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Symplectic Differential Reduction Algebras and Generalized Weyl Algebras

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Abstract. Given a map $\Xi\colon U(\mathfrak{g}) \to A$ of associative algebras, with $U(\mathfrak{g})$ the universal enveloping algebra of a (complex) finite-dimensional reductive Lie algebra $\mathfrak{g}$, the restriction functor from A-modules to $U(\mathfrak{g})$-modules is intimately tied to the representation theory of an A-subquotient known as the *reduction algebra* with respect to $(A, \mathfrak{g}, \Xi)$. Herlemont and Ogievetsky described differential reduction algebras for the general linear Lie algebra $\mathfrak{gl}(n)$ as algebras of deformed differential operators. Their map Ξ is a realization of $\mathfrak{gl}(n)$ in the N-fold tensor product of the n-th Weyl algebra tensored with $U(\mathfrak{gl}(n))$. In this paper, we further the study of differential reduction algebras by finding a presentation in the case when $\mathfrak{g}$ is the symplectic Lie algebra of rank two and Ξ is a canonical realization of $\mathfrak{g}$ inside the second Weyl algebra tensor the universal enveloping algebra of $\mathfrak{g}$, suitably localized. Furthermore, we prove that this differential reduction algebra is a generalized Weyl algebra (GWA), in the sense of Bavula, of a new type we term skew-affine. It is believed that symplectic differential reduction algebras are all skew-affine GWAs; then their irreducible weight modules could be obtained from standard GWA techniques.

Key words: Mickelsson algebras; Zhelobenko algebras; skew affine; quantum deformation; differential operators

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1 Introduction

1.1 Background

Mickelsson addressed the problem of reducing representations of Lie algebras over a subalgebra with the introduction of step algebras [25], which are now known as Mickelsson algebras. These algebras are part of a broader family, collectively called *reduction algebras* [27]. Specific applications and developments relying on reduction algebras are found in the decomposition of tensor product modules (via actions of diagonal reduction algebras) [13, 27], higher-order Fischer decompositions in harmonic analysis [9], conformal field theory [7, 8], and the construction of wave functions [2] in the realm of theoretical particle and nuclear physics, as examples. The last paper formalized extremal projectors, and Zhelobenko [35] is credited with realizing the importance of extremal projectors in a scheme to determine generators and relations of localized reduction algebras. See [37] for an overview of the construction of generalized Mickelsson algebras.

The crucial observation is that reduction algebras associated to a reductive Lie algebra $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+$ generally arise from a triple of data, that is, a Lie algebra homomorphism $\Xi\colon \mathfrak{g} \to A$ from $\mathfrak{g}$ to an associative algebra A or the resulting algebra map (perhaps also called Ξ) from the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ to A. We set I to be the left ideal in A generated by the positive nilpotent subalgebra $\mathfrak{g}_+ \subset \mathfrak{g}$ via the Ξ-induced action of $\mathfrak{g}$ on A. Then the reduction algebra $Z = Z(A, \mathfrak{g}, \Xi)$, up to a choice of localization, is the subquotient N/I, where the normalizer N is the largest subalgebra of A containing I as a two-sided ideal. A common case occurs when Ξ is an embedding of $\mathfrak{g}$ into the universal enveloping algebra $U(\mathfrak{t})$ of a Lie algebra $\mathfrak{t} \supset \mathfrak{g}$ containing $\mathfrak{g}$, and we write simply $Z = Z(\mathfrak{t}, \mathfrak{g})$. The same abbreviation of the map occurs in the super analogue or whenever the Lie (super)algebra homomorphism is understood. There is also a q-analog of this construction using Drinfeld–Jimbo quantum groups.

We now emphasize two points. The first is that reduction algebras are associative algebras with their own lanes of study. There are a great number of results on reduction algebras with many summarized at the end of Tolstoy's commemorative paper [29]. Khoroshkin and Ogievetsky furthered results by giving a complete presentation of the diagonal reduction algebra [27] of type A of any rank. With motivation drawn from the second author's work [32, 33] on finding explicit bases of tensor product representations of the orthosymplectic Lie superalgebras $\mathfrak{osp}(1|2n)$, we gave a complete presentation [13, Section 3.5] of the diagonal reduction superalgebra $Z(\mathfrak{osp}(1|2) \times \mathfrak{osp}(1|2), \mathfrak{osp}(1|2))$, the first recorded diagonal reduction algebra associated to a classical [21] Lie superalgebra. In the sequel [14], we classified finite-dimensional and certain infinite-dimensional representations of $Z(\mathfrak{osp}(1|2) \times \mathfrak{osp}(1|2), \mathfrak{osp}(1|2))$.

The second matter is that homomorphisms to the n-th Weyl algebra A_n (see the use of *canonical realizations* in [15, Definition 1]) yield examples of reduction algebras important to quantum deformations and noncommutative geometry. A particular example is the ring $\mathcal{D}_{\mathbf{h}}(n, N)$ of $\mathbf{h}$-deformed differential operators [17]. It is the reduction algebra defined from the diagonal map sending $\mathfrak{gl}(n)$ to (a localization of) the tensor product $U(\mathfrak{gl}(n)) \otimes A_{nN}$; the first factor relies on the definition of $U(\mathfrak{gl}(n))$, and the second factor uses N-copies of an oscillator realization of $\mathfrak{gl}(n)$.

In this paper, we combine the two points of emphasis above: We initiate the study of the symplectic analogue of $\mathcal{D}_{\mathbf{h}}(2, 1)$ by extending a classical oscillator realization of the rank two symplectic Lie algebra $\mathfrak{sp}(4)$ within A_2 such that the codomain algebra A is the tensor product $A_2 \otimes U(\mathfrak{sp}(4))$. We localize A to an algebra $\mathscr{A}$ in order to satisfy certain conditions allowing for the use of the $\mathfrak{sp}(4)$ extremal projector. After sorting out preliminaries in Section 2, we use the extremal projector to present a complete set of generators and relations of the *symplectic differential reduction algebra* $D(\mathfrak{sp}(4)) \cong Z(\mathscr{A}, \mathfrak{sp}(4))$ in Section 3. Furthermore, we remark that there is a homomorphism from the diagonal reduction algebra of $\mathfrak{sp}(4)$ to $D(\mathfrak{sp}(4))$, and therefore our presentation of the latter should be helpful in finding a presentation of the former. Lastly, we show in Section 4 that after a change of generators, $D(\mathfrak{sp}(4))$ is manifestly a generalized Weyl algebra (GWA) of a new and interesting type. In particular, the irreducible weight modules over $D(\mathfrak{sp}(4))$ can now be classified by understanding orbits and breaks in the maximal spectrum. See [4, 12, 24].

2 Preliminaries

The base field is $\mathbb{C}$ except when specified otherwise.

2.1 Oscillator realization and Chevalley generators

Weyl algebras. Let A_n denote the n-th Weyl algebra. Thus, A_n is the complex associative algebra with generators x_i, $\partial_i = \frac{\partial}{\partial x_i}$, for $i = 1, 2, \ldots, n$, and relations

$$x_i x_j - x_j x_i = 0, \qquad \partial_i \partial_j - \partial_j \partial_i = 0, \qquad \partial_i x_j - x_j \partial_i = \delta_{ij}, \tag{2.1}$$

where δ_{ij} is the Kronecker delta. The Weyl algebra A_n carries an involutive anti-automorphism[1] ϑ, called the symplectic Fourier transform, uniquely determined by

$$\vartheta(x_i) = \partial_i. \tag{2.2}$$

The rank two symplectic Lie algebra. Let $\mathfrak{g} = \mathfrak{sp}(4)$ be the 10-dimensional complex simple Lie algebra of Cartan type C_2. It consists of all 4×4 matrices of the form $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ where B and C are symmetric 2×2 matrices, A is any 2×2 matrix, and $D = -A^{\mathsf{T}}$. The oscillator realization of $\mathfrak{sp}(4)$ is a canonical algebra homomorphism $\omega\colon U(\mathfrak{sp}(4)) \to A_2$; see, for example, [10, Section 4.6] or [16]. The image of $\mathfrak{sp}(4)$ is $\mathrm{Span}_{\mathbb{C}}\{uv + vu \mid u, v \in \{x_1, x_2, \partial_1, \partial_2\}\}$. Since $\mathfrak{sp}(4)$ is simple, we may thus identify $\mathfrak{sp}(4)$ with the 10-dimensional Lie subalgebra of A_2 spanned by $\{a_{ij}, b_{ij}, c_{ij} \mid i = 1, 2\}$, where

$$a_{ij} = \frac{1}{2}(x_i\partial_j + \partial_j x_i) = x_i\partial_j + \frac{1}{2}\delta_{ij}, \qquad b_{ij} = b_{ji} = x_i x_j, \qquad c_{ij} = c_{ji} = \partial_i\partial_j. \tag{2.3}$$

Note that ϑ from (2.2) preserves the subspace $\mathfrak{sp}(4) \subset A_2$. We identify $\vartheta|_{\mathfrak{sp}(4)}$ with the involutive Lie algebra anti-automorphism τ given by $\tau(a_{ij}) = a_{ji}$, $\tau(b_{ij}) = c_{ij}$.

After this identification, the subspace $\mathfrak{h} = \mathbb{C}a_{11} \oplus \mathbb{C}a_{22}$ is a Cartan subalgebra of $\mathfrak{sp}(4)$. We use standard notation for the 8 roots of $\mathfrak{sp}(4)$, which are $\pm 2\epsilon_1$, $\pm 2\epsilon_2$, $\pm(\epsilon_1 - \epsilon_2)$, $\pm(\epsilon_1 + \epsilon_2)$, denoting the simple roots by $\alpha = \epsilon_1 - \epsilon_2$, $\beta = 2\epsilon_2$. The set of positive roots is $\Phi_+ = \{\beta, \beta + \alpha, \beta + 2\alpha, \alpha\}$. There are exactly two convex orderings [19] of the positive roots

$$\beta < \beta + \alpha < \beta + 2\alpha < \alpha, \qquad \alpha < \beta + 2\alpha < \beta + \alpha < \beta.$$

The corresponding root system is shown in Figure 1.

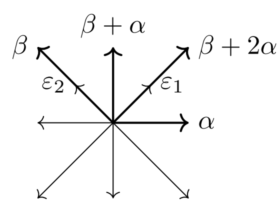

Figure 1. Root system of type C_2. Thick lines indicate a choice of positive roots.

Fix positive root vectors, with $\mathrm{i} = \sqrt{-1}$:

$$e_\alpha = a_{12}, \qquad e_\beta = \frac{\mathrm{i}}{2} b_{22}, \qquad e_{\beta+\alpha} = [e_\alpha, e_\beta], \qquad e_{\beta+2\alpha} = \frac{1}{2}[e_\alpha, e_{\beta+\alpha}].$$

These provide $\mathfrak{sl}(2)$-triples $(e_\gamma, f_\gamma, h'_\gamma)$, where $f_\gamma = \tau(e_\gamma)$ and $h'_\gamma = [e_\gamma, f_\gamma]$:

$$\begin{aligned}
&\alpha\colon\ (a_{12},\, a_{21},\, a_{11} - a_{22}) = (x_1\partial_2,\, x_2\partial_1,\, x_1\partial_1 - x_2\partial_2),\\
&\beta\colon\ \left(\frac{\mathrm{i}}{2} b_{22},\, \frac{\mathrm{i}}{2} c_{22},\, a_{22}\right) = \left(\frac{\mathrm{i}}{2} x_2^2 x_2,\, \frac{\mathrm{i}}{2}\partial_2^2,\, x_2\partial_2 + \frac{1}{2}\right),\\
&\beta + \alpha\colon\ (\mathrm{i}b_{12},\, \mathrm{i}c_{12},\, a_{11} + a_{22}) = (\mathrm{i}x_1 x_2,\, \mathrm{i}\partial_1\partial_2,\, x_1\partial_1 + x_2\partial_2 + 1),\\
&\beta + 2\alpha\colon\ \left(\frac{\mathrm{i}}{2} b_{11},\, \frac{\mathrm{i}}{2} c_{11},\, a_{11}\right) = \left(\frac{\mathrm{i}}{2} x_1 x_1,\, \frac{\mathrm{i}}{2}\partial_1\partial_1,\, x_1\partial_1 + \frac{1}{2}\right).
\end{aligned}$$

We reserve h_γ for the following integer-shifted coroots, which will simplify computations

$$h_\alpha = h'_\alpha, \qquad h_{\beta+2\alpha} = h'_{\beta+2\alpha} + 1, \qquad h_{\beta+\alpha} = h'_{\beta+\alpha} + 2, \qquad h_\beta = h'_\beta,$$

[1] That is, ϑ is $\mathbb{C}$-linear, $\vartheta(ab) = \vartheta(b)\vartheta(a)$, and $\vartheta^2 = \mathrm{Id}_{A_n}$.

where $h'_{\beta+\alpha} = h_\alpha + 2h_\beta$ and $h'_{\beta+2\alpha} = h_\alpha + h_\beta$. The simple coroots satisfy $\alpha(h_\beta) = -1$, $\beta(h_\alpha) = -2$. Also, note that any scalar shift of $h'_{\alpha+\beta}$ yields the total degree derivation on A_2 in the adjoint representation: $[x_1\partial_1 + x_2\partial_2 + 1, x_i] = x_i$ and $[x_1\partial_1 + x_2\partial_2 + 1, \partial_i] = -\partial_i$.

The map ζ. Composing the usual comultiplication in $U(\mathfrak{sp}(4))$ with the oscillator representation in the first tensor leg, we obtain an algebra homomorphism

$$\zeta\colon\ U(\mathfrak{sp}(4)) \to A_2 \otimes U(\mathfrak{sp}(4)), \qquad v \mapsto \omega(v) \otimes 1 + 1 \otimes v. \tag{2.4}$$

We denote the images of the $\mathfrak{sl}(2)$-triples under ζ by $(E_\gamma, F_\gamma, H'_\gamma)$. Similarly, put

$$H_\alpha = H'_\alpha, \qquad H_{\beta+2\alpha} = H'_{\beta+2\alpha} + 1, \qquad H_{\beta+\alpha} = H'_{\beta+\alpha} + 2, \qquad H_\beta = H'_\beta.$$

Remark 2.1. The tensoring of A_2 with $U(\mathfrak{sp}(4))$ is useful for the study of tensor product representations in which one tensor factor is an oscillator representation of $\mathfrak{sp}(4)$; however, the authors stress that there is no requirement in general for the codomain algebra to contain $U(\mathfrak{sp}(4))$ in order to study or apply reduction algebras. We observe that (2.3) does not yield an injective algebra homomorphism from $U(\mathfrak{sp}(4))$ to A_2 as, amongst many relations, $b_{12}c_{12} - a_{12}a_{21} + a_{11} - \frac{1}{2} = 0$. Yet one can study the associated reduction algebra, a topic we review next.

2.2 Reduction algebras

There is extensive literature [3, 14, 22, 26, 31, 35, 36] on reduction algebras and their applications. We provide a brief overview of constructions/perspectives and related notions for when $\mathfrak{g}$ is a complex finite-dimensional reductive Lie algebra $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+$ with fixed (maximal) nilpotent subalgebras $\mathfrak{g}_\pm$ and Cartan subalgebra $\mathfrak{h}$. See [13, Section 2] for details in the super case.

Subquotient algebra. Let A be an associative algebra and $\Xi\colon U(\mathfrak{g}) \to A$ a map of associative algebras. In particular, the map Ξ makes A into a $U(\mathfrak{g})$-bimodule via multiplication. Let $I = A \cdot \mathfrak{g}_+$ (written as $A\mathfrak{g}_+$ from hereon) be the left ideal in A generated by the image of $\mathfrak{g}_+$ under Ξ composed with the natural inclusion. The reduction algebra $Z = Z(A, \mathfrak{g}, \Xi)$ equals $N_A(I)/I$, where $N_A(I)$ is the normalizer of I in A. To clarify, the *normalizer* $N_A(I)$ of the left ideal I in A is the maximal subalgebra of A with respect to containment of I as a two-sided ideal. Hence as an A-subquotient the reduction algebra Z inherits its product from A. Note that normalizers may be termed *idealizers* in general ring theory [20, specifically Section 2 for historical context].

Remark 2.2. As mentioned in the introduction, the necessary data is encoded in a Lie algebra map $\Xi\colon \mathfrak{g} \to A$. Reduction algebras may be expressed as $Z(\mathfrak{t}, \mathfrak{g})$ (respectively, $Z(A, \mathfrak{g})$) if the map $\mathfrak{g} \to U(\mathfrak{t})$ (respectively, $\mathfrak{g} \to A$) is given in context.

Opposite endomorphism algebra. As $(\mathrm{End}_A(A))^{\mathrm{op}}$ is isomorphic to A, one might hope for a natural realization of the A-subquotient Z as endomorphisms of an A-module. Indeed, since $n \in N_A(I)$ implies $In \subset I$, right multiplication by $N_A(I)$ defines a right action on A/I such that $I \subset \mathrm{Ann}_{N_A(I)}(A/I)$. It follows that A/I is a right Z-module, in fact, an (A, Z)-bimodule. Then $\mathrm{End}_A(A/I)$ is a left Z-module isomorphic to the left Z-module Z, particularly since endomorphisms of A/I preserve Z and $1 + I \in Z$ generates A/I as an A-module. Now consider maps $1 + I \mapsto n + I$, $n \in N_A(I)$, which are precisely the elements of $\mathrm{End}_A(A/I)$ by the previous statement. Then opposite composition shows that the reduction algebra Z is isomorphic to $Z_{\mathrm{maps}} = (\mathrm{End}_A(A/I))^{\mathrm{op}}$, the opposite algebra of left A-module endomorphisms on A/I.

Invariants. For a left A-module V, the subspace of $\mathfrak{g}_+$-invariants is $V^+ = \{v \in V \mid ev = 0$ for all $e \in \mathfrak{g}_+\}$. Then since $N_A(I)V^+ \subset V^+$ and $IV^+ = \{0\}$, there is a natural action on V^+ by Z on the left. Taking the case where $V = A/I$, we have that $Z = (A/I)^+$ since $\mathfrak{g}_+(a + I) = \{0 + I\}$

if and only if $Ia \subset I$, the defining condition for $a \in N_A(I)$. We say then that A/I is the universal $\mathfrak{g}_+$-highest-weight A-module in the sense that given a pair (W, w) of a left A-module W containing a (singular) vector w (such as any $\mathfrak{g}_+$-highest-weight vector), there exists a unique A-module homomorphism $\Psi_w \colon A/I \to W$ sending $1 + I$ to w. Evaluation at $1 + I \in (A/I)^+$ determines Ψ_w completely; thus, the left Z-module $\mathrm{Hom}_A(A/I, W)$ and W^+ are isomorphic as left Z-modules, as in the special case in the previous paragraph.

Remark 2.3. Recasting W^+ as solutions to a system of equations puts Z as a symmetry algebra of the solution space. In particular, reduction algebras are useful in determining solutions to so-called extremal equations [36].

Coinvariants. There is a dual construction of reduction algebras based on $\mathfrak{g}_-$-coinvariants, the elements of $V_- = V/\mathfrak{g}_-V$ for a left A-module V (taking into account the action of $\mathfrak{g}_-$ induced by Ξ). We begin with the quotient space $(A/I)_- = (A/I)/\mathfrak{g}_-(A/I)$ that comprises the $\mathfrak{g}_-$-coinvariants of A/I. Note that A/I being a right Z-module and A being a $U(\mathfrak{g})$-bimodule implies $(A/I)_-$ is a right Z-module. By multiple applications of the fundamental theorem of homomorphisms and setting the right ideal $\mathfrak{g}_-A$ equal to J, we identify $(A/I)_-$ with the double coset space $J\backslash A/I = A/\mathrm{I\!I}$, where $\mathrm{I\!I}$ is the sum of abelian groups $I + J$. Explicitly, the mapping sending $(a + I) + \mathfrak{g}_-(A/I) \in (A/I)_-$ to $a + (I + J) \in a + \mathrm{I\!I}$ is an isomorphism of vector spaces. Then we have the composition $\nu = \pi \circ \iota \colon Z \to A/\mathrm{I\!I}$ of right Z-modules from the natural inclusion $\iota \colon Z \to A/I$ and the natural surjection $\pi \colon A/I \to A/\mathrm{I\!I}$.

Extremal projector. The use of $\mathrm{I\!I}$ for the sum $I + J$ of abelian groups is suggestive as the resulting quotient $A/\mathrm{I\!I}$ of $\mathfrak{g}_-$-coinvariants has the structure of an algebra isomorphic to Z via the extremal projector. Extremal projectors were introduced in [23] and subsequently studied by Asherova, Smirnov, Tolstoy [2]. Each finite-dimensional reductive Lie algebra $\mathfrak{g}$ over the complex numbers admits a unique extremal projector $P_{A,\mathfrak{g},\Xi} = P_{\mathfrak{g}} = P_{\mathfrak{g}_+} = P$ (varying only in expression) that projects the universal $\mathfrak{g}_+$-highest-weight A-module A/I (more generally, an A-module V) onto the space of invariants Z (respectively, V^+) along J (respectively, $\mathfrak{g}_-V$) when the input data $(A, \mathfrak{g}, \Xi)$ satisfies sufficient conditions:

(C1) With the adjoint action, the Lie algebra $\mathfrak{g}$ acts reductively on A via Ξ.

(C2) The left action of $\mathfrak{g}_+$ on A (or on V) is locally finite.

(C3) The image $\Xi(h - n)$ is invertible in A for any coroot h, integer n (*coroot condition*).

Using the categorical language of [1], we have $\nu^{-1} = (\pi \circ \iota)^{-1}$ is the (A, Z)-bimodule map coming by considering first the astriction of P to $A/\mathrm{I\!I}$ (descending P to $A/\mathrm{I\!I}$) before requiring the coastriction of the resulting map to its image, which is equal to $\mathrm{Im}(P) = Z$ (shrinking the codomain to Z): $A/I \xrightarrow{\pi} A/\mathrm{I\!I} \xrightarrow{\nu^{-1}} Z \xrightarrow{\iota} A/I$. Then, as noted by Khoroshkin and Ogievetsky, we get the *double coset algebra*:

Proposition 2.4 ([22, equation (3.7)]). *Under conditions (C1), (C2), and (C3), there is an associative binary product $\diamond$ ("diamond product") on $A/\mathrm{I\!I}$, defined by*

$$\bar{x} \diamond \bar{y} = \pi(xP(y + I)), \qquad \bar{x} = x + \mathrm{I\!I},\ \bar{y} = y + \mathrm{I\!I} \in A/\mathrm{I\!I},$$

which yields ν^{-1} as an isomorphism of associative algebras: $A/\mathrm{I\!I} \cong Z$.

Note: We use $xPy + \mathrm{I\!I}$ to mean the right-hand side of the diamond product $\bar{x} \diamond \bar{y}$ in $A/\mathrm{I\!I}$.

Remark 2.5 (on the diamond product). For ν^{-1} to be an algebra isomorphism, we require a product $\diamond$ such that $\nu^{-1}(\bar{x} \diamond \bar{y}) = \nu^{-1}(\bar{x})\nu^{-1}(\bar{y}) \in Z$. Then necessarily $\bar{x} \diamond \bar{y} = \bar{x}\nu^{-1}(\bar{y}) \in A/\mathrm{I\!I}$, where the right-hand side is interpreted through the right Z-module structure on $A/\mathrm{I\!I}$. Now considering the left A-module structure on A/I and the right Z-module map π, we have

$$\pi(xP(y + I)) = \pi(x(1 + I)P(y + I)) = \pi((x + I)P(y + I)) = \bar{x}\nu^{-1}(\bar{y}).$$

Ring of dynamical scalars. To ensure the coroot condition is satisfied, we follow Zhelobenko [36] to enact either of the equivalent steps of localizing the associative algebra A as input data of a reduction algebra or localizing the resulting reduction algebra, whose non-localized version we refer to as the *Mickelsson algebra*. The Mickelsson algebra may be regarded as a subalgebra of the corresponding (localized) reduction algebra. We also cite Zhelobenko [34, equation (4.16)] for the method of obtaining elements of the Mickelsson algebra from the localized reduction algebra by clearing denominators—the elements obtained by this method are called *normalized elements*. We will use normalized generators of Z in specific calculations.

In localizing A, we extend the complex scalars to *dynamical scalars*, which are (non-evaluated) rational functions in $\mathfrak{h}$. Precisely, consider the denominator set S generated by $\{h + n \mid h$ a coroot, $n \in \mathbb{Z}\}$, and write the localized A as $\mathscr{A} = S^{-1}U(\mathfrak{h}) \otimes_{U(\mathfrak{h})} A$. Then the reduction algebra associated to the data $(\mathscr{A}, \mathfrak{g}, \Xi)$ (where we use the same name for enlarging the codomain of Ξ from A to $\mathscr{A}$) is identified with the localization of $Z(A, \mathfrak{g}, \Xi)$, namely, $S^{-1}U(\mathfrak{h}) \otimes_{U(\mathfrak{h})} Z$.

In general, computing the normalizer of a left ideal is unwieldy, and the Mickelsson algebra is not a finitely generated $\mathbb{C}$-algebra. Let R be the ring of dynamical scalars $S^{-1}U(\mathfrak{h})$. The extension of the base ring to R yields a finitely-generated R-ring [6, Definition 2.1] $Z(\mathscr{A}, \mathfrak{g}, \Xi)$.

The thesis [17] of Herlemont (supervised by Ogievetsky) expands on the connections between extremal projectors and reduction algebras. A more geometric treatment is found in [28]. We proceed with the details for the construction of the differential reduction algebra of $\mathfrak{sp}(4)$.

3 Constructing a differential reduction algebra of $\mathfrak{sp}(4)$

From hereon, set $A = A_2 \otimes U(\mathfrak{sp}(4))$ and recall the algebra map ζ from (2.4). In this section, we write the extremal projector for $\mathfrak{sp}(4)$ in order to provide a finite presentation of the differential reduction algebra $D(\mathfrak{sp}(4))$ defined within.

3.1 The differential reduction algebra of $\mathfrak{sp}(4)$

Set the ring R of dynamical scalars to be

$$R = S^{-1}\zeta(U(\mathfrak{h})), \tag{3.1}$$

where $S \subset A$ is the multiplicative submonoid generated by $\{\zeta(h_\gamma)+n \mid \gamma \in \Phi_+, n \in \mathbb{Z}\}$. The set S is an Ore denominator set in A, so we may define $\mathscr{A}$ as the localization $\mathscr{A} = S^{-1}A$. Composing the canonical inclusion $A \to \mathscr{A}$ with ζ, we regard ζ as an algebra map $\zeta\colon U(\mathfrak{sp}(4)) \to \mathscr{A}$. Then $(\mathscr{A}, \mathfrak{sp}(4), \zeta)$ satisfies the conditions (C1), (C2), and (C3).

Let $I_+ = \mathscr{A}E_\alpha + \mathscr{A}E_\beta$ be the left ideal of $\mathscr{A}$ generated by the image of the positive nilpotent part of $\mathfrak{sp}(4)$ and $I_- = F_\alpha\mathscr{A} + F_\beta\mathscr{A}$ the right ideal generated by the image of the negative nilpotent part of $\mathfrak{sp}(4)$. Then as in Section 2.2, the space of double cosets with respect to the sum $\mathbb{I} = I_- + I_+$ in $\mathscr{A}$ is $D(\mathfrak{sp}(4)) = \mathscr{A}/\mathbb{I}$. Moreover, $D(\mathfrak{sp}(4))$ has the structure of an algebra under the diamond product. Following [35, Theorem 2], the extremal projector for $\mathfrak{sp}(4)$ is

$$P = P_\beta P_{\beta+\alpha} P_{\beta+2\alpha} P_\alpha = P_\alpha P_{\beta+2\alpha} P_{\beta+\alpha} P_\beta,$$

where, using the shifted coroots H_γ for $\gamma \in \Phi_+$,

$$P_\gamma = 1 - \frac{1}{H_\gamma + 2} F_\gamma E_\gamma + \cdots .$$

And $\bar{a} \diamond \bar{b} = aPb + \mathbb{I}$ for all $\bar{a} = a + \mathbb{I}$, $\bar{b} = b + \mathbb{I}$.

We call $D(\mathfrak{sp}(4))$ the *differential reduction algebra of* $\mathfrak{sp}(4)$, and $D(\mathfrak{sp}(4))$ is an associative algebra isomorphic to the reduction algebra $Z(\mathscr{A}, \mathfrak{sp}(4)) = N_{\mathscr{A}}(I_+)/I_+$. We introduce the following Weyl-algebra like elements. For $a \in \{\partial_1, \partial_2, x_1, x_2\}$, we put

$$\tilde{a} = a \otimes 1 \in \mathscr{A}, \qquad \bar{a} = \tilde{a} + \mathrm{II} \in D(\mathfrak{sp}(4)).$$

Lemma 3.1. *The algebra $D(\mathfrak{sp}(4))$ carries an involutive anti-automorphism Θ determined by*

$$\Theta(\bar{x}_i) = \bar{\partial}_i, \qquad i = 1, 2.$$

Proof. The Chevalley involution τ of $\mathfrak{sp}(4)$ extends to an involutive algebra anti-automorphism of $U(\mathfrak{sp}(4))$, also denoted τ. Recalling ϑ from (2.2), and ζ from (2.4), we have

$$\zeta(\tau(a)) = (\vartheta \otimes \tau) \circ \zeta(a), \qquad a \in \mathfrak{sp}(4).$$

The anti-automorphism $\vartheta \otimes \tau$ of $A_2 \otimes U(\mathfrak{sp}(4))$ uniquely extends to the localization $\mathscr{A}$ and preserves the double coset II and fixes the extremal projector P, hence induces an anti-automorphism Θ of the required form. ■

Lemma 3.1 allows us to cut computations almost in half.

3.2 A computational lemma for $D(\mathfrak{sp}(4))$

We provide some computations that will be needed in proof of Theorem 3.4. In $\mathscr{A}$, we have for any positive root γ,

$$aP_\gamma b \equiv ab + aF_\gamma \frac{-1}{H_\gamma} E_\gamma b + \cdots \equiv ab + [a, F_\gamma] \frac{-1}{H_\gamma} [E_\gamma, b] + \cdots \qquad \mathrm{mod}\ \mathrm{II}, \tag{3.2}$$

where $\cdots$ indicate higher-order terms from the extremal projector. However, we will only need the constant and linear terms in the calculations of this paper.

Lemma 3.2. *We have the following congruences modulo II:*

$$yP\widetilde{x}_1 \equiv y\widetilde{x}_1 \qquad \textit{for all} \quad y \in \mathscr{A}, \tag{3.3}$$

$$\widetilde{\partial}_2 P \widetilde{x}_2 \equiv \widetilde{\partial}_2 \widetilde{x}_2 + \frac{-1}{H_\alpha + 1} \widetilde{\partial}_1 \widetilde{x}_1, \tag{3.4}$$

$$\widetilde{x}_2 P \widetilde{\partial}_2 \equiv -1 + \frac{H_\beta}{(H_\beta + 1)(H_{\beta+\alpha} + 1)} \widetilde{\partial}_1 \widetilde{x}_1 + \left(1 + \frac{1}{H_\beta + 1}\right) \widetilde{\partial}_2 \widetilde{x}_2,$$

$$\begin{aligned} \widetilde{x}_1 P \widetilde{\partial}_1 \equiv {} & -1 + \frac{1}{H_\alpha + 1} + \left(1 + \frac{H_\alpha H_{\beta+\alpha} + H_{\beta+2\alpha} + 1}{(H_\alpha + 1)(H_{\beta+\alpha} + 1)(H_{\beta+2\alpha} + 1)}\right) \widetilde{\partial}_1 \widetilde{x}_1 \\ & + \frac{H_\alpha - H_{\beta+\alpha} - 2}{(H_\alpha + 1)(H_{\beta+\alpha} + 1)} \widetilde{\partial}_2 \widetilde{x}_2. \end{aligned} \tag{3.5}$$

Proof. The congruence (3.3) follows immediately from the fact that $[E_\gamma, \widetilde{x}_1] = 0$ for all positive roots γ. For the proof of (3.4), we have

$$\begin{aligned} \widetilde{\partial}_2 P \widetilde{x}_2 &\equiv \widetilde{\partial}_2 P_\alpha P_{\beta+2\alpha} P_{\beta+\alpha} P_\beta \widetilde{x}_2 && \text{definition of } P \\ &\equiv \widetilde{\partial}_2 P_\alpha \widetilde{x}_2 && \text{by } [E_{\beta+k\alpha}, \widetilde{x}_i] = 0 \\ &\equiv \widetilde{\partial}_2 \widetilde{x}_2 + \left[\widetilde{\partial}_2, F_\alpha\right] \frac{-1}{H_\alpha} [E_\alpha, \widetilde{x}_2] && \text{by (3.2)} \\ &\equiv \widetilde{\partial}_2 \widetilde{x}_2 + \widetilde{\partial}_1 \frac{-1}{H_\alpha} \widetilde{x}_1 && \text{by } [\widetilde{\partial}_2, F_\alpha] = \widetilde{\partial}_1 \\ &\equiv \widetilde{\partial}_2 \widetilde{x}_2 + \frac{-1}{H_\alpha + 1} \widetilde{\partial}_1 \widetilde{x}_1 && \text{by } H_\alpha \widetilde{\partial}_1 = \widetilde{\partial}_1 (H_\alpha - 1). \end{aligned}$$

We have proved (3.4). The remaining computations are similar in nature but also rely on the $\mathfrak{sp}(4)$ Lie algebra relations. ■

3.3 Finite presentation of $D(\mathfrak{sp}(4))$

We cite [27] as a reference on the discussion of producing generators of reduction algebras generally according to a weight-basis ordering and the use of extremal projectors.

Lemma 3.3 (generators)**.** *As a left R-module, $D(\mathfrak{sp}(4))$ is free with basis*

$$\big\{\bar\partial_1^{\diamond a}\diamond\bar\partial_2^{\diamond b}\diamond\bar x_2^{\diamond c}\diamond\bar x_1^{\diamond d}\mid a,b,c,d\in\mathbb{Z}_{\geq 0}\big\},$$

where an exponent $\diamond k$ signifies a k-fold diamond product. In particular, as an R-ring, $D(\mathfrak{sp}(4))$ is generated by the four elements $\bar x_1$, $\bar x_2$, $\bar\partial_1$, $\bar\partial_2$.

Proof. Let $\mathfrak{g}=\mathfrak{sp}(4)$ with triangular decomposition $\mathfrak{g}=\mathfrak{g}_-\oplus\mathfrak{h}\oplus\mathfrak{g}_+$. By the PBW theorem, $U(\mathfrak{g})$ is a free left $U(\mathfrak{h})$-module; therefore, we have an isomorphism of left $U(\mathfrak{h})$-modules relying on the usual comultiplication $\Delta(x)=x\otimes 1+1\otimes x$ on $U(\mathfrak{g})$,

$$\begin{aligned}
&U(\mathfrak{h})\otimes U(\mathfrak{g}_-)\otimes A_2\otimes U(\mathfrak{g}_+)\to A=A_2\otimes U(\mathfrak{g}),\\
&1\otimes w\otimes 1\otimes 1\mapsto w\otimes 1,\qquad w\in A_2,\qquad h\otimes 1\otimes 1\otimes 1\mapsto(\zeta\otimes\mathrm{Id})\circ\Delta(h),\qquad h\in U(\mathfrak{h}),\\
&1\otimes 1\otimes f\otimes 1\mapsto(\zeta\otimes\mathrm{Id})\circ\Delta(f),\qquad f\in U(\mathfrak{g}_-),\\
&1\otimes 1\otimes 1\otimes e\mapsto(\zeta\otimes\mathrm{Id})\circ\Delta(e),\qquad e\in U(\mathfrak{g}_+),
\end{aligned}$$

where ζ is the algebra map $U(\mathfrak{g})\to A_2\otimes U(\mathfrak{g})$ from (2.4). In particular, A is free as a left $U(\mathfrak{h})$-module. Consequently, the localized algebra $\mathscr{A}=S^{-1}A$ is free as a left R-module. The preceding sentences imply that $\mathscr{A}$ has a basis as a left R-module consisting of monomials FWE, where F and E are images under ζ of PBW monomials (with respect to any choice of ordered bases) for $U(\mathfrak{g}_-)$ and $U(\mathfrak{g}_+)$, respectively, and $W=\big(\tilde\partial_1\big)^a\big(\tilde\partial_2\big)^b(\tilde x_2)^c(\tilde x_1)^d$ is the image of the corresponding element without tildes in the $\mathbb{C}$-basis $\big\{(\partial_1)^a(\partial_2)^b(x_2)^c(x_1)^d\mid a,b,c,d\in\mathbb{Z}_{\geq 0}\big\}\subset A_2$ for the second Weyl algebra. A subset of the R-basis $\{FWE\}$ for $\mathscr{A}$ just described is an R-basis for the space $\mathrm{I\!I}=\zeta(\mathfrak{g}_-)\mathscr{A}+\mathscr{A}\zeta(\mathfrak{g}_+)$, namely the set of those FWE with either $F\neq\zeta(1)$ or $E\neq\zeta(1)$ (or $\zeta(z)$ if 1 is replaced in the PBW basis with another complex unit z). Therefore, we immediately get a basis for $D(\mathfrak{g})=\mathscr{A}/\mathrm{I\!I}$ as a left R-module

$$\big\{\big(\tilde\partial_1\big)^a\big(\tilde\partial_2\big)^b(\tilde x_2)^c(\tilde x_1)^d+\mathrm{I\!I}\mid a,b,c,d\in\mathbb{Z}_{\geq 0}\big\}.$$

We now order the single-element monomials $\tilde\partial_1$, $\tilde\partial_2$, $\tilde x_2$, $\tilde x_1$, writing $x\prec y$ for "x precedes y", based on their $\zeta(\mathfrak{h})$-weights in $\mathscr{A}$ (where $\prec$ is compatible with the standard root order on $\zeta(\mathfrak{h})^*$). Now x_1 is a highest weight vector since $[E_\alpha,x_1]=0$ and $[E_\beta,x_1]=0$. Thus ∂_1 is a lowest weight vector using Θ. We further have $[F_\alpha,x_1]\in\mathbb{C}x_2$ and $[F_\beta,x_1]=0$; $[E_\alpha,\partial_2]=0$ and $[E_\beta,\partial_2]\in\mathbb{C}x_2$; therefore, we order ∂_1, ∂_2, x_2, x_1 by

$$1\prec\partial_1\prec\partial_2\prec x_2\prec x_1 \tag{3.6}$$

and equip the corresponding set of monomials of the form $\big(\tilde\partial_1\big)^a\big(\tilde\partial_2\big)^b(\tilde x_2)^c(\tilde x_1)^d$ with the lexicographical ordering induced by (3.6). Then, by definition of the extremal projector and the Weyl algebra relations (2.1),

$$\begin{aligned}
\overline{\partial_1^a\partial_2^b x_2^c x_1^d}\diamond\overline{\partial_1^e\partial_2^f x_2^g x_1^h}&=\big(\tilde\partial_1\big)^a\big(\tilde\partial_2\big)^b(\tilde x_2)^c(\tilde x_1)^d\big(\tilde\partial_1\big)^e\big(\tilde\partial_2\big)^f(\tilde x_2)^g(\tilde x_1)^h\\
&\qquad+\text{lower order terms}+\mathrm{I\!I}\\
&=\big(\tilde\partial_1\big)^{a+e}\big(\tilde\partial_2\big)^{b+f}(\tilde x_2)^{c+g}(\tilde x_1)^{d+h}+\text{lower order terms}+\mathrm{I\!I};
\end{aligned}$$

and consequently, by induction,

$$\bar\partial_1^{\diamond a}\diamond\bar\partial_2^{\diamond b}\diamond\bar x_2^{\diamond c}\diamond\bar x_1^{\diamond d}=\big(\tilde\partial_1\big)^a\big(\tilde\partial_2\big)^b(\tilde x_2)^c(\tilde x_1)^d+\text{lower order terms}+\mathrm{I\!I}.$$

For example, see (3.4),

$$\bar{\partial}_2 \diamond \bar{x}_2 = \partial_2 x_2 + f(H_\alpha, H_\beta)\partial_1 x_1 + g(H_\alpha, H_\beta)1 + \mathrm{II},$$

where f and g are dynamical scalars in R, particularly, $f = f(H_\alpha) = \frac{-1}{H_\alpha+1}$ and $g = 1$.

The argument above proves that the two sets $\{(\tilde{\partial}_1)^a(\tilde{\partial}_2)^b(\tilde{x}_2)^c(\tilde{x}_1)^d + \mathrm{II} \mid a,b,c,d \in \mathbb{Z}_{\geq 0}\}$ and $\{\bar{\partial}_1^{\diamond a} \diamond \bar{\partial}_2^{\diamond b} \diamond \bar{x}_2^{\diamond c} \diamond \bar{x}_1^{\diamond d} \mid a,b,c,d \in \mathbb{Z}_{\geq 0}\}$ are related by a triangular matrix with 1's on the diagonal. Therefore, the latter is a basis for $\mathscr{A}/\mathrm{II}$ as a left R-module. Thus $D(\mathfrak{sp}(4))$ is the smallest R-ring containing $\bar{x}_1$, $\bar{x}_2$, $\bar{\partial}_1$, $\bar{\partial}_2$. ∎

Theorem 3.4 (finite presentation)**.** *The R-ring $D(\mathfrak{sp}(4))$ is generated by $\bar{x}_1$, $\bar{\partial}_1$, $\bar{x}_2$, $\bar{\partial}_2$ subject to the following finite set of relations:*

$$\bar{x}_1 H_\alpha = (H_\alpha - 1)\bar{x}_1, \qquad \bar{x}_1 H_\beta = H_\beta \bar{x}_1, \tag{3.7a}$$
$$\bar{\partial}_1 H_\alpha = (H_\alpha + 1)\bar{\partial}_1, \qquad \bar{\partial}_1 H_\beta = H_\beta \bar{\partial}_1,$$
$$\bar{x}_2 H_\alpha = (H_\alpha + 1)\bar{x}_2, \qquad \bar{x}_2 H_\beta = (H_\beta - 1)\bar{x}_2,$$
$$\bar{\partial}_2 H_\alpha = (H_\alpha - 1)\bar{\partial}_2, \qquad \bar{\partial}_2 H_\beta = (H_\beta + 1)\bar{\partial}_2, \tag{3.7b}$$
$$\bar{x}_1 \diamond \bar{x}_2 = \Big(1 + \frac{1}{H_\alpha + 1}\Big)\bar{x}_2 \diamond \bar{x}_1, \qquad \bar{\partial}_2 \diamond \bar{\partial}_1 = \bar{\partial}_1 \diamond \bar{\partial}_2\Big(1 + \frac{1}{H_\alpha + 1}\Big), \tag{3.7c}$$
$$\bar{x}_1 \diamond \bar{\partial}_2 = \Big(1 + \frac{1}{H_{\beta+\alpha} + 1}\Big)\bar{\partial}_2 \diamond \bar{x}_1, \qquad \bar{x}_2 \diamond \bar{\partial}_1 = \bar{\partial}_1 \diamond \bar{x}_2\Big(1 + \frac{1}{H_{\beta+\alpha} + 1}\Big), \tag{3.7d}$$
$$\bar{x}_1 \diamond \bar{\partial}_1 = -1 + \frac{1}{H_\alpha + 1} + f_{11}\bar{\partial}_1 \diamond \bar{x}_1 + f_{12}\bar{\partial}_2 \diamond \bar{x}_2, \tag{3.7e}$$
$$\bar{x}_2 \diamond \bar{\partial}_2 = -1 + f_{21}\bar{\partial}_1 \diamond \bar{x}_1 + f_{22}\bar{\partial}_2 \diamond \bar{x}_2, \tag{3.7f}$$

where $f_{ij} = f_{ij}(H_\alpha, H_\beta) \in R$ are given by

$$f_{11} = \frac{(a+1)(a-1)(b+1)}{a^2 b}, \qquad f_{12} = \frac{-(d+2)}{ac},$$
$$f_{21} = \frac{a(d-1) + c(d+1)}{acd}, \qquad f_{22} = \frac{d+1}{d}, \tag{3.8}$$

and we put $a = H_\alpha + 1$, $b = H_{\beta+2\alpha} + 1$, $c = H_{\beta+\alpha} + 1$, $d = H_\beta + 1$.

Proof. By Lemma 3.3, we are left to show (3.7) hold. We offer computational methods for the different type of relations: There are relations involving only the (R,R)-bimodule structure, (3.7a)–(3.7b); relations involving one diamond product, (3.7c) and (3.7d); and those involving two diamond products, (3.7e) and (3.7f).

Relations (3.7a)–(3.7b) follow from the corresponding identities in $\mathscr{A}$. For example,

$$\begin{aligned}\bar{x}_1 H_\alpha &= \tilde{x}_1 H_\alpha + \mathrm{II} = (x_1 \otimes 1)((x_1\partial_1 - x_2\partial_2) \otimes 1 + 1 \otimes h_\alpha) + \mathrm{II} \\ &= (x_1\partial_1 - x_2\partial_2 - 1)x_1 \otimes 1 + x_1 \otimes h_\alpha + \mathrm{II} = (H_\alpha - 1)\bar{x}_1,\end{aligned}$$

where we used the Weyl algebra relations of (2.1).

Next we consider relations involving one diamond product, namely, relations (3.7c) and (3.7d). The application of Θ produces one half of the results. For example, consider (3.7c) and recall (3.3). In particular, with $\equiv$ meaning congruence mod II, we have

$$\tilde{x}_2 P \tilde{x}_1 \equiv \tilde{x}_2 \tilde{x}_1 \mod \mathrm{II}. \tag{3.9}$$

Similarly, none of E_β, $E_{\beta+\alpha}$, $E_{\beta+2\alpha}$ involve ∂_2, which implies that $[E_\gamma, \tilde{x}_2] = 0$ for all $\gamma \in \Phi_+ \setminus \{\alpha\}$, but

$$[E_\alpha, \tilde{x}_2] = [x_1\partial_2 \otimes 1 + 1 \otimes e_\alpha, x_2 \otimes 1] = x_1 \otimes 1 = \tilde{x}_1.$$

Likewise, $[\tilde{x}_1, F_\alpha] = -\tilde{x}_2$. Consequently, using (3.7a), we have

$$\begin{aligned}\tilde{x}_1 P \tilde{x}_2 &\equiv \tilde{x}_1 P_\alpha P_{\beta+2\alpha} P_{\beta+\alpha} P_\beta \tilde{x}_2 \equiv \tilde{x}_1 P_\alpha \tilde{x}_2 \equiv \tilde{x}_1\tilde{x}_2 - \frac{1}{H_\alpha+1}[\tilde{x}_1, F_\alpha][E_\alpha, \tilde{x}_2] \\ &\equiv \tilde{x}_1\tilde{x}_2 + \frac{1}{H_\alpha+1}\tilde{x}_2\tilde{x}_1 \equiv \frac{H_\alpha+2}{H_\alpha+1}\tilde{x}_2\tilde{x}_1. \end{aligned} \tag{3.10}$$

Comparing (3.9) and (3.10), the first relation in (3.7c) is proved; moreover, Θ yields the second relation in (3.7c). Relation (3.7d) is proved in the same fashion with a simplification from $\tilde{x}_1 P \tilde{\partial}_2$ to $\tilde{x}_1 P_{\beta+\alpha}\tilde{\partial}_2$ and another application of (3.3).

Lastly, we analyze the relations involving two diamond products and use Lemma 3.2. Consider relation (3.7f): Solving for $\widetilde{\partial}_i\widetilde{x}_i$ in (3.3) and (3.4), we get

$$\widetilde{\partial}_1\widetilde{x}_1 + \mathrm{I\!I} = \bar{\partial}_1 \diamond \bar{x}_1, \tag{3.11}$$

$$\widetilde{\partial}_2\widetilde{x}_2 + \mathrm{I\!I} = \bar{\partial}_2 \diamond \bar{x}_2 + \frac{1}{H_\alpha+1}\bar{\partial}_1 \diamond \bar{x}_1. \tag{3.12}$$

Substituting equations (3.11) and (3.12) into the right-hand side of equation (3.4), we arrive at

$$\begin{aligned}\bar{x}_2 \diamond \bar{\partial}_2 &= -1 + \frac{H_\beta}{(H_\beta+1)(H_{\beta+\alpha}+1)}\bar{\partial}_1 \diamond \bar{x}_1 + \frac{H_\beta+2}{H_\beta+1}\left(\bar{\partial}_2 \diamond \bar{x} + 2 + \frac{1}{H_\alpha+1}\bar{\partial}_1 \diamond \bar{x}_1\right) \\ &= -1 + \frac{H_\beta+2}{H_\beta+1}\bar{\partial}_2 \diamond \bar{x}_2 + \frac{H_\beta(H_\alpha+1) + (H_\beta+2)(H_{\beta+\alpha}+1)}{(H_\alpha+1)(H_\beta+1)(H_{\beta+\alpha}+1)}\bar{\partial}_1 \diamond \bar{x}_1 \\ &= -1 + f_{22}\bar{\partial}_2 \diamond \bar{x}_2 + f_{21}\bar{\partial}_1 \diamond \bar{x}_1,\end{aligned}$$

proving relation (3.7f). To be sure, substituting equations (3.11) and (3.12) into (3.5), we obtain relation (3.7e). Thus we have a finite presentation of $D(\mathfrak{sp}(4))$ as a ring over the dynamical scalars R. ■

Remark 3.5 (quantum algebra and integral form)**.** Reduction algebras are quantum deformations of associative algebras [18]. In our case, if we formally send all $H_\gamma \to \infty$ we obtain the usual Weyl algebra relations. More precisely, consider the extension of scalars $D^*(\mathfrak{sp}(4)) = D(\mathfrak{sp}(4))\big[\hbar, \hbar^{-1}\big] \cong \mathbb{C}\big[\hbar, \hbar^{-1}\big] \otimes_{\mathbb{C}} D(\mathfrak{sp}(4))$. There is an "integral form" $D^0(\mathfrak{sp}(4))$ inside this algebra defined as the $\mathbb{C}[\hbar]$-subalgebra of $D^*(\mathfrak{sp}(4))$ generated by

$$\check{H}_\alpha = \hbar H_\alpha, \quad \check{H}_\beta = \hbar H_\beta, \quad \bar{x}_1, \quad \bar{x}_2, \quad \bar{\partial}_1, \quad \bar{\partial}_2.$$

Substituting $H_\gamma = \frac{1}{\hbar}\check{H}_\gamma$ in all relations above (and multiplying (3.7a)–(3.7b) by $\hbar$), we see that the quotient algebra of $D^0(\mathfrak{sp}(4))$ by the principal ideal $\langle\hbar\rangle$ is isomorphic to the Weyl algebra $A_2(\mathbb{C})$ tensored (over $\mathbb{C}$) by the ring R.

Corollary 3.6. *The ring $D(\mathfrak{sp}(4))$ is a domain (i.e., there are no left or right zero-divisors).*

Proof. One may use Lemma 3.3 to prove $D(\mathfrak{sp}(4))$ is a domain. Instead, we appeal to Remark 3.5. Since $D(\mathfrak{sp}(4))$ is a subalgebra of $D^*(\mathfrak{sp}(4))$, it suffices to show the latter is a domain. If $x, y \in D^*(\mathfrak{sp}(4))$ with $xy = 0$, then $\hbar^k x, \hbar^l y \in D^0(\mathfrak{sp}(4))$ for sufficiently large positive integers k, l, and $\big(\hbar^k x\big)\big(\hbar^l y\big) = \hbar^{k+l}xy = 0$. Thus it suffices to show that $D^0(\mathfrak{sp}(4))$ is a domain. The algebra $D^0(\mathfrak{sp}(4))$ has a filtration given by $D^0(\mathfrak{sp}(4))_{(k)} = D^0(\mathfrak{sp}(4)) + D^0(\mathfrak{sp}(4))\hbar + \cdots + D^0(\mathfrak{sp}(4))\hbar^k$ for $k \in \mathbb{Z}_{\geq 0}$, and the associated graded algebra is isomorphic to $D^0(\mathfrak{sp}(4))/\langle\hbar\rangle$. As mentioned in Remark 3.5, the quotient $D^0(\mathfrak{sp}(4))/\langle\hbar\rangle$ is isomorphic to $R \otimes_{\mathbb{C}} A_2(\mathbb{C})$, which is a domain. Therefore, $\operatorname{gr} D^0(\mathfrak{sp}(4))$ is a domain; thus $D^0(\mathfrak{sp}(4))$, and consequently, $D(\mathfrak{sp}(4))$ are domains. ■

4 Generalized Weyl algebras

In the 1990s, Bavula [4] described a class of algebras that extends notions of the n-th Weyl algebra A_n and called them generalized Weyl algebras (GWAs). GWAs include important examples from representation theory and ring theory, such as $U(\mathfrak{sl}(2))$, $U_q(\mathfrak{sl}(2))$, down-up algebras [5], skew Laurent-polynomial rings, to name a few. The class is closed under taking tensor products and certain skew polynomial extensions. See [11] for a recent survey on GWAs.

In [30], GWA presentations are given for certain subalgebras of Mickelsson step algebras $S(\mathfrak{g}_{n+1}, \mathfrak{g}_n)$, where $\mathfrak{g}_n = \mathfrak{sl}(n)$ or $\mathfrak{so}(n)$. In [24], certain reduction algebras were shown to be twisted GWAs. In this section, we recall the definition of GWAs and observe that $D(\mathfrak{sp}(4))$ is an example of an interesting subclass of GWAs that we term *skew-affine* GWAs.

We will rely on normalized generators of $D(\mathfrak{sp}(4))$ and their relations. The following statement follows from Theorem 3.4.

Proposition 4.1 (normalization)**.** *The normalized generators*

$$\hat{x}_1 = x_1, \qquad \hat{x}_2 = (H_\alpha + 2)\bar{x}_2, \qquad \hat{\partial}_1 = \bar{\partial}_1(H_\alpha + 1)(H_{\beta+\alpha} + 1), \qquad \hat{\partial}_2 = \bar{\partial}_2(H_{\beta+\alpha} + 1)$$

satisfy

$$[\hat{x}_1, \hat{x}_2]_\diamond = \big[\hat{\partial}_1, \hat{\partial}_2\big]_\diamond = \big[\hat{x}_1, \hat{\partial}_2\big]_\diamond = \big[\hat{x}_2, \hat{\partial}_1\big]_\diamond = 0, \tag{4.1}$$

where $[a, b]_\diamond = a \diamond b - b \diamond a$ is the diamond commutator.

4.1 The differential reduction algebra $D(\mathfrak{sp}(4))$ is a generalized Weyl algebra

Definition 4.2 (GWA)**.** Let $\mathbb{K}$ be a field. Let B be an associative $\mathbb{K}$-algebra, n be a positive integer, $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n) \in \mathrm{Aut}_{\mathbb{K}}(B)^n$ an n-tuple of commuting $\mathbb{K}$-algebra automorphisms of B, $t = (t_1, t_2, \dots, t_n) \in Z(B)^n$ an n-tuple of elements of the center of B. The associated *generalized Weyl algebra* (*GWA*) *of rank* (*or degree*) n, denoted $B(\sigma, t)$, is the B-ring generated by $X_1, Y_1, \dots, X_n, Y_n$ subject to the following relations, for $i, j = 1, 2, \dots, n$ and all $b \in B$:

$$X_i b = \sigma_i(b) X_i, \qquad b Y_i = Y_i \sigma_i(b), \tag{4.2a}$$

$$X_i X_j = X_j X_i, \qquad Y_i Y_j = Y_j Y_i, \qquad X_i Y_j = Y_j X_i \qquad \text{if} \quad i \neq j, \tag{4.2b}$$

$$Y_i X_i = t_i, \qquad X_i Y_i = \sigma_i(t_i).$$

Example 4.3 (revisiting the Weyl algebra)**.** Taking $\mathbb{K} = \mathbb{C}$, $B = \mathbb{C}[u_1, u_2, \dots, u_n]$, σ_i defined by $\sigma_i(u_j) = u_j - \delta_{ij}$, and $t_i = u_i$, the GWA $B(\sigma, t)$ is isomorphic to the n-th Weyl algebra $A_n(\mathbb{C})$ via $X_i \mapsto x_i$, $Y_i \mapsto \partial_i$.

Proposition 4.4 (automorphisms)**.** *Let $B = R[t_1, t_2]$, where R is the ring of dynamical scalars defined in* (3.1)*. The maps $\sigma_1, \sigma_2\colon B \to B$ are automorphisms of B when defined as follows*

$$\sigma_1(H_\alpha) = H_\alpha - 1, \qquad \sigma_2(H_\alpha) = H_\alpha + 1, \qquad \sigma_1(H_\beta) = H_\beta, \qquad \sigma_2(H_\beta) = H_\beta - 1,$$

$$\sigma_1(t_1) = \hat{c}_1 + \hat{f}_{11} t_1 + \hat{f}_{12} t_2, \qquad \sigma_2(t_1) = t_1,$$

$$\sigma_1(t_2) = t_2, \qquad \sigma_2(t_2) = \hat{c}_2 + \hat{f}_{21} t_1 + \hat{f}_{22} t_2,$$

where

$$\hat{c}_1 = -H_\alpha(H_{\beta+\alpha} + 1), \qquad \hat{c}_2 = -(H_\alpha + 2)(H_{\beta+\alpha} + 1),$$

$$\hat{f}_{i1} = f_{i1} \frac{(H_\alpha + i)(H_{\beta+\alpha} + 1)}{(H_\alpha + 2)(H_{\beta+\alpha} + 2)}, \qquad \hat{f}_{i2} = f_{i2} \frac{(H_\alpha + i)(H_{\beta+\alpha} + 1)}{(H_\alpha + 1)(H_{\beta+\alpha} + 2)}, \qquad i = 1, 2,$$

and $f_{ij} = f_{ij}(H_\alpha, H_\beta)$ were defined in (3.8)*.*

Remark 4.5 (commuting automorphisms)**.** It is far from obvious that the automorphisms σ_1 and σ_2 actually commute; however, we will show indirectly that σ_1 and σ_2 commute during an extended proof of Theorem 4.6 found in Appendix A.

Theorem 4.6 ($D(\mathfrak{sp}(4))$ is a GWA)**.** *With B and σ_i as in Proposition 4.1, there is a $\mathbb{C}$-algebra isomorphism $\phi\colon B(\sigma,t) \longrightarrow D(\mathfrak{sp}(4))$ satisfying*

$$\phi(H_\alpha) = H_\alpha, \qquad \phi(H_\beta) = H_\beta, \qquad \phi(X_i) = \hat{x}_i, \qquad \phi(Y_i) = \hat{\partial}_i, \qquad i = 1, 2,$$

where we use the normalized generators of $D(\mathfrak{sp}(4))$ from Proposition 4.4.

Proof. An extended argument is found in Appendix A. The proof relies on the relations of the normalized generators in (4.1) and the finite presentation of $D(\mathfrak{sp}(4))$ found in Theorem 3.4. ■

4.2 Rank two skew-affine GWAs

We consider a class of GWAs we call *skew-affine*, restricting to the case of rank two. These GWAs are believed to refine the connection [31] between reduction algebras and GWAs within the study of noncommutative rings.

Definition 4.7 (the skew-affine ansatz for GWAs of rank two)**.** Suppose T is a finitely generated commutative $\mathbb{C}$-algebra and let $D = T[t_1, t_2]$ be the polynomial algebra over T in two indeterminates t_i. Consider the following ansatz for $\mathbb{C}$-algebra automorphisms σ_i of D

$$\begin{aligned} &\sigma_i(t_i) = c_i + g_{i1}t_1 + g_{i2}t_2, \qquad i = 1, 2, \\ &\sigma_i(t_j) = t_j, \qquad i \neq j, \qquad \sigma_i|_T \in \operatorname{Aut}_{\mathbb{C}}(T), \end{aligned} \tag{4.3}$$

where c_i and g_{ij} are some fixed elements of T. A GWA defined by (4.3) is called a *skew-affine generalized Weyl algebra of rank two.*

As a corollary to Theorem 4.6, $D(\mathfrak{sp}(4))$ is a rank two skew-affine GWA. Its representation theory may be determined by suitable tools of GWAs; moreover, we conjecture that a family of skew-affine GWAs are found as differential reduction algebras of symplectic Lie algebras. We note that the defining relations of skew-affine GWAs are new to GWAs, and it will be useful to address higher-order generalizations in future work.

A Zero commutator of automorphisms in Theorem 4.6

The computations below show that σ_1 and σ_2 commute. We omit $\diamond$ for brevity.

Extended argument for proof of Theorem 4.6. Let $\phi\colon B \to D(\mathfrak{sp}(4))$ be the $\mathbb{C}$-algebra homomorphism determined by $\phi|_R = \mathrm{Id}_R$ and $\phi(t_i) = \hat{\partial}_i\hat{x}_i$ for $i = 1, 2$. Extend ϕ to $B \cup \{X_1, Y_1, X_2, Y_2\}$ by $\phi(X_i) = \hat{x}_i$, $\phi(Y_i) = \hat{\partial}_i$. We must show that the GWA relations (4.2) are satisfied by the images of ϕ. First, we prove that relations (4.2b) are preserved.

Using $\bar{x}_1 H_\alpha = (H_\alpha - 1)\bar{x}_1$,

$$\hat{x}_1\hat{x}_2 = \bar{x}_1(H_\alpha + 2)\bar{x}_2 = (H_\alpha + 1)\bar{x}_1\bar{x}_2 = (H_\alpha + 1)\frac{H_\alpha + 2}{H_\alpha + 1}\bar{x}_2\bar{x}_1 = \hat{x}_2\hat{x}_1.$$

Using $H_{\beta+\alpha}\bar{\partial}_1 = \bar{\partial}_1(H_{\beta+\alpha} - 1)$, $\bar{\partial}_2 H_\alpha = (H_\alpha - 1)\bar{\partial}_2$, and $\bar{\partial}_2 H_{\beta+\alpha} = (H_{\beta+\alpha} + 1)\bar{\partial}_2$, we get

$$\begin{aligned} \hat{\partial}_2\hat{\partial}_1 &= \bar{\partial}_2(H_{\beta+\alpha} + 1)\bar{\partial}_1(H_\alpha + 1)(H_{\beta+\alpha} + 1) = \bar{\partial}_2\bar{\partial}_1(H_\alpha + 1)H_{\beta+\alpha}(H_{\beta+\alpha} + 1) \\ &= \bar{\partial}_1\bar{\partial}_2\frac{H_\alpha + 2}{H_\alpha + 1}(H_\alpha + 1)H_{\beta+\alpha}(H_{\beta+\alpha} + 1) = \bar{\partial}_1\bar{\partial}_2(H_\alpha + 2)H_{\beta+\alpha}(H_{\beta+\alpha} + 1) \end{aligned}$$

$$= \bar{\partial}_1(H_\alpha+1)(H_{\beta+\alpha}+1)\bar{\partial}_2(H_{\beta+\alpha}+1) = \hat{\partial}_1\hat{\partial}_2.$$

Next, since $H_{\beta+\alpha}$ commutes with $\bar{x}_1\bar{\partial}_2$, and $H_{\beta+\alpha}\bar{\partial}_2 = \bar{\partial}_2(H_{\beta+\alpha}-1)$, we have

$$\begin{aligned}\hat{x}_1\hat{\partial}_2 &= \bar{x}_1\bar{\partial}_2(H_{\beta+\alpha}+1) = (H_{\beta+\alpha}+1)\bar{x}_1\bar{\partial}_2 = (H_{\beta+\alpha}+1)\frac{H_{\beta+\alpha}+2}{H_{\beta+\alpha}+1}\bar{\partial}_2\bar{x}_1\\ &= (H_{\beta+\alpha}+2)\bar{\partial}_2\bar{x}_1 = \bar{\partial}_2(H_{\beta+\alpha}+1)\bar{x}_1 = \hat{\partial}_2\hat{x}_1.\end{aligned}$$

From $H_\alpha\bar{x}_2 = \bar{x}_2(H_\alpha-1)$, $H_\alpha\bar{\partial}_1 = \bar{\partial}_1(H_\alpha-1)$, and $\bar{x}_2H_{\beta+\alpha} = (H_{\beta+\alpha}-1)\bar{x}_2$, we see

$$\begin{aligned}\hat{x}_2\hat{\partial}_1 &= (H_\alpha+2)\bar{x}_2\bar{\partial}_1(H_\alpha+1)(H_{\beta+\alpha}+1) = \bar{x}_2\bar{\partial}_1H_\alpha(H_\alpha+1)(H_{\beta+\alpha}+1)\\ &= \bar{\partial}_1\bar{x}_2\frac{H_{\beta+\alpha}+2}{H_{\beta+\alpha}+1}H_\alpha(H_\alpha+1)(H_{\beta+\alpha}+1) = \bar{\partial}_1\bar{x}_2H_\alpha(H_\alpha+1)(H_{\beta+\alpha}+2)\\ &= \bar{\partial}_1(H_\alpha+1)(H_\alpha+2)(H_{\beta+\alpha}+1)\bar{x}_2 = \hat{\partial}_1\hat{x}_2.\end{aligned}$$

We have proved that the four relations in (4.2b) are preserved by ϕ.

Next, to show that (4.2a) holds for all $b \in B$, it suffices to show (4.2a) holds for $b \in \{H_\alpha, H_\beta, t_1, t_2\}$ since σ_i are $\mathbb{C}$-algebra homomorphisms. For $b = H_\alpha$ and $b = H_\beta$, (4.2a) follows directly from (3.7a)–(3.7b) and the definition of σ_i.

For $b = t_i$, the identities are immediately verified once we prove that $\phi(t_i) = \phi(Y_i)\phi(X_i)$ and $\phi\big(\sigma_i(t_i)\big) = \phi(X_i)\phi(Y_i)$, because then

$$\phi(X_i)\phi(t_i) = \phi(X_i)\phi(Y_i)\phi(X_i) = \phi\big(\sigma_i(t_i)\big)\phi(X_i)$$

and for $j \neq i$

$$\phi(X_i)\phi(t_j) = \phi(X_i)\phi(Y_j)\phi(X_j) = \hat{x}_i\hat{\partial}_j\hat{x}_j = \hat{x}_j\hat{\partial}_j\hat{x}_i = \phi\big(\sigma_i(t_j)\big)\phi(X_i);$$

likewise for Y_i.

Actually, that $\phi(t_i) = \phi(Y_i)\phi(X_i)$ is immediate by definition of $\phi(t_i)$. So, what remains is to prove that

$$\phi\big(\sigma_i(t_i)\big) = \phi(X_i)\phi(Y_i) \tag{A.1}$$

for $i = 1, 2$.

Now the left-hand side of (A.1) equals

$$\begin{aligned}\phi(\sigma_i(t_i)) &= \phi\big(c_i + \hat{f}_{i1}t_1 + \hat{f}_{i2}t_2\big) = c_i + \hat{f}_{i1}\hat{\partial}_1\hat{x}_1 + \hat{f}_{i2}\hat{\partial}_2\hat{x}_2\\ &= c_i + \hat{f}_{i1}\cdot(H_\alpha+2)(H_{\beta+\alpha}+2)\bar{\partial}_1\bar{x}_1 + \hat{f}_{i2}\cdot(H_\alpha+1)(H_{\beta+\alpha}+2)\bar{\partial}_2\bar{x}_2\\ &= c_i + (H_\alpha+i)(H_{\beta+\alpha}+1)\big(f_{i1}\bar{\partial}_1\bar{x}_1 + f_{i2}\bar{\partial}_2\bar{x}_2\big);\end{aligned} \tag{A.2}$$

indeed, the right-hand side of (A.1) equals

$$\hat{x}_i\hat{\partial}_i = (H_\alpha+i)(H_{\beta+\alpha}+1)\bar{x}_i\bar{\partial}_i,$$

which equals (A.2), by (3.7e).

At this point we can establish that σ_1 and σ_2 actually commute. We have for any $b \in B$,

$$0 = (\hat{x}_1\hat{x}_2 - \hat{x}_2\hat{x}_1)b = (\sigma_1(\sigma_2(b)) - \sigma_2(\sigma_1(b)))\hat{x}_1\hat{x}_2.$$

By Corollary 3.6, it follows that σ_1 and σ_2 commute.

We have shown all GWA relations (4.2) are preserved by ϕ. Therefore, there exists a well-defined $\mathbb{C}$-algebra homomorphism $\phi\colon B(\sigma,t) \to D(\mathfrak{sp}(4))$ as in the statement of Theorem 4.6. Moreover, ϕ is a homomorphism of left R-modules. By [4], when $B = R[t_1, t_2, \dots, t_n]$ for some subring R, the monomials $Y_1^aY_2^bX_1^cX_2^d$ $(a, b, c, d \in \mathbb{Z}_{\geq 0})$ form a basis for $B(\sigma, t)$ as a left R-module. These are mapped under ϕ to the monomials $\hat{\partial}_1^a\hat{\partial}_2^b\hat{x}_1^c\hat{x}_2^d$ which form a left R-basis for $D(\mathfrak{sp}(4))$ by Lemma 3.3. Therefore, ϕ is bijective. ∎

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