,3,0)}\Delta\}\ ,$$

and is isomorphic to W_{31}. Finally, the degree 0 component is generated by $\partial^{(0,1,2,3)}\Delta$. The graded characteristic is therefore

$$q^3\, s_{211} + (q + q^2)\, s_{31} + q^2\, s_{22} + s_4 = q^3\, Q'_{211}(q^{-1}) = \tilde{Q}'_{211}\ .$$

Note that the interpretation of the $F_\mu(X;q)$ as graded characteristics yields the positivity of the Kostka-Foulkes polynomials.

7 Kostka-Foulkes polynomials and the charge of Young tableaux

In [**Fo1**], Foulkes asked for a combinatorial explanation of the positivity of Kostka polynomials. He conjectured the existence of a natural statistics $c(t)$ on the set of Young tableaux such that

$$K_{\lambda\mu}(q) = \sum_{t \in \mathrm{Tab}\,(\lambda,\mu)} q^{c(t)} \ .$$

This conjecture was settled by Lascoux and Schützenberger [**LS2**], [**Sc**], who identified $c(t)$ to be the rank of a poset structure on $\mathrm{Tab}\,(\cdot,\mu)$.

The definition of the *charge* $c(t)$ rests upon the interpretation of Young tableaux as elements of a monoid, whose definition and basic properties will be recalled now [**LS1**], [**La**]. Let $A = \{a_1 < a_2 < \ldots < a_n\}$ denote a totally ordered alphabet of noncommutative indeterminates, and consider the free monoid A^* generated by A. The Robinson-Schensted bijection associates with a word $w \in A^*$ a couple $(P(w), Q(w))$ of Young tableaux of the same shape. It was shown by Knuth [**Kn**] that the equivalence on A^* defined by

$$w \equiv w' \Longleftrightarrow P(w) = P(w')$$

is generated by the relations

$$zxy \equiv xzy \ , \quad yxz \equiv yzx \ , \quad yxx \equiv xyx \ , \quad yxy \equiv yyx \ ,$$

for any $x<y<z$ in A. The quotient set $A^*/\!\equiv$, which by definition is in one to one correspondence with the set of Young tableaux on A, is therefore endowed with a multiplicative structure reflecting the Robinson-Schensted construction.

This multiplication being noncommutative, Lascoux and Schützenberger introduced the operation of *cyclage*, a monoid analogue of the conjugation $h \to g^{-1}hg$ in noncommutative groups [**LS1**], [**La**]. To be precise, given t, t' in the plactic monoid $A^*/\!\equiv$ and $i \geq 2$, write

$$t \xrightarrow{\ i\ } t'$$

if and only if there exists u in $A^*/\!\equiv$ such that $t = a_i\, u$ and $t' = u\, a_i$. In this way, $\mathrm{Tab}\,(\cdot,\mu)$ is given the structure of a connected oriented graph, whose transitive closure is a partial order with minimal element the row tableau $t_\mu := a_1^{\mu_1} a_2^{\mu_2} \cdots a_n^{\mu_n}$. For example, if $\mu = (2,2,1)$, one obtains the ranked poset shown in Figure 1, in which for convenience the letters a_i in the tableaux have been replaced by their indices i. The cocharge $co(t)$ of a tableau t is defined as its rank in the poset $\mathrm{Tab}\,(\cdot\,,\,\mu)$, that is, the number of cyclages needed to transform t into the row tableau t_μ. Denote by $m(\mu)$ the maximal value of the cocharge on $\mathrm{Tab}\,(\cdot\,,\,\mu)$ (if μ is a partition, $m(\mu) = n(\mu) = \sum_i (i-1)\mu_i$). The *charge* of t is $c(t) = m(\mu) - co(t)$ and one has

Theorem 7.1 *The Kostka-Foulkes polynomial $K_{\lambda\mu}(q)$ is the generating function of the charge on* $\mathrm{Tab}\,(\lambda,\mu)$*:*

$$K_{\lambda\mu}(q) = \sum_{t \in \mathrm{Tab}\,(\lambda,\mu)} q^{c(t)} \ .$$

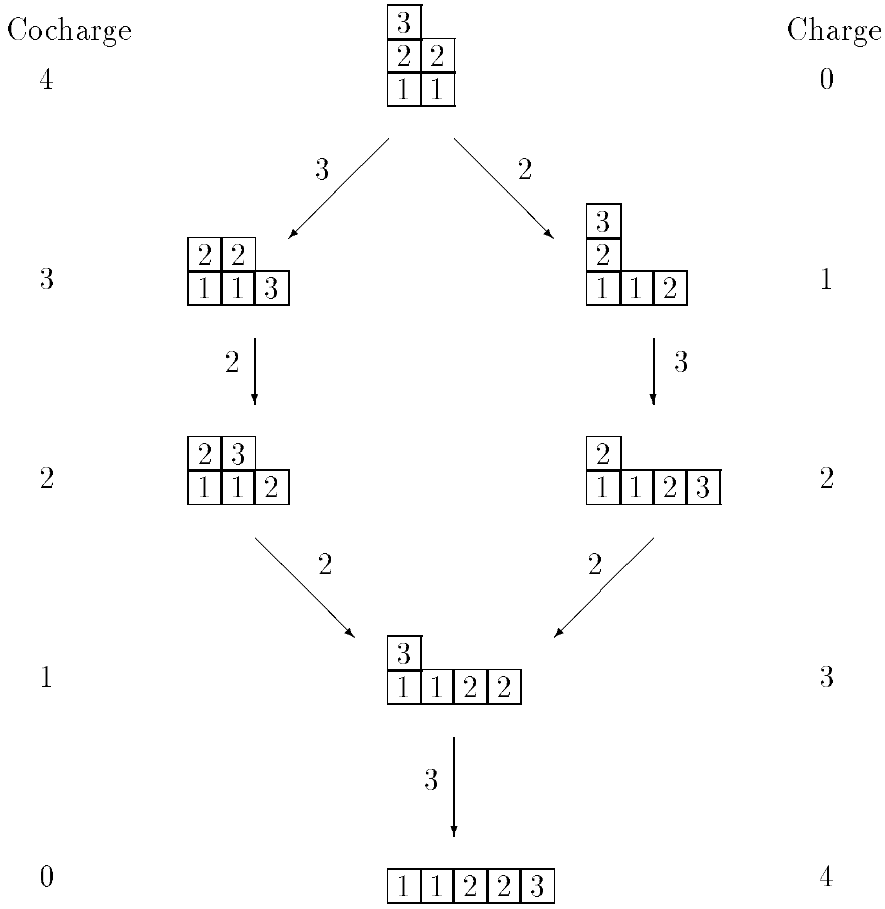


Figure 1:

Given a tableau t of shape λ, put $s_t := s_\lambda$. With this notation, the theorem can be stated as follows.

Corollary 7.2 *The modified Hall-Littlewood function Q'_μ is equal to*

$$Q'_\mu = \sum_{t \in \mathrm{Tab}(\cdot\,,\,\mu)} q^{c(t)}\, s_t \ .$$

Example 7.3 The description given in Figure 1 of the ranked poset Tab $(.\,,\,(2,2,1))$ yields the expansion

$$Q'_{221} = s_{221} + (q + q^2)s_{32} + qs_{311} + (q^2 + q^3)s_{41} + q^4 s_5 \ .$$

Alternatively, one can compute the charge of a Young tableau t whose weight is a partition μ using the following algorithm.

Suppose first that $\mu = (1^n)$. Let w denote the word obtained by reading t from left to right and top to bottom. Attach an index to each letter of w as follows; the letter a_1 has index 0, and if a_r has index i, then a_{r+1} has index i or $i+1$ according as it lies to the left or right of a_r. The charge of t (or w) is then the sum of the indices.

Now if μ is any partition, the row-reading w of t is decomposed into a sequence of standard words w_i. To obtain w_1, read w from the right and choose the first letter a_1, then the first letter a_2 to the left of the a_1 chosen, and so on. If at any stage there is no a_{r+1} to the left of the a_r chosen, go back to the right end of w. This procedure extracts a standard subword w_1 of w of length μ'_1. Now erase w_1 from w and repeat the procedure to get a standard subword w_2 of length μ'_2, and so on. The charge of t is the sum of the charges of the w_i.

Example 7.4 Take

$$t = \begin{array}{|c|c|ccc}\cline{1-2} 6 & 6 \\ \cline{1-2} 4 & 5 \\ \cline{1-3} 2 & 4 & \multicolumn{1}{c|}{5} \\ \hline 1 & 1 & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{3} \\ \hline \end{array}$$

where for short the letter a_r is simply written r. One has $w = 6\,\mathbf{6}\,4\,\mathbf{5}\,\mathbf{2}\,\mathbf{4}\,5\,\mathbf{1}\,1\,2\,3\,\mathbf{3}$ and $w_1 = 6\,5\,2\,4\,1\,3$, $w_2 = 6\,4\,5\,1\,2\,3$. Hence, $c(t) = c(w_1) + c(w_2) = 4 + 11 = 15$.

The combinatorial description of Hall-Littlewood functions and Kostka polynomials provides some new methods of investigation. Thus, it enabled Han to prove the following conjecture of Gupta for the growth of Kostka polynomials [**Gu3**], [**Han**].

Given a partition λ and a positive integer a, denote by $\lambda \cup a$ the partition obtained by inserting in λ a new part equal to a.

Theorem 7.5 *For any partitions λ, μ and integer a, the difference*

$$K_{\lambda\cup a,\mu\cup a}(q) - K_{\lambda\mu}(q)$$

is a polynomial with nonnegative coefficients.

Han's proof consists in the construction of an injective map from Tab (λ, μ) to Tab $(\lambda \cup a, \mu \cup a)$ which preserves the charge of Young tableaux.

Another proof of Theorem 7.5 has been given by Kirillov [**Ki4**]. It is based on the theory of rigged configurations that we shall now present.

8 Kostka-Foulkes polynomials, Bethe Ansatz and configurations

A different description of Kostka-Foulkes polynomials was obtained by Kirillov and Reshetikhin [**KR**] as a consequence of their investigation of combinatorial aspects of the quantum inverse scattering method.

The state space of certain integrable quantum systems, called $\mathfrak{gl}\,(n, \mathbf{C})$-invariant models, is equal to the tensor product

$$\mathfrak{h} = S^{\mu_1}(V) \otimes \cdots \otimes S^{\mu_r}(V)$$

of symmetric powers of the defining representation V of $\mathfrak{gl}\,(n, \mathbf{C})$. The eigenvectors of the Hamiltonian of the model can be constructed by means of a substitution known as the *Bethe Ansatz* [**Fa**]. They are parametrized by the solutions of a system of algebraic equations (Bethe equations) which in turn are in one to one correspondence with combinatorial objects named *rigged configurations* [**TF**]. It can be shown analytically that the Bethe vectors form a complete system of highest weight vectors for the $\mathfrak{gl}\,(n, \mathbf{C})$-module $\mathfrak{h}$ and hence can also be parametrized by Young tableaux of weight $\mu = (\mu_1, \ldots, \mu_r)$ [**Ki1**][**Ki2**]. In [**KKR**] [**KR**] an explicit bijection is constructed between Young tableaux and rigged configurations, which sends the charge of Young tableaux on a natural invariant of rigged configurations, namely, the sum of their *quantum numbers*. This yields a very efficient formula for evaluating Kostka-Foulkes polynomials as sums of products of q-binomial coefficients.

Before explaining the Kirillov-Reshetikhin bijection, we introduce the necessary definitions concerning configurations. Let λ, μ be partitions such that $|\lambda| = |\mu|$. It is convenient to regard λ, μ as infinite nonincreasing sequences by putting $\lambda_k = 0$ for $k > \ell(\lambda)$, $\mu_k = 0$ for $k > \ell(\mu)$. A *matrix of type* $(\lambda,\, \mu)$ is a matrix $M = (m_{ij})_{i,j\geq 1}$ with only a finite number of nonzero integer entries, such that

$$\sum_{j\geq 1} m_{ij} = \lambda_i \ , \quad \sum_{i\geq 1} m_{ij} = \mu'_j \ .$$

Here and in the sequel, we denote by μ'_j the jth part of the conjugate partition μ' of μ.

With such a matrix M are associated two matrices $P = (P_{ij})_{i,j\geq 1}$, $Q = (Q_{ij})_{i,j\geq 1}$ defined by

$$P_{ij} = \sum_{k\leq j} (m_{ik} - m_{i+1,k}) \ , \quad Q_{ij} = \sum_{k\geq i+1} (m_{kj} - m_{k,j+1}) \ .$$

The matrix M is an *admissible matrix of type* $(\lambda,\, \mu)$ if and only if P_{ij} and Q_{ij} are nonnegative for all $i, j \geq 1$. The numbers P_{ij} such that $Q_{ij} > 0$ are called the *vacancy numbers* of M.

An admissible matrix is conveniently pictured as a sequence $\nu = (\nu^0, \nu^1, \nu^2, \ldots)$ of partitions, or *configuration*, defined by

$$(\nu^i)'_j = \sum_{k\geq i+1} m_{kj} \ .$$

Obviously, one can recover the admissible matrix M from the configuration ν, so that the two combinatorial datas M or ν are equivalent. In particular, the type $(\lambda,\, \mu)$ is computed from ν by $\lambda_i = |\nu^i| - |\nu^{i-1}|$, for $i \geq 1$, and $\mu = \nu^0$.

The rows of the diagram of ν^i are called *strings of type* i. By construction the nonzero integers Q_{ij} in the ith row of Q are equal to the multiplicities of the partition ν^i, *i.e.* to the successive widths of the *blocks* of strings of equal length in ν^i, so that a vacancy number P_{ij} is in fact associated with each block of strings of type $i \geq 1$. It is written to the right of the block.

Example 8.1 Let us illustrate all these definitions. The matrix

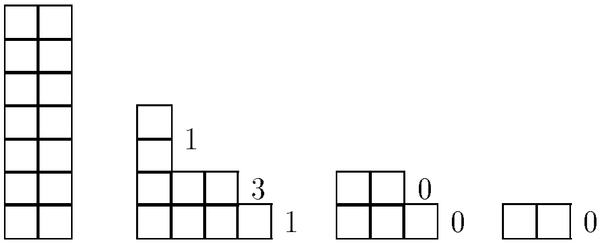
$$M = \begin{pmatrix} 3 & 5 & -2 & -1 \\ 2 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

has type (λ, μ), where $\lambda = (5,4,3,2)$ and $\mu = (2,2,2,2,2,2,2)$. The associated matrices P and Q are equal to

$$P = \begin{pmatrix} \mathbf{1} & 6 & \mathbf{3} & \mathbf{1} \\ 1 & \mathbf{0} & \mathbf{0} & 1 \\ 0 & \mathbf{0} & 1 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} \mathbf{2} & 0 & \mathbf{1} & \mathbf{1} \\ 0 & \mathbf{1} & \mathbf{1} & 0 \\ 0 & \mathbf{1} & 0 & 0 \end{pmatrix}.$$

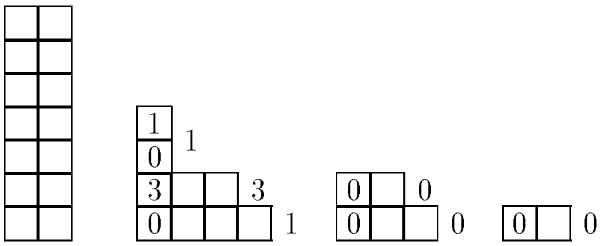
The nonzero entries of Q and the corresponding vacancy numbers in P have been printed in bold type.

The representation of M as a configuration $\nu = (\nu^0, \nu^1, \nu^2, \nu^3)$ is



The last definition is that of a *rigged configuration*, by which we mean a configuration ν together with a family of *quantum numbers*. These are nonnegative integers $\mathcal{I}^i_j$ attached to each string ν^i_j of type $i \geq 1$, which must not exceed the vacancy number associated with their block. Quantum numbers are written in the leftmost cell of the string to which they belong. The order of the quantum numbers inside a given block is irrelevant. In other words, permutations of the strings of a same block are allowed, so that we can always assume that quantum numbers are weakly increasing (or decreasing) from bottom to top within each block.

Example 8.2 The following is a rigging of the configuration ν of Example 8.1.



It is clear from the definition that the number of rigged configurations $(\nu, \mathcal{I})$ corresponding to a given configuration ν is equal to the product of binomial coefficients

$$N = \prod_{i,j \geq 1} \binom{P_{ij} + Q_{ij}}{Q_{ij}}$$

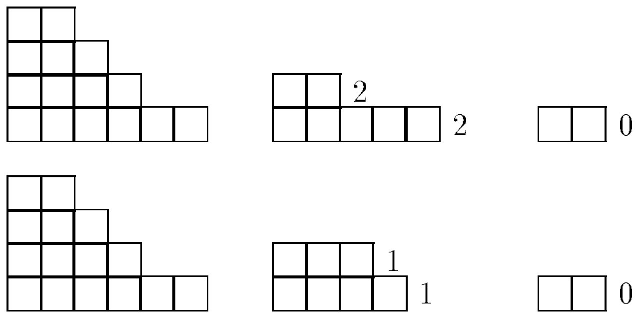
(with the usual convention $\binom{m}{0} = 1$ for $m \in \mathbf{Z}$). As stated above, the Bethe vectors of $\mathfrak{h}$ are highest weight vectors parametrized by rigged configurations. More precisely, the Bethe vectors of weight λ in $S^{\mu_1}(V) \otimes \cdots \otimes S^{\mu_r}(V)$ are labelled by rigged configurations of type $(\lambda,\, \mu)$. It follows from the completeness of the Bethe system that

Theorem 8.3 *The Kostka number $K_{\lambda\mu}$ is equal to*

$$K_{\lambda\mu} = \sum_{\nu} N(\nu) = \sum_{\nu} \prod_{i,j\geq 1} \binom{P_{ij}(\nu) + Q_{ij}(\nu)}{Q_{ij}(\nu)}$$

the sum running over all configurations ν of type $(\lambda,\, \mu)$.

Example 8.4 Take $\lambda = (8,5,2)$, $\mu = (6,4,3,2)$. There are two configurations ν of type $(\lambda,\, \mu)$, viz.



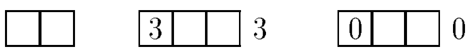
Thus $K_{\lambda\mu} = \binom{2+1}{1}\binom{2+1}{1}\binom{0+1}{1} + \binom{1+1}{1}\binom{1+1}{1}\binom{0+1}{1} = 9 + 4 = 13.$

We shall now explain the bijection devised by Kirillov and Reshetikhin for proving combinatorially Theorem 8.3.

A string ν^i_j of type $i \geq 1$ in a rigged configuration $(\nu, \mathcal{I})$ is called *singular* when its quantum number $\mathcal{I}^i_j$ is maximal, *i.e.* is equal to the corresponding vacancy number. By convention, the only singular string of type 0 is the upper one. Consider now chains of weakly increasing singular strings

$$\nu^0_{i_0} \leq \nu^1_{i_1} \leq \cdots \leq \nu^s_{i_s}$$

in $(\nu, \mathcal{I})$. Define the *rank* of $(\nu, \mathcal{I})$ as the maximal length of such a chain. For example, the rank of the rigged configuration shown in Example 8.2 is equal to 3, the longest chain being



Let t be a Young tableau of shape λ and weight μ. The cells of t can be ordered by first reading the 1's from left to right, then the 2's from left to right, and so on (this is the usual standardization of t). The rigged configuration $(\nu, \mathcal{I})$ corresponding to t is now constructed inductively as follows.

The last cell of t belongs to row number r, say. Let $\overline{t}$ be the tableau obtained from t by erasing this cell, and assume that the corresponding rigged configuration $(\overline{\nu},\, \overline{\mathcal{I}})$ is known.

Construction of ν: Add one cell to one of the longest singular strings of type $r-1$ in $(\overline{\nu},\, \overline{\mathcal{I}})$ (if there is none, create a new string of type $r-1$ and length 1). Let m be the length of this new string. Similarly, add one cell to one of the longest singular strings of type $r-2$ and length $\leq m-1$ (if there is none, create a new string of type $r-2$ and length 1). And so on. If necessary, reorder the strings inside each diagram in order to get a sequence of partitions. In this way a new configuration ν of type (λ, μ) is obtained, which differs from $\overline{\nu}$ by a single chain C composed of r strings.

Construction of $\mathcal{I}$: Compute the vacancy numbers of ν. The r strings composing C are given their maximal quantum numbers, while the quantum numbers of all the other strings remain unchanged.

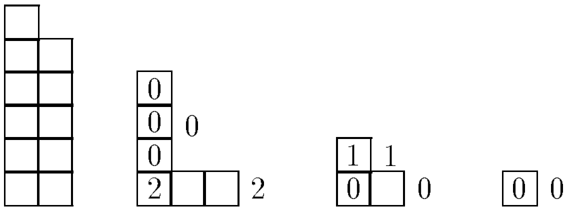
Example 8.5 We take

$$t = \begin{array}{|c|c|c|c|c|} \hline 6 & 6 \\ \cline{1-2} 4 & 5 \\ \cline{1-3} 2 & 4 & 5 \\ \hline 1 & 1 & 2 & 3 & 3 \\ \hline \end{array}$$

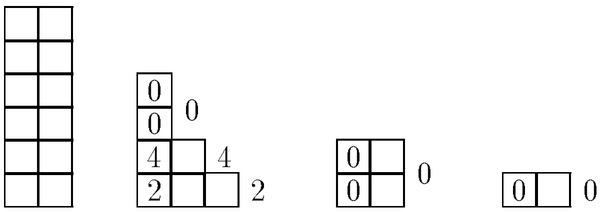
The configuration $(\overline{\nu}, \overline{\mathcal{I}})$ associated to

$$\overline{t} = \begin{array}{|c|c|c|c|c|} \cline{1-1} 6 \\ \cline{1-2} 4 & 5 \\ \cline{1-3} 2 & 4 & 5 \\ \hline 1 & 1 & 2 & 3 & 3 \\ \hline \end{array}$$

is supposed to be known, and is equal to



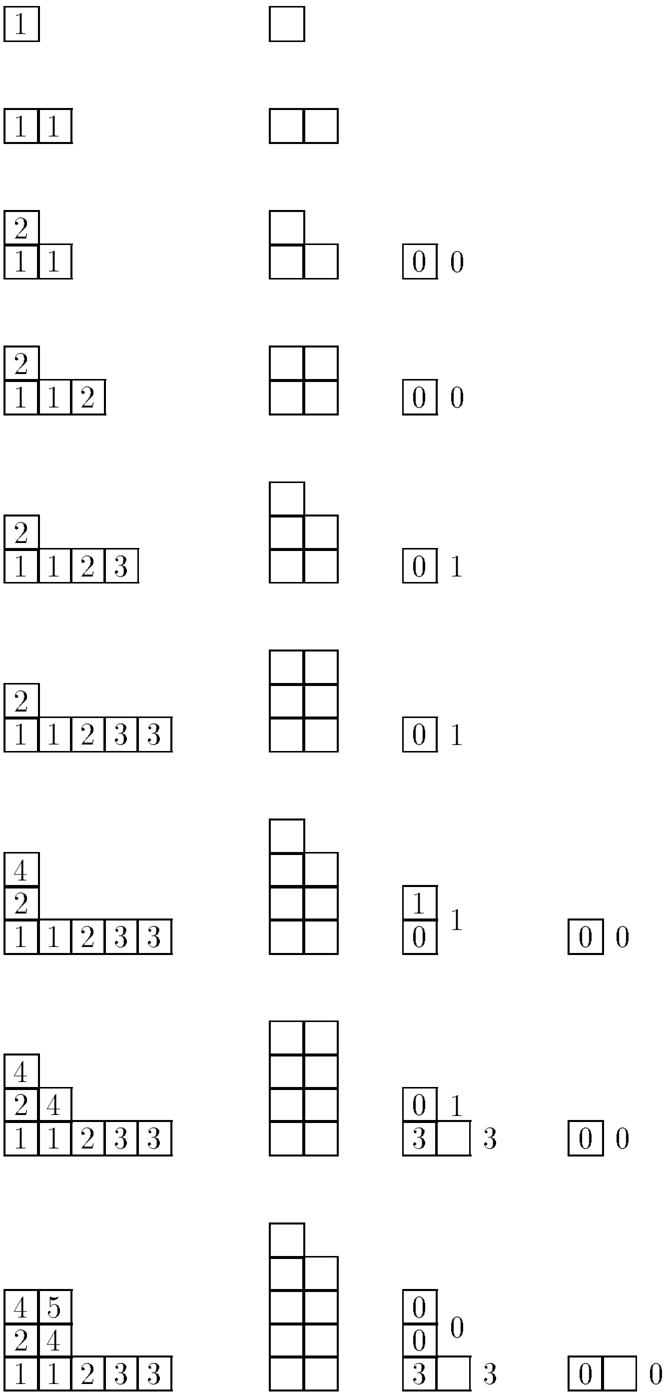
Here, $r = 4$ and the rigged configuration $(\nu, \mathcal{I})$ computed by application of the rules described above is

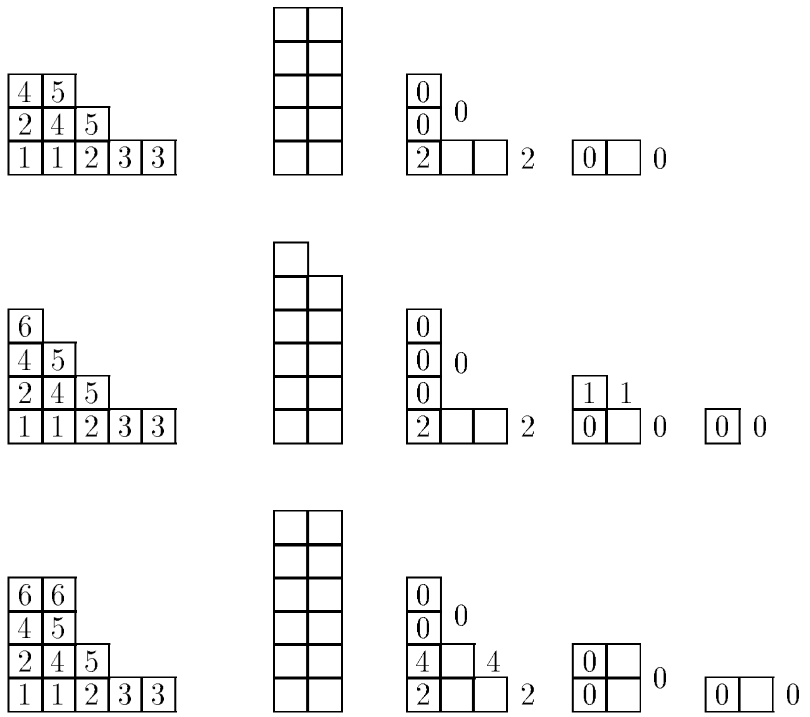


the associated matrices being

$$M = \begin{pmatrix} 2 & 4 & -1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} , \quad P = \begin{pmatrix} \mathbf{0} & \mathbf{4} & \mathbf{2} \\ 1 & \mathbf{0} & 1 \\ 0 & \mathbf{0} & 0 \end{pmatrix} , \quad Q = \begin{pmatrix} \mathbf{2} & \mathbf{1} & \mathbf{1} \\ 0 & \mathbf{2} & 0 \\ 0 & \mathbf{1} & 0 \end{pmatrix} .$$

Example 8.6 We give now the complete construction of the rigged configuration associated to the tableau t of Example 8.5. One passes from one step to the following by application of the previous rules. For the sake of brevity, the matrices M, P, Q necessary at each step for the determination of the vacancy and quantum numbers have not been reproduced.





The most important property of the Kirillov-Reshetikhin bijection is that the charge of a Young tableau t can be computed very easily from the rigged configuration $(\nu, \mathcal{I})$ to which it corresponds.

Given a configuration ν with admissible matrix $M = (m_{ij})$, define the *charge of* ν (or M) by

$$c(\nu) = c(M) = \frac{1}{2} \sum_{i,j} m_{ij}(m_{ij} - 1) \ .$$

Then one can show that

$$c(t) = c(\nu) + \sum_{i,j} \mathcal{I}^i_j \ ,$$

which yields the following q-analogue of 8.3

Theorem 8.7 *The Kostka polynomial $K_{\lambda\mu}(q)$ is equal to*

$$K_{\lambda\mu}(q) = \sum_{\nu} \prod_{i,j \geq 1} q^{c(\nu)} \begin{bmatrix} P_{ij}(\nu) + Q_{ij}(\nu) \\ Q_{ij}(\nu) \end{bmatrix}_q \ ,$$

the sum running over all configurations ν of type (λ, μ).

Example 8.8 Take $\lambda = (8, 5, 2)$, $\mu = (6, 4, 3, 2)$ as in Example 8.4. The admissible matrices corresponding to the two configurations given in this example are respectively

$$\begin{pmatrix} 2 & 2 & 2 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 2 & 2 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \ .$$

Their charges are both equal to 3, hence

$$K_{\lambda\mu}(q) = q^3 \begin{bmatrix} 3 \\ 1 \end{bmatrix}_q \begin{bmatrix} 3 \\ 1 \end{bmatrix}_q + q^3 \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q = 2q^3 + 4q^4 + 4q^5 + 2q^6 + q^7 \ .$$

An example of application of Theorem 8.7 is the proof by Kirillov of the unimodality of generalized Gaussian coefficients [**Ki5**]. In fact, when λ consists of only two parts, 8.7 gives a generalization of the KOH formula for q-binomial coefficients.

9 Specializations at roots of unity

When $q = -1$, the Hall-Littlewood functions coincide with those introduced by Schur in his theory of projective representations of the symmetric group.

The investigation of Hall-Littlewood functions at other roots of unity has been initiated by Morris [**Mo2**], in connection with problems in the modular representation theory of the symmetric group. Let p be a prime, ζ a primitive pth root of 1, and μ a partition with at least one multiplicity $m_i(\mu) \geq p$. It follows from (16) that $Q_\mu(X;\zeta) = 0$. On the other hand,

$$Q_\mu(X;q) = \sum_\lambda K_{\lambda,\mu}(q)\, s_\lambda((1-q)X) = \sum_{\lambda,\rho} K_{\lambda,\mu}(q)\, \frac{\chi^\lambda_\rho}{z_\rho}\, p_\rho((1-q)X)$$

$$= \sum_\rho \left(\sum_\lambda K_{\lambda,\mu}(q)\, \chi^\lambda_\rho \right) \frac{\prod_i (1-q^{\rho_i})}{z_\rho}\, p_\rho(X) \ .$$

Hence, for all partitions ρ such that $\rho_i \not\equiv 0 \bmod p$, one has

$$\sum_\lambda K_{\lambda,\mu}(\zeta)\, \chi^\lambda_\rho = 0 \ . \tag{35}$$

Denote by $\pi^p(n)$ the set of partitions ρ of n whose parts are not divisible by p, and by $\pi_p(n)$ the set of partitions μ whose multiplicities are less than p. It is well known that $p^*(n) := |\pi^p(n)| = |\pi_p(n)|$. The elements of $\pi^p(n)$ correspond to p-regular conjugacy classes in $\mathbf{S}_n$, that is, classes whose elements have an order prime to p, while $\pi_p(n)$ parametrizes the p-modular irreducibles characters. A basic problem in the p-modular representation theory of $\mathbf{S}_n$ is to find a complete set of linearly independent relations between the characters χ^λ restricted to p-regular classes ρ. The number of such independent relations being equal to $p(n) - p^*(n)$, the system of identities (35) provides a solution to this question. Thus, in the case $n = 4$, $p = 2$, one obtains the three relations

$$\chi^{1^4}_\rho - \chi^{211}_\rho + 2\chi^{22}_\rho - \chi^{31}_\rho + \chi^4_\rho = 0 \ , \quad \chi^{211}_\rho - \chi^{22}_\rho - \chi^4_\rho = 0 \ , \quad \chi^{22}_\rho - \chi^{31}_\rho + \chi^4_\rho = 0 \ ,$$

for $\rho = (1^4),\ (3,1)$.

The problem of specializing Hall-Littlewood functions at roots of unity is also related to some combinatorial properties of the so-called *cyclic characters* introduced by Foulkes [**Fo2**]. These symmetric functions correspond to characters of the symmetric group induced by irreducible representations of cyclic subgroups generated by a full cycle. A combinatorial interpretation of their coefficients in the basis of Schur functions has been given by Kraskiewicz and Weyman [**KW**] in terms of congruence properties of the major index of permutations, or, which amounts to the same, charge of standard tableaux. A generalization of this result has been proposed in [**LLT2**], in which the coefficients of certain plethysms with the cyclic characters are interpreted in terms of the charge of more general tableaux. This result can also be viewed as an isomorphism theorem for certain submodules of the cohomology of a generalized flag manifold, or also in terms of harmonic polynomials relative to the symmetric group.

Finally, these theorems can be interpreted in terms of congruence properties of Kostka-Foulkes polynomials modulo cyclotomic polynomials.

We shall review in this Section the main results of [**LLT1**] and [**LLT2**].

The first one is a factorization property of the functions $Q'_\lambda(X,q)$ when q is specialized to a primitive root of unity. This is to be seen as a generalization of the fact that when q is specialized to 1 the function $Q'_\lambda(X;q)$ reduces to $h_\lambda(X) = \prod_i h_{\lambda_i}(X)$.

Theorem 9.1 *Let $\lambda = (1^{m_1}2^{m_2}\dots n^{m_n})$ be a partition written multiplicatively. Set $m_i = kq_i + r_i$ with $0 \le r_i < k$, and $\mu = (1^{r_1}2^{r_2}\dots n^{r_n})$. Then, ζ being a primitive k-th root of unity,*

$$Q'_\lambda(X;\zeta) = Q'_\mu(X;\zeta)\prod_{i\ge 1}[Q'_{(i^k)}(X;\zeta)]^{q_i} \ . \tag{36}$$

The functions $Q'_{(i^k)}(X;\zeta)$ appearing in the right-hand side of (36) can be given a more convenient expression.

Theorem 9.2 *Let $p_k \circ h_n$ denote the plethysm of the complete function h_n by the power-sum p_k, which is defined by the generating series $\sum_n p_k \circ h_n(X)\, z^n = \prod_{x\in X}(1-zx^k)^{-1}$. Then, if ζ is as above a primitive k-th root of unity, one has*

$$Q'_{(n^k)}(X;\zeta) = (-1)^{(k-1)n} p_k \circ h_n(X).$$

Example 9.3 With $k=3$ ($\zeta = e^{2i\pi/3}$), we have

$$Q'_{444433311}(X;\zeta) = Q'_{411}(X;\zeta)\,Q'_{333}(X;\zeta)\,Q'_{444}(X;\zeta) = Q'_{411}(X;\zeta)\,p_4\circ h_{43} \ .$$

One can derive from 9.1, 9.2 the following results for Green polynomials at roots of unity, which had been conjectured by Morris and Sultana [**MS**] [**Su**]:

Theorem 9.4 *Let $\lambda = (1^{m_1}2^{m_2}\dots n^{m_n})$ be a partition, with at least one part i with multiplicity $m_i \ge k$, and let $\mu = (\mu_1,\dots,\mu_r)$ be another partition, with at least one part $\mu_h \equiv 0 \pmod k$. Then,*

$$X^\lambda_\mu(\zeta) \equiv 0 \pmod k \ .$$

Theorem 9.5 *Let $\lambda = (1^{m_1}\dots i^{m_i}\dots n^{m_n})$, where for some i, $m_i \ge 1$, $|\lambda| = n$, $\lambda^* = (1^{m_1}\dots i^{m_i-1}\dots n^{m_n})$ and suppose $n = kr$. Then,*

$$X^{(r^k)}_\lambda(\zeta) = (-1)^{(k-1)j} k\ X^{(r-j)^k}_{\lambda^*}(\zeta)$$

if $i = jk$, and $X^{(r^k)}_\lambda(\zeta) = 0$ otherwise.

Example 9.6 With $k = 4$ ($\zeta = i = \sqrt{-1}$), we have

$$X_{422}^{41111}(i) = \langle Q'_{1111} Q'_4 \, , \, p_4 p_{22} \rangle = (-1)^3 \langle p_4 Q'_4 \, , \, p_4 p_{22} \rangle$$

$$= -\langle Q'_4 \, , \, D_{p_4}(p_4 p_{22}) \rangle = -\langle Q'_4 \, , \, 4 p_{22} \rangle = -4 \langle h_4 \, , \, p_{22} \rangle = -4 \; .$$

Theorem 9.5 may in fact be generalized as follows.

Given two partitions λ and μ, we denote by $\lambda \vee \mu$ the partition obtained by reordering the concatenation of λ and ν, *e.g.* $(2,\, 2,\, 1) \vee (5,\, 2,\, 1,\, 1) = (5,\, 2^3,\, 1^3)$. We write $\mu^k = \mu \vee \mu \vee \cdots \vee \mu$ (k factors). If $\mu = (\mu_1, \ldots, \mu_r)$, we set $k\mu = (k\mu_1, \ldots, k\mu_r)$. Let $\lambda = \mu^k$, and let ν be another partition. The Green polynomial $X_\nu^\lambda(\zeta)$ can then be expressed by means of the value of a permutation character of S_n:

$$X_\nu^{\mu^k}(\zeta) = (-1)^{(k-1)|\mu|} \langle p_\nu ,\; p_k(h_\mu) \rangle = (-1)^{(k-1)|\mu|} \langle \phi_k(p_\nu),\; h_\mu \rangle \; ,$$

where ϕ_k is the adjoint of the linear operator $\psi^k : \; F \mapsto p_k \circ F$ (Adams operator), so that:

Theorem 9.7 $X_\nu^{\mu^k}(\zeta) = 0$ *when ν is not of the form $k\pi$ for some partition π, and*

$$X_{k\pi}^{\mu^k}(\zeta) = (-1)^{(k-1)|\mu|} k^{\ell(\pi)} \langle p_\pi \, , \, h_\mu \rangle \; .$$

In particular, $X_{k\pi}^{(r^k)}(\zeta) = (-1)^{(k-1)r} k^{\ell(\pi)}$.

Example 9.8 *(i)* With $k = 4$ ($\zeta = i$), $X_{422}^{2222}(i) = \langle p_4(h_2),\; p_{422} \rangle = 0$ and

$$X_{44}^{42}(i) = \langle h_2,\; \phi_4(p_{44}) \rangle = \langle h_2,\; 16 p_{11} \rangle = 16 \; .$$

(ii) With $k = 3$ ($\zeta = e^{2\pi i/3}$),

$$X_{933}^{333222}(\zeta) = +3^3 \langle p_{311},\; h_{23} \rangle = 3^3 \langle p_{11},\; h_2 D_{p_3} h_3 \rangle = 27 \; .$$

To state our next result, we recall some definitions from number theory. For $k, n \in \mathbf{N}$, the *Ramanujan* (or *Von Sterneck*) *sum* $c(k,n)$ (also denoted $\Phi(k,n)$) is the sum of the k-th powers of the *primitive* n-th roots of unity. Its value is given by *Hölder's formula*: if $(k,n) = d$ and $n = md$, then $c(k,n) = \mu(m)\phi(n)/\phi(m)$, where μ is the Moebius function and ϕ is the Euler totient function (see *e.g.* [**HW**] or [**NV**]).

Let $P(q) = \sum_{k=0}^{n-1} a_k \, q^k \; \in \mathbf{Z}[q]$ be a polynomial of degree $\le n-1$. P is said to be *even modulo* n if $(i,n) = (j,n) \; \Rightarrow a_i = a_j$. The following result, due to E. Cohen ([**Co**], see also [**De1**] for a simpler proof), provides a generalization of the Moebius inversion formula:

Lemma 9.9 *The polynomial P is even modulo n iff for every divisor d of n, the residue of $P(q)$ modulo the cyclotomic polynomial $\Phi_d(q)$ is a constant $r_d \in Z$. In this case, one has*

$$a_k = \frac{1}{n} \sum_{d|n} c(k,d) r_d \; , \qquad r_d = \sum_{t|n} c(n/d, t) a_{n/t} \; .$$

With the aid of Ramanujan sums, we define the symmetric functions

$$\ell_n^{(k)} = \frac{1}{n} \sum_{d|n} c(k,d) p_d^{n/d} \ . \tag{37}$$

These functions were first obtained by Foulkes as Frobenius characteristics of the representations of the symmetric group induced by irreducible representations of a cyclic subgroup generated by an n-cycle [**Fo2**]. A combinatorial interpretation of the multiplicity $\langle s_\lambda \,,\, \ell_n^{(k)} \rangle$ has been given by Kraskiewicz and Weyman [**KW**]. This result is equivalent to the congruence

$$Q'_{1^n}(X;q) \equiv \sum_{0 \le k \le n-1} q^k \ell_n^{(k)} \pmod{1-q^n} \ .$$

A proof using Cohen's formula can be found in [**De1**]. Adapting this argument to our case, we arrive at the following generalization of the result of [**KW**]:

Theorem 9.10 *Let e_i be the i-th elementary symmetric function, and for $\lambda = (\lambda_1, \ldots, \lambda_m)$, $e_\lambda = e_{\lambda_1} \cdots e_{\lambda_r}$. Then, the multiplicity $\langle s_\mu \,,\, \ell_k^{(r)} \circ e_\lambda \rangle$ of the Schur function s_μ in the plethysm $\ell_k^{(r)} \circ e_\lambda$ is equal to the number of Young tableaux of shape μ' (conjugate partition) and weight λ^k whose charge is congruent to r modulo k.*

Example 9.11 With $k=4$, $r=2$ and $\lambda=(2)$,

$$\ell_4^{(2)} \circ e_2 = s_{431} + s_{422} + s_{41111} + 2s_{3311} + 2s_{3221}$$
$$+2s_{32111} + s_{2222} + s_{22211} + 2s_{221111} + s_{2111111} \ .$$

To compute the coefficient $\langle s_{32111} \,,\, \ell_4^{(2)} \circ e_2 \rangle = 2$, we have to find the number of tableaux of shape $(3, 2, 1, 1, 1)' = (5, 2, 1)$, weight $(2, 2, 2, 2)$ and charge $\equiv 2 \pmod 4$. The two tableaux satisfying these constraints are:

$$\begin{array}{|c|cccc}\cline{1-1} 3 \\ \cline{1-2} 2 & \multicolumn{1}{|c|}{4} \\ \hline 1 & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{4} \\ \hline \end{array} \qquad\qquad \begin{array}{|c|cccc}\cline{1-1} 4 \\ \cline{1-2} 2 & \multicolumn{1}{|c|}{3} \\ \hline 1 & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{4} \\ \hline \end{array}$$

which both have charge equal to 6.

Similarly, the reader can check that $\langle s_{732} \,,\, \ell_4^{(2)} \circ e_{21} \rangle = 5$ is the number of tableaux with shape $(3, 3, 2, 1, 1, 1, 1)$, weight $(2, 2, 2, 2, 1, 1, 1, 1)$ and charge $\equiv 2 \pmod 4$.

A related result (see *e.g.* [**MT**]), gives an interpretation of the cyclic characters $\ell_n^{(k)}$ in terms of $\mathbf{S}_n$-harmonic polynomials: let H^k be the Frobenius characteristic of the homogeneous component $\mathcal{H}^k$ of degree k of the module $\mathcal{H}$ of $\mathbf{S}_n$-harmonic polynomials. Then,

$$\ell_n^{(k)} = \sum_{i \equiv k \pmod n} H^i \ .$$

More generally, the plethysm $\ell_k^{(j)} \circ e_\lambda$ can be expressed as a positive sum of Frobenius characteristics of modules of harmonic polynomials. Recall that to each partition μ of n, it is possible to associate a certain submodule L_μ of $\mathcal{H}_n$ with graded characteristic

$$\mathcal{F}_q(L_\mu) = \tilde{Q}'_\mu(X;q) = \sum_k q^k H_\mu^k(X) \ . \tag{38}$$

Theorem 9.12 *Let $\mu = \lambda^k$. Then,*

$$\ell_k^{(j)} \circ e_\lambda = \sum_{i \equiv j \pmod{k}} H_\mu^i \ .$$

Example 9.13 The character of the preceding example decomposes into three parts

$$\ell_4^{(2)} \circ e_2 = H_{2222}^2 + H_{2222}^6 + H_{2222}^{10},$$

the three components being

$$H_{2222}^2 = s_{221111} + s_{2111111} \ ,$$

$$H_{2222}^6 = s_{3311} + 2s_{3221} + s_{2222} + s_{41111} + 2s_{32111} + s_{22211} + s_{221111} \ ,$$

$$H_{2222}^{10} = s_{431} + s_{422} + s_{3311} \ .$$

Theorems 9.1, 9.2 may also be stated in terms of specialization of Kostka-Foulkes polynomials at roots of unity, or equivalently, in terms of congruence properties of the $K_{\lambda,\mu}(q)$ modulo cyclotomic polynomials.

Theorem 9.14 *For any partition λ with $|\lambda| = kn$,*

$$K_{\lambda,n^k}(q) \equiv (-1)^{(k-1)n}\, s_\lambda(1,\, q,\, q^2,\, \ldots,\, q^{k-1}) \pmod{\Phi_k(q)}$$

and more generally, for any partition μ with $|\mu| = n$,

$$K_{\lambda,\mu^k}(q) \equiv (-1)^{(k-1)|\mu|}\, (h_{k\mu} * s_\lambda)(1,\, q,\, q^2,\, \ldots,\, q^{k-1}) \pmod{\Phi_k(q)} \ .$$

We mention that a different type of congruence follows from results of Stanley [**Sta**] and Gupta [**Gu1**], that is, for $\beta_1 = n$ and

$$\lambda = [\alpha,\, \beta]_k := (\alpha_1 + n, \ldots, \alpha_s + n, n, \ldots, n, n - \beta_t, n - \beta_{t-1}, \ldots, n - \beta_2, 0)$$

then (*cf.* [**Ki4**])

$$K_{\lambda,(n^k)}(q) \equiv (s_\alpha * s_\beta)(q,\, \ldots,\, q^{k-1}) \pmod{q^k} \ .$$

Theorem 9.14 gives some congruences modulo cyclotomic polynomials for certain columns of the q-Kostka matrix. We can also derive from 9.1, 9.2 other congruences for certain rows of the same matrix.

Let ρ_n be the staircase partition $(n, \ldots, 2, 1)$. It is well-known that for $k = 2$ ($\zeta = -1$), one has $s_{\rho_n}(X) = P_{\rho_n}(X; -1)$, the staircase P-Schur function. More generally, it can be shown that if ζ is a primitive k-th root of unity, $P_{\rho_n^{k-1}}(X;\zeta) = s_{\rho_n^{k-1}}(X)$. This is a consequence of the following result:

Theorem 9.15 *Let ζ be a primitive k-th root of 1, and λ be a partition such that $\ell(\lambda') \le n$. Then,*

$$s_{\lambda^k \vee \rho_n^{k-1}}(X) = \sum_\mu K_{\lambda\mu}\ P_{\mu^k \vee \rho_n^{k-1}}(X;\zeta)$$

where $K_{\lambda\mu}$ are the usual Kostka numbers. In other words, we have the following congruence modulo the cyclotomic polynomial $\Phi_k(q)$:

$$K_{\lambda^k \vee \rho_n^{k-1},\ \nu}(q) \equiv \begin{cases} K_{\lambda\mu} & \textit{if} \quad \nu = \mu^k \vee \rho_n^{k-1} \\ 0 & \textit{otherwise.} \end{cases}$$

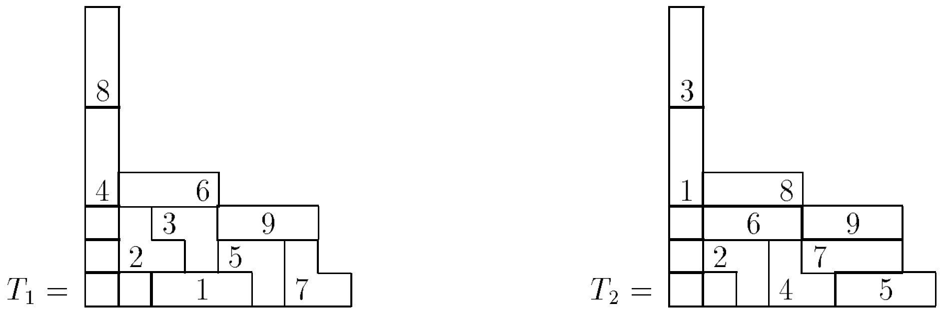


Figure 2:

Example 9.16 Take $k = 3$, $n = 3$, $I = (3, 2)$ and $J = (2, 2, 1)$. Then,

$$K_{333332222211,332222222211111}(q) = q^3(1 + 2q + 6q^2 + 13q^3 + 24q^4 + 39q^5$$

$$+59q^6 + 81q^7 + 105q^8 + 128q^9 + 148q^{10} + 163q^{11} + 171q^{12} + 172q^{13} + 165q^{14} + 153q^{15} + 134q^{16}$$

$$+114q^{17} + 92q^{18} + 72q^{19} + 53q^{20} + 38q^{21} + 25q^{22} + 16q^{23} + 9q^{24} + 5q^{25} + 2q^{26} + q^{27})$$

$$\equiv K_{32,221} = 2 \pmod{1 + q + q^2} .$$

An alternative and more combinatorial formulation of Theorems 9.1 and 9.2 may be presented by introducing the notion of *ribbon tableau.*

Recall that to a partition λ is associated a k-core $\lambda_{(k)}$ and a k-quotient $\lambda^{(k)}$ [**JK**]. The k-core is the unique partition obtained by successively removing k-ribbons (or skew hooks) from λ. The different possible ways of doing so can be distinguished from one another by labelling 1 the last ribbon removed, 2 the penultimate, and so on. Thus Figure 2 shows two different ways of reaching the 3-core $\lambda_{(3)} = (2,\, 1^2)$ of $\lambda = (8,\, 7^2,\, 4,\, 1^5)$. These pictures represent two 3-ribbon tableaux T_1, T_2 of shape $\lambda/\lambda_{(3)}$ and weight $\mu = (1^9)$.

To define k-ribbon tableaux of general weight and shape, we need some terminology. The *initial cell* of a k-ribbon R is its rightmost and bottommost cell. Let $\theta = \beta/\alpha$ be a skew shape, and set $\alpha_+ = (\beta_1) \vee \alpha$, so that α_+/α is the horizontal strip made of the bottom cells of the columns of θ. We say that θ is a *horizontal k-ribbon strip* of weight m, if it can be tiled by m k-ribbons the initial cells of which lie in α_+/α. (One can check that if such a tiling exists, it is unique).

Now, a *k-ribbon tableau* T of shape λ/ν and weight $\mu = (\mu_1, \ldots, \mu_r)$ is defined as a chain of partitions

$$\nu = \alpha^0 \subset \alpha^1 \subset \cdots \subset \alpha^r = \lambda$$

such that α^i/α^{i-1} is a horizontal k-ribbon strip of weight μ_i. Graphically, T may be described by numbering each k-ribbon of α^i/α^{i-1} with the number i. We denote by $\mathrm{Tab}_k(\lambda/\nu,\, \mu)$ the set of k-ribbon tableaux of shape λ/ν and weight μ, and we set

$$K^{(k)}_{\lambda/\nu,\,\mu} = |\mathrm{Tab}_k(\lambda/\nu,\, \mu)| \; .$$

Finally we recall the definition of the k-sign $\epsilon_k(\lambda/\nu)$. Define the sign of a ribbon R as $(-1)^{h-1}$, where h is the height of R. The k-sign $\epsilon_k(\lambda/\nu)$ is the product of the signs of all the ribbons of a k-ribbon tableau of shape λ/ν (this does not depend on the particular tableau chosen, but only on the shape).

The origin of these combinatorial definitions is best understood by analyzing carefully the operation of multiplying a Schur function s_ν by a plethysm of the form $p_k \circ h_\mu$. Equivalently, thanks to the involution ω, one may rather consider a product of the type $s_\nu\,[p_k \circ e_\mu]$. To this end, since

$$p_k \circ e_\mu = (e_{\mu_1} \circ p_k) \cdots (e_{\mu_n} \circ p_k) = m_{k^{\mu_1}} \cdots m_{k^{\mu_n}}$$

one needs only to apply repeatedly the following multiplication rule due to Muir [**Mu**] (see also [**Li3**]):

$$s_\nu\, m_\alpha = \sum_\beta s_{\nu+\beta}\ ,$$

sum over all distinct permutations β of $(\alpha_1, \alpha_2, \dots, \alpha_n, 0, \dots)$. Here, as in the right-hand side of (6), the Schur functions $s_{\nu+\beta}$ are not necessary indexed by partitions and have therefore to be standardized, this reduction yielding only a finite number of nonzero summands. For example,

$$s_{31}\, m_3 = s_{61} + s_{313} + s_{31003} = s_{61} - s_{322} + s_{314}\ .$$

Other terms such as s_{34} or s_{3103} reduce to 0. It is easy to deduce from this rule that the multiplicity

$$\langle s_\nu\, m_{k^{\mu_i}}\ ,\ s_\lambda \rangle$$

is nonzero iff λ'/μ' is a horizontal k-ribbon strip of weight μ_i, in which case it is equal to $\epsilon_k(\lambda/\nu)$. Hence, applying ω we arrive at the expansion

$$s_\nu\,[p_k \circ h_\mu] = \sum_\lambda \epsilon_k(\lambda/\mu)\, K^{(k)}_{\lambda/\nu\,,\mu}\, s_\lambda$$

from which we deduce by 9.1, 9.2 that

$$K^{(k)}_{\lambda\,\mu} = (-1)^{(k-1)|\mu|}\, \epsilon_k(\lambda)\, K_{\lambda\,\mu^k}(\zeta)$$

and more generally, defining as in [**KR**] the skew Kostka-Foulkes polynomial $K_{\lambda/\nu\,,\alpha}(q)$ by

$$K_{\lambda/\nu\,,\alpha}(q) = \langle s_{\lambda/\nu}\ ,\ Q'_\alpha(q)\rangle$$

we can write

$$K^{(k)}_{\lambda/\nu\,,\mu} = (-1)^{(k-1)|\mu|}\, \epsilon_k(\lambda/\nu)\, K_{\lambda/\nu\,,\mu^k}(\zeta)\ .$$

It turns out that enumerating k-ribbon tableaux is equivalent to enumerating k-uples of ordinary Young tableaux, as shown by the correspondence to be described now. This bijection was first studied by Stanton and White [**SW**] in the case of ribbon tableaux of right shape λ (without k-core) and standard weight $\mu = (1^n)$ (see also [**FS**]). We need some additional definitions.

Let R be a k-ribbon of a k-ribbon tableau. R contains a unique cell with coordinates $(x,\, y)$ such that $y - x \equiv 0 \pmod{k}$. We decide to write in this cell the number attached to R, and we define the *type* $i \in \{0,\, 1,\, \dots,\, k-1\}$ of R as the distance between this cell and the initial cell of R. For example, the 3-ribbons of T_1 are divided up into three classes:

- 4, 6, 8, of type 0;
- 1, 2, 7, 9, of type 1;
- 3, 5, of type 2.

Define the *diagonals* of a k-ribbon tableau as the sequences of integers read along the straight lines $D_i : y - x = ki$. Thus T_1 has the sequence of diagonals

$$((8),\, (4),\, (2,\, 3,\, 6),\, (1,\, 5,\, 9),\, (7)) \ .$$

This definition applies in particular to 1-ribbon tableaux, *i.e.* ordinary Young tableaux. It is obvious that a Young tableau is uniquely determined by its sequence of diagonals. Hence, we can associate to a given k-ribbon tableau T of shape λ/ν a k-uple $(t_0,\, t_1,\, \ldots,\, t_{k-1})$ of Young tableaux defined as follows; the diagonals of t_i are obtained by erasing in the diagonals of T the labels of all the ribbons of type $\neq i$. For instance, if $T = T_1$ the first ribbon tableau of Figure 2, the sequence of diagonals of t_1 is $((2),\, (1,\, 9),\, (7))$, and

$$t_1 = \begin{array}{|c|c|}\hline 2 & 9 \\ \hline 1 & 7 \\ \hline \end{array}$$

The complete triple $(t_0,\, t_1,\, t_2)$ of Young tableaux associated to T_1 is

$$\tau^1 = \left(\begin{array}{|c|c}\cline{1-1} 8 & \\ \hline 4 & \multicolumn{1}{c|}{6} \\ \hline \end{array}\ ,\ \begin{array}{|c|c|}\hline 2 & 9 \\ \hline 1 & 7 \\ \hline \end{array}\ ,\ \begin{array}{|c|c|}\hline 3 & 5 \\ \hline \end{array} \right)$$

whereas that corresponding to T_2 is

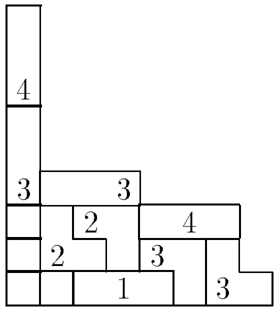
$$\tau^2 = \left(\begin{array}{|c|c}\cline{1-1} 3 & \\ \hline 1 & \multicolumn{1}{c|}{8} \\ \hline \end{array}\ ,\ \begin{array}{|c|c|}\hline 6 & 9 \\ \hline 4 & 5 \\ \hline \end{array}\ ,\ \begin{array}{|c|c|}\hline 2 & 7 \\ \hline \end{array} \right)$$

One can show that if $\nu = \lambda_{(k)}$, the k-core of λ, the k-uple of shapes $(\lambda^0,\, \lambda^1,\, \ldots,\, \lambda^{k-1})$ of $(t_0,\, t_1,\, \ldots,\, t_{k-1})$ depends only on the shape λ of T, and is equal to the k-quotient $\lambda^{(k)}$ of λ. Moreover the correspondence $T \longrightarrow (t_0,\, t_1,\, \ldots,\, t_{k-1})$ establishes a bijection between the set of k-ribbon tableaux of shape $\lambda/\lambda_{(k)}$ and weight μ, and the set of k-uples of Young tableaux of shapes $(\lambda^0,\, \ldots,\, \lambda^{k-1})$ and weights $(\mu^0,\, \ldots,\, \mu^{k-1})$ with $\mu_i = \sum_j \mu_i^j$. (See [**SW**] or [**FS**] for a proof in the case when $\lambda_{(k)} = (0)$ and $\mu = (1^n)$).

For example, keeping $\lambda = (8,\, 7^2,\, 4,\, 1^5)$, the triple

$$\tau = \left(\begin{array}{|c|c}\cline{1-1} 4 & \\ \hline 3 & \multicolumn{1}{c|}{3} \\ \hline \end{array}\ ,\ \begin{array}{|c|c|}\hline 2 & 4 \\ \hline 1 & 3 \\ \hline \end{array}\ ,\ \begin{array}{|c|c|}\hline 2 & 3 \\ \hline \end{array} \right)$$

with weights $((0,\, 0,\, 2,\, 1),\, (1,\, 1,\, 1,\, 1),\, (0,\, 1,\, 1,\, 0))$ corresponds to the 3-ribbon tableau

$T =$ 

of weight $\mu = (1,\,2,\,4,\,2)$.

As before, the significance of this combinatorial construction becomes clearer once interpreted in terms of symmetric functions. Recall the definition of ϕ_k, the adjoint of the linear operator $\psi^k:\ F \mapsto p_k \circ F$ acting on the space of symmetric functions. In other words, ϕ_k is characterized by

$$\langle \phi_k(F)\,,\,G\rangle = \langle F\,,\,p_k \circ G\rangle\ , \quad F,\,G \in Sym\ .$$

Littlewood has shown [**Li3**] that if λ is a partition whose k-core $\lambda_{(k)}$ is null, then

$$\phi_k(s_\lambda) = \epsilon_k(\lambda)\, s_{\lambda^0}\, s_{\lambda^1}\,\cdots\, s_{\lambda^{k-1}} \tag{39}$$

where $\lambda^{(k)} = (\lambda^0,\,\ldots\,,\,\lambda^{k-1})$ is the k-quotient. Therefore,

$$K^{(k)}_{\lambda\,\mu} = \epsilon_k(\lambda)\,\langle p_k \circ h_\mu\,,\,s_\lambda\rangle = \epsilon_k(\lambda)\,\langle \phi_k(s_\lambda)\,,\,h_\mu\rangle = \langle s_{\lambda^0}\, s_{\lambda^1}\,\cdots\, s_{\lambda^{k-1}}\,,\,h_\mu\rangle$$

is the multiplicity of the weight μ in the product of Schur functions $s_{\lambda^0}\cdots s_{\lambda^{k-1}}$, that is, is equal to the number of k-uples of Young tableaux of shapes $(\lambda^0,\,\ldots\,,\,\lambda^{k-1})$ and weights $(\mu^0,\,\ldots\,,\,\mu^{k-1})$ with $\mu_i = \sum_j \mu_i^j$. Thus, the bijection described above gives a combinatorial proof of (39).

More generally, if λ is replaced by a skew partition λ/ν, (39) becomes [**KSW**]

$$\phi_k(s_{\lambda/\nu}) = \epsilon_k(\lambda/\nu)\, s_{\lambda^0/\nu^0}\, s_{\lambda^1/\nu^1}\,\cdots\, s_{\lambda^{k-1}/\nu^{k-1}}$$

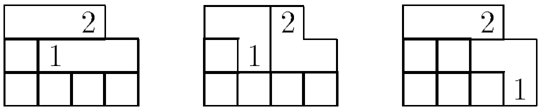
if $\lambda_{(k)} = \nu_{(k)}$, and 0 otherwise. This can also be deduced from the previous combinatorial correspondence, but we shall not go into further details.

Returning to Kostka polynomials, we may summarize this discussion by stating Theorems 9.1 and 9.2 in the following way:

Theorem 9.17 *Let λ and ν be partitions and set $\nu = \mu^k \vee \alpha$ with $m_i(\alpha) < k$. Denoting by ζ a primitive kth root of unity, one has*

$$K_{\lambda,\,\nu}(\zeta) = (-1)^{(k-1)|\mu|} \sum_\beta \epsilon_k(\lambda/\beta)\, K^{(k)}_{\lambda/\beta,\,\mu}\, K_{\beta,\,\alpha}(\zeta)\ . \tag{40}$$

Example 9.18 We take $\lambda = (4^2,\,3)$, $\nu = (2^2,\,1^7)$ and $k = 3$ ($\zeta = e^{2i\pi/3}$). In this case, $\nu = \mu^k \vee \alpha$ with $\mu = (1^2)$ and $\alpha = (2^2,\,1)$. The summands of (40) are parametrized by the 3-ribbon tableaux of external shape λ and weight μ. Here we have three such tableaux:



so that

$$K_{443,\,221111111}(\zeta) = 2K_{41,\,221}(\zeta) - K_{32,\,221}(\zeta) = 2(\zeta^2+\zeta^3) - (\zeta+\zeta^2) = 2\zeta^2 + 3 \ .$$

When $|\alpha| \le |\lambda_{(k)}|$, (40) becomes simpler. For if $|\alpha| < |\lambda_{(k)}|$ then $K_{\lambda,\,\nu}(\zeta) = 0$, and otherwise the sum reduces to one single term

$$K_{\lambda,\,\nu}(\zeta) = (-1)^{(k-1)|\mu|}\epsilon_k(\lambda/\lambda_{(k)})\, K^{(k)}_{\lambda/\lambda_{(k)},\,\mu}\, K_{\lambda_{(k)},\,\alpha}(\zeta) \ .$$

In particular, if $\nu = (1^n)$, one recovers the following theorem of Morris and Sultana [**MS**].

Theorem 9.19 *Let λ be a partition of n and ζ a primitive kth root of unity. Denote by $H(\lambda^{(k)})$ the product of the hook-lengths of the k partitions $\lambda^0, \ldots, \lambda^{k-1}$, and by $|\lambda^{(k)}|$ the sum of their weights. Set $n = kq + r$, $0 \le r < k$. If $r \ne |\lambda_{(k)}|$, then $K_{\lambda,\,(1^n)}(\zeta) = 0$, otherwise,*

$$K_{\lambda,\,(1^n)}(\zeta) = (-1)^{(k-1)q}\epsilon_k(\lambda/\lambda_{(k)})\,\frac{|\lambda^{(k)}|!}{H(\lambda^{(k)})}\, K_{\lambda_{(k)},\,1^r}(\zeta) \ .$$

Indeed, the correspondence we have just described between k-ribbon tableaux and k-uples of Young tableaux shows at once, in view of the classical hook-formula [**JK**], that

$$K^{(k)}_{\lambda/\lambda_{(k)},\,1^q} = \frac{|\lambda^{(k)}|!}{H(\lambda^{(k)})} \ .$$

10 Schur P-functions

As mentionned before, the specialization of Hall-Littlewood functions at $q = -1$ is of particular interest, since, for a strict partition λ in $\pi_2(n)$, the symmetric functions $P_\lambda(X;-1)$ and $Q_\lambda(X;-1)$ coincide with those introduced by Schur in his study of the projective representations of $\mathbf{S}_n$.

A strict partition λ of length m can be represented in the form $\lambda = \rho_m + \nu$, where $\rho_m = (m,\, m-1,\, \ldots,\, 1)$. It is then known that, if X is a finite set of variables of cardinality $\le m+1$, the following factorization holds:

$$P_{\rho_m+\nu}(X;-1) = P_{\rho_m}(X;-1)\, s_\nu(X) = s_{\rho_m}(X)\, s_\nu(X) \ . \tag{41}$$

This formula due to Stanley has been used by Pragacz to show that the $\mathbf{Z}$-linear combinations of Schur P-functions are characterized among symmetric polynomials by a cancellation property [**Pr**]. The reader is referred to [**Pr**] for some geometric applications.

Another interesting property of P-functions is the staircase multiplication formula [**W**]

$$P_{\rho_r}\, P_{\rho_s} = P_{\rho_r+\rho_s} \ , \tag{42}$$

valid irrespective of the number (finite or infinite) of variables. The following identity, established in [**LLT3**], provides a common generalization of (41) and (42). It gives a quadratic expression of any P-function in terms of the S-functions.

Theorem 10.1 *Let λ be a partition of length at most n. Then*

$$P_{\rho_n+\lambda} = \sum_{\mu} \omega\left(s_{\rho_n+2\mu}\right) D_{p_2 \circ s_\mu} s_\lambda \tag{43}$$

the summation being over all partitions μ.

Example 10.2 With $\lambda = (4, 3, 1)$ we have

$$P_{752} = P_{(321)+(431)} = s_{321}\, s_{431} + s_{32111}\, s_{411} + s_{3211111}\, (-s_4) + s_{32221}\, s_{211}$$
$$+ s_{3222111}\, (-s_2) + s_{3222221}\, (-s_0)\ .$$

When $\lambda = \rho_m$ it is well known that

$$\frac{\partial}{\partial p_{2i}} s_\lambda = 0$$

for all i, so that $D_{p_2 \circ s_\mu} s_\lambda = 0$ for $\mu \neq 0$, and since $s_\lambda = P_{\rho_m}$, (43) reduces to (42). Also, when dealing with a set of variables X of cardinality $n+1$, one has

$$\omega\left(s_{\rho_n+2\mu}\right)(X) = 0$$

as soon as $\mu \neq 0$, and (43) reduces to (41).

Other interesting special cases can be explicited. For example, when $\lambda = (i)$ is a row partition,

$$P_{\rho_n+(i)} = \sum_{0 \leq j \leq [i/2]} \omega(s_{\rho_n+(2j)})\, h_{i-2j}\ . \tag{44}$$

More generally, in the case where $\lambda = \rho_m + 2\nu$ with $\ell(\nu) \leq m$, the derivatives can be expressed as

$$D_{p_2 \circ s_\mu}\, s_{\rho_m+2\nu} = \sum_{\alpha} \langle s_\nu\ ,\ s_\mu\, s_\alpha \rangle\, s_{\rho_m+2\alpha} \tag{45}$$

so that, for $\ell(\nu) \leq m \leq n$,

$$P_{\rho_n+\rho_m+2\nu} = \sum_{\mu,\,\alpha} c^{\nu}_{\mu,\,\alpha}\, \omega(s_{\rho_n+2\mu})\, s_{\rho_m+2\alpha} \tag{46}$$

where the $c^{\nu}_{\mu,\,\alpha}$ are the Littlewood-Richardson multiplicities.

In the general case, the multiplicity

$$\langle D_{p_2 \circ s_\mu}\, s_\lambda\ ,\ s_\gamma \rangle = \langle s_\lambda\ ,\ s_\gamma\, [p_2 \circ s_\mu] \rangle$$

of the product $\omega(s_{\rho_n+2\mu})\, s_\gamma$ in (43) can be combinatorially described by means of an analogue of the Littlewood-Richardson rule, in which the usual Young tableaux are to be replaced by 2-ribbons tableaux (or *domino tableaux*) [**CL**].

Surprisingly, the Schur P-functions happen to be connected to a family of orthogonal polynomials known as *Bessel polynomials*. These polynomials can be defined as follows. The continued fraction expansion of $\tanh z^{-1}$ has for successive convergents

$$F_1 = \frac{1}{z}\ ,\quad F_2 = \frac{3z}{3z^2+1}\ ,\quad F_3 = \frac{15z^2+1}{15z^3+6z}\ ,\quad F_4 = \frac{105z^3+10z}{105z^4+45z^2+1}\ ,\ \ldots$$

The sum of the numerator and denominator of F_n is a polynomial $y_n(z)$ of degree n, called the nth Bessel polynomial.

A symmetric function $F(X;T)$ (where T is some set of variables) is said to be a symmetric analogue of a formal power series $f(T)$ if there exists a specialization $X = A$ of X such that $F(A;T) = f(T)$. The set of variables A can be *virtual* which means that the specialization is defined by specifying (in an arbitrary way) the values of a sequence of algebraically independent generators of *Sym*. For example, $\sigma_t(X)$ is a symmetric analogue of the exponential function $e^t = \sigma_t(E)$, where the virtual set E is defined by $h_n(E) = 1/n!$, or equivalently by $p_1(E) = 1$ and $p_k(E) = 0$ for $k > 1$.

A more substantial example is given by the symmetric Eulerian polynomials $\mathcal{A}_n(X;t)$, with generating series [**De2**]

$$\mathcal{A}_X(t) = \sum_{n\geq 0} \mathcal{A}_n(X;t) = \frac{1-t}{1-t\,\sigma_{1-t}(X)} \ .$$

The usual Eulerian polynomials are then given by

$$A_n(t) = n!\,\mathcal{A}_n(E;t) \ .$$

The Schur P or Q-functions provide symmetric analogues of the Bessel polynomials. Set once for all $q = -1$, and

$$Y_n(X;z) = \frac{Q_{\rho_n}(X+z)}{Q_{\rho_n}(X)}$$

(in λ-ring notation). Then the vertex operator formula (23) shows that if μ is the linear functional defined by

$$\mu(z^n) = (-1)^{n+1} Q_{n+1}(X) \ ,$$

then [**LT**],

Theorem 10.3 *1. The $Y_n(X;z)$ are orthogonal for μ, and more precisely,*

$$\mu(z^k Y_n(X;z)) = (-1)^{k+1} \frac{Q_{\rho_n \vee (k+1)}(X)}{Q_{\rho_n}(X)} \ ,$$

which is zero for $0 \leq k \leq n-1$.

2. $Y_n(E;z) = y_n(z)$

The staircase multiplication formula (42) written in the form

$$Q_{\rho_m}\, Q_{\rho_n} = 2^m\, Q_{\rho_m+\rho_n}$$

($m \leq n$), gives in view of (23)

$$\mu(z^k\, Y_m(X;z)\, Y_n(X;z)) = (-1)^{k+1} \frac{Q_{\rho_m+\rho_n \vee (k+1)}(X)}{Q_{\rho_m+\rho_n}(X)} \ ,$$

and specializing again to $X = E$, one obtains a formula conjectured by Favreau in his thesis [**Fa**] for computing the linearization coefficients of Bessel polynomials. (This was proved independently by J. Zeng).

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