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ENUMERATIVE APPLICATIONS OF SYMMETRIC FUNCTIONS

BY

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1. Introduction. — This paper consists of two related parts. In the first part the theory of D-finite power series in several variables and the theory of symmetric functions are used to prove P-recursiveness for regular graphs and digraphs and related objects, that is, that their counting sequences satisfy linear homogeneous recurrences with polynomial coefficients. Previously this has been accomplished only for small degrees. See, for example, GOULDEN, JACKSON, and REILLY [7], GOULDEN and JACKSON [6], and READ [16, 18]. These authors found the recurrences satisfied by the sequences in question. Although the methods used here are in principle constructive, we are concerned here only with the question of existence of these recurrences and we do not find them.

In the second part we consider a generalization of symmetric functions in several sets of variables, first studied by MACMAHON [13; 14, Vol. 2, pp. 280–326]. MacMahon's generalized symmetric functions can be used to find explicit formulas and prove P-recursiveness for some objects to which the theory of ordinary symmetric functions does not apply, such as Latin rectangles and 0-1 matrices with zeros on the diagonal and given row and column sums.

I. Symmetric functions and P-recursiveness

2. D-finite power series and P-recursive functions. — A formal power series $f(x)$ is said to be *D-finite* (or *differentiably finite*) if f satisfies a linear homogeneous differential equation with polynomial coefficients. An equivalent condition is that the set of derivatives of f spans a finite-dimensional vector space over the field of rational functions in x. A function $a(n)$ defined on the nonnegative integers is said to be *P-recursive* (or

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polynomially recursive) if there exist polynomials $p_0(n)$, $p_1(n)$, ..., $p_k(n)$ such that

$$\sum_{i=0}^{k} p_i(n)a(n+i) = 0$$

for all nonnegative integers n.

The fundamental fact relating these two concepts is that $a(n)$ is P-recursive if and only if its generating function $\sum_{n=0}^{\infty} a(n)x^n$ is D-finite. We refer the reader to STANLEY [20] for the proof of this and other basic facts.

In this paper we show that counting sequences for certain combinatorial problems which can be expressed as coefficients of symmetric functions are P-recursive. To do this we need a multivariable generalization of the theory of D-finiteness and P-recursiveness. Such a generalization was first given by ZEILBERGER [22]. However, Zeilberger's definition of multivariable P-recursiveness is not suitable for our purposes. A more useful definition of multivariable P-recursiveness has been given by LIPSHITZ [10], but we shall work only with multivariable D-finiteness.

The theory of D-finiteness generalizes easily to the multivariable case. In the next section we define multivariable D-finiteness and describe some of its properties.

3. D-finite power series in several variables. — First we discuss the theory of D-finite power series. Let I be an integral domain and let F be the quotient field of $I[x_1, x_2, \ldots, x_n]$. Let $f(x_1, x_2, \ldots, x_n)$ be a formal power series in $I[[x_1, x_2, \ldots, x_n]]$. We say that $f(x)$ is *D-finite* in the variables x_1, x_2, ..., x_n if the set of all partial derivatives $\dfrac{\partial^{i_1+\cdots+i_n} f}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}}$ spans a finite-dimensional vector space over F (as a subspace of the tensor product $F \otimes_{I[x_1,\ldots,x_n]} I[[x_1, x_2, \ldots, x_n]]$).

The following lemma contains some of the basic facts about D-finite power series in several variables that we will need :

LEMMA 1.

(i) *The set of D-finite power series forms an I-subalgebra of $I[[x_1, \ldots, x_n]]$.*

(ii) *If f is D-finite in $x_1, x_2, \ldots, x_n$ then f is D-finite in any subset of $x_1, x_2, \ldots, x_n$.*

(iii) *If $f(x_1, x_2, \ldots, x_n)$ is D-finite in $x_1, x_2, \ldots, x_n$ and for each i, r_i is a polynomial in the variables y_1, y_2, ..., y_m, (which may include some or all of the x_i) then $f(r_1, r_2, \ldots, r_n)$ is D-finite in y_1, y_2, ..., y_m, as long as it is well-defined as a formal power series.*

(iv) *If $P(x)$ is a polynomial in $x_1, x_2, \ldots, x_n$ then $e^{P(x)}$ is D-finite.*

The proofs of these statements are straightforward, and are similar to proofs for the one-variable case given by STANLEY [20]. (See also LIPSHITZ [10].) We need one further fact about D-finite power series in several variables, due to LIPSHITZ [9], which is somewhat harder to prove. If $A(x) = \sum a(i_1, \ldots, i_n)x_1^{i_1} \cdots x_n^{i_n}$ and $B(x) = \sum b(i_1, \ldots, i_n)x_1^{i_1} \cdots x_n^{i_n}$, then the Hadamard product $A(x) \odot B(x)$ with respect to the variables $x_1, x_2, \ldots, x_n$ is defined to be

$$\sum a(i_1, \ldots, i_n)b(i_1, \ldots, i_n)x_1^{i_1} \cdots x_n^{i_n}.$$

Note that the a's and b's may involve other variables.

LEMMA 2 (LIPSHITZ [9]). — *Suppose that A and B are D-finite in the variables $x_1, x_2, \ldots, x_{m+n}$. Then the Hadamard product $A \odot B$ with respect to the variables $x_1, x_2, \ldots, x_n$ is D-finite in $x_1, x_2, \ldots, x_{m+n}$.*

Now suppose that f is a formal power series in an infinite set X of variables. For any subset S of X let f_S be the formal power series in the variables in S obtained by setting to zero all the variables in $X - S$. We shall say that f is D-finite in X if f_S is D-finite in S for every finite subset S of X. With this definition, all the properties of D-finite series in finitely many variables are easily seen to remain valid, except that in Lemma 1(iii) we may only substitute for finitely many variables.

4. Symmetric functions. — We recall some facts about symmetric functions. We refer the reader to MACDONALD [12] for proofs and details.

We work with symmetric functions in the infinitely many variables $x_1, x_2, \ldots$ with coefficients in a field of characteristic zero. We will be concerned with the following particular symmetric functions :

The *power sum* symmetric function p_n is defined by

$$p_n = \sum_i x_i^n.$$

More generally, if $\lambda = \lambda_1 \lambda_2 \cdots \lambda_k$ is a partition, we define $p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_k}$.

The *elementary* symmetric function e_n is defined by

$$e_n = \sum_{i_1 < i_2 < \cdots < i_n} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

If $\lambda = \lambda_1 \lambda_2 \cdots \lambda_k$ is a partition, we define $e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_k}$.

The *complete* symmetric function h_n is defined by

$$h_n = \sum_{i_1 \leq i_2 \leq \cdots \leq i_n} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

It is convenient to define h_λ to be $h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_k}$ for any sequence $\lambda = \lambda_1\lambda_2\cdots\lambda_k$ of nonnegative integers, not necessarily a partition. We set $h = \sum_{n=0}^\infty h_n$ and $e = \sum_{n=0}^\infty e_n$, where $h_0 = e_0 = 1$.

The *monomial* symmetric function m_λ is the sum of all distinct monomials of the form $x_{i_1}^{\lambda_1}\cdots x_{i_k}^{\lambda_k}$, where $i_1, \ldots, i_k$ are distinct.

It is known that each of the sets $\{e_\lambda\}$, $\{h_\lambda\}$, $\{p_\lambda\}$, and $\{m_\lambda\}$, where λ ranges over all partitions of n, is a basis for the vector space of symmetric functions homogeneous of degree n.

If λ has r_i parts equal to i for each i, then we define z_λ to be $1^{r_1}2^{r_2}\cdots k^{r_k}\, r_1!\, r_2! \cdots r_k!$.

There is a symmetric scalar product $\langle\ ,\ \rangle$ defined on symmetric functions that has the following properties :

$$\langle m_\lambda, h_\mu\rangle = \delta_{\lambda\mu} \tag{4.1}$$

and

$$\langle p_\lambda, p_\mu\rangle = z_\lambda \delta_{\lambda\mu} \tag{4.2}$$

where $\delta_{\lambda\mu}$ is 1 if $\lambda = \mu$ and 0 otherwise.

This scalar product was introduced by Redfield [19] in 1927 in his then-ignored but now-famous paper on what later became known as Pólya theory. Redfield called it the “cap product.” The scalar product was rediscovered by Hall [8] in 1957 and is often attributed to him. It is equivalent to the usual scalar product on characters of symmetric groups.

Note that (4.1) implies that if f is a symmetric function, then the coefficient of $x_{i_1}^{\lambda_1}x_{i_2}^{\lambda_2}\cdots x_{i_k}^{\lambda_k}$ in f is $\langle f, h_\lambda\rangle$. To evaluate scalar products of symmetric functions, we shall express them in terms of power sum symmetric functions and use (4.2). Thus, we need to express the complete homogeneous symmetric functions in terms of power sum symmetric functions, and this is accomplished by the formula

$$\sum_{n=0}^\infty h_n = \exp\left(\sum_{k=1}^\infty \frac{p_k}{k}\right), \tag{4.3}$$

which implies that

$$h_n = \sum_\lambda \frac{p_\lambda}{z_\lambda}$$

where the sum is over all partitions λ of n.

Next we recall the operation of *internal* (also called *inner*) product on symmetric functions which is defined by

$$p_\lambda * p_\mu = \delta_{\lambda\mu} z_\lambda p_\lambda \tag{4.4}$$

and extended by linearity to all symmetric functions. The internal product was discovered by REDFIELD [19] in 1927, who called it the "cup product," and it was rediscovered by LITTLEWOOD [11] in 1956. It is equivalent to pointwise multiplication of characters of symmetric groups, which corresponds to the tensor (or Kronecker) product of representations.

5. D-finite symmetric functions. — We shall say that a symmetric function is D-finite if it is D-finite when considered as a power series in the p_n. We shall show that functions obtained from coefficients of D-finite symmetric functions are P-recursive.

THEOREM 3. — *Suppose that f and g are symmetric functions which are D-finite in the p_i and possibly in some other variables. Then $f * g$ is D-finite in these variables.*

Proof. — Note that $f*g = f \odot g \odot u$, where $\odot$ is the Hadamard product in the p_i and u is the symmetric function given by

$$u = \sum_{\lambda} z_\lambda p_\lambda,$$

where the sum is over all partitions λ. Now

$$\begin{aligned} u &= \sum_{r_1, r_2, \dots} 1^{r_1} 2^{r_2} \cdots r_1!\, r_2! \cdots p_1^{r_1} p_2^{r_2} \cdots \\ &= \left(\sum_{r_1} r_1!\,(1p_1)^{r_1}\right)\left(\sum_{r_2} r_2!\,(2p_2)^{r_2}\right)\cdots \\ &= A(1p_1)A(2p_2)\cdots, \end{aligned}$$

where $A(y) = \sum_{n=0}^{\infty} n!\, y^n$. Since u is easily seen to be D-finite, $f * g$ is a Hadamard product of three D-finite power series, and is thus D-finite by Lipshitz's theorem.

COROLLARY 4. — *Let f and g be symmetric functions which are D-finite in the p_i and in another variable t, and suppose that g involves only finitely many of the p_i. Then $\langle f, g\rangle$ is D-finite in t as long as it is well-defined as a formal power series.*

Proof. — By the previous theorem, $f * g$ is D-finite in the p_i and in t, and involves only finitely many of the p_i. Then $\langle f, g\rangle$ is obtained from $f * g$ by setting each p_i equal to 1, and thus the conclusion follows from Lemma 1(iii).

Note that without the restriction on g the theorem would not be true : according to our definitions, $\sum_{n=0}^{\infty} c_n p_n$ is D-finite for any coefficients c_n

and thus any power series in t can be obtained as a scalar product of two D-finite symmetric functions.

COROLLARY 5. — *Let f be a D-finite symmetric function and let S be a finite set of integers. Define integers b_n as follows : b_n is the sum over all n-tuples $(\lambda_1, \lambda_2, \dots, \lambda_n) \in S^n$ of the coefficient of $x_1^{\lambda_1} \cdots x_n^{\lambda_n}$ in f. Then $b(t) = \sum_{n=0}^{\infty} b_n t^n$ is D-finite.*

Proof. — The coefficient of $x_1^{\lambda_1} \cdots x_n^{\lambda_n}$ in f is $\langle f, h_\lambda \rangle$, and so $b(t) = \langle f, g \rangle$, where

$$g = \sum_{n=0}^{\infty} \left(t \sum_{i \in S} h_i \right)^n = \left(1 - t \sum_{i \in S} h_i \right)^{-1}$$

and the assertion follows from the previous corollary.

In particular, it will follow that the generating functions for various types of graphs and hypergraphs on n vertices whose degrees are constrained to a finite set are D-finite. This proves a conjecture of GOULDEN and JACKSON [6].

Next we need to consider the operation of *composition* (also called *plethysm*) for symmetric functions. First, suppose that g is a symmetric function which can can be expressed in the form $t_1 + t_2 + \cdots$, where each t_i is of the form $x_1^{i_1} x_2^{i_2} \cdots x_k^{i_k}$. (The terms t_i need not be distinct.) Then for any symmetric function $f = f(x_1, x_2, \dots)$ the composition $f(g)$ is defined to be $f(t_1, t_2, \dots)$.

In the general case, composition may be defined as follows : If f_1 and f_2 are symmetric functions then $(f_1 + f_2)(g) = f_1(g) + f_2(g)$ and $(f_1 f_2)(g) = f_1(g) f_2(g)$ so it is sufficient to define $p_n(g)$. This is accomplished by the formula $p_n(g) = g(p_n)$, where $g(p_n)$ is determined by the special case given in the previous paragraph, or by the formula $p_m(p_n) = p_{mn}$.

THEOREM 6. — *Suppose that g is a polynomial in the p_n. Then $h(g)$ is D-finite.*

Proof. — To show that $h(g)$ is D-finite we need only show that $h(g)$ is D-finite in the variables $p_1, p_2, \dots, p_n$ for each n.

By (4.3) we have

$$\begin{aligned} h(g) &= \exp\left(\sum_{k=1}^{\infty} \frac{p_k(g)}{k} \right) = \exp\left(\sum_{k=1}^{\infty} \frac{g(p_k)}{k} \right) \\ &= \exp\left(\sum_{k=1}^{n} \frac{g(p_k)}{k} \right) \exp\left(\sum_{k=n+1}^{\infty} \frac{g(p_k)}{k} \right), \end{aligned}$$

where the second factor on the right does not involve $p_1, p_2, \dots, p_n$. Then by Lemma 1(iv), $h(g)$ is D-finite in $p_1, p_2, \dots, p_n$.

The same reasoning shows that $e(g)$ is also D-finite, where $e = \sum_{n=0}^{\infty} e_n$.

Let us now give some examples of Theorem 6. Consider the products

$$h(h_2) = \prod_{i \le j} \frac{1}{1 - x_i x_j} \tag{5.1}$$

$$h(e_2) = \prod_{i < j} \frac{1}{1 - x_i x_j} \tag{5.2}$$

$$e(h_2) = \prod_{i \le j} (1 + x_i x_j) \tag{5.3}$$

$$e(e_2) = \prod_{i < j} (1 + x_i x_j). \tag{5.4}$$

By Theorem 6, they are all D-finite. Each counts a class of graphs. Thus the coefficient of $x_1^{\lambda_1} x_2^{\lambda_2} \cdots$ in (5.1) is the number of graphs on the vertex set $\{1, 2, \ldots\}$, with multiple edges and loops allowed, such that the degree of vertex i is λ_i, where a loop contributes 2 to the degree of its vertex. Similarly, (5.2) counts graphs with multiple edges but no loops, (5.3) counts graphs with loops allowed, but not multiple edges, and (5.4) counts graphs without loops or multiple edges. Graphs with loops in which a loop contributes only 1 to the degree of its vertex are counted by $h(e_1 + e_2)$ (multiple edges allowed) and $e(e_1 + e_2)$ (multiple edges not allowed). Similarly, k-uniform hypergraphs are counted by $e(e_k)$, and so on.

6. Symmetric functions in several sets of variables. — In some applications it is necessary to work with symmetric functions in two or more sets of variables. For simplicity, we consider here only the case of two sets of variables, which we use to count nonnegative integer matrices with prescribed row and column sums (or equivalently, digraphs with prescribed indegrees and outdegrees or two-colored graphs with prescribed degrees).

Let x_1, x_2, ... and y_1, y_2, ... be two disjoint sets of variables. We shall consider power series in these variables which are symmetric in the x's and symmetric in the y's. It is easy to see that such a symmetric function can be expressed in the form

$$\sum_{\lambda,\, \mu} a_{\lambda\mu} p_\lambda(x) p_\mu(y)$$

where $p_\lambda(x)$ means $p_\lambda(x_1, x_2, \ldots)$ and similarly for $p_\mu(y)$. We call these series D-finite if they are D-finite in the $p_i(x)$ and $p_j(y)$.

We may extend the scalar product $\langle\ ,\ \rangle$ to symmetric functions in two sets of variables by setting

$$\langle f_1(x) f_2(y), g_1(x) g_2(y) \rangle = \langle f_1(x), g_1(x) \rangle \langle f_2(y), g_2(y) \rangle.$$

If f is a symmetric function, by $f(xy)$ we mean $f(x_1y_1, x_1y_2, \ldots, x_iy_j, \ldots)$. Thus, for example, we have

$$h(xy) = \prod_{i,j} \frac{1}{1 - x_iy_j}.$$

This product is easily seen to be D-finite using the fact that $p_n(xy) = p_n(x)p_n(y)$. It is clear that the coefficient of $x_1^{\lambda_1} \cdots x_n^{\lambda_n} y_1^{\mu_1} \cdots y_n^{\mu_n}$ in $h(xy)$ is the number of digraphs on $\{1, 2, \cdots, n\}$, with multiple edges allowed, in which vertex i has outdegree λ_i and indegree μ_i, or equivalently, the number of $n \times n$ matrices of nonnegative integers in which the sum of the ith row is λ_i and the sum of the jth column is μ_j. This coefficient is easily seen to be equal to $\langle h(xy), h_\lambda(x)h_\mu(y)\rangle$, which is also equal to $\langle h_\lambda(x), h_\mu(x)\rangle$.

Now let b_n be the number of $n \times n$ nonnegative integer matrices with every row and column sum equal to k. It follows that

$$b(t) = \sum_{n=0}^{\infty} b_n t^n = \langle h(xy), g(x, y)\rangle, \tag{6.1}$$

where $g(x, y)$ is given by $g(x, y) = \big(1 - t h_k(x)h_k(y)\big)^{-1}$, and by reasoning as before, $b(t)$ is D-finite. Similarly, $e(xy)$ counts 0-1 matrices with prescribed row and column sums, or equivalently, digraphs without multiple edges with prescribed indegrees and outdegrees.

7. Explicit formulas and asymptotics. — In all of our examples, we have actually shown something stronger than P-recursiveness—we have shown that there exists an explicit formula for the numbers in question as a sum of fixed multiplicity. (In the terminology of Zeilberger, these sums are "multi-hypergeometric.") Although these formulas are complicated, they can be used to derive asymptotic approximations.

For example, the number of $n \times n$ nonnegative integer matrices with every row and column sum two is

$$\begin{aligned}
\langle h(xy), h_2^n(x)h_2^n(y)\rangle = \langle h_2^n, h_2^n\rangle &= \left\langle \left(\frac{p_1^2 + p_2}{2}\right)^n, \left(\frac{p_1^2 + p_2}{2}\right)^n \right\rangle \\
&= 2^{-2n} \sum_{i,j} \binom{n}{i}\binom{n}{j} \langle p_1^{2n-2i} p_2^i, p_1^{2n-2j} p_2^j\rangle \\
&= 2^{-2n} \sum_{i} \binom{n}{i}^2 \langle p_1^{2n-2i} p_2^i, p_1^{2n-2i} p_2^i\rangle \\
&= \sum_i 2^{-(2n-i)} \binom{n}{i}^2 (2n - 2i)!\, i!\,.
\end{aligned}$$

It can be shown that asymptotically we may replace each summand by its limit as $n \to \infty$, and thus the sum is asymptotic to

$$2^{-2n}(2n)! \sum_{i=0}^{\infty} \frac{2^i}{i!} \left(\frac{1}{2}\right)^{2i} = 2^{-2n}(2n)!\, e^{1/2}.$$

A more detailed analysis yields a complete asymptotic expansion.

More generally, the number of $n \times n$ nonnegative integer matrices with every row and column sum k is $\langle h_k^n, h_k^n \rangle$, and it can be shown that the major contribution to this scalar product comes from the terms in

$$\left\langle \left(\frac{p_1^k}{k!} + \frac{p_1^{k-2}}{(k-2)!}\frac{p_2}{2}\right)^n, \left(\frac{p_1^k}{k!} + \frac{p_1^{k-2}}{(k-2)!}\frac{p_2}{2}\right)^n \right\rangle$$
$$= \frac{(kn)!}{k!^{2n}} \sum_{i=0}^{n} \frac{\big(k(k-1)\big)^{2i}}{i!\, 2^i} \frac{\big(n(n-1)\cdots(n-i+1)\big)^2}{kn(kn-1)\cdots(kn-2i+1)}.$$

The sum is asymptotic to

$$\frac{(kn)!}{k!^{2n}} \sum_{i=0}^{\infty} \frac{\big(k^2(k-1)^2/2\big)^i}{i!} \left(\frac{1}{k}\right)^{2i} = \frac{(kn)!}{k!^{2n}} e^{(k-1)^2/2},$$

as found by EVERETT and STEIN [2], who also used symmetric functions. A similar analysis can be used to obtain asymptotic expansions for related problems, since although in general the formulas have many terms, nearly all are asymptotically insignificant.

II. MacMahon's symmetric functions of several systems of quantities

8. MacMahon's symmetric functions. — Some enumeration problems involve generating functions which are almost, but not quite, symmetric. Here are three examples :

Example 1. — The number of $n \times n$ 0-1 matrices with zeros on the diagonal with row sums $r_1, r_2, \ldots, r_m$ and column sums $c_1, c_2, \ldots, c_n$ is the coefficient of $x_1^{r_1} \cdots x_m^{r_m} y_1^{c_1} \cdots y_n^{c_n}$ in

$$\prod_{i \neq j} (1 + x_i y_j).$$

Example 2. — The number of $3 \times n$ Latin rectangles is the coefficient of $x_1 x_2 \cdots x_n y_1 y_2 \cdots y_n z_1 z_2 \cdots z_n$ in

$$\left(\sum_{i,j,k} x_i y_j z_k\right)^n,$$

where the sum is over all triples of distinct integers i, j, k.

Example 3. — Consider the monoid freely generating by the letters $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$, subject only to the commutation relations $a_i b_i = b_i a_i$. By the Cartier-Foata theory of free partially commutative monoids [1], the number of equivalence classes of words in this monoid with u_i occurrences of a_i and v_i occurrences of b_i is the coefficient of $x_1^{u_1} \ldots x_n^{u_n} y_1^{v_1} \ldots y_n^{v_n}$ in

$$(1 - x_1 - \cdots x_n - y_1 - \cdots - y_n + x_1 y_1 + \cdots + x_n y_n)^{-1}.$$

One can show that this coefficient is also the number of words in the letters $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$, with u_i occurrences of a_i and v_i occurrences of b_i, containing no consecutive $a_i b_i$.

These generating functions all have the property that they are symmetric under any permutation of the subscripts which acts the same on x's and y's, i.e., the coefficient of $x_1^u y_2^v$ is equal to the coefficient of $x_a^u y_b^v$ as long as $a \neq b$, but need not be equal to the coefficient of $x_1^u y_1^v$. (The usual theory of symmetric functions in two sets of variables applies to symmetric functions which are symmetric independently in the x's and the y's.) These more general symmetric functions were studied by MacMahon [13; 14, Vol. 2, pp. 280–326] who called them "symmetric functions of several systems of quantities." MacMahon applied to them his favorite tool for manipulating symmetric functions, Hammond operators, and claimed to have solved the problem of counting Latin rectangles with these operators. His work on these symmetric functions seems to have been ignored, and his claimed solution to the problem of counting Latin rectangles dismissed as impractical and useless.

In previous sections we showed how the theory of symmetric functions can be applied to get "useful" formulas from symmetric function generating functions, and in particular, to show that certain sequences are P-recursive. We now do the same for MacMahon's symmetric functions of several systems of quantities, which we henceforth call MacMahon symmetric functions. First we discuss the fundamental bases for MacMahon symmetric functions and the formulas relating them, which are straightforward generalizations of those for ordinary symmetric functions. For simplicity, we discuss here only MacMahon symmetric functions in two sets of variables. The generalization to more than two presents no difficulties.

9. Bases. — We take two sets of variables, $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$. A formal power series f in these variables is a *MacMahon symmetric function* if whenever $i_1, i_2, \ldots, i_n$ are distinct positive integers, and $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are nonnegative integers, the coefficient of $x_{i_1}^{a_1} y_{i_1}^{b_1} x_{i_2}^{a_2} y_{i_2}^{b_2} \cdots$ in f is equal to the coefficient of $x_1^{a_1} y_1^{b_1} x_2^{a_2} y_2^{b_2} \cdots$ in f.

Just as bases for ordinary partitions are indexed by partitions of integers, bases for the MacMahon symmetric functions are indexed by bipartite partitions. A *bipartite number* is an element of $\mathbf{N} \times \mathbf{N} - \{(0,0)\}$, where $\mathbf{N}$ is the set of nonnegative integers. A *bipartite partition* of the bipartite number (a,b) is a multiset of bipartite numbers with (componentwise) sum (a,b). Thus $\{(0,1),(0,1),(1,0)\}$ is a bipartite partition of $(1,2)$. For simplicity we write $\{(0,1),(0,1),(1,0)\}$ as $(0,1)(0,1)(1,0)$ or as $(0,1)^2(1,0)$.

Now if $\lambda = (a_1,b_1)(a_2,b_2)\cdots$ is a bipartite partition, we define the *monomial* symmetric function m_λ to be the sum of all monomials of the form

$$x_{i_1}^{a_1} y_{i_1}^{b_1} x_{i_2}^{a_2} y_{i_2}^{b_2} \cdots$$

where $i_1, i_2, \ldots$ are distinct. For example,

$$m_{(1,0)(0,1)} = \sum_{i \neq j} x_i y_j$$

$$m_{(1,1)} = \sum_{i} x_i y_i$$

$$m_{(1,1)(1,1)} = \sum_{i<j} x_i y_i x_j y_j.$$

Note that $m_{(1,1)(1,1)}$ is not equal to $\sum_{i \neq j} x_i y_i x_j y_j$ since the latter sum contains every monomial twice. It is clear that the m_λ over all bipartite partitions λ of (a,b) constitute a basis for the vector space of all MacMahon symmetric functions of degree (a,b).

Next we define the three "multiplicative bases" : the *elementary* symmetric functions e_λ, the *complete* symmetric functions h_λ, and the *power sum* symmetric functions p_λ. These bases are multiplicative in the sense that if $\lambda = (a_1,b_1)(a_2,b_2)\cdots$ then $e_\lambda = e_{(a_1,b_1)} e_{(a_2,b_2)} \cdots$ and similarly for the other bases. We define $e_{(a,b)}$ by

$$1 + \sum_{a,\,b} e_{(a,b)} s^a t^b = \prod_i (1 + x_i s + y_i t),$$

so that $e_{(a,b)} = m_{(1,0)^a(0,1)^b}$, and we define $h_{(a,b)}$ by

$$1 + \sum_{a,\,b} h_{(a,b)} s^a t^b = \prod_i \frac{1}{1 - x_i s - y_i t}.$$

Note that in general $h_{(a,b)}$ is not a sum of m_λ's with unit coefficients. For example,

$$h_{(1,1)} = x_1y_1 + y_1x_1 + x_1y_2 + y_1x_2 + \cdots = 2m_{(1,1)} + m_{(1,0)(0,1)}.$$

We define $p_{(a,b)}$ by

$$p_{(a,b)} = \sum_i x_i^a y_i^b = m_{(a,b)}.$$

By taking logarithms and exponentiating we obtain

$$(9.1) \quad 1 + \sum_{a,b} e_{(a,b)} s^a t^b = \exp\Bigl(\sum_{k+l>0} (-1)^{k+l-1} \frac{1}{k+l} \binom{k+l}{l} p_{(k,l)} s^k t^l\Bigr)$$

and

$$(9.2) \qquad 1 + \sum_{a,b} h_{(a,b)} s^a t^b = \exp\Bigl(\sum_{k+l>0} \frac{1}{k+l} \binom{k+l}{l} p_{(k,l)} s^k t^l\Bigr).$$

We now show that $\{e_\lambda\}$, $\{h_\lambda\}$, and $\{p_\lambda\}$ are in fact bases. Since they have the right cardinality, it is sufficient to show that they span, and in view of (9.1) and (9.2) it is sufficient to show that the p_λ span.

We may define a partial order on the bipartite partitions of (a,b) by saying that μ covers λ if λ can be obtained from μ by replacing two parts of μ by their sum. Thus $(2,0)(1,3) < (1,0)(1,0)(1,0)(0,3)$ since $(2,0) = (1,0) + (1,0)$ and $(1,3) = (1,0) + (0,3)$.

It is easy to see that

$$p_\mu = \sum_{\lambda \le \mu} c_\lambda m_\lambda$$

for some integers c_λ, with $c_\mu \ne 0$. Thus, for example,

$$\begin{aligned} p_{(1,1)(1,1)} &= \Bigl(\sum_i x_i y_i\Bigr)\Bigl(\sum_j x_j y_j\Bigr) \\ &= \sum_{i \ne j} x_i y_i x_j y_j + \sum_i x_i^2 y_i^2 \\ &= 2m_{(1,1)(1,1)} + m_{(2,2)} \end{aligned}$$

It follows that these equations can be solved to express the m_λ as linear combinations of the p_λ, and thus the p_λ form a basis.

If every part of λ is of the form $(a_i, 0)$, then all of these MacMahon symmetric functions reduce to the corresponding ordinary symmetric functions.

Now let $\bar{x}_1, \bar{x}_2, \ldots, \bar{y}_1, \bar{y}_2, \ldots$ be new variables. If f is a MacMahon symmetric function let us write $f(x, y)$ for f and $f(\bar{x}, \bar{y})$ for f with $\bar{x}_i$ replacing x_i and $\bar{y}_i$ replacing y_i.

Suppose that λ has r_{ij} parts equal to (i, j) for each i and j. Then set

$$z_\lambda = \prod_{i,j} r_{ij}! \left(\frac{i!\, j!}{(i+j-1)!} \right)^{r_{ij}} .$$

Note that unlike the case of ordinary symmetric functions, the z_λ are not in general integers; for example, $z_{(2,2)} = 2/3$. The following formulas are proved similarly to their analogs for ordinary symmetric functions :

$$\prod_{i,j} \frac{1}{1 - x_i \bar{x}_j - y_i \bar{y}_j} = \sum_\lambda h_\lambda(x, y) m_\lambda(\bar{x}, \bar{y}) \tag{9.3}$$

$$= \sum_\lambda z_\lambda^{-1} p_\lambda(x, y) p_\lambda(\bar{x}, \bar{y}) \tag{9.4}$$

MacMahon [14, Vol. 2, pp. 286–291] proved the following “law of symmetry” : The coefficient of $x_1^{a_1} y_1^{b_1} x_2^{a_2} y_2^{b_2} \cdots$ in $h_{(c_1,d_1)(c_2,d_2)\cdots}$ is equal to the coefficient of $x_1^{c_1} y_1^{d_1} x_2^{c_2} y_2^{d_2} \cdots$ in $h_{(a_1,b_1)(a_2,b_2)\cdots}$. MacMahon’s law of symmetry follows easily from (9.3).

Just as in the case of ordinary symmetric functions, we may define a scalar product on MacMahon symmetric functions by $\langle h_\lambda, m_\mu \rangle = \delta_{\lambda\mu}$. Equivalently, for any symmetric function f, $\langle h_\lambda, f \rangle$ is the coefficient of $x_1^{a_1} y_1^{b_1} x_2^{a_2} y_2^{b_2} \cdots$ in f. MacMahon’s law of symmetry is then equivalent to the formula $\langle h_\lambda, h_\mu \rangle = \langle h_\mu, h_\lambda \rangle$, which implies that $\langle\ ,\ \rangle$ is symmetric. It follows from (9.4) by a standard linear algebra argument that $\langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda\mu}$.

In the author’s opinion MacMahon’s work on symmetric functions failed to achieve what it might have because of his ignorance of the scalar product, which is understandable, since linear algebra was not well-known in MacMahon’s day. Instead of the scalar product, MacMahon used what he called “Hammond operators,” which can be used for the same purposes. Hammond operators, as explained elegantly and concisely by Macdonald [12, p. 45], are adjoints of multiplication operators : if f is a symmetric function then the Hammond operator θ_f is defined by

$$\langle \theta_f(g), h \rangle = \langle g, fh \rangle$$

for all symmetric functions g and h. Thus in particular, $\langle f, g \rangle = \langle f \cdot 1, g \rangle = \langle 1, \theta_f(g) \rangle$, so if f and g are homogeneous of the same degree, $\langle f, g \rangle = \theta_f(g)$. But Hammond operators are undesirable for two reasons. First they disguise the symmetry of the scalar product. Second, they can be represented as differential operators. Although this might seem like an advantage, it seems to be of little use, but misleads by directing attention in the wrong direction.

10. D-finiteness and P-recursiveness. — The theory of D-finiteness for symmetric functions generalizes easily to MacMahon symmetric functions. We call a MacMahon symmetric function D-finite if it is D-finite in the $p_{(a,b)}$. For example,

$$\prod_{i\neq j}(1+x_iy_j)$$

is D-finite because it is equal to

$$\exp\biggl(\sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{j}(p_{(j,0)}p_{(0,j)}-p_{(j,j)})\biggr).$$

It follows that for fixed k, the number of $n\times n$ 0-1 matrices with zeros on the diagonal and every row and column sum k is P-recursive as as a function of n.

By using MacMahon symmetric functions in k sets of variables, one can show that for fixed k, the number of $k\times n$ Latin rectangles is P-recursive as a function of n. In GESSEL [4] a combinatorial derivation is given of a formula for $k\times n$ Latin rectangles which implies P-recursiveness. This formula can also be obtained from MacMahon symmetric functions.

11. Explicit formulas. — We now give two simple examples of the use of MacMahon symmetric functions to derive explicit formulas. First we find the number of $2\times n$ Latin rectangles. This number is easily seen to be the coefficient of $x_1\cdots x_ny_1\cdots y_n$ in $\left(\sum_{i\neq j}x_iy_j\right)^n = e_{(1,1)}^n$. Thus the desired number is $\left\langle h_{(1,1)}^n, e_{(1,1)}^n\right\rangle$. We have

$$h_{(1,1)}=p_{(1,0)(0,1)}+p_{(1,1)} \qquad \text{and} \qquad e_{(1,1)}=p_{(1,0)(0,1)}-p_{(1,1)}.$$

Therefore, the number of $2\times n$ Latin rectangles is

$$\begin{aligned}\left\langle h_{(1,1)}^n, e_{(1,1)}^n\right\rangle &= \left\langle (p_{(1,0)(0,1)}+p_{(1,1)})^n, (p_{(1,0)(0,1)}-p_{(1,1)})^n\right\rangle\\ &= \biggl\langle \sum_{i=0}^{n}\binom{n}{i}p_{(1,0)}^ip_{(0,1)}^ip_{(1,1)}^{n-i},\\ &\qquad\qquad \sum_{j=0}^{n}\binom{n}{j}p_{(1,0)}^jp_{(0,1)}^j(-1)^{n-j}p_{(1,1)}^{n-j}\biggr\rangle\end{aligned}$$

$$= \sum_{i=0}^{n} \binom{n}{i}^2 (-1)^{n-i} \big\langle p_{(1,0)}^i p_{(0,1)}^i p_{(1,1)}^{n-i}, p_{(1,0)}^i p_{(0,1)}^i p_{(1,1)}^{n-i} \big\rangle$$
$$= \sum_{i=0}^{n} \binom{n}{i}^2 (-1)^{n-i} i!\, i!\, (n-i)!$$
$$= n! \sum_{i=0}^{n} (-1)^{n-i} \frac{n!}{(n-i)!} = n!\, D_n,$$

where D_n is the nth derangement number.

Next we consider Example 3 of Section 7, which involves the coefficient of $x_1^{u_1} \dots x_n^{u_n} y_1^{v_1} \dots y_n^{v_n}$ in

$$(1 - x_1 - \cdots x_n - y_1 - \cdots - y_n + x_1 y_1 + \cdots + x_n y_n)^{-1}.$$

This coefficient is then $\big\langle h_{(u_1,v_1)(u_2,v_2)\cdots}, f \big\rangle$, where

$$f = \big(1 - p_{(1,0)} - p_{(0,1)} + p_{(1,1)}\big)^{-1}$$
$$= \sum_{i,j,k} (-1)^i p_{(1,1)^i (1,0)^j (0,1)^k} \frac{(i+j+k)!}{i!\, j!\, k!}$$

It follows that if $\lambda = (1,1)^i (1,0)^j (0,1)^k$ then

$$\langle p_\lambda, f \rangle = (-1)^i (i+j+k)! \tag{11.1}$$

and $\langle p_\lambda, f \rangle = 0$ for λ not of this form. Now let θ be the homomorphism from the MacMahon symmetric functions to polynomials in z defined by

$$\theta(p_{(1,1)}) = -z, \quad \theta(p_{(1,0)}) = \theta(p_{(0,1)}) = z,$$

and $\theta(p_{(a,b)}) = 0$ for other (a,b). Let L be the linear functional on polyomials in z defined by $L(z^n) = n!$, so that L has the integral representation

$$L\big(r(z)\big) = \int_0^\infty e^{-z} r(z)\, dz.$$

It follows from (11.1) that for any symmetric function g,

$$\langle g, f \rangle = L\big(\theta(g)\big).$$

Now let

$$r_{u,v}(z) = \theta(h_{(u,v)}) = \sum_{i=0}^{\min\{u,v\}} (-1)^i \frac{z^{u+v-i}}{i!\,(u-i)!\,(v-i)!}.$$

Then the coefficient we want is

$$L\Bigl(\prod_{j=1}^{n} r_{u_j,v_j}(z)\Bigr). \tag{11.2}$$

We note that this result can also be obtained by an argument like that used in the theory of rook polynomials.

For the special case $u_i = v_i = 1$ we obtain

$$\Bigl\langle h^n_{(1,1)}, f \Bigr\rangle = L\bigl((z^2 - z)^n\bigr) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} (2n-i)!,$$

as is well-known. (See, for example, Stanley [21, Exercise 10, p. 89; Solution, p. 93].)

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