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Bailey Pairs and an Identity of Chern–Li–Stanton–Xue–Yee

Shashank KANADE [a] *and Jeremy LOVEJOY* [b]

a) *Department of Mathematics, University of Denver, 2390 South York Street, Denver, Colorado 80210, USA*
E-mail: *shashank.kanade@du.edu*
URL: https://cs.du.edu/~shakanad/

b) *CNRS, Université Paris Cité, Bâtiment Sophie Germain, Case Courier 7014, 8 Place Aurélie Nemours, 75205 Paris Cedex 13, France*
E-mail: *lovejoy@math.cnrs.fr*
URL: https://lovejoy.perso.math.cnrs.fr

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Abstract. We show how Bailey pairs can be used to give a simple proof of an identity of Chern, Li, Stanton, Xue, and Yee. The same method yields a number of related identities as well as false theta companions.

Key words: Bailey pairs, q-series identities, false theta functions

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Dedicated to James Lepowsky on the occasion of his 80th birthday and Stephen Milne on the occasion of his 75th birthday

1 Introduction

Recall the usual q-series notation,

$$(a_1, a_2, \ldots, a_k)_\infty = (a_1, a_2, \ldots, a_k; q)_\infty = \prod_{j=0}^{\infty} \big(1 - a_1 q^j\big)\big(1 - a_2 q^j\big) \cdots \big(1 - a_k q^j\big)$$

and

$$(a_1, a_2, \ldots, a_k)_n = (a_1, a_2, \ldots, a_k; q)_n = \prod_{j=0}^{n-1} \big(1 - a_1 q^j\big)\big(1 - a_2 q^j\big) \cdots \big(1 - a_k q^j\big),$$

valid for $n \geq 0$, along with the q-binomial coefficient

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \dfrac{(q)_n}{(q)_{n-k}(q)_k} & \text{if } 0 \leq k \leq n, \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

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In a recent study of q-series and partitions related to Ariki–Koike algebras, Chern, Li, Stanton, Xue, and Yee [8] established the following family of q-multisum identities.

Theorem 1.1. *Let $m \geq 1$ and $0 \leq a \leq m-1$. Then we have*

$$\sum_{n_m,\ldots,n_1 \geq 0} \frac{q^{\binom{n_m+1}{2}+\cdots+\binom{n_1+1}{2}}}{(q)_{n_m}} \prod_{i=1}^{m-1} \begin{bmatrix} n_{i+1}+\delta_{a,i} \\ n_i \end{bmatrix} = \frac{\left(q^{a+1}, q^{m+1-a}, q^{m+2}; q^{m+2}\right)_\infty}{(q)_\infty \left(q;q^2\right)_\infty}. \tag{1.2}$$

This generalizes a classical identity in the theory of partitions [4, equation (2.26), $t = q$],

$$\sum_{n \geq 0} \frac{q^{\binom{n+1}{2}}}{(q)_n} = \frac{1}{\left(q;q^2\right)_\infty} = (-q)_\infty.$$

The proof of (1.2) in [8] is lengthy and impressive, involving a symmetry property, a q-binomial coefficient multisum transformation formula, and two identities of Andrews [5] and Kim–Yee [14].

In the first part of this paper, we give a streamlined proof of (1.2) using the Bailey pair machinery. All that we require are the classical Bailey lemma and Bailey lattice along with a Bailey-type lemma from [16] – see Lemma 2.3. In fact, we establish a much more general result, which allows us to prove (1.2) and many more families of identities like it.

Theorem 1.2. *If (α_n, β_n) is a Bailey pair relative to q and f_n is a sequence defined for all integers n which satisfies*

$$\alpha_n = \frac{1-q^{2n+1}}{1-q} f_n, \qquad n \geq 0 \tag{1.3}$$

and

$$q^n f_n = -q^{-n-1} f_{-n-1}, \qquad n \in \mathbb{Z}, \tag{1.4}$$

then for all $m \geq 1$ and $0 \leq a \leq m$, we have

$$\begin{aligned} \sum_{n_m,\ldots,n_1 \geq 0} &\frac{q^{\binom{n_m+1}{2}+\cdots+\binom{n_1+1}{2}}\left(q^2;q^2\right)_{n_1+\delta_{a,0}}}{(q)_{n_m}} \beta_{n_1+\delta_{a,0}} \prod_{i=1}^{m-1} \begin{bmatrix} n_{i+1}+\delta_{a,i} \\ n_i \end{bmatrix} \\ &= \frac{(-q)_\infty}{(q)_\infty} \sum_{n \in \mathbb{Z}} q^{m\binom{n}{2}+an} f_n. \end{aligned} \tag{1.5}$$

Note that using (1.4), the condition (1.3) could be written alternatively as:

$$\alpha_n = \frac{f_n + f_{-n-1}}{1-q}, \qquad n \geq 0.$$

Theorem 1.1 now follows using the classical Bailey pair relative to q [20, E(3)],

$$\alpha_n = \frac{1-q^{2n+1}}{1-q}(-1)^n q^{n^2} \tag{1.6}$$

and

$$\beta_n = \frac{1}{\left(q^2;q^2\right)_n}, \tag{1.7}$$

together with the triple product identity (2.2). Other Bailey pairs lead to similar families of identities, and we give a number of examples in Section 3 – see Theorems 3.2, 3.3, and 3.5.

In the second part of the paper, we prove a result similar to Theorem 1.2 but involving false theta functions instead of theta functions. To state it, we define the function

$$\operatorname{sgn}(n) = \begin{cases} 1 & \text{if } n \geq 0, \\ -1 & \text{if } n < 0. \end{cases}$$

Theorem 1.3. *If (α_n, β_n) is a Bailey pair relative to q and f_n is as in (1.3) and (1.4), then for all $m \geq 1$ and $0 \leq a \leq m$, we have*

$$\sum_{n_m,\dots,n_1\geq 0} \frac{(-1)^{n_m} q^{\binom{n_m+1}{2}+\cdots+\binom{n_1+1}{2}} (q^2;q^2)_{n_1+\delta_{a,0}}}{(-q)_{n_m}} \beta_{n_1+\delta_{a,0}} \prod_{i=1}^{m-1} \begin{bmatrix} n_{i+1}+\delta_{a,i} \\ n_i \end{bmatrix}$$
$$= \begin{cases} \sum_{n\in\mathbb{Z}} \mathrm{sgn}(-n)(-1)^n q^{m\binom{n}{2}+an} f_n & \text{if } 0 \leq a \leq m-1, \\ \sum_{n\in\mathbb{Z}} \mathrm{sgn}(n)(-1)^n q^{m\binom{n+1}{2}} f_n & \text{if } a = m. \end{cases} \tag{1.8}$$

Each family of identities arising from Theorem 1.2 then has a false theta counterpart using Theorem 1.3. In the case of (1.2), using (1.6) and (1.7) this is the following.

Theorem 1.4. *For $m \geq 1$ and $0 \leq a \leq m-1$, we have*

$$\sum_{n_m,\dots,n_1\geq 0} \frac{(-1)^{n_m} q^{\binom{n_m+1}{2}+\cdots+\binom{n_1+1}{2}}}{(-q)_{n_m}} \prod_{i=1}^{m-1} \begin{bmatrix} n_{i+1}+\delta_{a,i} \\ n_i \end{bmatrix} = \sum_{n\in\mathbb{Z}} \mathrm{sgn}(-n) q^{(m+2)\binom{n}{2}+(a+1)n}.$$

The base case $m = 1$ and $a = 0$ is the well-known

$$\sum_{n\geq 0} \frac{(-1)^n q^{\binom{n+1}{2}}}{(-q)_n} = \sum_{n\geq 0} q^{n(3n+1)/2}\big(1-q^{2n+1}\big).$$

See Theorems 4.5–4.7 for further examples.

In the final part of the paper we are motivated by the series $\mathcal{S}_{m,a}$, defined for $m \geq 1$ and $0 \leq a \leq m$ by

$$\mathcal{S}_{m,a} = \sum_{n_m,\dots,n_1\geq 0} \frac{q^{n_m^2+\cdots+n_1^2}}{(q^2;q^2)_{n_m}} \prod_{i=1}^{m-1} \begin{bmatrix} n_{i+1}+\delta_{a,i} \\ n_i \end{bmatrix}_{q^2}.$$

These are naturally dilated versions of the series in (1.2), and it is known that we have

$$\mathcal{S}_{m,0} = \mathcal{S}_{m,m} = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} \big(q^{m+1}, q^{m+3}, q^{2m+4}; q^{2m+4}\big)_\infty. \tag{1.9}$$

See [22, Corollary 1.5 (b)] for general m and [7, Section 5] for m even. While it appears that the $\mathcal{S}_{m,a}$ are not infinite products for $a \notin \{0, m\}$, we prove the following result on the differences of these series.

Theorem 1.5. *For $1 \leq a \leq m$, we have*

$$\mathcal{S}_{m,a} - \mathcal{S}_{m,a-1} = q^a \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} \big(q^{m+1-2a}, q^{m+3+2a}, q^{2m+4}; q^{2m+4}\big)_\infty.$$

This and several similar families of identities will follow as special cases of our third main theorem.

Theorem 1.6. *Suppose that (α_n, β_n) is a Bailey pair relative to 1 and that there is a sequence g_n, defined for all integers n, with $g_0 = 1$,*

$$\alpha_n = \begin{cases} 1 & \text{if } n = 0, \\ (1+q^n) g_n & \text{if } n > 1, \end{cases} \tag{1.10}$$

and

$$g_{-n} = q^n g_n. \tag{1.11}$$

Then for all $m \geq 0$ and $1 \leq a \leq m$, we have

$$\sum_{n_m,\dots,n_1\geq 0} \frac{q^{\frac{1}{2}(n_m^2+n_{m-1}^2+\cdots+n_1^2)}\left(-q^{\frac{1}{2}},q\right)_{n_1}}{(q)_{n_m}} \beta_{n_1} \prod_{i=1}^{m-1} \begin{bmatrix} n_{i+1}+\delta_{a,i} \\ n_i \end{bmatrix}$$
$$- \sum_{n_m,\dots,n_1\geq 0} \frac{q^{\frac{1}{2}(n_m^2+n_{m-1}^2+\cdots+n_1^2)}\left(-q^{\frac{1}{2}},q\right)_{n_1+\delta_{a-1,0}}}{(q)_{n_m}} \beta_{n_1+\delta_{a-1,0}} \prod_{i=1}^{m-1} \begin{bmatrix} n_{i+1}+\delta_{a-1,i} \\ n_i \end{bmatrix}$$
$$= -q^{\frac{a-1}{2}} \frac{\left(-q^{\frac{1}{2}}\right)_\infty}{(q)_\infty} \sum_{n\in\mathbb{Z}} q^{\frac{m}{2}n^2+na-n} g_{n+1}. \tag{1.12}$$

Note that with (1.11), the condition (1.10) could be written alternatively as

$$\alpha_n = \begin{cases} 1 & \text{if } n = 0, \\ g_n + g_{-n} & \text{if } n > 1. \end{cases}$$

The paper is organized as follows. In the next section, we collect some basic facts about Bailey pairs. In Section 3, we prove Theorem 1.2 and give several applications, including Theorem 1.1. In Section 4, we prove Theorem 1.3 and give false theta companions for each family of identities in Section 3. In Section 5, we prove Theorem 1.6, which allows us to deduce Theorem 1.5 and other similar identities. Wherever possible, we include historical remarks about the "base cases" of our identities.

While it may be striking that identities involving products of q-binomial coefficients

$$\prod_{i=1}^{m-1} \begin{bmatrix} n_{i+1}+\delta_{a,i} \\ n_i \end{bmatrix} \tag{1.13}$$

are so widespread, the fact that the Bailey pair machinery can be used to prove such identities should not be a surprise. This is the most powerful and systematic technique for treating q-multisum identities, and a number of identities with products of q-binomial coefficients like (1.13) have already appeared in the literature in connection to Bailey pairs. We discuss these briefly in the concluding remarks at the end of the paper.

2 Bailey pairs

In this section, we review the necessary background on Bailey pairs. A Bailey pair relative to x [3] is a pair of sequences (α_n, β_n), $n \in \mathbb{Z}_{\geq 0}$, satisfying

$$\beta_n = \sum_{k=0}^{n} \frac{\alpha_k}{(q)_{n-k}(xq)_{n+k}}.$$

Note that in the limit, this gives (subject to convergence conditions)

$$\lim_{n\to\infty} \beta_n = \frac{1}{(q,xq)_\infty} \sum_{k=0}^{\infty} \alpha_k. \tag{2.1}$$

In practice, the above sum often becomes an infinite product by an appeal to the Jacobi triple product identity [4, equation (2.2.10)],

$$\sum_{n\in\mathbb{Z}}(-1)^n z^n q^{n^2} = \left(q^2, zq, z^{-1}q; q^2\right)_\infty, \tag{2.2}$$

or the quintuple product identity [9],

$$\left(\sum_{\substack{n\in\mathbb{Z}\\ n\equiv 0 \pmod 3}} - \sum_{\substack{n\in\mathbb{Z}\\ n\equiv 2 \pmod 3}}\right) z^n q^{\frac{1}{3}\binom{n+1}{2}} = \left(q, zq, z^{-1}\right)_\infty \left(z^2q, z^{-2}q; q^2\right)_\infty. \tag{2.3}$$

The most important fact about Bailey pairs is the following, which is known as the Bailey lemma. From a given Bailey pair relative to x, it produces new Bailey pairs relative to x.

Lemma 2.1 ([3]). *If (α_n, β_n) is a Bailey pair relative to x, then so is (α_n', β_n'), where*

$$\alpha_n' = \frac{(b,c)_n (xq/bc)^n}{(xq/b, xq/c)_n}\alpha_n \qquad \text{and} \qquad \beta_n' = \frac{1}{(xq/b, xq/c)_n}\sum_{j=0}^{n}\frac{(b,c)_j (xq/bc)_{n-j}(xq/bc)^j}{(q)_{n-j}}\beta_j.$$

A result similar to Lemma 2.1 takes a Bailey pair relative to x and produces Bailey pairs relative to x/q.

Lemma 2.2 ([1]). *If (α_n, β_n) is a Bailey pair relative to x, then (α_n', β_n') is a Bailey pair relative to x/q, where*

$$\alpha_n' = (1-x)\left(\frac{x}{bc}\right)\frac{(b,c)_n (x/bc)^n}{(x/b, x/c)_n}\left(\frac{\alpha_n}{1-xq^{2n}} - \frac{xq^{2n-2}\alpha_{n-1}}{1-xq^{2n-2}}\right)$$

and

$$\beta_n' = \frac{1}{(x/b, x/c)_n}\sum_{j=0}^{n}\frac{(b,c)_j (x/bc)_{n-j}(x/bc)^j}{(q)_{n-j}}\beta_j.$$

Here by convention we take

$$\alpha_{-1} = 0. \tag{2.4}$$

Finally, we have two lemmas that change a Bailey pair relative to x to one relative to xq. The first will be key to introducing q-binomial coefficients of the form $\left[\begin{smallmatrix} n_{i+1}+1 \\ n_i \end{smallmatrix}\right]$. The second is not needed for the proof of Theorem 1.1 but will be used to establish a certain Bailey pair later in the paper.

Lemma 2.3 ([16]). *If (α_n, β_n) is a Bailey pair relative to x, then (α_n', β_n') is a Bailey pair relative to xq, where*

$$\alpha_n' = \frac{1}{1-xq}\left(\frac{1-q^{n+1}}{1-xq^{2n+2}}\alpha_{n+1} + \frac{q^n(1-xq^n)}{1-xq^{2n}}\alpha_n\right) \qquad \text{and} \qquad \beta_n' = \left(1-q^{n+1}\right)\beta_{n+1}.$$

Lemma 2.4 ([15]). *If (α_n, β_n) is a Bailey pair relative to x, then (α_n', β_n') is a Bailey pair relative to xq, where*

$$\alpha_n' = \frac{\left(1-xq^{2n+1}\right)(xq/b)_n(-b)^n q^{\binom{n}{2}}}{(1-xq)(bq)_n}\sum_{j=0}^{n}\frac{(b)_j}{(xq/b)_j}(-b)^{-j}q^{-\binom{j}{2}}\alpha_j \qquad \text{and} \qquad \beta_n' = \frac{(b)_n}{(bq)_n}\beta_n.$$

3 Proof of Theorem 1.2 and applications

We begin this section by establishing a key Bailey lemma for the proof of Theorem 1.2.

Lemma 3.1. *Let $m \geq 1$ and $0 \leq a \leq m$. Suppose that (α_n, β_n) is a Bailey pair relative to q and that there is a sequence $(f_n)_{n\geq -1}$ such that*

$$\alpha_n = \frac{1-q^{2n+1}}{1-q} f_n, \tag{3.1}$$

where

$$q^{-1} f_{-1} = -f_0. \tag{3.2}$$

Then (α'_n, β'_n) is a Bailey pair relative to q, where

$$\alpha'_n = \begin{cases} \frac{1}{1-q} q^{m\binom{n+1}{2}} \left(q^{a(n+1)} f_{n+1} - q^{-an+2n-1} f_{n-1}\right) & \text{if } 0 \leq a \leq m-1, \\ \frac{1}{1-q} q^{m\binom{n+1}{2}} \left(1-q^{2n+1}\right) f_n & \text{if } a = m, \end{cases}$$

and

$$\begin{aligned} \beta'_n &= \beta'_{n_{m+1}} \\ &= \frac{1}{(-q)_{n_{m+1}}} \sum_{n_m,\dots,n_1 \geq 0} \frac{q^{\binom{n_m+1}{2}+\cdots+\binom{n_1+1}{2}} (q^2;q^2)_{n_1+\delta_{a,0}}}{(q)_{n_{m+1}-n_m}(q)_{n_m}} \beta_{n_1+\delta_{a,0}} \prod_{i=1}^{m-1} \begin{bmatrix} n_{i+1}+\delta_{a,i} \\ n_i \end{bmatrix}. \end{aligned} \tag{3.3}$$

Proof. *Step 1*: First assume $a > 0$. Iterate Lemma 2.1 a times with $x \mapsto q$, $b \mapsto -q$, and $c \to \infty$ to obtain another Bailey pair relative to q. The resulting α_n and β_n are

$$\alpha_n = q^{a\binom{n+1}{2}} \frac{1-q^{2n+1}}{1-q} f_n \tag{3.4}$$

and

$$\beta_n = \beta_{n_{a+1}} = \frac{1}{(-q)_{n_{a+1}}} \sum_{n_a,\dots,n_1 \geq 0} \frac{q^{\binom{n_a+1}{2}+\cdots+\binom{n_1+1}{2}} (-q)_{n_1}}{(q)_{n_{a+1}-n_a} \cdots (q)_{n_2-n_1}} \beta_{n_1}. \tag{3.5}$$

Here and throughout the paper we use the convention that $1/(q)_n = 0$ if $n < 0$. If $a = m$, we stop here. Multiplying the numerator and denominator of the above sum by $(q)_{n_1} \cdots (q)_{n_m}$ and rewriting in terms of q-binomial coefficients using (1.1) gives (3.3) for $a = m$.

Step 2: Apply Lemma 2.3 with $x \mapsto q$ to obtain a Bailey pair relative to q^2:

$$\begin{aligned} \alpha_n &= \frac{1-q^{n+1}}{(q)_2} \left(q^{a\binom{n+2}{2}} f_{n+1} + q^{n+a\binom{n+1}{2}} f_n\right) \\ &= \frac{1-q^{n+1}}{(q)_2} q^{(a+2)\binom{n+1}{2}-n^2} \left(f_n + q^{a(n+1)-n} f_{n+1}\right) \end{aligned}$$

and

$$\beta_n = \beta_{n_{a+1}} = \frac{1-q^{n_{a+1}+1}}{(-q)_{n_{a+1}+1}} \sum_{n_a,\dots,n_1 \geq 0} \frac{q^{\binom{n_a+1}{2}+\cdots+\binom{n_1+1}{2}} (-q)_{n_1}}{(q)_{n_{a+1}+1-n_a} \cdots (q)_{n_2-n_1}} \beta_{n_1}.$$

Step 3: Use Lemma 2.1 with $x \mapsto q^2$, $b \mapsto -q^2$, and $c \to \infty$ to obtain another Bailey pair relative to q^2. The resulting pair is

$$\alpha_n = \frac{1-q^{2n+2}}{(1+q)(q)_2} q^{(a+3)\binom{n+1}{2}-n^2}\left(f_n + q^{a(n+1)-n} f_{n+1}\right)$$

and

$$\beta_n = \beta_{n_{a+2}} = \frac{1}{(1+q)(-q)_{n_{a+2}}} \sum_{n_{a+1},\dots,n_1 \geq 0} \frac{q^{\binom{n_{a+1}+1}{2}+\cdots+\binom{n_1+1}{2}}\left(1-q^{n_{a+1}+1}\right)(-q)_{n_1}}{(q)_{n_{a+2}-n_{a+1}}(q)_{n_{a+1}+1-n_a}\cdots(q)_{n_2-n_1}} \beta_{n_1}.$$

At this point, we multiply both α_n and β_n by $1+q$.

Step 4: Use Lemma 2.2 with $x \mapsto q^2$ and $b, c \mapsto q$ to obtain a Bailey pair relative to q. This retains the β_n, but does change the α_n. When $n \geq 1$, we have

$$\begin{aligned}
\alpha_n &= \frac{1}{1-q}\Big(q^{(a+3)\binom{n+1}{2}-n^2}\left(f_n + q^{a(n+1)-n} f_{n+1}\right) \\
&\qquad - q^{2n+(a+3)\binom{n}{2}-(n-1)^2}\left(f_{n-1} + q^{an-n+1} f_n\right)\Big) \\
&= \frac{1}{1-q}\left(q^{(a+3)\binom{n+1}{2}-n^2+a(n+1)-n} f_{n+1} - q^{2n+(a+3)\binom{n}{2}-(n-1)^2} f_{n-1}\right) \\
&= \frac{1}{1-q} q^{(a+1)\binom{n+1}{2}}\left(q^{a(n+1)} f_{n+1} - q^{-an+2n-1} f_{n-1}\right).
\end{aligned}$$

When $n = 0$, the above is also valid, using (2.4) and (3.2).

Step 5: Iterate Lemma 2.1 $m-a-1$ times with $x \mapsto q$, $b \mapsto -q$, and $c \to \infty$ to arrive at the final Bailey pair relative to q:

$$\alpha_n = \frac{1}{1-q} q^{m\binom{n+1}{2}}\left(q^{a(n+1)} f_{n+1} - q^{-an+2n-1} f_{n-1}\right)$$

and

$$\beta_n = \beta_{n_{m+1}} = \frac{1}{(-q)_{n_{m+1}}} \sum_{n_m,\dots,n_1 \geq 0} \frac{q^{\binom{n_m+1}{2}+\cdots+\binom{n_1+1}{2}}\left(1-q^{n_{a+1}+1}\right)(-q)_{n_1}}{(q)_{n_{m+1}-n_m}\cdots(q)_{n_{a+1}+1-n_a}\cdots(q)_{n_2-n_1}} \beta_{n_1}.$$

Rewriting the sum in terms of q-binomial coefficients gives

$$\beta_n = \beta_{n_{m+1}} = \frac{1}{(-q)_{n_{m+1}}} \sum_{n_m,\dots,n_1 \geq 0} \frac{q^{\binom{n_m+1}{2}+\cdots+\binom{n_1+1}{2}}\left(q^2;q^2\right)_{n_1}}{(q)_{n_{m+1}-n_m}(q)_{n_m}} \beta_{n_1} \prod_{i=1}^{m-1} \begin{bmatrix} n_{i+1}+\delta_{a,i} \\ n_i \end{bmatrix},$$

which coincides with (3.3).

Step 6: If $a = 0$, performing Steps 2–5 leads to the following β_n (while the formula for α_n is the same as in Step 5 with $a \mapsto 0$):

$$\beta_n = \beta_{n_{m+1}} = \frac{1}{(-q)_{n_{m+1}}} \sum_{n_m,\dots,n_1 \geq 0} \frac{q^{\binom{n_m+1}{2}+\cdots+\binom{n_1+1}{2}}\left(1-q^{n_1+1}\right)(-q)_{n_1+1}}{(q)_{n_{m+1}-n_m}\cdots(q)_{n_2-n_1}} \beta_{n_1+1}.$$

Rewriting this in terms of q-binomial coefficients gives

$$\beta_n = \beta_{n_{m+1}} = \frac{1}{(-q)_{n_{m+1}}} \sum_{n_m,\dots,n_1 \geq 0} \frac{q^{\binom{n_m+1}{2}+\cdots+\binom{n_1+1}{2}}\left(q^2;q^2\right)_{n_1+1}}{(q)_{n_{m+1}-n_m}(q)_{n_m}} \beta_{n_1+1} \prod_{i=1}^{m-1} \begin{bmatrix} n_{i+1}+\delta_{a,i} \\ n_i \end{bmatrix},$$

which again matches (3.3). This completes the proof. ∎

Armed with this Bailey pair, we are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. Let (α_n, β_n) be a Bailey pair relative to q satisfying (3.1) and (3.2). Applying (2.1) to the Bailey pair in Lemma 3.1, we obtain

$$\sum_{n_m,\dots,n_1\geq 0} \frac{q^{\binom{n_m+1}{2}+\cdots+\binom{n_1+1}{2}}(q^2;q^2)_{n_1+\delta_{a,0}}}{(q)_{n_m}} \beta_{n_1+\delta_{a,0}} \prod_{i=1}^{m-1} \begin{bmatrix} n_{i+1}+\delta_{a,i} \\ n_i \end{bmatrix}$$
$$= \frac{(-q)_\infty}{(q)_\infty} \cdot \begin{cases} \sum_{n\geq 0} q^{m\binom{n+1}{2}} \left(q^{a(n+1)} f_{n+1} - q^{-an+2n-1} f_{n-1}\right) & \text{if } 0 \leq a \leq m-1, \\ \sum_{n\geq 0} q^{m\binom{n+1}{2}} \left(1-q^{2n+1}\right) f_n & \text{if } a = m. \end{cases} \tag{3.6}$$

If f_n also satisfies (1.4), then we obtain the right-hand side of (1.5) from the right-hand side of (3.6) when $a < m$ by computing as follows:

$$\begin{aligned}
&\sum_{n\geq 0} q^{m\binom{n+1}{2}} \left(q^{a(n+1)} f_{n+1} - q^{-an+2n-1} f_{n-1}\right) \\
&\quad = \sum_{n\geq 1} q^{m\binom{n}{2}+an} f_n - \sum_{n\geq 0} q^{m\binom{n+1}{2}-an+2n-1} f_{n-1} \\
&\quad = \sum_{n\geq 1} q^{m\binom{n}{2}+an} f_n - \sum_{n\leq 0} q^{m\binom{n}{2}+an-2n-1} f_{-n-1} \\
&\quad = \sum_{n\geq 1} q^{m\binom{n}{2}+an} f_n + \sum_{n\leq 0} q^{m\binom{n}{2}+an} f_n.
\end{aligned}$$

The case $a = m$ is similar. ■

We now give several applications of Theorem 1.2, beginning with Theorem 1.1.

Proof of Theorem 1.1. Using the Bailey pair in (1.6) and (1.7), we have

$$f_n = (-1)^n q^{n^2},$$

which is readily seen to satisfy (1.4). Applying Theorem 1.2, the left-hand side of (1.5) becomes the left-hand side of (1.2), while the sum on the right-hand side is

$$\sum_{n\in\mathbb{Z}} q^{m\binom{n}{2}+an} f_n = \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{m+2}{2}n^2+(a-\frac{m}{2})n} = \left(q^{a+1}, q^{m+1-a}, q^{m+2}; q^{m+2}\right)_\infty,$$

by (2.2). ■

Our next two applications of Theorem 1.2 are contained in the following theorems.

Theorem 3.2. *For $m \geq 1$ and $0 \leq a \leq m$, we have*

$$\sum_{n_m,\dots,n_1\geq 0} \frac{q^{n_m^2+n_m+\cdots+n_1^2+n_1}}{(q^2;q^2)_{n_m}(-q;q^2)_{n_1+\delta_{a,0}}} \prod_{i=1}^{m-1} \begin{bmatrix} n_{i+1}+\delta_{a,i} \\ n_i \end{bmatrix}_{q^2}$$
$$= \frac{(-q^2;q^2)_\infty \left(q^{2a+1}, q^{2m-2a+2}, q^{2m+3}; q^{2m+3}\right)_\infty}{(q^2;q^2)_\infty}. \tag{3.7}$$

Theorem 3.3. *For $m \geq 1$ and $0 \leq a \leq m$, we have*

$$\sum_{n_m,\dots,n_1\geq 0} \frac{q^{\binom{n_m+1}{2}+\cdots+\binom{n_1+1}{2}}\left(-1;q^2\right)_{n_1+\delta_{a,0}}}{(q)_{n_m}\left(q;q^2\right)_{n_1+\delta_{a,0}}} \prod_{i=1}^{m-1} \begin{bmatrix} n_{i+1}+\delta_{a,i} \\ n_i \end{bmatrix}$$
$$= \frac{(-q)_\infty}{(q)_\infty}\left(-q^a, q^{m+1+a}, q^{m+1-a}, -q^{2m+2-a}, -q^{m+1}, q^{2m+2}; q^{2m+2}\right)_\infty. \tag{3.8}$$

For $m = 1$ and $a = 0, 1$, Theorem 3.2 gives the identities

$$\sum_{n\geq 0} \frac{q^{n^2+n}}{\left(q^2;q^2\right)_n\left(-q;q^2\right)_{n+1}} = \frac{\left(-q^2;q^2\right)_\infty\left(q,q^4,q^5;q^5\right)_\infty}{\left(q^2;q^2\right)_\infty}$$

and

$$\sum_{n\geq 0} \frac{q^{n^2+n}}{\left(q^2;q^2\right)_n\left(-q;q^2\right)_n} = \frac{\left(-q^2;q^2\right)_\infty\left(q^2,q^3,q^5;q^5\right)_\infty}{\left(q^2;q^2\right)_\infty}.$$

The first of these appears in Slater's compendium of Rogers–Ramanujan type identities [21, equation (17)], while the second appears there in an equivalent form [21, equation (99)]. Theorem 3.3 generalizes

$$\sum_{n\geq 0} \frac{q^{\binom{n+1}{2}}\left(-q^2;q^2\right)_n}{(q)_n\left(q;q^2\right)_{n+1}} = (-q)_\infty^2\left(-q^4;q^4\right)_\infty$$

and

$$\sum_{n\geq 0} \frac{q^{\binom{n+1}{2}}\left(-1;q^2\right)_n}{(q)_n\left(q;q^2\right)_n} = (-q)_\infty^2\left(-q^2;q^4\right)_\infty,$$

which are the cases $(a,b) = (\mathrm{i}q, -\mathrm{i}q)$ and $(\mathrm{i}, -\mathrm{i})$ of the q-analogue of Gauss' second theorem (see [2, equation (1.8)] or [10, Appendix II, equation (II.11)]),

$$\sum_{n\geq 0} \frac{(a,b)_n q^{\binom{n+1}{2}}}{(q)_n\left(abq;q^2\right)_n} = \frac{\left(aq,bq;q^2\right)_\infty}{\left(q,abq;q^2\right)_\infty}.$$

Proof of Theorem 3.2. We will prove this theorem with $q \mapsto q^{1/2}$. We start with the seed pair relative to q [23, equation (4.4)] determined by

$$f_n = (-1)^n q^{\frac{1}{4}n(3n-1)}, \tag{3.9}$$

which satisfies (1.4), and

$$\beta_n = \frac{1}{\left(q^2;q^2\right)_n\left(-q^{\frac{1}{2}};q\right)_n}.$$

The sum is obtained immediately from (1.5), and on the product side we have

$$\sum_{n\in\mathbb{Z}} q^{m\binom{n}{2}+an} f_n = \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{2m+3}{4}n^2+\left(\frac{4a-2m-1}{4}\right)n} = \left(q^{a+\frac{1}{2}}, q^{m+1-a}, q^{m+\frac{3}{2}}; q^{m+\frac{3}{2}}\right)_\infty,$$

by (2.2). ∎

To prove Theorem 3.3, we will need the following Bailey pair relative to q, which does not appear to be in the literature.

Lemma 3.4. *The pair (α_n, β_n) is a Bailey pair relative to q, where*

$$\alpha_n = \frac{(-1)^{\binom{n}{2}} q^{\binom{n}{2}} \left(1 - q^{2n+1}\right)}{1-q} \qquad \text{and} \qquad \beta_n = \frac{\left(-1; q^2\right)_n}{(q)_{2n}}.$$

Proof. We start with the Bailey pair relative to 1 [20, p. 470, line 7],

$$A_n = \begin{cases} 1 & \text{if } n = 0, \\ (-1)^r q^{2r^2} \left(q^r + q^{-r}\right) & \text{if } n = 2r, r > 0, \\ (-1)^r q^{2r^2} \left(q^r - q^{3r+1}\right) & \text{if } n = 2r+1 \end{cases} \qquad \text{and} \qquad B_n = \frac{\left(-1; q^2\right)_n}{(q)_{2n}}.$$

Using the case $b = 0$ of Lemma 2.4, we have that (α_n, β_n) is a Bailey pair relative to q, where $\beta_n = B_n$ and

$$\alpha_n = \frac{\left(1 - q^{2n+1}\right) q^{n^2}}{1-q} \sum_{j=0}^{n} q^{-j^2} A_j.$$

To prove the lemma, then it suffices to show that for all n we have

$$\sum_{j=0}^{n} q^{-j^2} A_j = (-1)^{\binom{n}{2}} q^{-\binom{n+1}{2}}.$$

This is easily established by mathematical induction. ∎

Proof of Theorem 3.3. Corresponding to the α_n in the Bailey pair in Lemma 3.4, we have

$$f_n = (-1)^{\binom{n}{2}} q^{\binom{n}{2}}, \tag{3.10}$$

which can again be seen to satisfy (1.4). We obtain the product using (2.2)

$$\sum_{n \in \mathbb{Z}} q^{m\binom{n}{2} + an} f_n = \sum_{n \in \mathbb{Z}} (-1)^{\binom{n}{2}} q^{\frac{m+1}{2} n^2 + \frac{2a-m-1}{2} n} = \left(-q^a, q^{m+1-a}, -q^{m+1}; -q^{m+1}\right)_\infty.$$

Here, while invoking (2.2) with $q \mapsto \mathrm{i} q^{\frac{m+1}{2}}$ and $z \mapsto \mathrm{i} q^{\frac{2a-m-1}{2}}$, we use that

$$(-1)^n \mathrm{i}^{n+n^2} = (-1)^{n + \frac{n^2+n}{2}} = (-1)^{\binom{n}{2}}.$$ ∎

The final result in this section collects three further applications of Theorem 1.2, where the infinite product now arises from the quintuple product identity (2.3) instead of the triple product identity.

Theorem 3.5. *For $m \geq 1$ and $0 \leq a \leq m$, we have the following identities:*

$$\sum_{n_m, \ldots, n_1 \geq 0} \frac{q^{\binom{n_m+1}{2} + \cdots + \binom{n_1+1}{2}}}{(q)_{n_m} \left(q; q^2\right)_{n_1 + \delta_{a,0}}} \prod_{i=1}^{m-1} \begin{bmatrix} n_{i+1} + \delta_{a,i} \\ n_i \end{bmatrix}$$
$$= \frac{(-q)_\infty}{(q)_\infty} \left(q^{m+1-a}, q^{2m+3+a}, q^{3m+4}; q^{3m+4}\right)_\infty \left(q^{m+2+2a}, q^{5m+6-2a}; q^{6m+8}\right)_\infty, \tag{3.11}$$

$$\sum_{n_m, \ldots, n_1 \geq 0} \frac{q^{\binom{n_m+1}{2} + \cdots + \binom{n_1+1}{2} + (n_1 + \delta_{a,0})^2 - (n_1 + \delta_{a,0})}}{(q)_{n_m} \left(q; q^2\right)_{n_1 + \delta_{a,0}}} \prod_{i=1}^{m-1} \begin{bmatrix} n_{i+1} + \delta_{a,i} \\ n_i \end{bmatrix}$$

$$= \frac{(-q)_\infty}{(q)_\infty} \left(q^{m+1-a}, q^{2m+1+a}, q^{3m+2}; q^{3m+2}\right)_\infty \left(q^{m+2a}, q^{-2a+5m+4}; q^{6m+4}\right)_\infty, \tag{3.12}$$

$$\sum_{n_m,\dots,n_1\geq 0} \frac{q^{\binom{n_m+1}{2}+\cdots+\binom{n_1+1}{2}} \left(-1;q^3\right)_{n_1+\delta_{a,0}}}{(q)_{n_m} \left(q;q^2\right)_{n_1+\delta_{a,0}} (-1;q)_{n_1+\delta_{a,0}}} \prod_{i=1}^{m-1} \begin{bmatrix} n_{i+1}+\delta_{a,i} \\ n_i \end{bmatrix}$$

$$= \frac{(-q)_\infty}{(q)_\infty} \left(q^{m+1-a}, q^{2m+2+a}, q^{3m+3}; q^{3m+3}\right)_\infty \left(q^{m+2a+1}, q^{-2a+5m+5}; q^{6m+6}\right)_\infty. \tag{3.13}$$

Proof. For each of these identities, we will use a Bailey pair with relative to q that has the form (1.3) with f_n of the following shape for some t:

$$f_n = \begin{cases} q^{\frac{t}{3}\binom{n+1}{2}-n} & \text{if } n \equiv 0 \pmod 3, \\ 0 & \text{if } n \equiv 1 \pmod 3, \\ -q^{\frac{t}{3}\binom{n+1}{2}-n} & \text{if } n \equiv 2 \pmod 3. \end{cases} \tag{3.14}$$

It can be seen that this f_n satisfies (1.4) for any t, however, for our identities, we will only require $t \in \{2,3,4\}$.

For (3.11), we use $t=4$, and the corresponding Bailey pair is [23, equation (4.6)], with

$$\beta_n = \frac{1}{(q)_{2n}}. \tag{3.15}$$

For (3.12), we use $t=2$, and the Bailey pair is given right before [23, Theorem 4.1], with

$$\beta_n = \frac{q^{n^2-n}}{(q)_{2n}}. \tag{3.16}$$

For (3.13), we use $t=3$, and the Bailey pair is [18, P(6)]. This can also be obtained from the pair $P(1)$ of [18], by using Lemma 2.4 to retain the β_n and change the α_n so that $x=q$; see also [13]. This pair has

$$\beta_n = \frac{\left(-1;q^3\right)_n}{(q)_{2n}(-1;q)_n}. \tag{3.17}$$

Using (3.15) and (3.16), and (3.17) in (1.5), we immediately obtain the requisite sum-sides of (3.11)–(3.13). For the products, using (3.14) we obtain

$$\sum_{n\in\mathbb{Z}} q^{m\binom{n}{2}+an} f_n = \left(\sum_{\substack{n\in\mathbb{Z} \\ n\equiv 0 \ (\mathrm{mod}\ 3)}} - \sum_{\substack{n\in\mathbb{Z} \\ n\equiv 2 \ (\mathrm{mod}\ 3)}} \right) q^{m\binom{n}{2}+an+\frac{t}{3}\binom{n+1}{2}-n}$$

$$= \left(\sum_{\substack{n\in\mathbb{Z} \\ n\equiv 0 \ (\mathrm{mod}\ 3)}} - \sum_{\substack{n\in\mathbb{Z} \\ n\equiv 2 \ (\mathrm{mod}\ 3)}} \right) q^{\frac{3m+t}{3}\binom{n+1}{2}+(-m+a-1)n},$$

and then substituting $(z,q) \mapsto \left(q^{-m+a-1}, q^{3m+t}\right)$ in (2.3) completes the proof. ■

Remark 3.6. The base cases of (3.11), corresponding to $m=1$ and $a=0,1$ are recorded with the right-hand side in the form of a theta series by Rogers [19, p. 331, equation (1)]. These same cases of (3.12) are also given by Rogers [19, p. 330, equation (2)]. The cases $m=1$ and $a=0,1$ of (3.13) could be deduced by using the limiting case of the Bailey lemma (with $b=-q$, $c\to\infty$) applied to the Bailey pairs relative to q given in [6, equations (3.6) and (3.11)], respectively.

Remark 3.7. Note that the products in all three of these identities are closely related to the principal characters of standard $A_2^{(2)}$ modules. Alternate sum-sides for the same products may be found in [13]. It would be interesting to investigate whether the sums appearing in Theorem 3.5 have any connection to the Ariki–Koike algebras or to Kleshchev multipartitions.

4 Proof of Theorem 1.3 and applications

We begin this section by establishing a key Bailey lemma by following steps similar to Steps 1–6 in Section 3 with appropriate modifications. Only slight modifications are required for $a = m$ or when $a < m-1$, but for the case $a = m-1$ we will use an analogue of the case $x = q$ of Lemma 2.3 with $\beta'_n = \big(1-q^{n+1}\big)\beta_{n+1}$ replaced by $\beta'_n = \big(1+q^{n+1}\big)\beta_{n+1}$. This will be the third of a sequence of three lemmas. The first of these is a transcription of [16, Lemma 3.3].

Lemma 4.1. *If (α_n, β_n) is a Bailey pair relative to x with $\alpha_0 = \beta_0 = 0$, then (α'_n, β'_n) is a Bailey pair relative to xq, where*

$$\alpha'_n = \frac{1}{1-xq}\left(\frac{1}{1-xq^{2n+2}}\alpha_{n+1} - \frac{xq^{2n}}{1-xq^{2n}}\alpha_n\right) \qquad \textit{and} \qquad \beta'_n = \beta_{n+1}.$$

The next lemma removes the hypothesis $\alpha_0 = \beta_0 = 0$ when $x = q$.

Lemma 4.2. *If (α_n, β_n) is a Bailey pair relative to q, then (α'_n, β'_n) is a Bailey pair relative to q^2, where*

$$\alpha'_n = \frac{1}{1-q^2}\left(\frac{1}{1-q^{2n+3}}\alpha_{n+1} - \frac{q^{2n+1}}{1-q^{2n+1}}\alpha_n\right) + \frac{\beta_0}{(q)_2}\big(1+q^{n+1}\big)(-1)^n q^{\binom{n+1}{2}}$$

and

$$\beta'_n = \beta_{n+1}.$$

Proof. Recall the unit Bailey pair relative to q [3, equations (2.12) and (2.13), $a \mapsto q$]:

$$A_n = \frac{1-q^{2n+1}}{1-q}(-1)^n q^{\binom{n}{2}} \qquad \text{and} \qquad B_n = \delta_{n,0}.$$

Using the linearity of Bailey pairs, we have that $(\alpha_n - A_n\beta_0, \beta_n - B_n\beta_0)$ is a Bailey pair relative to q satisfying the hypothesis of Lemma 4.1. A short computation then gives the result. ■

The third lemma is the necessary lemma for our proof of the case $a = m-1$.

Lemma 4.3. *If (α_n, β_n) is a Bailey pair relative to q, then (α'_n, β'_n) is a Bailey pair relative to q^2, where*

$$\alpha'_n = \frac{1}{1-q^2}\left(\frac{1+q^{n+1}}{1-q^{2n+3}}\alpha_{n+1} - \frac{q^n\big(1+q^{n+1}\big)}{1-q^{2n+1}}\alpha_n\right) + \frac{2\beta_0}{(q)_2}\big(1+q^{n+1}\big)(-1)^n q^{\binom{n+1}{2}}$$

and

$$\beta'_n = \big(1+q^{n+1}\big)\beta_{n+1}.$$

Proof. The result follows from taking the negative of the Bailey pair in Lemma 2.3 with $x = q$ plus twice the Bailey pair in Lemma 4.2. ■

We are now ready to prove the key result on Bailey pairs.

Lemma 4.4. *Let $m \geq 1$ and $0 \leq a \leq m$. Suppose (α_n, β_n) is a Bailey pair relative to q and let f_n be as in Lemma 3.1. Then (α'_n, β'_n) is a Bailey pair relative to q, where*

$$\alpha'_n = \begin{cases} \dfrac{1}{1-q}(-1)^n q^{m\binom{n+1}{2}}\big(q^{a(n+1)}f_{n+1} - q^{-an+2n-1}f_{n-1}\big) & \textit{if } 0 \leq a \leq m-1, \\ \dfrac{1}{1-q}(-1)^n q^{m\binom{n+1}{2}}\big(1-q^{2n+1}\big)f_n & \textit{if } a = m, \end{cases}$$

and

$$\beta_n' = \beta_{n_{m+1}}' \tag{4.1}$$
$$= \frac{1}{(q)_{n_{m+1}}} \sum_{n_m,\dots,n_1\geq 0} \frac{(-1)^{n_m} q^{\binom{n_m+1}{2}+\cdots+\binom{n_1+1}{2}} \left(q^2;q^2\right)_{n_1+\delta_{a,0}}}{(q)_{n_{m+1}-n_m}(-q)_{n_m}} \beta_{n_1+\delta_{a,0}} \prod_{i=1}^{m-1} \begin{bmatrix} n_{i+1}+\delta_{a,i} \\ n_i \end{bmatrix}.$$

Proof. The proof is broken up into three cases. First suppose that $a = m$. In this case, we apply Step 1 as in the proof of Lemma 3.1, except that at the final iteration of Lemma 2.1 we use $b \mapsto q$ and $c \to \infty$ instead of $b \mapsto -q$ and $c \to \infty$. The resulting pair is

$$\alpha_n = (-1)^n q^{m\binom{n+1}{2}} \frac{1-q^{2n+1}}{1-q} f_n$$

and

$$\beta_n = \beta_{n_{m+1}} = \frac{1}{(q)_{n_{m+1}}} \sum_{n_m,\dots,n_1\geq 0} \frac{(q)_{n_m}(-1)^{n_m} q^{\binom{n_m+1}{2}+\cdots+\binom{n_1+1}{2}}(-q)_{n_1}}{(-q)_{n_m}(q)_{n_{m+1}-n_m}\cdots(q)_{n_2-n_1}} \beta_{n_1}.$$

Multiplying the numerator and denominator of the above sum by $(q)_{n_1}\cdots(q)_{n_{m-1}}$ and converting to q-binomial coefficients gives (4.1).

Next assume that $a < m-1$. In this case, we follow the same steps as in the proof of Lemma 3.1 except that in the very last iteration of Step 5 we use $b = q$ and $c \to \infty$ instead of $b = -q$ and $c \to \infty$. The result is

$$\alpha_n = \frac{1}{1-q}(-1)^n q^{m\binom{n+1}{2}} \left(q^{a(n+1)} f_{n+1} - q^{-an+2n-1} f_{n-1}\right)$$

and either

$$\beta_n = \beta_{n_{m+1}} = \frac{1}{(q)_{n_{m+1}}} \sum_{n_m,\dots,n_1\geq 0} \frac{(q)_{n_m}(-1)^{n_m} q^{\binom{n_m+1}{2}+\cdots+\binom{n_1+1}{2}}\left(1-q^{n_{a+1}+1}\right)(-q)_{n_1}}{(-q)_{n_m}(q)_{n_{m+1}-n_m}\cdots(q)_{n_{a+1}+1-n_a}\cdots(q)_{n_2-n_1}} \beta_{n_1}$$

if $a > 0$, or

$$\beta_n = \beta_{n_{m+1}} = \frac{1}{(q)_{n_{m+1}}} \sum_{n_m,\dots,n_1\geq 0} \frac{(q)_{n_m}(-1)^{n_m} q^{\binom{n_m+1}{2}+\cdots+\binom{n_1+1}{2}}\left(1-q^{n_1+1}\right)(-q)_{n_1+1}}{(-q)_{n_m}(q)_{n_{m+1}-n_m}\cdots(q)_{n_2-n_1}} \beta_{n_1+1}$$

if $a = 0$. Rewriting in terms of q-binomial coefficients gives the result.

Finally, we consider the case $a = m-1$. If $a > 0$, then we begin with the Bailey pair in (3.4) and (3.5), then redo Steps 2–4 with modifications as follows.

Step 2: Applying Lemma 4.3, we obtain a Bailey pair relative to q^2,

$$\alpha_n = \frac{1+q^{n+1}}{(q)_2}\left(q^{a\binom{n+2}{2}} f_{n+1} - q^{n+a\binom{n+1}{2}} f_n\right) + 2\frac{\left(1+q^{n+1}\right)}{(q)_2}(-1)^n q^{\binom{n+1}{2}}$$
$$= \frac{1+q^{n+1}}{(q)_2} q^{(a+2)\binom{n+1}{2}-n^2}\left(-f_n + q^{a(n+1)-n} f_{n+1}\right) + 2\frac{\left(1+q^{n+1}\right)}{(q)_2}(-1)^n q^{\binom{n+1}{2}}$$

and

$$\beta_n = \beta_{n_{a+1}} = \frac{1}{(-q)_{n_{a+1}}} \sum_{n_a,\dots,n_1\geq 0} \frac{q^{\binom{n_a+1}{2}+\cdots+\binom{n_1+1}{2}}(-q)_{n_1}}{(q)_{n_{a+1}+1-n_a}\cdots(q)_{n_2-n_1}} \beta_{n_1}.$$

Step 3: Using Lemma 2.1 with $x \mapsto q^2$, $b \mapsto q^2$ and $c \to \infty$, we obtain another Bailey pair relative to q^2,

$$\alpha_n = \frac{1-q^{2n+2}}{(1-q)(q)_2}(-1)^n q^{(a+3)\binom{n+1}{2}-n^2}\left(-f_n + q^{a(n+1)-n} f_{n+1}\right) + 2\frac{(1-q^{2n+2})}{(1-q)(q)_2} q^{n^2+n}$$

and

$$\beta_n = \beta_{n_{a+2}} = \frac{1}{(1-q)(q)_{n_{a+2}}} \sum_{n_{a+1},\dots,n_1 \geq 0} \frac{(-1)^{n_{a+1}}(q)_{n_{a+1}+1} q^{\binom{n_{a+1}+1}{2}+\cdots+\binom{n_1+1}{2}}(-q)_{n_1}}{(-q)_{n_{a+1}}(q)_{n_{a+2}-n_{a+1}}(q)_{n_{a+1}+1-n_a}\cdots(q)_{n_2-n_1}} \beta_{n_1}.$$

The factor $(1-q)$ cancels.

Step 4: Use Lemma 2.2 with $x \mapsto q^2$ and $b, c \mapsto q$ to obtain a Bailey pair relative to q. This does not change the β_n, which, after rewriting in terms of q-binomial coefficients, coincides with (4.1) for $a = m-1$ and $a > 0$. As for the α_n, for $n > 0$ we have

$$\begin{aligned}\alpha_n &= \frac{1}{1-q}\left((-1)^n q^{(a+3)\binom{n+1}{2}-n^2}\left(-f_n + q^{a(n+1)-n} f_{n+1}\right)\right) + \frac{2q^{n^2+n}}{1-q} \\ &\quad - \frac{1}{1-q}\left((-1)^{n-1} q^{2n+(a+3)\binom{n}{2}-(n-1)^2}\left(-f_{n-1} + q^{an-(n-1)} f_n\right)\right) - \frac{2q^{2n}q^{n^2-n}}{1-q} \\ &= \frac{1}{1-q}\left((-1)^n q^{(a+1)\binom{n+1}{2}}\left(q^{a(n+1)} f_{n+1} - q^{-an+2n-1} f_{n-1}\right)\right).\end{aligned}$$

Using (2.4) and (3.2), the above is also valid for $n = 0$. This completes the proof except for $m = 1$, $a = 0$. For this case, the argument is similar. One applies Steps 2–4 beginning with the Bailey pair (α_n, β_n) in the statement of the lemma. ■

With the Bailey pair from Lemma 4.4 in hand, we are now ready to deduce Theorem 1.3.

Proof of Theorem 1.3. Let (α_n, β_n) be a Bailey pair relative to q satisfying (3.1) and (3.2). Applying (2.1) to the Bailey pair in Lemma 4.4, we obtain

$$\begin{aligned}&\sum_{n_m,\dots,n_1 \geq 0} \frac{(-1)^{n_m} q^{\binom{n_m+1}{2}+\cdots+\binom{n_1+1}{2}}(q^2;q^2)_{n_1+\delta_{a,0}}}{(-q)_{n_m}} \beta_{n_1+\delta_{a,0}} \prod_{i=1}^{m-1} \begin{bmatrix} n_{i+1}+\delta_{a,i} \\ n_i \end{bmatrix} \\ &= \begin{cases} \displaystyle\sum_{n \geq 0} (-1)^n q^{m\binom{n+1}{2}}\left(q^{a(n+1)} f_{n+1} - q^{-an+2n-1} f_{n-1}\right) & \text{if } 0 \leq a \leq m-1, \\ \displaystyle\sum_{n \geq 0} (-1)^n q^{m\binom{n+1}{2}}\left(1-q^{2n+1}\right) f_n & \text{if } a = m. \end{cases}\end{aligned} \tag{4.2}$$

If f_n also satisfies (1.4), then a short computation as in the proof of Theorem 1.2 converts the right-hand side of (4.2) into the right-hand side of (1.8). ■

Using Theorem 1.3, we obtain false theta companions for each family of identities established in Section 3. The proofs use the same Bailey pairs but in (1.8) instead of (1.5). For example, as noted in the introduction, Theorem 1.4 follows upon inserting (1.6) and (1.7) in (1.8).

The false theta companions to (3.7) and (3.8), obtained using the Bailey pairs corresponding to (3.9) and (3.10) in Theorem 1.3 are collected in the following two results.

Theorem 4.5. *For $m \geq 1$ and $0 \leq a \leq m$, we have*

$$\sum_{n_m,\dots,n_1 \geq 0} \frac{(-1)^{n_m} q^{n_m^2+n_m+\cdots+n_1^2+n_1}}{(-q^2;q^2)_{n_m}(-q;q^2)_{n_1+\delta_{a,0}}} \prod_{i=1}^{m-1} \begin{bmatrix} n_{i+1}+\delta_{a,i} \\ n_i \end{bmatrix}_{q^2}$$

$$= \begin{cases} \sum_{n\in\mathbb{Z}} \operatorname{sgn}(-n) q^{m(n^2-n)+n(3n-1)/2+2an} & \text{if } 0 \le a \le m-1, \\ \sum_{n\in\mathbb{Z}} \operatorname{sgn}(n) q^{m(n^2+n)+n(3n-1)/2} & \text{if } a = m. \end{cases}$$

Theorem 4.6. *For $m \geq 1$ and $0 \leq a \leq m$, we have*

$$\sum_{n_m,\dots,n_1\ge 0} \frac{(-1)^{n_m} q^{\binom{n_m+1}{2}+\cdots+\binom{n_1+1}{2}} \left(-1;q^2\right)_{n_1+\delta_{a,0}}}{(-q)_{n_m}\left(q;q^2\right)_{n_1+\delta_{a,0}}} \prod_{i=1}^{m-1} \begin{bmatrix} n_{i+1}+\delta_{a,i} \\ n_i \end{bmatrix}$$
$$= \begin{cases} \sum_{n\in\mathbb{Z}} \operatorname{sgn}(-n)(-1)^{\binom{n+1}{2}} q^{(m+1)\binom{n}{2}+an} & \text{if } 0 \le a \le m-1, \\ \sum_{n\in\mathbb{Z}} \operatorname{sgn}(n)(-1)^{\binom{n+1}{2}} q^{(m+1)\binom{n+1}{2}-n} & \text{if } a = m. \end{cases}$$

Finally, we record the false theta companions to the identities in Theorem 3.5. To keep the expressions concise, we use the Legendre symbol

$$\left(\frac{n}{3}\right) = \begin{cases} 1 & \text{if } n \equiv 1 \pmod 3, \\ -1 & \text{if } n \equiv -1 \pmod 3, \\ 0 & \text{if } n \equiv 0 \pmod 3. \end{cases}$$

Theorem 4.7. *For $m \geq 1$ and $0 \leq a \leq m$, we have*

$$\sum_{n_m,\dots,n_1\ge 0} \frac{(-1)^{n_m} q^{\binom{n_m+1}{2}+\cdots+\binom{n_1+1}{2}}}{(-q)_{n_m}\left(q;q^2\right)_{n_1+\delta_{a,0}}} \prod_{i=1}^{m-1} \begin{bmatrix} n_{i+1}+\delta_{a,i} \\ n_i \end{bmatrix}$$
$$= \begin{cases} \sum_{n\in\mathbb{Z}} \operatorname{sgn}(-n)(-1)^n \left(\frac{1-n}{3}\right) q^{\frac{3m+4}{3}\binom{n+1}{2}+(a-m-1)n} & \text{if } 0 \le a \le m-1, \\ \sum_{n\in\mathbb{Z}} \operatorname{sgn}(n)(-1)^n \left(\frac{1-n}{3}\right) q^{\frac{3m+4}{3}\binom{n+1}{2}-n} & \text{if } a = m, \end{cases} \tag{4.3}$$

$$\sum_{n_m,\dots,n_1\ge 0} \frac{(-1)^{n_m} q^{\binom{n_m+1}{2}+\cdots+\binom{n_1+1}{2}+(n_1+\delta_{a,0})^2-(n_1+\delta_{a,0})}}{(-q)_{n_m}\left(q;q^2\right)_{n_1+\delta_{a,0}}} \prod_{i=1}^{m-1} \begin{bmatrix} n_{i+1}+\delta_{a,i} \\ n_i \end{bmatrix}$$
$$= \begin{cases} \sum_{n\in\mathbb{Z}} \operatorname{sgn}(-n)(-1)^n \left(\frac{1-n}{3}\right) q^{\frac{3m+2}{3}\binom{n+1}{2}+(a-m-1)n} & \text{if } 0 \le a \le m-1, \\ \sum_{n\in\mathbb{Z}} \operatorname{sgn}(n)(-1)^n \left(\frac{1-n}{3}\right) q^{\frac{3m+2}{3}\binom{n+1}{2}-n} & \text{if } a = m, \end{cases} \tag{4.4}$$

$$\sum_{n_m,\dots,n_1\ge 0} \frac{(-1)^{n_m} q^{\binom{n_m+1}{2}+\cdots+\binom{n_1+1}{2}} \left(-1;q^3\right)_{n_1+\delta_{a,0}}}{(-q)_{n_m}\left(q;q^2\right)_{n_1+\delta_{a,0}}(-1;q)_{n_1+\delta_{a,0}}} \prod_{i=1}^{m-1} \begin{bmatrix} n_{i+1}+\delta_{a,i} \\ n_i \end{bmatrix}$$
$$= \begin{cases} \sum_{n\in\mathbb{Z}} \operatorname{sgn}(-n)(-1)^n \left(\frac{1-n}{3}\right) q^{(m+1)\binom{n+1}{2}+(a-m-1)n} & \text{if } 0 \le a \le m-1, \\ \sum_{n\in\mathbb{Z}} \operatorname{sgn}(n)(-1)^n \left(\frac{1-n}{3}\right) q^{(m+1)\binom{n+1}{2}-n} & \text{if } a = m. \end{cases}$$

We close this section by noting that a number of the bases cases of Theorems 4.5–4.7 appear in classical work of Rogers [19]. For example, the base cases corresponding to $m = 1$ of Theorem 4.5 are found at [19, p. 334, equation (7)]. The case $m = 1$ and $a = 0$ of Theorem 4.6 is found at [19, p. 333, equation (5)]. Finally, the two cases of (4.3) and (4.4) corresponding to $m = 1$ are at [19, p. 332, equation (1)] and [19, p. 332, equation (2)], respectively.

5 Proof of Theorem 1.6 and applications

We begin this section with the following Bailey lemma.

Lemma 5.1. *Let $m \geq 1$ and $0 \leq a \leq m$. Suppose that (α_n, β_n) is a Bailey pair relative to 1 and that there is a sequence $(g_n)_{n\geq 0}$ with $g_0 = 1$ and*

$$\alpha_n = \begin{cases} 1 & \text{if } n = 0, \\ (1+q^n)g_n & \text{if } n > 1. \end{cases}$$

Then (α'_n, β'_n) is a Bailey pair relative to 1, where

$$\alpha'_n = \begin{cases} \dfrac{1+q^{\frac{a}{2}}g_1}{1-q^{\frac{1}{2}}} & \text{if } n = 0, \\ q^{\frac{m}{2}n^2+n}\left(\dfrac{\left(g_n + q^{n(a-1)+\frac{a}{2}}g_{n+1}\right)}{1-q^{n+\frac{1}{2}}} - \dfrac{q^{-\frac{1}{2}}\left(g_n + q^{-n(a-1)+\frac{a-2}{2}}g_{n-1}\right)}{1-q^{n-\frac{1}{2}}}\right) & \text{if } n > 0, \end{cases} \tag{5.1}$$

and

$$\begin{aligned}\beta'_n &= \beta_{n_{m+1}} \\ &= \frac{1}{\left(-q^{\frac{1}{2}}, q\right)_{n_{m+1}}} \sum_{n_m,\ldots,n_1\geq 0} q^{\frac{1}{2}(n_m^2+n_{m-1}^2+\cdots+n_1^2)}\left(-q^{\frac{1}{2}}, q\right)_{n_1+\delta_{a,0}}\beta_{n_1+\delta_{a,0}} \prod_{i=1}^{m} \begin{bmatrix} n_{i+1}+\delta_{a,i} \\ n_i \end{bmatrix}.\end{aligned} \tag{5.2}$$

Proof. As in the two previous sections, we break the proof into several steps.

Step 1: We first assume that $a > 0$, postponing the case $a = 0$ case until later (see Step 6). Iterating Lemma 2.1 a times with $x \mapsto 1$, $b \mapsto -q^{\frac{1}{2}}$, and $c \to \infty$, we obtain the following pair relative to 1:

$$\alpha_n = \begin{cases} 1 & \text{if } n = 0, \\ q^{\frac{a}{2}n^2}(1+q^n)g_n & \text{if } n > 0, \end{cases} \tag{5.3}$$

and

$$\beta_n = \beta_{n_{a+1}} = \frac{1}{\left(-q^{\frac{1}{2}}\right)_{n_{a+1}}} \sum_{n_a,\ldots,n_1\geq 0} \frac{q^{\frac{1}{2}(n_a^2+n_{a-1}^2+\cdots+n_1^2)}\left(-q^{\frac{1}{2}}\right)_{n_1}}{(q)_{n_{a+1}-n_a}(q)_{n_a-n_{a-1}}\cdots(q)_{n_2-n_1}}\beta_{n_1}. \tag{5.4}$$

Step 2: Now we apply Lemma 2.3 with $x \mapsto 1$ to obtain a Bailey pair relative to q:

$$\begin{aligned}\alpha_n &= \begin{cases} \dfrac{1+q^{\frac{a}{2}}g_1}{1-q} & \text{if } n = 0, \\ \dfrac{q^{\frac{a}{2}n^2+n}}{1-q}\left(g_n + q^{n(a-1)+\frac{a}{2}}g_{n+1}\right) & \text{if } n > 0 \end{cases} \\ &= \frac{q^{\frac{a}{2}n^2+n}}{1-q}\left(g_n + q^{n(a-1)+\frac{a}{2}}g_{n+1}\right),\end{aligned}$$

and

$$\beta_n = \beta_{n_{a+1}} = \frac{\left(1-q^{n_{a+1}+1}\right)}{\left(-q^{\frac{1}{2}}\right)_{n_{a+1}+1}} \sum_{n_a,\ldots,n_1\geq 0} \frac{q^{\frac{1}{2}(n_a^2+n_{a-1}^2+\cdots+n_1^2)}\left(-q^{\frac{1}{2}}\right)_{n_1}}{(q)_{n_{a+1}+1-n_a}(q)_{n_a-n_{a-1}}\cdots(q)_{n_2-n_1}}\beta_{n_1}.$$

Step 3: Next we apply Lemma 2.1 with $x \mapsto q$, $b \mapsto -q^{\frac{3}{2}}$, and $c \to \infty$, and then cancel a factor of $\big(1+q^{\frac{1}{2}}\big)^{-1}$ in the resulting α_n and β_n to obtain the following Bailey pair relative to $x = q$:

$$\alpha_n = \frac{q^{\frac{a+1}{2}n^2+n}}{1-q}\big(g_n + q^{n(a-1)+\frac{a}{2}}g_{n+1}\big)\big(1+q^{n+\frac{1}{2}}\big)$$

and

$$\beta_n = \beta_{n_{a+2}} = \frac{1}{\big(-q^{\frac{1}{2}}\big)_{n_{a+2}}} \sum_{n_{a+1},\dots,n_1\ge 0} \frac{\big(1-q^{n_{a+1}+1}\big)q^{\frac{1}{2}(n_{a+1}^2+n_a^2+\cdots+n_1^2)}\big(-q^{\frac{1}{2}}\big)_{n_1}}{(q)_{n_{a+2}-n_{a+1}}(q)_{n_{a+1}+1-n_a}(q)_{n_a-n_{a-1}}\cdots(q)_{n_2-n_1}}\beta_{n_1}.$$

Step 4: In the next step, we retain the β_n, but change α_n so that the new Bailey pair is relative to $x = 1$, by using Lemma 2.2 with $x \mapsto q$ and $b, c \mapsto q^{\frac{1}{2}}$. Keeping in mind (2.4) for the case $n = 0$, we have

$$\alpha_n = \begin{cases} \dfrac{1+q^{\frac{a}{2}}g_1}{1-q^{\frac{1}{2}}} & \text{if } n = 0\\[2ex] q^{\frac{a+1}{2}n^2+n}\left(\dfrac{\big(g_n+q^{n(a-1)+\frac{a}{2}}g_{n+1}\big)}{1-q^{n+\frac{1}{2}}} - \dfrac{q^{-\frac{1}{2}}\big(g_n+q^{-n(a-1)+\frac{a-2}{2}}g_{n-1}\big)}{1-q^{n-\frac{1}{2}}}\right) & \text{if } n > 0, \end{cases}$$

and

$$\beta_n = \beta_{n_{a+2}} = \frac{1}{\big(-q^{\frac{1}{2}}\big)_{n_{a+2}}} \sum_{n_{a+1},\dots,n_1\ge 0} \frac{\big(1-q^{n_{a+1}+1}\big)q^{\frac{1}{2}(n_{a+1}^2+n_a^2+\cdots+n_1^2)}\big(-q^{\frac{1}{2}}\big)_{n_1}}{(q)_{n_{a+2}-n_{a+1}}(q)_{n_{a+1}-n_a+1}(q)_{n_a-n_{a-1}}\cdots(q)_{n_2-n_1}}\beta_{n_1}.$$

Step 5: If $a \neq m$, we use Lemma 2.1 $m-a-1$ times with $x \mapsto 1$, $b \mapsto -q^{\frac{1}{2}}$ and $c \to \infty$. Converting the β_n to q-binomial notation then gives the required Bailey pair relative to $x = 1$.

If $a = m$, we use Lemma 2.1 with $x \mapsto 1$, $b \mapsto -q^{\frac{1}{2}}$, and $c \to 0$ to obtain a new Bailey pair relative to $x = 1$. This gives the correct α_n in (5.1), while

$$\begin{aligned}\beta_n &= \frac{q^{-\frac{n^2}{2}}}{\big(-q^{\frac{1}{2}}\big)_n}\sum_{n_{m+2}=0}^{n}(-1)^{n-n_{m+2}}q^{\binom{n-n_{m+2}}{2}}\frac{\big(-q^{\frac{1}{2}}\big)_{n_{m+2}}}{(q)_{n-n_{m+2}}}\\ &\quad\times\frac{1}{\big(-q^{\frac{1}{2}}\big)_{n_{m+2}}}\sum_{n_{m+1},\dots,n_1\ge 0}\frac{\big(1-q^{n_{m+1}+1}\big)q^{\frac{1}{2}(n_{m+1}^2+n_m^2+\cdots+n_1^2)}\big(-q^{\frac{1}{2}}\big)_{n_1}}{(q)_{n_{m+2}-n_{m+1}}(q)_{n_{m+1}-n_m+1}(q)_{n_m-n_{m-1}}\cdots(q)_{n_2-n_1}}\beta_{n_1}.\end{aligned}$$

We now sum over n_{m+2} first, using the q-binomial theorem [4, equation (3.3.6)]:

$$\sum_{n_{m+2}=0}^{n}(-1)^{n-n_{m+2}}q^{\binom{n-n_{m+2}}{2}}\frac{1}{(q)_{n-n_{m+2}}(q)_{n_{m+2}-n_{m+1}}} = \delta_{n,n_{m+1}}.$$

Thus, the β_n simplifies to

$$\beta_n = \beta_{n_{m+1}} = \frac{1}{\big(-q^{\frac{1}{2}}\big)_{n_{m+1}}}\sum_{n_m,\dots,n_1\ge 0}\frac{\big(1-q^{n_{m+1}+1}\big)q^{\frac{1}{2}(n_m^2+\cdots+n_1^2)}\big(-q^{\frac{1}{2}}\big)_{n_1}}{(q)_{n_{m+1}-n_m+1}(q)_{n_m-n_{m-1}}\cdots(q)_{n_2-n_1}}\beta_{n_1},$$

and converting to q-binomial coefficients gives the required (5.2).

Step 6: If $a = 0$, we skip Step 1 and use Steps 2–5. We obtain the Bailey pair with α_n given by (5.1) for $a = 0$ and

$$\beta_n = \beta_{n_{m+1}} = \frac{1}{\big(-q^{\frac{1}{2}}\big)_{n_{m+1}}}\sum_{n_m,\dots,n_1\ge 0}\frac{q^{\frac{1}{2}(n_m^2+\cdots+n_1^2)}\big(1-q^{n_1+1}\big)\big(-q^{\frac{1}{2}}\big)_{n_1+1}}{(q)_{n_{m+1}-n_m}\cdots(q)_{n_2-n_1}}\beta_{n_1+1}.$$

Converting to q-binomial notation completes the proof. ∎

Armed with the Bailey pair in Lemma 5.1, we now prove Theorem 1.6.

Proof of Theorem 1.6. Subtracting the case $a-1$ from the case a of Lemma 5.1, we obtain a new Bailey pair (α_n, β_n) relative to 1. For $n = 0$, we have

$$\alpha_0 = \frac{1+q^{\frac{a}{2}}g_1}{1-q^{\frac{1}{2}}} - \frac{1+q^{\frac{a-1}{2}}g_1}{1-q^{\frac{1}{2}}} = -q^{\frac{a-1}{2}}g_1,$$

and for $n > 0$ we have

$$\begin{aligned}\alpha_n &= q^{\frac{m}{2}n^2+n}\left(\frac{\left(g_n + q^{n(a-1)+\frac{a}{2}}g_{n+1}\right)}{1-q^{n+\frac{1}{2}}} - \frac{q^{-\frac{1}{2}}\left(g_n + q^{-n(a-1)+\frac{a-2}{2}}g_{n-1}\right)}{1-q^{n-\frac{1}{2}}}\right)\\ &\quad - q^{\frac{m}{2}n^2+n}\left(\frac{\left(g_n + q^{n(a-2)+\frac{a-1}{2}}g_{n+1}\right)}{1-q^{n+\frac{1}{2}}} - \frac{q^{-\frac{1}{2}}\left(g_n + q^{-n(a-2)+\frac{a-3}{2}}g_{n-1}\right)}{1-q^{n-\frac{1}{2}}}\right)\\ &= -q^{\frac{m}{2}n^2+n}\left(q^{n(a-2)+\frac{a-1}{2}}g_{n+1} + q^{-n(a-1)+\frac{a-3}{2}}g_{n-1}\right).\end{aligned}$$

Now using (5.2) and (2.1) along with the fact that

$$\lim_{n_{m+1}\to\infty}\begin{bmatrix} n_{m+1}+\delta_{a,m} \\ n_m \end{bmatrix} = \frac{1}{(q)_{n_m}},$$

we obtain

$$\text{l.h.s. of (1.12)} = \frac{(-q^{\frac{1}{2}})_\infty}{(q)_\infty}\left(-q^{\frac{a-1}{2}}g_1 - \sum_{n\geq 1} q^{\frac{m}{2}n^2+n}\left(q^{n(a-2)+\frac{a-1}{2}}g_{n+1} + q^{-n(a-1)+\frac{a-3}{2}}g_{n-1}\right)\right).$$

If g_n satisfies (1.11), then we compute

$$\begin{aligned}\text{l.h.s. of (1.12)} &= -q^{\frac{a-1}{2}}\frac{(-q^{\frac{1}{2}})_\infty}{(q)_\infty}\left(\sum_{n\geq 0} q^{\frac{m}{2}n^2+an-n}g_{n+1} + \sum_{n\geq 1} q^{\frac{m}{2}n^2-an+2n-1}g_{n-1}\right)\\ &= -q^{\frac{a-1}{2}}\frac{(-q^{\frac{1}{2}})_\infty}{(q)_\infty}\left(\sum_{n\geq 0} q^{\frac{m}{2}n^2+an-n}g_{n+1} + \sum_{n\leq -1} q^{\frac{m}{2}n^2+an-2n-1}g_{-n-1}\right)\\ &= -q^{\frac{a-1}{2}}\frac{(-q^{\frac{1}{2}})_\infty}{(q)_\infty}\sum_{n\in\mathbb{Z}} q^{\frac{m}{2}n^2+na-n}g_{n+1},\end{aligned}$$

as desired. ∎

We now give several applications of Theorem 1.6, beginning with Theorem 1.5.

Proof of Theorem 1.5. We use the Bailey pair relative to 1 [20, p. 468],

$$\alpha_n = \begin{cases} 1 & \text{if } n = 0, \\ (-1)^n q^{n^2-\frac{1}{2}n}(1+q^n) & \text{if } n > 0, \end{cases} \qquad \text{and} \qquad \beta_n = \frac{1}{(-q^{\frac{1}{2}})_n (q)_n},$$

corresponding to $g_n = (-1)^n q^{n^2-\frac{n}{2}}$. Using this in Theorem 1.6 with $q = q^2$, we obtain

$$\begin{aligned}\mathcal{S}_{m,a} - \mathcal{S}_{m,a-1} &= \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} q^a \sum_{n\in\mathbb{Z}} (-1)^n q^{(m+2)n^2-2na-n}\\ &= q^a \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\cdot\left(q^{m+1-2a}, q^{m+3+2a}, q^{2m+4}; q^{2m+4}\right)_\infty,\end{aligned}$$

by the triple product identity (2.2). ∎

Next, for $m \geq 1$ and $0 \leq a \leq m$ define

$$\mathcal{R}_{m,a} = \sum_{n_m,\ldots,n_1 \geq 0} \frac{q^{n_m^2+\cdots+n_1^2}\left(-q;q^2\right)_{n_1+\delta_{a,0}}}{\left(q^2;q^2\right)_{n_m}} \prod_{i=1}^{m-1} \begin{bmatrix} n_{i+1}+\delta_{a,i} \\ n_i \end{bmatrix}_{q^2}.$$

Theorem 5.2. *For $1 \leq a \leq m$, we have*

$$\mathcal{R}_{m,a} - \mathcal{R}_{m,a-1} = q^{a+1} \frac{\left(-q;q^2\right)_\infty}{\left(q^2;q^2\right)_\infty} \left(q^{m+6+2a}, q^{m-2a}, q^{2m+6}; q^{2m+6}\right)_\infty.$$

Proof. We use the Bailey pair relative to 1 [20, p. 468, B(1)],

$$\alpha_n = \begin{cases} 1 & \text{if } n = 0, \\ (-1)^n q^{n(3n-1)/2}(1+q^n) & \text{if } n > 0, \end{cases} \tag{5.5}$$

and

$$\beta_n = \frac{1}{(q)_n}. \tag{5.6}$$

Applying Theorem 1.6 and then letting $q \mapsto q^2$ gives

$$\begin{aligned}\mathcal{R}_{m,a} - \mathcal{R}_{m,a-1} &= -q^{a-1} \frac{\left(-q;q^2\right)_\infty}{\left(q^2;q^2\right)_\infty} \sum_{n\in\mathbb{Z}} q^{mn^2+2an-2n}(-1)^{n+1} q^{(n+1)(3n+2)} \\ &= q^{a+1} \frac{\left(-q;q^2\right)_\infty}{\left(q^2;q^2\right)_\infty} \sum_{n\in\mathbb{Z}} q^{(m+3)n^2+(2a+3)n}(-1)^n,\end{aligned}$$

and the result follows from the triple product identity (2.2). ■

Analogous to (1.9), we have that $\mathcal{R}_{m,m}$ are products for $m \geq 1$.

Theorem 5.3. *For $m \geq 1$, we have*

$$\mathcal{R}_{m,m} = \frac{\left(-q;q^2\right)_\infty}{\left(q^2;q^2\right)_\infty} \left(q^{m+4}, q^{m+2}, q^{2m+6}; q^{2m+6}\right)_\infty.$$

Proof. We start with the Bailey pair in (5.5) and (5.6), and then consider the Bailey pair given by (5.3) and (5.4) with $a = m$. We then apply (2.1). Observing that

$$\begin{aligned}\sum_{n\geq 0} \alpha_n &= 1 + \sum_{n\geq 1} q^{\frac{m}{2}n^2}(1+q^n)(-1)^n q^{n(3n-1)/2} = \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{m+3}{2}n^2 - \frac{n}{2}} \\ &= \left(q^{\frac{m}{2}+2}, q^{\frac{m}{2}+1}, q^{m+3}; q^{m+3}\right)_\infty\end{aligned}$$

by (2.2), the result follows after dilating by $q \mapsto q^2$. ■

For our final application, for $m \geq 1$ and $0 \leq a \leq m$ define

$$\mathcal{T}_{m,a} = \sum_{n_m,\ldots,n_1 \geq 0} \frac{q^{n_m^2+\cdots+n_1^2}\left(-1;q^2\right)_{n_1+\delta_{a,0}}}{\left(q^2;q^2\right)_{n_m}\left(q,q^2\right)_{n_1+\delta_{a,0}}} \prod_{i=1}^{m-1} \begin{bmatrix} n_{i+1}+\delta_{a,i} \\ n_i \end{bmatrix}_{q^2}.$$

Theorem 5.4. *For $1 \leq a \leq m$, we have*

$$\mathcal{T}_{m,a} - \mathcal{T}_{m,a-1} = -q^{a-1} \frac{\left(-q;q^2\right)_\infty}{\left(q^2;q^2\right)_\infty} \left(-q^{m+2a}, -q^{m+2-2a}, q^{2m+2}; q^{2m+2}\right)_\infty.$$

Proof. In Theorem 1.6, we argue as usual using the Bailey pair relative to 1 [20, p. 468, corrected],

$$\alpha_n = \begin{cases} 1 & \text{if } n = 0, \\ q^{\binom{n}{2}}(1+q^n) & \text{if } n > 0, \end{cases} \tag{5.7}$$

and

$$\beta_n = \frac{(-1)_n}{\left(-q^{\frac{1}{2}}, q^{\frac{1}{2}}, q\right)_n}. \tag{5.8}$$

This concludes the proof. ∎

Again, the $\mathcal{T}_{m,m}$ are products for $m \geq 1$.

Theorem 5.5. *For $m \geq 1$, we have*

$$\mathcal{T}_{m,m} = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\left(-q^m, -q^{m+2}, q^{2m+2}; q^{2m+2}\right)_\infty.$$

Proof. We start with the Bailey pair in (5.7) and (5.8), and then consider the Bailey pair given by (5.3) and (5.4) with $a = m$. We then apply (2.1). Observing that

$$\sum_{n\geq 0} \alpha_n = 1 + \sum_{n\geq 1} q^{\frac{m}{2}n^2}(1+q^n)q^{\frac{n(n-1)}{2}} = \sum_{n\in\mathbb{Z}} q^{\frac{m+1}{2}n^2 - \frac{n}{2}} = \left(-q^{\frac{m}{2}}, -q^{\frac{m}{2}+1}, q^{m+1}; q^{m+1}\right)_\infty$$

by (2.2) gives the result after dilating by $q \mapsto q^2$. ∎

6 Concluding remarks

In this paper, we have shown how the families of q-series identities involving products of q-binomial coefficients like the one in (1.13) fit naturally into the theory of Bailey pairs. As noted in the introduction, this is not a surprise, given that several similar identities have already appeared in the literature in relation to Bailey pairs. For example, Hikami's variant of the Andrews–Gordon identities [11],

$$\sum_{n_{m-1},\dots,n_1\geq 0} \frac{q^{n_1^2+\cdots+n_{m-1}^2+n_{a+1}+\cdots+n_{m-1}}}{(q)_{n_{m-1}}} \prod_{i=1}^{m-2} \begin{bmatrix} n_{i+1}+\delta_{a,i} \\ n_i \end{bmatrix} = \frac{\left(q^{a+1}, q^{2m-a}, q^{2m+1}; q^{2m+1}\right)_\infty}{(q)_\infty},$$

valid for $m \geq 2$ and $0 \leq a \leq m-1$, can be proved using the theory of Bailey pairs, as can several other similar families of infinite product and false theta identities, [11, 16, 17].

Identities involving products of q-binomial coefficients similar to (1.13) also arose in connection with Schur's indices of certain $4d$ $N = 2$ Argyres–Douglas theories in [12]. For $t \geq 1$, $1 \leq s \leq t+1$ define (with the convention that $n_{t+1} = 0$):

$$\mathcal{D}_{t,s} = \sum_{n_1,\dots,n_t\geq 0} q^{n_s} \prod_{r=1}^{t} \frac{q^{n_r n_{r+1}+n_r}}{(q)_{n_r}^2}.$$

Then, for $k \geq 1$, $0 \leq i \leq k$, it was proved in [12, Theorems 6.1 and 7.1] that

$$(-1)^{k-i}\mathcal{D}_{2k,k+i+1} + 2\sum_{j=i+1}^{k} (-1)^{k-j}\mathcal{D}_{2k,k+j+1} = \frac{\left(q^{k-i+1}, q^{k+i+2}, q^{2k+3}; q^{2k+3}\right)_\infty}{(q)_\infty^{2k+1}}, \tag{6.1}$$

$$(-1)^{k-i}\mathcal{D}_{2k-1,k+i} + 2\sum_{j=i+1}^{k}(-1)^{k-j}\mathcal{D}_{2k-1,k+j} = \frac{1}{(q)_\infty^{2k}}\sum_{n\in\mathbb{Z}}\operatorname{sgn}(n)q^{(k+1)n^2+in}. \tag{6.2}$$

The first step in the proof of these identities involves using quantum dilogarithm to deduce

$$\mathcal{D}_{t,s} = \frac{1}{(q)_\infty^t}\sum_{m_1,\dots,m_{t-1}\geq 0}(-1)^{\sum_{j=2}^{t-1}m_j}q^{\binom{m_1+1}{2}+\sum_{j=1}^{t-1}\binom{m_j+1}{2}}\frac{(q)_{m_{t-1}+\delta_{t,s}}}{(q)_{m_1}^2}\prod_{i=1}^{t-2}\begin{bmatrix} m_i+\delta_{i,s-1} \\ m_{i+1}\end{bmatrix} \tag{6.3}$$

for $t\geq 2$, $2\leq s\leq t+1$. Here we have taken the liberty to rewrite the expressions in [12, Proposition 4.2] to more transparently exhibit the q-binomial coefficients of the shape (1.13). Crucially, the next step in the proof of (6.1) and (6.2) uses the theory of Bailey pairs to go from (6.3) to the theta and false theta counterparts, as appropriate [12, Section 5]. Note the slight change in the order of the variables: The summation variables in Theorem 1.1 are ordered subject to $n_{i+1}+\delta_{a,i}\geq n_i$, however in (6.3), they are ordered $m_i+\delta_{i,s-1}\geq m_{i+1}$. We omit the discussion of two further families of identities involving w-deformed $\mathcal{D}$ functions proved in [12].

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