

Elliptic Stable Envelopes for Certain Non-Symplectic Varieties and Dynamical R -Matrices for Superspin Chains from the Bethe/Gauge Correspondence

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Abstract. We generalize Aganagic–Okounkov’s theory of elliptic stable envelopes, and its physical realization in Dedushenko–Nekrasov’s and Bullimore–Zhang’s works, to certain varieties without holomorphic symplectic structure or polarization. These classes of varieties include, in particular, classical Higgs branches of 3d $\mathcal{N} = 2$ quiver gauge theories. The Bethe/gauge correspondence relates such a gauge theory to an anisotropic/elliptic superspin chain, and the stable envelopes compute the R -matrix that solves the dynamical Yang–Baxter equation (dYBE) for this spin chain. As an illustrative example, we solve the dYBE for the elliptic $\mathfrak{sl}(1|1)$ spin chain with fundamental representations using the corresponding 3d $\mathcal{N} = 2$ SQCD whose classical Higgs branch is the Lascoux resolution of a determinantal variety. Certain Janus partition functions of this theory on $I \times \mathbb{E}$ for an interval I and an elliptic curve \mathbb{E} compute the elliptic stable envelopes, and in turn the geometric elliptic R -matrix, of the anisotropic $\mathfrak{sl}(1|1)$ spin chain. Furthermore, we consider the 2d and 1d reductions of elliptic stable envelopes and the R -matrix. The reduction to 2d gives the K-theoretic stable envelopes and the trigonometric R -matrix, and a further reduction to 1d produces the cohomological stable envelopes and the rational R -matrix. The latter recovers Rimányi–Rozansky’s results that appeared recently in the mathematical literature.

Key words: equivariant elliptic cohomology; elliptic stable envelope; 3d $\mathcal{N} = 2$ theory; Janus interfaces; elliptic genus

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Contents

1	Introduction	3
1.1	Background and motivation	3
1.2	Summary of the results	9
1.3	Future directions	12
1.4	Organization of the paper	13
1.5	Glossary of notations	13
I	Stable envelopes: beyond symplectic varieties	15
2	Classical Higgs branches of 3d $\mathcal{N} = 2$ gauge theories	15
2.1	Example: GIT quotients	15
2.2	Example: quiver varieties	17
3	Elliptic stable envelopes for partially-polarized varieties	20
3.1	Chambers and attracting sets	23
3.2	Attractive line bundle	25
3.3	Universal line bundle, Kähler torus and resonant locus	25
3.4	Elliptic stable envelope	26
3.5	Triangle lemma and duality	29
3.6	Abelianization	29
3.7	R -matrices and dynamical Yang–Baxter equations	31
3.8	R -matrices for quiver varieties	31
3.9	K-theory and cohomology limit	34
II	The solution to dYBE for $\mathfrak{sl}(1 1)$ from 3d $\mathcal{N} = 2$ SQCD	37
4	Stable envelopes and R-matrix from gauge theory: the setup	37
4.1	The Bethe/Gauge correspondence	38
4.2	3d $\mathcal{N} = 2$ SQCD and its parameters	40
4.3	Branes and the Bethe/Gauge correspondence	42
4.4	Gauge-theoretic definition of elliptic stable envelope	44
5	Boundaries and interfaces in 3d $\mathcal{N} = 2$ SQCD	46
5.1	Vacua and BPS equations in 3d $\mathcal{N} = 2$ Gauge theories	46
5.1.1	Vacuum equations	46
5.1.2	BPS equations	47
5.2	Classical Higgs branch	48
5.2.1	Classical Higgs branches of 3d $\mathcal{N} = 2$ SQCDs	49
5.2.2	Fixed points	51
5.2.3	Attracting sets and Morse flow	52
5.3	Boundaries and interfaces	54
5.3.1	Thimble (exceptional Dirichlet), $\mathcal{D}_{\mathfrak{C}}(p)$	55
5.3.2	Enriched Neumann, $\mathcal{N}_{\mathfrak{C}}(p)$	56
5.3.3	Massless boundary condition, $\mathcal{B}_{\mathbb{L}(p)}(p)$	58
5.3.4	Janus interface, $\mathcal{J}(m_{\mathfrak{C}}, 0)$	59

6	Stable envelopes and R-matrix from gauge theory: the computation	60
6.1	Elliptic stable envelope	60
6.2	R -matrix for elliptic $\mathfrak{sl}(1 1)$ spin chains	63
7	2d and 1d avatars of elliptic stable envelopes and the R-matrix	65
7.1	The K-theory limit	65
7.2	The cohomology limit	67
A	Equivariant elliptic cohomology	69
A.1	Chern class	70
A.2	Gysin map	71
A.3	Supports	71
A.4	Correspondences	71
A.5	Degree of a line bundle	72
B	Notations and conventions for supersymmetry	72
B.1	Conventions for spinors	72
B.2	3d $\mathcal{N} = 2$ supersymmetry	73
B.3	Localizing supercharge	73
C	$\mathfrak{sl}(1 1)$ rational R-matrix from the A-model localization	75
C.1	Construction of geometric rational R -matrix	75
C.2	Further details on two-point functions of 2d GLSM observables	84
C.2.1	Brief recap of Closset–Cremonesi–Park localization formula	84
C.2.2	Explicit computation of two-point functions	87
	References	89

1 Introduction

1.1 Background and motivation

In this paper, we study some mathematical and physical aspects of one-dimensional quantum integrable spin chains with anisotropy based on Lie superalgebras \mathfrak{g} of type A. Given a Lie algebra/superalgebra \mathfrak{g} , there is a family of integrable spin chains with varying spectral curves and spectrum-generating algebras:

Spin chain	Spectral curve	Spectrum generating algebra
XXX/Rational	\mathbb{C}	Yangian, $Y_{\hbar}(\mathfrak{g})$
XXZ/Trigonometric	\mathbb{C}^{\times}	Quantum affine, $\mathcal{U}_{\hbar}(\hat{\mathfrak{g}})$
XYZ/Elliptic	\mathbb{E}_{τ}	Elliptic dynamical quantum group, $E_{\tau, \hbar}(\mathfrak{g})$

Table 1. One-dimensional integrable quantum spin chains based on the Lie algebra \mathfrak{g} . Here \hbar is the usual quantization parameter, τ denotes the complex structure of the elliptic curve \mathbb{E}_{τ} , and $\hat{\mathfrak{g}}$ is the affine Lie algebra of \mathfrak{g} .

Given one of these choices, to complete the description of a specific spin chain, we need to choose the number of sites L in the chain, along with the \mathfrak{g} -module V_i and the spectral curve valued inhomogeneity v_i at the i th site for $i = 1, \dots, L$. As an example, the XYZ \mathfrak{sl}_2 spin chain

with $V_i = \mathbb{C}^2$ being the fundamental representation for all i can be defined by the following Hamiltonian [7]:

$$H_{\text{XYZ-}\frac{1}{2}} = -\frac{1}{2} \sum_{i=1}^L \sum_{j=1}^3 J_j \sigma_{(i)}^j \sigma_{(i+1)}^j, \quad (1.1)$$

where $\sigma^1, \sigma^2, \sigma^3$ are the standard Pauli matrices, $\sigma_{(i)}$ acts on V_i , and J_1, J_2, J_3 are three mutually distinct coupling constants. We consider periodic spin chains by identifying the $(L+1)$ st site with the first one. Starting from the Hamiltonian, one can construct a large commutative algebra of operators containing the Hamiltonian, called the Bethe subalgebra of the spectrum-generating algebra. These operators provide the large number of conserved charges for the spin chain, which is the source of its integrability. The wave function and energy spectrum of such spin chains are obtained using its equivalence to the 2d statistical eight-vertex model and the Bethe ansatz [7, 50, 79, 129, 131, 132, 136, 138, 149, 150].

It became apparent from the pioneering works [5, 6, 90, 128, 149, 150] that the more general and fundamental approach to defining integrable spin chains is utilizing the *R-matrix* and the *Yang-Baxter equation* (YBE). For any two \mathfrak{g} -modules U and V with inhomogeneities u and v , the *R-matrix* is an operator $R(u-v) \in \text{End}(U \otimes V)$ depending meromorphically on the difference $u-v$.¹ It is sometimes convenient to use a diagrammatic notation for the *R-matrix*, as in Figure 1.

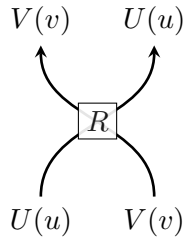


Figure 1. Diagrammatic representation of the *R-matrix* $R(u-v): U \otimes V \rightarrow U \otimes V$. The notation $U(u)$ means that at the site with the module U the inhomogeneity is u . For rational spin chains, given a \mathfrak{g} -module U , $U(u)$ becomes the evaluation module for the Yangian $\mathcal{Y}_{\hbar}(\mathfrak{g})$ with the evaluation parameter u .

The defining property of the *R-matrix* is that it satisfies the YBE. For \mathfrak{g} -modules U, V and W with inhomogeneities u, v and w , the YBE can be written as

$$R_{UV}(u-v)R_{UW}(u-w)R_{VW}(v-w) = R_{VW}(v-w)R_{UW}(u-w)R_{UV}(u-v). \quad (1.2)$$

It is an equality of operators in $U \otimes V \otimes W$. Here R_{UV} is the *R-matrix* in $\text{End}(U \otimes V)$ and its action is extended to $U \otimes V \otimes W$ by identity, similarly for R_{VW} and R_{UW} . Figure 2 shows the diagrammatic YBE.

The *R-matrix* is used to construct the commuting charges of the spin chain as follows. Given the spin chain Hilbert space $\mathcal{H} = \bigotimes_{i=1}^L V_i$, we add an auxiliary site U to it with inhomogeneity u .² Then we define the transfer matrix

$$t(u) := \text{tr}_U [R_{UV_1}(u-v_1)R_{UV_2}(u-v_2) \cdots R_{UV_L}(u-v_L)] \in \text{End}(\mathcal{H}),$$

¹The difference is taken in the appropriate abelian group, $\mathbb{C}, \mathbb{C}^\times$ or \mathbb{E}_τ .

²Inhomogeneity of the auxiliary site is also referred to as the spectral parameter in literature. We use the terms inhomogeneity of a site and its spectral parameter interchangeably.

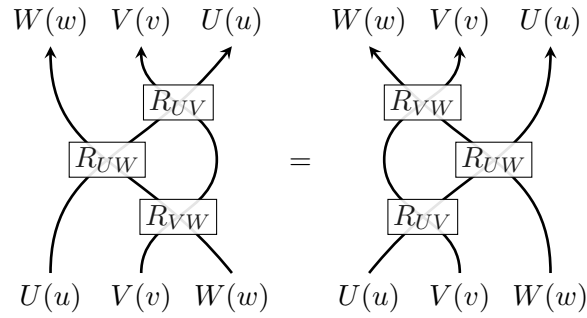
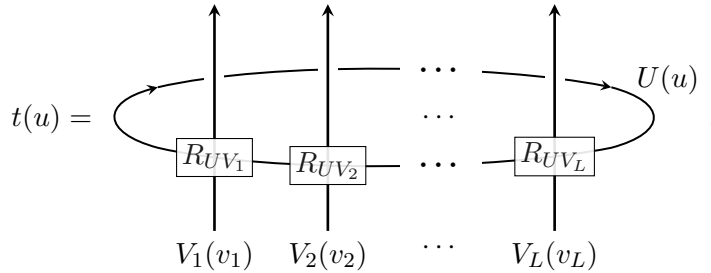


Figure 2. Diagrammatic Yang–Baxter equation. A line joining two R -matrices is a contraction of indices for the vector space labeling the line. In particular, the order of matrix multiplication in the YBE (1.2) is read off from the diagrams by reading from bottom to top.

which can also be depicted in a diagram as



As a consequence of the YBE, these transfer matrices commute with each other for arbitrary inhomogeneities $[t(u), t(v)] = 0$. Therefore, if we expand the transfer matrix as $t(u) = \sum_{n=0}^{\infty} u^n h_n$, then the coefficients $h_n \in \text{End}(\mathcal{H})$ form a large commutative algebra of operators: $[h_m, h_n] = 0$. This becomes the Bethe subalgebra of the spin chain. Any element of this subalgebra can be taken to be the Hamiltonian of the quantum mechanical system.

The R -matrix, as a solution of the YBE, has been computed in all cases from Table 1 for bosonic Lie [8, 9, 11, 12, 13, 14, 81, 82, 116, 130, 135] and affine Lie algebra [71, 116]. For Lie superalgebras, the first examples of solutions to YBE appeared in [82]. Certain rational solutions were also obtained in [80]. The classical trigonometric solutions to YBE are constructed in [85] while the explicit trigonometric quantum R -matrices first appeared in [10]. Furthermore, the rational solution for $\mathfrak{g} = \mathfrak{sl}(1|1)$ is constructed using the notion of stable envelopes in [117]. In this paper, we propose a general construction of elliptic R -matrices for Lie superalgebras of type A and we explicitly compute the elliptic R -matrix for $\mathfrak{g} = \mathfrak{sl}(1|1)$. Reduction of the elliptic R -matrix automatically gives the trigonometric R -matrix. Further reduction of the trigonometric R -matrix gives the rational R -matrix. The reduction of our elliptic $\mathfrak{sl}(1|1)$ R -matrix down to the rational case matches with the previously computed rational $\mathfrak{sl}(1|1)$ R -matrix from [117].

The tool that we use to construct the elliptic R -matrices is a conjectural relationship between quantum integrable systems and supersymmetric gauge theories, called the *Bethe–Gauge correspondence*, proposed by Nekrasov and Shatashvili [102, 103, 104]. Part of their conjecture is an action of the spectrum-generating algebra of a quantum integrable system, Yangian for example, on the cohomology (or generalized cohomologies) of the Higgs branch of the corresponding gauge theory. The correspondence and its string-theory origin, when \mathfrak{g} is a bosonic Lie algebra, have been extensively studied [23, 27, 33, 54, 57, 74, 77, 97, 98, 102, 103, 104, 105, 110, 111, 113], generalized to higher dimensions [44, 45, 69, 70, 84, 101, 102, 105], and the mathematical formulation has been established [1, 2, 25, 88, 106, 107]. For \mathfrak{g} being a Lie superalgebra, the correspondence and its string-theory origin were studied recently [20, 66, 99, 112] but despite progress [26, 117], a full mathematical treatment was lacking. Providing such a mathematical

formulation for \mathfrak{g} being a Lie superalgebra of type A has been one of the central motivations of this work.

Notably, for a 3d gauge theory with $\mathcal{N} = 4$ supersymmetry, the Higgs branch is a complex symplectic variety. A particular example is the 3d $\mathcal{N} = 4$ quiver gauge theory, whose Higgs branch is the *Nakajima quiver variety* [94], and in this case, the works of Nakajima [95] and Varagnolo [141] establish the quantum affine algebra $\mathcal{U}_\hbar(\hat{\mathfrak{g}}_Q)$ (resp. Yangian $Y_\hbar(\mathfrak{g}_Q)$) actions on equivariant cohomology (resp. equivariant K-theory) of the Nakajima quiver variety \mathcal{M}_Q , where \mathfrak{g}_Q is the Kac–Moody algebra associated to the quiver Q . Their construction relies on the generators and relations of the Yangian $Y_\hbar(\mathfrak{g}_Q)$ or quantum affine algebra $\mathcal{U}_\hbar(\hat{\mathfrak{g}}_Q)$.

From the quantum integrability point of view, the spectrum-generating algebras are better explained in the framework of R -matrices and Yang–Baxter equations (1.2), instead of generators and relations. In the context of the Bethe–Gauge correspondence, the vector spaces on which the R -matrices act are generalized cohomologies of the Higgs branches of the corresponding gauge theories.

Providing the mathematical formulation of the Bethe–Gauge correspondence has been one of the main motivations of Maulik and Okounkov in their fundamental work [88]. To construct R -matrices acting on the cohomologies, Maulik and Okounkov [88] came up with a map, called the *stable envelope*, from the equivariant cohomology of the torus fixed point set to the equivariant cohomology of the whole Higgs branch

$$\text{Stab}: H_{\mathbb{T}}(\mathcal{X}^A) \rightarrow H_{\mathbb{T}}(\mathcal{X}). \quad (1.3)$$

Here \mathcal{X} is assumed to be a smooth complex symplectic variety, with a torus \mathbb{T} action such that the subtorus $A \subset \mathbb{T}$ preserves the symplectic structure. Their construction depends on a choice of chamber \mathfrak{C} which is a certain subset of the real Lie algebra $\text{Lie}(A)_{\mathbb{R}}$, and the R -matrix is defined as

$$R_{\mathfrak{C}_2 \leftarrow \mathfrak{C}_1} := \text{Stab}_{\mathfrak{C}_2}^{-1} \circ \text{Stab}_{\mathfrak{C}_1}: H_{\mathbb{T}}(\mathcal{X}^A) \rightarrow H_{\mathbb{T}}(\mathcal{X}^A). \quad (1.4)$$

Then it follows from the definition that R -matrices satisfy the braiding relation

$$R_{\mathfrak{C}_1 \leftarrow \mathfrak{C}_n} \circ R_{\mathfrak{C}_n \leftarrow \mathfrak{C}_{n-1}} \circ \cdots \circ R_{\mathfrak{C}_2 \leftarrow \mathfrak{C}_1} = 1, \quad (1.5)$$

which implies the Yang–Baxter equation. The choice of chamber \mathfrak{C} induces a partial order \succ on the set of connected components of \mathcal{X}^A such that $\text{Stab}_{\mathfrak{C}}$ is triangular with respect to \succ and $R_{-\mathfrak{C} \leftarrow \mathfrak{C}} = \text{Stab}_{-\mathfrak{C}}^{-1} \circ \text{Stab}_{\mathfrak{C}}$ gives a Gauss decomposition of $R_{-\mathfrak{C} \leftarrow \mathfrak{C}}$. A similar kind of Gauss decomposition of R -matrices has been discussed by Khoroshkin and Tolstoy in the algebraic setting [76, 115]. Since the paper of Maulik and Okounkov came out, the construction of stable envelopes has been extended to K-theory [106] and elliptic cohomology [2], giving rise to trigonometric and elliptic solutions to the Yang–Baxter equations respectively.

From the physics point of view, a strategy for the construction of cohomological stable envelopes was suggested by Nekrasov [97, 98], and was successfully implemented for gauge theories associated with $\mathfrak{sl}(2)$ spin chains in [23]. The latter also provides a recipe for the computation of the R -matrix by an A-model localization computation. The systematic gauge-theoretic construction of elliptic stable envelopes was developed in [24, 40]. In these works the elliptic stable envelope was identified as certain Janus partition functions of 3d $\mathcal{N} = 4$ theories on $I \times \mathbb{E}_\tau$ where I is some interval and \mathbb{E}_τ is the elliptic curve. The Janus interface interpolates between certain boundary conditions corresponding to the vacua of massless and massive 3d theories. These boundary conditions correspond to equivariant elliptic cohomology classes of the Higgs branch of the 3d theory and the (flavor) torus fixed points of the Higgs branch respectively [39]. The elliptic stable envelope is then seen as a map between these cohomologies (the elliptic version of (1.3)). Dedushenko and Nekrasov [40] construct Janus interfaces for more general 3d $\mathcal{N} = 2$

theories, though they provide concrete boundary conditions corresponding to vacua of 3d $\mathcal{N} = 4$ theories and they compute the stable envelopes for complex symplectic varieties corresponding to 3d $\mathcal{N} = 4$ Higgs branches. In this paper we build on the techniques of [40] and generalize the boundary conditions to vacua of 3d $\mathcal{N} = 2$ theories and use them to compute the elliptic stable envelopes of non-symplectic varieties corresponding to 3d $\mathcal{N} = 2$ Higgs branches. Some related topics such as vertex functions for Higgs branches of 3d $\mathcal{N} = 4$ (and $\mathcal{N} = 2$) theories and the role of elliptic stable envelopes in the transformation of vertex functions under mirror symmetry/symplectic duality have been discussed in [35, 41].

A key ingredient in the mathematical construction of elliptic stable envelope [2] is a *polarization*, i.e., a decomposition of the tangent bundle in the K-theory

$$T_X = T_X^{1/2} + \hbar^{-1}(T_X^{1/2})^\vee \in K_{\mathbb{T}}(X). \quad (1.6)$$

This kind of decomposition shows up naturally on the Higgs branches of 3d $\mathcal{N} = 4$ gauge theories, for example, hypertoric varieties and Nakajima quiver varieties. Computations of stable envelopes have been done for many examples of these types of varieties [46, 47, 53, 121, 122, 123, 125]. However, for more general 3d $\mathcal{N} = 2$ theories, their Higgs branches do not have complex symplectic structure or polarization, even worse they can be singular. There have been efforts to construct stable envelopes for varieties without complex symplectic structure or polarization [117], the authors constructed cohomological stable envelopes using the *super weight functions*, which generalized their previous works on weight functions and stable envelopes on cotangent bundles of flag varieties [121, 142]. They found the R -matrix of $Y_{\hbar}(\mathfrak{sl}(1|1))$ as the geometric R -matrix of the total space of the bundle $\mathcal{O}(-1)^{\oplus 2} \rightarrow \mathbb{P}^1$. The super Lie algebra $\mathfrak{sl}(1|1)$ justifies the terminology “super” weight functions.

Another approach to generalizing the construction of stable envelopes was proposed by Okounkov in [109], with a focus on the elliptic version. An observation in [109] is that polarization is not crucial for the existence of elliptic stable envelopes. A technical alternative to polarization is a notion called an *attractive line bundle* \mathcal{S}_X on the \mathbb{T} -equivariant elliptic cohomology $\text{Ell}_{\mathbb{T}}(X)$ (see Appendix A for a review on equivariant elliptic cohomology). Roughly speaking, \mathcal{S}_X provides a “square root” of the elliptic Thom line bundle $\Theta(T_X)$, in parallel to a polarization being “half” of the tangent bundle. For a precise definition, see Section 3.2. It is shown in [109] that if X admits a polarization (1.6), then the elliptic Thom line bundle $\Theta(T_X^{1/2})$ is an attractive line bundle, which makes the construction of elliptic stable envelopes in [2] a special case of the more general one in [109].

Part of the objectives in this paper is to use the idea developed in [109] to extend the construction of elliptic stable envelopes in [2] to the classical Higgs branches of a certain class of 3d $\mathcal{N} = 2$ gauge theories which do not have $\mathcal{N} = 4$ supersymmetry. In mathematical terminology, we construct the elliptic stable envelopes for certain smooth varieties which do not necessarily admit complex symplectic structure or polarization. The replacement for the polarization in our construction is a certain structure called the *partial polarization*, which means a decomposition of the tangent bundle in the K-theory $T_X = \text{Pol}_X + \hbar^{-1}\text{Pol}_X^\vee + \mathcal{E} \in K_{\mathbb{T}}(X)$, where $\mathcal{E} \in K_{\mathbb{T}}(X)$ is a K-theory class that is required to have the following properties:

- $\mathcal{E} = \mathcal{E}^\vee$ in $K_{\mathbb{T}}(X)$, and that
- the elliptic Thom bundle $\Theta(\mathcal{E})$ admits a square root.

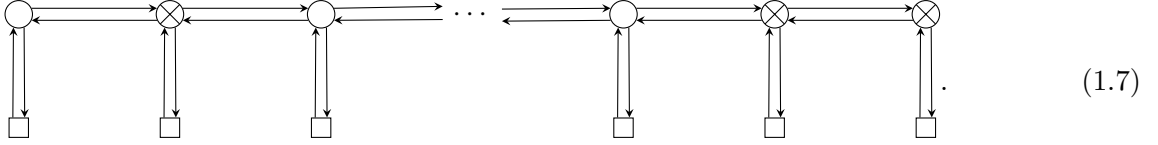
Then it turns out that $\Theta(\text{Pol}_X) \otimes \sqrt{\Theta(\mathcal{E})}$ is an attractive line bundle, thus the machinery developed in [109] is applicable to our situation.

We will give a sufficient condition for the existence of partial polarization in Proposition 3.1. A class of examples that satisfy the criterion in Proposition 3.1 are quiver varieties. They are similar to Nakajima quiver varieties, namely our quiver variety is built from the cotangent bundle

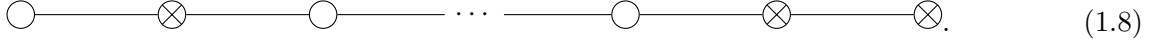
to the representation of a quiver by imposing some relations and stability conditions and then taking the quotient by the gauge group; but the difference is that we allow some of the nodes to be free from imposing complex moment map equations, these nodes will be called *odd*, and the ordinary nodes where complex moment map equations are imposed will be called *even*. We use the following notation for the odd/even nodes

$$\begin{array}{cc} \text{odd} & \text{even} \\ \otimes & \circ \end{array} .$$

The following is an example of a type A quiver (framed and doubled)



Our notation might remind the reader of Kac–Dynkin diagrams for Lie superalgebras [55], for example, the Kac–Dynkin diagram corresponding to (1.7) is the following



Indeed our choice of notation is on purpose, due to the following conjecture we make.

Conjecture 1.1. *Let Q be a finite or affine type A quiver with nodes decorated as above, and let \mathfrak{g}_Q be the Lie superalgebra associated with the corresponding Kac–Dynkin diagram. For a generic stability parameter ζ , and a pair of dimension vectors $\mathbf{v}, \mathbf{w} \in \mathbb{N}^{Q_0}$, denote by $\mathcal{M}^\zeta(\mathbf{v}, \mathbf{w})$ the quiver variety of the given data. There exists an action of elliptic dynamical quantum super group $E_{\hbar, \tau}(\mathfrak{g}_Q)$ on the extended equivariant elliptic cohomology scheme*

$$E_{\mathbb{T}}(\mathcal{M}^\zeta(\mathbf{w})) = \bigsqcup_{\mathbf{v}} E_{\mathbb{T}}(\mathcal{M}^\zeta(\mathbf{v}, \mathbf{w})).$$

Similarly, there exists an action of quantum affine super algebra $\mathcal{U}_{\hbar}(\widehat{\mathfrak{g}}_Q)$ on the equivariant K -theory

$$K_{\mathbb{T}}(\mathcal{M}^\zeta(\mathbf{w})) = \bigoplus_{\mathbf{v}} K_{\mathbb{T}}(\mathcal{M}^\zeta(\mathbf{v}, \mathbf{w})),$$

and an action of super Yangian $\mathcal{Y}_{\hbar}(\mathfrak{g}_Q)$ on the equivariant cohomology

$$H_{\mathbb{T}}(\mathcal{M}^\zeta(\mathbf{w})) = \bigoplus_{\mathbf{v}} H_{\mathbb{T}}(\mathcal{M}^\zeta(\mathbf{v}, \mathbf{w})).$$

Moreover, all the actions factor through the corresponding Maulik–Okounkov quantum groups constructed via the stable envelopes for $\mathcal{M}^\zeta(\mathbf{w})$.

In anisotropic spin chains, as in (1.1), when all coupling constants are distinct, the R -matrix depends non-trivially on them. This dependence is manifested as a dependence of the R -matrix on an additional spectral curve valued parameter called the Dynamical parameter. Thus the elliptic R -matrix, and the corresponding YBE which involves nontrivial shifts of the spectral parameters, are also called the dynamical R -matrix and the dynamical Yang–Baxter equation (dYBE) [51, 52]. Our Conjecture 1.1 is supported by the evidence of the weight structure occurring in the shifts in the dYBE. In fact, we will show (see (3.17)) that the R -matrices constructed from the elliptic stable envelopes for quiver varieties satisfy the following dynamical Yang–Baxter equation

$$\mathbf{R}_{12}(z)\mathbf{R}_{13}(z - \hbar\boldsymbol{\mu}^{(2)})\mathbf{R}_{23}(z) = \mathbf{R}_{23}(z - \hbar\boldsymbol{\mu}^{(1)})\mathbf{R}_{13}(z)\mathbf{R}_{12}(z - \hbar\boldsymbol{\mu}^{(3)}), \quad (1.9)$$

where z is the dynamical parameter and we have suppressed the spectral parameter dependence of R -matrices. Here the dynamical shifts $\boldsymbol{\mu}$ are weights in a certain highest-weight module of $\widehat{\mathfrak{sl}}(m|n)$ or $\mathfrak{sl}(m|n)$.

1.2 Summary of the results

Let us now briefly summarize the main contributions of this work.

Elliptic stable envelopes for partially polarized varieties.

- We define the notion of partial polarization on a smooth variety X with a torus T action (see Definition 3.1), and give a sufficient condition for the existence of partial polarization (see Proposition 3.1). It turns out that a large class of GIT quotients admits partial polarization (see Example 3.1), this includes all abelian gauge group cases, and all the quiver varieties considered in Section 2.2, and the total spaces of vector bundles on partial flag varieties (see Example 3.3).
- We show that a partial polarization induces an attractive line bundle on the equivariant elliptic cohomology $\text{Ell}_T(X)$, a notion defined by Okounkov in [109]. Then the existence and uniqueness of elliptic stable envelope for partially polarized varieties follow from the general result in [109], see Theorem 3.1. In particular, elliptic stable envelopes exist for all cases considered in Examples 3.1, 3.2, 3.3 and 3.4. The last example was also considered in the recent work of Rimányi and Rozansky [117, Section 2.3.3], and our result implies the existence of stable envelopes in this example.
- We compute the elliptic stable envelope for the quiver with a single node which is odd, i.e., the total space of direct sum of L copies of tautological bundles on $\text{Gr}(N, L)$, see equation (3.6).
- The triangle lemma and the duality for the elliptic stable envelopes on partially polarized varieties are obtained verbatim to that in [2], see Section 3.5.
- We extend the abelianization procedure described by Aganagic and Okounkov [2] to the partially polarized case in Section 3.6. This is potentially helpful in the computation of stable envelopes for more general quivers.
- In Section 3.7, the R -matrices are defined in the same way as [2, 88], and dynamical Yang–Baxter equations (3.15) follows from the braiding relations. The R -matrices and dynamical Yang–Baxter equations for the quiver varieties are discussed in Section 3.8. For general quivers of finite or affine type A, we observe that the dynamical shift μ is closely related to a highest-weight module of the corresponding Lie superalgebra.
- In Section 3.9, we show that the analog of the limit procedure in [2] also works for the partially polarized varieties and produces K-theoretic and cohomological stable envelopes, see Corollary 3.2. We present the explicit K-theoretic and cohomological stable envelopes for the quiver with a single node which is odd, see (3.26) and (3.27). The cohomological stable envelope (3.27) recovers the recent result of Rimányi and Rozansky [117, Theorem 7.3].

A couple of distinctions between $3d \mathcal{N} = 2$ and $3d \mathcal{N} = 4$ cases. For the purpose of illustration, we solve the dYBE for the Lie superalgebra $\mathfrak{g} = \mathfrak{sl}(1|1)$ from the corresponding gauge theory, which is a $3d \mathcal{N} = 2$ SQCD. While rather similar in computation, there are a few important conceptual differences from the computation of $\mathfrak{sl}(2)$ elliptic stable envelopes from $3d \mathcal{N} = 4$ SQCDs.

- **Higgs branch: Non-hyperkähler and quantum correction.** Higgs branches of $3d \mathcal{N} = 2$ theories without $\mathcal{N} = 4$ supersymmetry are Kähler manifolds, unlike $3d \mathcal{N} = 4$ Higgs branches which are hyperkähler. For example, in the case of $3d \mathcal{N} = 2$ SQCD, the classical Higgs branch $\mathcal{M}_H(N, L)$ is given by the direct sum of L copies of the tautological bundle over $\text{Gr}(N, L)$, the so-called resolved determinantal variety [83]. Also, unlike in case of the $3d \mathcal{N} = 4$ Higgs branches, $3d \mathcal{N} = 2$ Higgs branches receive quantum corrections [3, 36]. In this paper, unless explicitly mentioned otherwise, all mentions of $3d \mathcal{N} = 2$ Higgs branches will refer to the *classical* Higgs branches.

- **Non-isolated fixed points.** Another departure from the 3d $\mathcal{N} = 4$ SQCD case is that there can still be continuous Higgs branch moduli in 3d $\mathcal{N} = 2$ SQCDs in the presence of generic flavor twisted masses (even at the quantum level [3, 36]). As a result, we can not have a general correspondence between fully massive vacua and Bethe eigenstates of integrable superspin chains, as is more familiar in the bosonic Bethe/Gauge correspondence [102, 103]. More specifically, in the context of interval partition functions computing stable envelopes or R -matrices for bosonic spin chains, boundary conditions are chosen representing these fully massive vacua [23, 40]. On the Higgs branch, these fully massive vacua correspond to isolated fixed points of the (flavor) torus action. In contrast, the fixed point set of the 3d $\mathcal{N} = 2$ Higgs branches decomposes into connected components of nonzero dimensions (5.26). So we introduce brane-type boundary conditions preserving $\mathcal{N} = (0, 2)$ supersymmetry mimicking these connected components (see Section 5.3.1), rather than isolated points.

The solution to dYBE for $\mathfrak{sl}(1|1)$ from 3d $\mathcal{N} = 2$ SQCD. Let us now present the formulas.

- **Elliptic stable envelopes and the elliptic R -matrix for $\mathfrak{sl}(1|1)$.** We compute the stable envelopes as Janus partition functions in 3d $\mathcal{N} = 2$ SQCD on $I \times \mathbb{E}_\tau$. The theory has $U(N)$ gauge group and $SU(L) \times U(1)_\hbar$ flavor symmetry. There is also a topological $U(1)_{\text{top}}$ symmetry acting only on monopoles. A Janus interface is created by varying $U(1)^{L-1}$ twisted masses m_1, \dots, m_L (satisfying $m_1 + \dots + m_L = 0$) along I . The masses interpolate between nonzero generic values satisfying $m_1 < \dots < m_L$ and zero. A choice of such ordering creates a chamber \mathfrak{C} in \mathbb{R}^{L-1} . The formula for the stable envelope is (see Section 6.1 for details)

$$\begin{aligned} \mathbf{Stab}_{\mathfrak{C}}(p) &:= (-1)^{\sum_a \#(i < p(a))} \\ &\times \text{Sym}_{S_N} \left[\left(\prod_a \mathbf{f}_{\mathfrak{C}, p(a)}(s_a, x, \hbar, z) \right) \left(\prod_{p(a) > p(b)} \frac{1}{\vartheta(s_a s_b^{-1})} \right) \right], \end{aligned} \quad (1.10)$$

where ϑ is the elliptic theta function (defined in (5.43)), Sym_{S_N} denotes the symmetrization with respect to the gauge holonomies s_a , and

$$\mathbf{f}_{\mathfrak{C}, n}(s, \mathbf{x}, \hbar, z) := \left(\prod_{i < n} \vartheta(sx_i) \right) \frac{\vartheta(sx_n z \hbar^{n-L})}{\vartheta(z \hbar^{n-L})} \left(\prod_{i > n} \vartheta(sx_i \hbar) \right).$$

The elliptic equivariant parameters s , x , \hbar and z correspond to background holonomies for the $U(N)$, $SU(L)$, $U(1)_\hbar$, and $U(1)_{\text{top}}$ connections respectively.

The corresponding R -matrix, which satisfies dYBE, is given by (see Section 6.2 for details)

$$\mathbf{R}_{-\mathfrak{C} \leftarrow \mathfrak{C}}(u, z) = \left(\begin{array}{c|cc|c} 1 & 0 & 0 & 0 \\ 0 & \frac{\vartheta(u)\vartheta(z)\vartheta(z\hbar^{-2})}{\vartheta(u^{-1}\hbar)\vartheta(z\hbar^{-1})^2} & \frac{\vartheta(\hbar)\vartheta(uz\hbar^{-1})}{\vartheta(u^{-1}\hbar)\vartheta(z\hbar^{-1})} & 0 \\ 0 & \frac{\vartheta(\hbar)\vartheta(u^{-1}z\hbar^{-1})}{\vartheta(u^{-1}\hbar)\vartheta(z\hbar^{-1})} & \frac{\vartheta(u)}{\vartheta(u^{-1}\hbar)} & 0 \\ \hline 0 & 0 & 0 & \frac{\vartheta(u\hbar)}{\vartheta(u^{-1}\hbar)} \end{array} \right), \quad (1.11)$$

where $-\mathfrak{C}$ denotes the opposite chamber, and the diagonal blocks from top to bottom correspond to zero, one, and two-magnon sectors, respectively. To the best of our knowledge,

the elliptic stable envelopes and the elliptic R -matrix for $\mathfrak{sl}(1|1)$ have not been constructed in the literature before. Furthermore, this is also the first time that a new solution to dYBE has been constructed directly from the gauge theory using the Bethe/Gauge correspondence that we could find.

- **K-theoretic stable envelopes and the trigonometric R -matrix for $\mathfrak{sl}(1|1)$.** The $3d \rightarrow 2d$ reduction of elliptic stable envelopes gives K-theoretic stable envelopes and the trigonometric R -matrix. The precise way to get the K-theoretic stable envelope is described in Section 3.9. First, we define the slope parameter s associated to the Kähler parameter z by

$$s := \lim_{\substack{\ln z \rightarrow \infty \\ \ln q \rightarrow \infty}} \operatorname{Re} \left(-\frac{\ln z}{\ln q} \right) \in \mathbb{R} \setminus \mathbb{Z},$$

where $q := e^{2\pi i \tau}$. Then, the K-theoretic stable envelope, as the limit of (1.10), is given by (see Example 3.10 and Section 7.1 for details)

$$\begin{aligned} \operatorname{Stab}_{\mathfrak{e},s}(p) &= (-1)^{\sum_a \#(i < p(a))} \hbar^{\frac{1}{2} \sum_a \#(i > p(a))} \\ &\times \operatorname{Sym}_{S_N} \left[\left(\prod_{a=1}^N f_{\mathfrak{e},s,p(a)}(s_a, \mathbf{x}, \hbar) \right) \cdot \left(\prod_{a>b} \frac{1}{(s_a s_b^{-1})^{\frac{1}{2}} - (s_a s_b^{-1})^{-\frac{1}{2}}} \right) \right], \end{aligned} \quad (1.12)$$

where

$$f_{\mathfrak{e},s,n}(s, \mathbf{x}, \hbar) \equiv (s x_n)^{\lfloor s \rfloor} \prod_{i < n} (1 - (s x_i)^{-1}) \prod_{i > n} (1 - (s x_i \hbar)^{-1}).$$

The corresponding trigonometric R -matrix, as the reduction of (1.11), is $G(x) := x^{\frac{1}{2}} - x^{-\frac{1}{2}}$

$$R_{-\mathfrak{e} \leftarrow \mathfrak{e}}(u) = \left(\begin{array}{c|cc|c} 1 & 0 & 0 & 0 \\ \hline 0 & \frac{G(u)}{G(u^{-1}\hbar)} & u^{\lfloor s \rfloor + \frac{1}{2}} \frac{G(\hbar)}{G(u^{-1}\hbar)} & 0 \\ 0 & u^{-\lfloor s \rfloor - \frac{1}{2}} \frac{G(\hbar)}{G(u^{-1}\hbar)} & \frac{G(u)}{G(u^{-1}\hbar)} & 0 \\ \hline 0 & 0 & 0 & \frac{G(u\hbar)}{G(u^{-1}\hbar)} \end{array} \right). \quad (1.13)$$

Note that all holonomies are still multiplicative but are now \mathbb{C}^\times -valued.

- **Cohomological stable envelopes and the rational R -matrix for $\mathfrak{sl}(1|1)$.** A further $2d \rightarrow 1d$ reduction of (1.12) gives cohomological stable envelopes (see Section 3.9 for the precise mathematical procedure)

$$\operatorname{Stab}_{\mathfrak{e}}(p) = (-1)^{\sum_a \#(i < p(a))} \operatorname{Sym}_{S_N} \left[\left(\prod_{a=1}^N f_{\mathfrak{e},m}(s, \mathbf{x}, \hbar) \right) \cdot \left(\prod_{a>b} \frac{1}{s_a - s_b} \right) \right],$$

where $f_{\mathfrak{e},n}(s, \mathbf{x}, \hbar) := \prod_{i < n} (s + x_i) \prod_{i > n} (s + x_i + \hbar)$, while the reduction of (1.13) produces the rational R -matrix

$$R_{-\mathfrak{e} \leftarrow \mathfrak{e}}(u) = \left(\begin{array}{c|cc|c} 1 & 0 & 0 & 0 \\ \hline 0 & \frac{u}{\hbar - u} & \frac{\hbar}{\hbar - u} & 0 \\ 0 & \frac{\hbar}{\hbar - u} & \frac{u}{\hbar - u} & 0 \\ \hline 0 & 0 & 0 & \frac{\hbar + u}{\hbar - u} \end{array} \right).$$

All holonomies are now additive and \mathbb{C} -valued. These reproduce the results of Rimányi and Rozansky [117] (see Example 3.10 and Section 7.2 for details).

Dynamical R -matrices for the fundamental representation of $\mathfrak{sl}(n|1)$. We write down explicit elliptic dynamical R -matrices for the fundamental representation of $\mathfrak{sl}(n|1)$. Details are in Example 3.9. Let us label the basis for \mathbb{C}^{n+1} by v_α where $\alpha \in \{0, 1, \dots, n\}$. Then there exists a dynamical R -matrix for the tensor product $\mathbb{C}^{n+1} \otimes \mathbb{C}^{n+1}$ which reads as follows (see Example 3.9 for details)

$$\begin{aligned} & \mathbf{R}(u, z_1, \dots, z_n)(v_\alpha \otimes v_\beta) \\ &= \begin{cases} v_\alpha \otimes v_\beta, & \alpha = \beta < n, \\ D(u)v_\alpha \otimes v_\beta, & \alpha = \beta = n, \\ C(u)v_\alpha \otimes v_\beta + B\left(u, \hbar^{-\delta_{\beta,n}} \prod_{i=\alpha+1}^{\beta} z_i\right) v_\beta \otimes v_\alpha, & \alpha < \beta, \\ A\left(u, \hbar^{-\delta_{\alpha,n}} \prod_{i=\beta+1}^{\alpha} z_i\right) v_\alpha \otimes v_\beta + B\left(u, \hbar^{\delta_{\alpha,n}} \prod_{i=\beta+1}^{\alpha} z_i^{-1}\right) v_\beta \otimes v_\alpha, & \beta < \alpha. \end{cases} \end{aligned} \quad (1.14)$$

Here we use the shorthand notations for some special functions

$$\begin{aligned} A(u, z) &= \frac{\vartheta(z\hbar)\vartheta(z\hbar^{-1})\vartheta(u)}{\vartheta(z)^2\vartheta(u\hbar^{-1})}, & B(u, z) &= \frac{\vartheta(\hbar)\vartheta(uz)}{\vartheta(z)\vartheta(u^{-1}\hbar)}, & C(u) &= \frac{\vartheta(u)}{\vartheta(u\hbar^{-1})}, \\ D(u) &= \frac{\vartheta(u\hbar)}{\vartheta(u^{-1}\hbar)}. \end{aligned}$$

Then \mathbf{R} satisfies the dynamical Yang–Baxter equation (1.9) for $\boldsymbol{\mu} = (\mu_1, \dots, \mu_n)$ given by weights for the fundamental representation of $\mathfrak{sl}(n|1)$ (see (3.21) for details on the weights). This suggests that we should actually treat \mathbb{C}^{n+1} as $\mathbb{C}^{n|1}$, with basis elements v_0, v_1, \dots, v_{n-1} being even and v_n being odd.

The rational limit of (1.2) is

$$\mathbf{R}(u) = \mathbf{P} \left(\frac{u}{u - \hbar} \mathbf{\Pi} - \frac{\hbar}{u - \hbar} \mathbf{1} \right).$$

Here \mathbf{P} is the usual swapping-tensor operator, i.e., $\mathbf{P}(v_\alpha \otimes v_\beta) = v_\beta \otimes v_\alpha$, and $\mathbf{\Pi}$ is the super swapping-tensor operator, i.e., $\mathbf{\Pi}(v_\alpha \otimes v_\beta) = (-1)^{|v_\alpha||v_\beta|} v_\beta \otimes v_\alpha$, where $|v|$ is the parity of the vector v which equals to 0 if v is even and 1 if v is odd. Note that we can rewrite

$$(u - \hbar)\mathbf{\Pi}\mathbf{P}\mathbf{R}(u) = u\mathbf{1} - \hbar\mathbf{\Pi},$$

the right hand side is the more familiar rational R -matrix for the fundamental representation of $\mathfrak{sl}(n|1)$ in the literature [114].

1.3 Future directions

Let us mention some directions related to the study of this paper that we find interesting:

Stable envelopes from 4d Chern–Simons theory. A unifying theme for $(1+1)d$ integrable systems has emerged in recent years. 4d Chern–Simons theory describes quantum integrable spin-chain models and a large class of classical 2d integrable field theories [29, 30, 31, 32, 34]. In the particular case of integrable spin-chain models, the R -matrix can be constructed by a relatively straightforward Feynman diagram calculation. However, the mathematical formulation of the BGC shows that more basic objects to study are stable envelopes through which

R -matrix can be constructed easily. It would be very interesting to provide a 4d Chern–Simons perspective of the mathematical works [2, 88, 106, 107] and specially the construction of various stable envelopes. The string-theory realization of the Bethe–Gauge correspondence in which 4d Chern–Simons theory plays a crucial role should be a very useful guide in providing such a perspective (see [33, 38, 66] for some related works in this direction).

3d $\mathcal{N} = 2$ mirror symmetry for stable envelopes. It is believed that the stable envelopes for two mirror-dual pairs (X, X^\vee) must match after certain identification of parameters. This conjecture has been proven in many special cases related to 3d $\mathcal{N} = 4$ mirror symmetry: cotangent bundle of Grassmannian [120], complete flag variety of type- A_n [119] and arbitrary type [124], hypertoric varieties [127, 134], and many examples of Bow varieties [118]. We expect that a similar conclusion holds for the case of 3d $\mathcal{N} = 2$ mirror symmetry. Therefore, the first step in proving this conjecture would be to construct the elliptic stable envelope for the Higgs branch of the mirror-dual of the gauge theory we considered here. It is expected that the physical setup of [40] which we employed here in the presence of a Janus interface in the space of FI parameters (instead of a mass-Janus interface) produces the elliptic stable envelopes for the mirror-dual of Higgs branches considered in this paper. This would provide a solid ground for progress towards the mathematical formulation of 3d $\mathcal{N} = 2$ mirror symmetry (or symplectic duality).

1.4 Organization of the paper

The mathematical and physical contents of the paper are mainly organized in Part I and Part II respectively. In Section 2, we characterize the varieties we are interested in, including the classical Higgs branches of 3d $\mathcal{N} = 2$ theories, and in Section 3, we lay down the mathematical foundation of elliptic stable envelopes for these spaces. In Section 4, we give motivations, in the context of the Bethe/Gauge correspondence, for computing the stable envelopes as certain mass Janus interfaces. In particular, we argue that certain mass Janus partition functions for 3d $\mathcal{N} = 2$ SQCDs on $I \times \mathbb{E}_\tau$ should give us the $\mathfrak{sl}(1|1)$ elliptic stable envelopes. To compute the Janus partition functions for the 3d theories on intervals we need specific half-BPS boundary conditions corresponding to the vacua of the theory – we describe these in Section 5. In Section 6, we do the explicit computation of the Janus partition functions and find the formulas for the elliptic $\mathfrak{sl}(1|1)$ stable envelopes and the R -matrix satisfying the dYBE. In the last main section, Section 7, we take certain limits of the elliptic stable envelopes and recover the K-theoretic and the cohomological stable envelopes and their respective R -matrices for the $\mathfrak{sl}(1|1)$ superspin chains.

We include a brief review of equivariant elliptic cohomology in Appendix A. Our conventions regarding supersymmetry are summarized in Appendix B. Lastly, in Appendix C, we provide an alternative computation of the rational R -matrix for $\mathfrak{sl}(1|1)$ spin chains based on the observations in [23, 97, 98] using the A-model computation.

Note added. After the appearance of this work, a related work appeared in the context of stable envelopes and 3d $\mathcal{N} = 2$ gauge theories in [139]. This work does not have any overlap with the present paper.

1.5 Glossary of notations

Some often-used symbols:

Latin

- A Maximal torus of F_A .
- \mathfrak{a} Lie algebra of A.
- \mathfrak{c} A chamber for the A-twisted masses,
for $F_A = \mathrm{SU}(L)$ it corresponds to a choice of ordering for the m_i s.

\mathbb{E}_τ	An elliptic curve with complex moduli τ , part of the 3d space-time.
$\text{Ell}_T(\mathbf{X})$	T-equivariant elliptic cohomology of \mathbf{X} .
$\text{E}_T(\mathbf{X})$	Extended T-equivariant elliptic cohomology of \mathbf{X} .
\mathcal{E}_T	T-equivariant elliptic cohomology of a point.
\mathcal{E}_K	K-equivariant elliptic cohomology of a point.
F	Total flavor symmetry, $F = F_A \times \mathbb{C}_h^\times$ in Part I and $F = F_A \times \text{U}(1)_h$ in Part II.
F_A	Part of the flavor symmetry, the framing in quivers, complex in Part I and real in Part II. In SQCDs $F_A = \text{SU}(L)$.
\mathcal{F}_p	Connected component of the A-fixed point set in \mathcal{M}_H labeled by p , $\mathcal{M}_H^A = \bigsqcup_p \mathcal{F}_p$. In SQCDs p is a choice of N integers out of $\{1, \dots, L\}$.
$\widehat{\mathcal{F}}_p$	Lift of \mathcal{F}_p to the space \mathbf{M} of chirals, projection: $\pi: \mathbf{M}^s \rightarrow \mathcal{M}_H$, $\pi(\widehat{\mathcal{F}}_p) = \mathcal{F}_p$.
\mathfrak{f}	Lie algebra of F .
\mathfrak{f}_A	Lie algebra of F_A .
G	Gauge group, complex in Part I and real in Part II, in SQCDs $G = \text{SU}(N)$.
\mathfrak{g}	Lie algebra of G .
\mathbf{H}	Maximal torus of G
\mathfrak{h}	Lie algebra of \mathbf{H} .
\hbar	Equivariant elliptic parameter/fugacity for $\text{U}(1)_h$.
I	A finite interval $[y_-, y_+]$, part of the 3d space-time.
\mathbf{K}	Kähler torus.
\mathbf{M}	Space of chiral multiplets. $\mathbf{M} = T^*\text{Hom}(\mathbb{C}^L, \mathbb{C}^N)$ for SQCDs.
\mathbf{M}^s	Stable locus of \mathbf{M} .
\mathbf{M}_0	Space of massless chirals in the vacuum p .
$\mathbf{M}_{\mathfrak{C};\pm}(p)$	Space of $+/-$ -vely massive chirals in the vacuum p and chamber \mathfrak{C} .
\mathcal{M}_H	Classical Higgs branch.
$m_i, m_i^{\mathbb{C}}$	A-twisted (in 3d) and $\mathbb{A}_{\mathbb{C}}$ -twisted (in 2d) masses.
m_i^I	Triplet of A-twisted masses in 1d, $I = 1, 2, 3$.
Pol_X	Partial polarization of \mathbf{X} .
Pol_X^{op}	Opposite partial polarization of \mathbf{X} .
Q	Quiver/Kac–Dynkin diagram (e.g., (1.8)).
\overline{Q}	Framed and doubled quiver (e.g., (1.7)).
\widetilde{Q}	Tripled/gauge theory quiver (e.g., (4.1)).
$\mathcal{Q}, \widetilde{\mathcal{Q}}$	Chiral multiplets in SQCDs valued in $\text{Hom}(\mathbb{C}^L, \mathbb{C}^N)$ and $\text{Hom}(\mathbb{C}^N, \mathbb{C}^L)$.
$\mathcal{Q}, Q, \mathbf{Q}$	Supercharges in 3d, 2d, and 1d.
$\mathcal{R}, \widetilde{\mathcal{R}}$	Space of the Q s and \widetilde{Q} s in SQCD.
$\mathcal{R}_0(p), \widetilde{\mathcal{R}}(p)$	Spaces of massless Q s and \widetilde{Q} s in the vacuum p .
$\mathcal{R}_{\mathfrak{C};\pm}(p), \widetilde{\mathcal{R}}_{\mathfrak{C};\pm}(p)$	$+/-$ -ve mass subspaces of $\mathcal{R}, \widetilde{\mathcal{R}}$ in the vacuum p and chamber \mathfrak{C} .
$\mathbf{R}, \mathbf{R}^s, \mathbf{R}$	Elliptic, K-theoretic, and rational R -matrices.
$\mathbb{R}_h, \mathbb{R}_{\text{top}}$	Lie algebras of $\text{U}(1)_h$, and $\text{U}(1)_{\text{top}}$.
s	Slope in the K-theoretic stable envelope.
s_a	Equivariant elliptic parameter/fugacity for \mathbf{H} . In SQCDs $a = 1, \dots, N$.
S_A^1, S_B^1	Non-contractible cycles of $\mathbb{E}_\tau = S_A^1 \times S_B^1$, we go from 3d \rightarrow 2d \rightarrow 1d by reducing S_B^1 first and then S_A^1 .
Stab , Stab ^s , Stab	Elliptic, K-theoretic, and cohomological stable envelopes.
\mathbf{T}	Maximal torus of F .
\mathfrak{t}	Lie algebra of \mathbf{T} .

T_X	Tangent bundle of X .
\mathcal{U}	Universal line bundle on elliptic cohomology scheme.
$U(1)_h, U(1)_{\text{top}}$	Parts of the flavor symmetry. $U(1)_{\text{top}}$ acts only on monopoles in 3d.
X	Smooth quasi-projective complex variety with T action.
x_i	Equivariant elliptic parameter/fugacity for A . In SQCDs $i = 1, \dots, L$.
y	Real coordinate along I .
z	Equivariant elliptic parameter/fugacity for $U(1)_{\text{top}}$.

Greek

$\zeta, \zeta^{\mathbb{C}}$	Real (in 3d) and complex in 2d FI parameters.
ζ^I	Triplet of real FI parameters in 1d, $I = 1, 2, 3$.
μ	Complex moment map for the gauge group G action.
μ_{ev}	Complex moment map for the even part G_{ev} of the gauge group action.
$\mu_{\mathfrak{g}}$	Real moment map for the gauge group G action.
σ	Real adjoint scalar in 3d $\mathcal{N} = 2$ vector multiplet.
$\sigma^{\mathbb{C}}$	Complex adjoint scalar in 2d $\mathcal{N} = (2, 2)$ vector multiplet.

Part I

Stable envelopes: beyond symplectic varieties

2 Classical Higgs branches of 3d $\mathcal{N} = 2$ gauge theories

The essential data extracted from a 3d $\mathcal{N} = 2$ gauge theory is a complex algebraic group G , a complex G -representation \mathbf{M} , with a G -invariant algebraic function $\mathcal{W}: \mathbf{M} \rightarrow \mathbb{C}$, and a character $\zeta: G \rightarrow \mathbb{C}^\times$. The Higgs branch of the 3d $\mathcal{N} = 2$ gauge theory associated to $(G, \mathbf{M}, \mathcal{W}, \zeta)$ is then defined as the GIT quotient

$$\mathcal{M}_H(G, \mathbf{M}, \mathcal{W}, \zeta) := \text{Crit}(\mathcal{W})^{\zeta^{-ss}}/G, \quad (2.1)$$

where $\text{Crit}(\mathcal{W})^{\zeta^{-ss}}$ is the ζ -semistable locus of the critical locus $\text{Crit}(\mathcal{W})$ which is the solution to the equation $d\mathcal{W} = 0$.

Assumption. We assume that the semistable locus $\text{Crit}(\mathcal{W})^{\zeta^{-ss}}$ is smooth and the action of G on it is free so that we get a smooth Higgs branch $\mathcal{M}_H(G, \mathbf{M}, \mathcal{W}, \zeta)$. As we will see later, this assumption is satisfied in many examples that we consider. This setting is not the most general one for generic 3d $\mathcal{N} = 2$ gauge theories, but this is enough for our purpose.

2.1 Example: GIT quotients

A typical example of $\mathcal{M}_H(G, \mathbf{M}, \mathcal{W}, \zeta)$ is as follows. Let $G = G_{\text{ev}} \times G_{\text{odd}}$ that acts on a representation \mathcal{R} and then we take $\mathbf{M} := \mathcal{R} \oplus \mathcal{R}^\vee \oplus \mathfrak{g}_{\text{ev}}$. We choose a complex moment map $\mu: \mathcal{R} \oplus \mathcal{R}^\vee \rightarrow \mathfrak{g}$ for the G action, and define $\mu_{\text{ev}}: \mathcal{R} \oplus \mathcal{R}^\vee \rightarrow \mathfrak{g}_{\text{ev}}^\vee$ to be the composition $\text{pr}_{\text{ev}} \circ \mu$, where $\text{pr}_{\text{ev}}: \mathfrak{g}^\vee \rightarrow \mathfrak{g}_{\text{ev}}^\vee$ is the projection to the even part. Note that μ_{ev} is the moment map for the action of G_{ev} on \mathbf{M} . We take $\mathcal{W} = \langle X, \mu_{\text{ev}} \rangle$ where X is the coordinate on \mathfrak{g}_{ev} and $\langle \cdot, \cdot \rangle$ is the pairing between \mathfrak{g}_{ev} and $\mathfrak{g}_{\text{ev}}^\vee$. Then we choose a character $\zeta: G \rightarrow \mathbb{C}^\times$. In this case, the Higgs branch is then isomorphic to $\mathcal{M}_H(G, \mathbf{M}, \mathcal{W}, \zeta) \cong \mu_{\text{ev}}^{-1}(0)^{\zeta^{-ss}}//G$, where $\mu_{\text{ev}}^{-1}(0)^{\zeta^{-ss}}$ is the ζ -semistable locus of

the $\mu_{\text{ev}}^{-1}(0)$. In other words, $\mathcal{M}_H(G, \mathbf{M}, \mathcal{W}, \zeta)$ is a further quotient of a Hamiltonian reduction by a group action

$$\mathcal{M}_H(G, \mathbf{M}, \mathcal{W}, \zeta) \cong ((\mathcal{R} \oplus \mathcal{R}^\vee)^{\zeta-ss} // G_{\text{ev}}) // G_{\text{odd}}. \quad (2.2)$$

Notice that we have a G -equivariant closed embedding $\mu^{-1}(0) \hookrightarrow \mu^{-1}(\mathfrak{g}_{\text{ev}}^\perp) = \mu_{\text{ev}}^{-1}(0)$, which leads to a closed embedding

$$\mu^{-1}(0)^{\zeta-ss} // G \hookrightarrow \mu_{\text{ev}}^{-1}(0)^{\zeta-ss} // G.$$

In other words, if we consider the moment map μ for the whole gauge group G , and perform the Hamiltonian reduction for the G -action, then the resulting variety is a smooth and closed (but could be empty) subvariety of $\mathcal{M}_H(G, \mathbf{M}, \mathcal{W}, \zeta)$.

Example 2.1. When G is abelian, G acts trivially on its Lie algebra \mathfrak{g} , and we have a commutative diagram

$$\begin{array}{ccccc} \mu^{-1}(0)^{\zeta-ss} // G & \hookrightarrow & \mu_{\text{ev}}^{-1}(0)^{\zeta-ss} // G & \hookrightarrow & (\mathcal{R} \oplus \mathcal{R}^\vee)^{\zeta-ss} // G \\ \downarrow & & \downarrow & & \downarrow \\ \{0\} & \hookrightarrow & \mathfrak{g}_{\text{ev}}^\perp & \hookrightarrow & \mathfrak{g}^\vee \end{array}$$

such that every square is Cartesian. The whole diagram is equivariant under the action of the flavour torus $\mathbf{A} = (\mathbb{C}^\times)^{\text{rk } \mathcal{R}} / G$. The central fiber $\mu^{-1}(0)^{\zeta-ss} // G$ is a hypertoric variety and the GIT quotient $(\mathcal{R} \oplus \mathcal{R}^\vee)^{\zeta-ss} // G$ is known as the Lawrence toric variety [64]. For a generic choice of ζ , we have $(\mathcal{R} \oplus \mathcal{R}^\vee)^{\zeta-ss} = (\mathcal{R} \oplus \mathcal{R}^\vee)^{\zeta-s}$ and the latter is smooth over the base \mathfrak{g}^\vee , therefore $(\mathcal{R} \oplus \mathcal{R}^\vee)^{\zeta-ss} // G$ is flat over the base \mathfrak{g}^\vee and $\mu_{\text{ev}}^{-1}(0)^{\zeta-ss} // G$ is flat over $\mathfrak{g}_{\text{ev}}^\perp$. In the case when the charge matrix $A: \mathbb{Z}^{\text{rk } \mathcal{R}} \rightarrow \text{Char}(G)$ is surjective and unimodular,³ the action of G on $(\mathcal{R} \oplus \mathcal{R}^\vee)^{\zeta-ss}$ is free,⁴ so $(\mathcal{R} \oplus \mathcal{R}^\vee)^{\zeta-ss} // G$ is smooth over \mathfrak{g}^\vee , thus $\mu_{\text{ev}}^{-1}(0)^{\zeta-ss} // G$ is a smooth deformation of the hypertoric variety $\mu^{-1}(0)^{\zeta-ss} // G$ over the base $\mathfrak{g}_{\text{ev}}^\perp$.

In the abelian gauge group example, we have the implication

$$\mu_{\text{ev}}^{-1}(0)^{\zeta-ss} \neq \emptyset \Rightarrow \mu^{-1}(0)^{\zeta-ss} \neq \emptyset.$$

In fact, if $\mu_{\text{ev}}^{-1}(0)^{\zeta-ss}$ is nonempty, then $\mu_{\text{ev}}^{-1}(0)^{\zeta-ss} // G$ is projective over the affine scheme $\mu_{\text{ev}}^{-1}(0) // G$. The projection is equivariant under the \mathbb{C}_r^\times which acts on $\mathcal{R} \oplus \mathcal{R}^\vee$ with weight r and acts on $\mathfrak{g}_{\text{ev}}^\perp$ and \mathfrak{g}^\vee with weight r^2 . \mathbb{C}_r^\times contracts $\mu_{\text{ev}}^{-1}(0) // G$ to a unique point, which is the image of $0 \in \mathcal{R} \oplus \mathcal{R}^\vee$, therefore the image of $\mu_{\text{ev}}^{-1}(0)^{\zeta-ss} // G$ in $\mu_{\text{ev}}^{-1}(0) // G$, which is a \mathbb{C}_h^\times -invariant closed subset, must contain the image of 0. In particular, $\mu^{-1}(0)^{\zeta-ss} // G$ is nonempty.

To summarize, in the case when the gauge group is abelian and under the unimodular assumption, we always get a smooth deformation of the corresponding hypertoric variety over the base $\mathfrak{g}_{\text{ev}}^\perp$, and the deformation is equivariant under the action of $\mathbf{A} \times \mathbb{C}_r^\times$, where \mathbf{A} is the flavour torus $(\mathbb{C}^\times)^{\text{rk } \mathcal{R}} / G$. The smooth deformation gives rise to an isomorphism between cohomology groups $H^*(\mu^{-1}(0)^{\zeta-ss} // G) \cong H^*(\mu_{\text{ev}}^{-1}(0)^{\zeta-ss} // G)$, and we will see that the stable envelopes for $\mu^{-1}(0)^{\zeta-ss} // G$ and $\mu_{\text{ev}}^{-1}(0)^{\zeta-ss} // G$ are the same (see Example 3.6).

However, when G is nonabelian, the cohomologies of $\mu^{-1}(0)^{\zeta-ss} // G$ and $\mu_{\text{ev}}^{-1}(0)^{\zeta-ss} // G$ are not isomorphic in general; more drastically, it is possible that $\mu_{\text{ev}}^{-1}(0)^{\zeta-ss}$ is nonempty but $\mu^{-1}(0)^{\zeta-ss}$ is empty. See Example 2.4 below.

³The action of G on \mathcal{R} gives a homomorphism $G \rightarrow (\mathbb{C}^\times)^{\text{rk } \mathcal{R}}$, the charge matrix A is the induced map on characters. A is called unimodular if and only if every $\text{rk } G \times \text{rk } G$ submatrix has determinant $\in \{0, \pm 1\}$.

⁴In fact, G acts on $(\mathcal{R} \oplus \mathcal{R}^\vee)^{\zeta-ss}$ freely $\iff G$ acts on $\mu^{-1}(0)^{\zeta-ss}$ freely \iff charge matrix is surjective and unimodular [64].

Remark 2.1 (tautological generation). We say that the K-theory $K(\mu_{\text{ev}}^{-1}(0)^{\zeta^{-ss}}//G)$ is generated by tautological classes if the natural map $K_G(\text{pt}) \rightarrow K(\mu_{\text{ev}}^{-1}(0)^{\zeta^{-ss}}//G)$ is surjective. This property is also known as Kirwan surjectivity. For hyperkähler quotients (i.e., $G_{\text{odd}} = 1$), it is known that both hypertoric varieties (i.e., G is torus) and the Nakajima quiver varieties have the Kirwan surjectivity property, see [63, 89]. Let ζ_b be the restriction of ζ to G_{ev} , then we have a natural open embedding $(\mathcal{R} \oplus \mathcal{R}^\vee)^{\zeta^{-ss}} \subset (\mathcal{R} \oplus \mathcal{R}^\vee)^{\zeta_b^{-ss}}$, which induces open embedding:

$$(\mathcal{R} \oplus \mathcal{R}^\vee)^{\zeta^{-ss}} //_{G_{\text{ev}}} \hookrightarrow (\mathcal{R} \oplus \mathcal{R}^\vee) //_{\zeta_b} G_{\text{ev}},$$

where the right-hand-side is the usual hyperkähler reduction. Assume that $K_{G_{\text{odd}}}((\mathcal{R} \oplus \mathcal{R}^\vee) //_{\zeta_b} G_{\text{ev}})$ is generated by $K_{G_{\text{ev}} \times G_{\text{odd}}}(\text{pt}) = K_G(\text{pt})$, then $K_{G_{\text{odd}}}((\mathcal{R} \oplus \mathcal{R}^\vee)^{\zeta^{-ss}} //_{G_{\text{ev}}})$ is also generated by $K_G(\text{pt})$, whence (2.2) implies that $K(\mu_{\text{ev}}^{-1}(0)^{\zeta^{-ss}}//G)$ is generated by $K_G(\text{pt})$. In other words,

- G_{odd} -equivariant Kirwan surjectivity of $(\mathcal{R} \oplus \mathcal{R}^\vee) //_{\zeta_b} G_{\text{ev}}$ implies the Kirwan surjectivity of $\mu_{\text{ev}}^{-1}(0)^{\zeta^{-ss}}//G$.

Since for the hypertoric varieties and the Nakajima quiver varieties, the Kirwan surjectivity holds for flavour group equivariant K-theory, we conclude that: if either G is a torus or (\mathcal{R}, G) is constructed from a quiver representation (see Section 2.2), then $\mu_{\text{ev}}^{-1}(0)^{\zeta^{-ss}}//G$ has the Kirwan surjectivity.

2.2 Example: quiver varieties

A large class of GIT quotients comes from 3d $\mathcal{N} = 2$ quiver gauge theories, and we recall the construction of the Higgs branches of such theories here.

Let $Q = (Q_0, Q_1)$ be a quiver, i.e., Q_0 and Q_1 are finite sets (known as the set of nodes and edges respectively) together with two maps $h, t: Q_1 \rightarrow Q_0$ sending an edge to its head and tail respectively. We add one more structure to the quiver Q , namely, we separate Q_0 into two parts $Q_0 = Q_0^{\text{ev}} \sqcup Q_0^{\text{odd}}$, which will be called even and odd respectively. Our notation for the even and odd nodes is as follows

$$\begin{array}{cc} \text{odd} & \text{even} \\ \otimes & \circ \end{array} .$$

Let $\mathbf{w}, \mathbf{v} \in \mathbb{N}^{Q_0}$ be Q_0 -tuples of natural numbers, then the gauge group and the chiral multiplets are built from the above data as follows.

The gauge group $G = G_{\text{ev}} \times G_{\text{odd}}$ is such that

$$G_{\text{ev}} = \prod_{i \in Q_0^{\text{ev}}} \text{GL}(\mathbf{v}_i), \quad G_{\text{odd}} = \prod_{i \in Q_0^{\text{odd}}} \text{GL}(\mathbf{v}_i).$$

And the space of chiral multiplets $\mathbf{M} = \mathcal{R} \oplus \mathcal{R}^\vee \oplus \mathfrak{g}_{\text{ev}}$ is such that

$$\mathcal{R} = \bigoplus_{i \in Q_0} \text{Hom}(W_i, W_i) \oplus \bigoplus_{a \in Q_1} \text{Hom}(V_{t(a)}, V_{h(a)}),$$

where W_i is the \mathbf{w}_i -dimensional complex vector space and V_i is the \mathbf{v}_i -dimensional complex vector space. The action of G on \mathcal{R} is the obvious one, namely $\text{GL}(\mathbf{v}_i)$ acts on V_i as the fundamental representation. We give notations to the field contents of \mathcal{R} by letting $\alpha_i \in \text{Hom}(V_i, W_i)$ for $i \in Q_0$ and $x_a \in \text{Hom}(V_{t(a)}, V_{h(a)})$ for $a \in Q_1$; for the dual representation \mathcal{R}^\vee , we use the notation $\tilde{\alpha}_i \in \text{Hom}(W_i, V_i)$ and $\tilde{x}_a \in \text{Hom}(V_{h(a)}, V_{t(a)})$ for $a \in Q_1$.

The holomorphic symplectic form on $\mathcal{R} \oplus \mathcal{R}^\vee$ is

$$\omega = \sum_{a \in Q_1} dx_a \wedge d\tilde{x}_a + \sum_{i \in Q_0} d\alpha_i \wedge d\tilde{\alpha}_i,$$

so there is a moment map $\mu: \mathcal{R} \oplus \mathcal{R}^\vee \rightarrow \mathfrak{g}^\vee$ such that

$$\mu(x_a, \tilde{x}_a, \alpha_i, \tilde{\alpha}_i) = \sum_{a \in Q_1} [x_a, \tilde{x}_a] + \sum_{i \in Q_0} \tilde{\alpha}_i \alpha_i.$$

We define $\mu_{\text{ev}}: \mathcal{R} \oplus \mathcal{R}^\vee \rightarrow \mathfrak{g}_{\text{ev}}^\vee$ to be the composition of μ with the projection $\mathfrak{g}^\vee \rightarrow \mathfrak{g}_{\text{ev}}^\vee$.

We introduce the flavour symmetry group $F = G_W \times \mathbb{C}_\hbar^\times$, such that $G_W = \prod_{i \in Q_0} \text{GL}(\mathbf{w}_i)$, $\text{GL}(\mathbf{w}_i)$ acts on W_i by fundamental representation, and \mathbb{C}_\hbar^\times acts on \mathbf{M} by scaling \mathcal{R}^\vee with weight \hbar^{-1} and scaling \mathfrak{g}_{ev} with weight \hbar and fixing \mathcal{R} . Note that the action of F on \mathbf{M} commutes with the action of G . We also fix a maximal torus \mathbb{T} of F , namely $\mathbb{T} = T_W \times \mathbb{C}_\hbar^\times$ such that $T_W = \prod_{i \in Q_0} (\mathbb{C}^\times)^{\mathbf{w}_i}$ is a maximal torus of G_W .

Finally, we choose a character $\zeta: G \rightarrow \mathbb{C}^\times$, ζ can be written as $\zeta(g) = \prod_{i \in Q_0} \det(g_i)^{\zeta_i}$. Then the Higgs branch of the 3d $\mathcal{N} = 2$ quiver gauge theory associated to the quiver Q with gauge rank \mathbf{v} and flavour rank \mathbf{w} is isomorphic to the GIT quotient

$$\mathcal{M}^\zeta(\mathbf{v}, \mathbf{w}) = \mu_{\text{ev}}^{-1}(0)^{\zeta-ss} // G, \quad (2.3)$$

and we write

$$\mathcal{M}^\zeta(\mathbf{w}) := \bigsqcup_{\mathbf{v} \in \mathbb{N}^{Q_0}} \mathcal{M}^\zeta(\mathbf{v}, \mathbf{w}),$$

for a fixed \mathbf{w} . In other words, we impose complex moment map equations only for even nodes.

If ζ equals to $\zeta_+ := (1, \dots, 1)$ or $\zeta_- := (-1, \dots, -1)$, then according to the King's criterion for the stability [78], a quiver representation $(V, x_a, \tilde{x}_a, \alpha_i, \tilde{\alpha}_i) \in \mathcal{R} \oplus \mathcal{R}^\vee$ is ζ -semistable if and only if the following condition is satisfied respectively:

- (ζ_+) If $S_i \subset V_i$ are subspaces such that S is preserved under the maps (x_a, \tilde{x}_a) , and that $S_i \supset \text{Im}(\tilde{\alpha}_i)$ for all $i \in Q_0$, then $S = V$.
- (ζ_-) If $T_i \subset V_i$ are subspaces such that T is preserved under the maps (x_a, \tilde{x}_a) , and that $T_i \subset \text{Ker}(\alpha_i)$ for all $i \in Q_0$, then $T = 0$.

In the following discussion, we always assume that

- ζ is generic, i.e., $(\mathcal{R} \oplus \mathcal{R}^\vee)^{\zeta-s} = (\mathcal{R} \oplus \mathcal{R}^\vee)^{\zeta-ss}$.

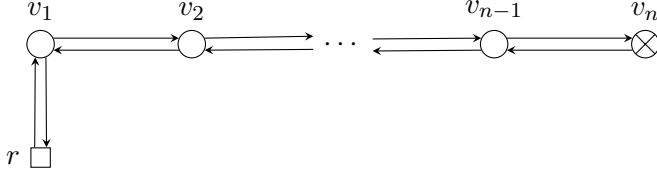
For example, ζ_\pm are generic.

It is well known that if ζ is generic, then $\mu: (\mathcal{R} \oplus \mathcal{R}^\vee)^{\zeta-ss} \rightarrow \mathfrak{g}^\vee$ is smooth, and the action of G on $(\mathcal{R} \oplus \mathcal{R}^\vee)^{\zeta-ss}$ is free. Since the projection $\text{pr}_{\text{ev}}: \mathfrak{g}^\vee \rightarrow \mathfrak{g}_{\text{ev}}^\vee$ is smooth, the composition $\mu_{\text{ev}} = \text{pr}_{\text{ev}} \circ \mu$ is smooth. Therefore, we have the following lemma.

Lemma 2.1. *Assume that ζ is generic then, $\mathcal{M}^\zeta(\mathbf{v}, \mathbf{w})$ is a smooth variety and the quotient map $\mu_{\text{ev}}^{-1}(0)^{\zeta-ss} \rightarrow \mathcal{M}^\zeta(\mathbf{v}, \mathbf{w})$ is a principal G -bundle.*

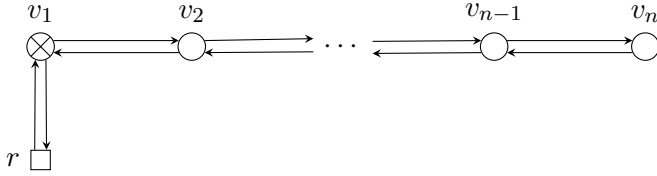
Example 2.2. If there is no odd node, i.e., Q_0^{odd} is empty, then $\mu_{\text{ev}} = \mu$ and in this case $\mathcal{M}^\zeta(\mathbf{v}, \mathbf{w})$ is a Nakajima quiver variety [93, 94].

Example 2.3. Let Q be an A_n quiver with $Q_0^{\text{ev}} = \{1, \dots, n-1\}$ and $Q_0^{\text{odd}} = \{n\}$, take framing dimension $\mathbf{w} = (r, 0, \dots, 0)$ and gauge dimension $\mathbf{v} = (v_1, \dots, v_n)$. Below is the doubled quiver \overline{Q} ,



$\mathcal{M}^{\zeta^+}(\mathbf{v}, \mathbf{w})$ is nonempty if and only if $r \geq v_1 \geq \dots \geq v_n$, and if nonempty then $\mathcal{M}^{\zeta^+}(\mathbf{v}, \mathbf{w})$ is isomorphic to $\text{GL}_r \times^P \mathfrak{m}$, where P is the parabolic subgroup of GL_r that stabilizes a fixed flag $F_\bullet = F_1 \subset F_2 \subset \dots \subset F_n \subset F_{n+1} = \mathbb{C}^r$ such that $\dim F_{n+1}/F_i = v_i$, and \mathfrak{m} is the Lie algebra of the subgroup $M \subset P$ that acts on F_{i+1}/F_i as identity for all i except for $i = n$. $\mathcal{M}^{\zeta^+}(\mathbf{v}, \mathbf{w})$ is a vector bundle over the flag variety $\text{Fl}_{\mathbf{v}} = \text{GL}_r/P$. Note that M contains the unipotent radical of P , therefore $\mathcal{M}^{\zeta^+}(\mathbf{v}, \mathbf{w})$ contains $T^*\text{Fl}_{\mathbf{v}}$ as a closed subvariety.

Example 2.4. Let Q be an A_n quiver with $Q_0^{\text{ev}} = \{2, \dots, n\}$ and $Q_0^{\text{odd}} = \{1\}$, take framing dimension $\mathbf{w} = (r, 0, \dots, 0)$ and gauge dimension $\mathbf{v} = (v_1, \dots, v_n)$. Below is the doubled quiver \overline{Q} ,



$\mathcal{M}^{\zeta^+}(\mathbf{v}, \mathbf{w})$ is nonempty if and only if $v_n \leq r$ and $v_{i+1} \leq v_i \leq v_{i+1} + r$. $\mathcal{M}^{\zeta^+}(\mathbf{v}, \mathbf{w})$ can be described as follows. Let $d_i = v_i - v_{i+1}$ for $1 \leq i \leq n-1$ and $d_n = v_n$, then $\mathcal{M}^{\zeta^+}(\mathbf{v}, \mathbf{w})$ is a vector bundle over the convolution of affine Grassmannian orbits

$$\widetilde{\text{Gr}}_{\text{GL}_r}^{\vec{\lambda}} = \text{Gr}_{\text{GL}_r}^{\omega_{d_n}} \widetilde{\times} \text{Gr}_{\text{GL}_r}^{\omega_{d_{n-1}}} \widetilde{\times} \dots \widetilde{\times} \text{Gr}_{\text{GL}_r}^{\omega_{d_1}},$$

where $\vec{\lambda} = (\omega_{d_n}, \dots, \omega_{d_1})$ and ω_i is the i -th fundamental coweight of GL_r . The convolution of the affine Grassmannian gives a map $m: \widetilde{\text{Gr}}_{\text{GL}_r}^{\vec{\lambda}} \rightarrow \overline{\text{Gr}}_{\text{GL}_r}^{v_1 \omega_1}$, and the latter represents sub-lattices \mathcal{L} in $\mathbb{C}[z]^{\oplus r}$ of codimension v_1 , so it has a v_1 -dimensional vector bundle \mathcal{V} which is the universal quotient $\mathbb{C}[z]^{\oplus r} / \mathcal{L}_{\text{univ}}$. $\mathcal{M}^{\zeta^+}(\mathbf{v}, \mathbf{w})$ is the total space of the vector bundle $\text{Hom}(\mathcal{V}, \mathcal{O}^{\oplus r})$ on the $\widetilde{\text{Gr}}_{\text{GL}_r}^{\vec{\lambda}}$.

In the last example, if we replace the odd node with a even one, then $\mathcal{M}^{\zeta^+}(\mathbf{v}, \mathbf{w})$ is nonempty if and only if $r \geq v_1 \geq v_2 \geq \dots \geq v_n$. In particular, if $n \geq 2$ then there exists \mathbf{v} such that $\mu^{-1}(0)^{\zeta^+ - ss}$ is empty but $\mu_{\text{ev}}^{-1}(0)^{\zeta^+ - ss}$ is nonempty, for instance $\mathbf{v} = (nr, (n-1)r, \dots, r)$.

Let $\mathbf{A} \subset T_W$ be a subtorus, such that W decomposes as eigenspaces

$$W = \bigoplus_{\lambda \in \text{Char}(\mathbf{A})} W^\lambda,$$

and we write $\mathbf{w} = \sum_{\lambda} \mathbf{w}^\lambda$ for the dimension vector, then it is well known that the \mathbf{A} -fixed points set $\mathcal{M}^{\zeta}(\mathbf{w})^{\mathbf{A}}$ is isomorphic to the product

$$\mathcal{M}^{\zeta}(\mathbf{w})^{\mathbf{A}} \cong \prod_{\lambda \in \text{Cochar}(\mathbf{A})} \mathcal{M}(\mathbf{w}^\lambda).$$

3 Elliptic stable envelopes for partially-polarized varieties

Aganagic–Okounkov defined the elliptic stable envelope and showed their existence and uniqueness for hypertoric varieties and Nakajima quiver varieties [2], later Okounkov extended the result on the existence and uniqueness of elliptic stable envelope to algebraic stacks with a polarization [108] and to varieties with an attractive line bundle on its equivariant elliptic cohomology [109].

A polarization, namely a decomposition of tangent bundle $T_X = T_X^{1/2} + (T_X^{1/2})^\vee$ in the K-theory class of a variety or stack X , is manifested for the Higgs branches of 3d $\mathcal{N} = 4$ gauge theories, since the moduli space is holomorphic symplectic by construction.

However, in the 3d $\mathcal{N} = 2$ setting, we do not have a polarization in general. In fact, the Higgs branch can be odd-dimensional, for example, the resolved conifold in the case of SQED with two flavours (see equation (5.15)). In this section, we show that the condition of having a polarization can be weakened to having a partial polarization (defined below). We basically follow the line of arguments in [109], the main point is that we can always construct an attractive line bundle from a partial polarization.

Throughout this section, we denote by X a smooth quasi-projective complex variety with a torus \mathbb{T} action, we fix a nontrivial group homomorphism $\mathbb{T} \rightarrow \mathbb{C}_\hbar^\times$ and a subtorus $A \subset \ker(\mathbb{T} \rightarrow \mathbb{C}_\hbar^\times)$.

Definition 3.1. A *partial polarization* on X is the following data:

- a decomposition of the tangent bundle

$$T_X = \text{Pol}_X^+ + \hbar^{-1}(\text{Pol}_X^+)^{\vee} + \text{Pol}_X^- + \hbar(\text{Pol}_X^-)^{\vee} + \mathcal{E} \in K_{\mathbb{T}}(X),$$

for some \mathbb{T} -equivariant K-theory classes $\text{Pol}_X := \text{Pol}_X^+ + \text{Pol}_X^-$ and \mathcal{E} ,

such that

- (1) $\mathcal{E} = \mathcal{E}^\vee$ in $K_{\mathbb{T}}(X)$, and that
- (2) the elliptic Thom line bundle $\Theta(\mathcal{E})$ admits a square root.⁵

We call such X a partially-polarized variety, with partial polarization Pol_X . And we define the opposite partial polarization to be $\text{Pol}_X^{\text{op}} = \text{Pol}_X^{\text{op},+} + \text{Pol}_X^{\text{op},-} = \hbar^{-1}(\text{Pol}_X^+)^{\vee} + \hbar(\text{Pol}_X^-)^{\vee}$.

Remark 3.1. In particular, the existence of partial polarization implies that T_X equals to T_X^\vee in $K_A(X)$.

We give a criterion when a variety X has a partial polarization in the following.

Proposition 3.1. Let $P_X^\pm \in K_{\mathbb{T}}(X)$, and moreover assume that there exist

- (1) a finite set of pairs $(G_i, n_i)_{i \in I}$, where $n_i \in \mathbb{Z}_{\neq 0}$ and G_i is a reductive group whose derived subgroup $[G_i, G_i]$ is simply connected and every simple constituent is of type A, C, D, E₆, or G₂,
- (2) for every $i \in I$, a \mathbb{T} -equivariant principal G_i bundle \mathcal{P}_i on X ,

such that the tangent bundle T_X decomposes in the \mathbb{T} -equivariant K-theory into

$$T_X = P_X^+ + \hbar^{-1}(P_X^+)^{\vee} + P_X^- + \hbar(P_X^-)^{\vee} + \sum_{i \in I} n_i \text{adj}(\mathcal{P}_i) \in K_{\mathbb{T}}(X),$$

where $\text{adj}(\mathcal{P}_i) := \mathcal{P}_i \times^{G_i} \mathfrak{g}_i$ is the adjoint bundle associated to \mathcal{P}_i . Then $P_X = P_X^+ + P_X^-$ is a partial polarization on X .

⁵For a review of elliptic cohomology and Theta bundles, see Appendix A.

The above proposition is the direct consequence of the following lemma.

Lemma 3.1. *Let G be a reductive group whose derived subgroup is simply connected and every simple constituent is of type A, C, D, E_6 , or G_2 . Let \mathcal{P} be a \mathbb{T} -equivariant principal G -bundle on X , then the elliptic Thom line bundle $\Theta(\text{adj}(\mathcal{P}))$ has a square root in $\text{Pic}(\text{Ell}_{\mathbb{T}}(X))$.*

Proof. It suffices to prove the existence of a square root in the universal case, i.e., $\Theta(\mathfrak{g})$ on $\mathbb{E} \otimes_{\mathbb{Z}} \mathbb{X}_*/W$ has a square root, where \mathbb{X}_* is the cocharacter lattice of G and W is the Weyl group and for any G -representation V , $\Theta(V)$ is the pullback of the Θ -bundle along the induced map $\mathbb{E} \otimes_{\mathbb{Z}} \mathbb{X}_*/W \rightarrow S^{\text{rk} V} \mathbb{E}$ between the moduli of semistable bundles on \mathbb{E} .

We first assume that G is simple, then the cocharacter lattice \mathbb{X}_* agrees with the coroot lattice Q^\vee . According to the result of Looijenga [86], $\mathbb{E} \otimes_{\mathbb{Z}} Q^\vee/W$ is isomorphic to the weighted projective space $\mathbb{P}(1, g_1, \dots, g_r)$ such that

$$\theta^\vee = \sum_{i=1}^r g_i \alpha_i^\vee,$$

is the decomposition of the dual of the highest root in terms of simple coroots. It is well known that the Picard group of $\mathbb{P}(1, g_1, \dots, g_r)$ is freely generated by $\mathcal{O}(m)$, where $m = \text{lcm}(g_1, \dots, g_r)$ [15], and Looijenga also shows that $\Theta(\mathfrak{g})$ is isomorphic to $\mathcal{O}(2h^\vee)$ where $h^\vee = 1 + \sum_{i=1}^r g_i$ is the dual Coxeter number [86], therefore $\Theta(\mathfrak{g})$ has a square root if and only if h^\vee is divisible by m , which is true if and only if G is of type A, C, D, E_6 or G_2 .

In the general case, G is the central quotient of $\tilde{G} := H \times G'$ by a finite abelian group Γ , where H is a torus and G' is the derived subgroup of G . Then $\mathbb{E} \otimes_{\mathbb{Z}} \mathbb{X}_*/W$ is the quotient of $\mathbb{E} \otimes_{\mathbb{Z}} \tilde{\mathbb{X}}_*/W$ by $H^1(\mathbb{E}, \Gamma)$, where $\tilde{\mathbb{X}}_*$ is the cocharacter lattice of \tilde{G} , so there is a fiber sequence

$$H^1(\mathbb{E}, \Gamma)^\vee \longrightarrow \text{Pic}(\mathbb{E} \otimes_{\mathbb{Z}} \mathbb{X}_*/W) \longrightarrow \text{Pic}(\mathbb{E} \otimes_{\mathbb{Z}} \tilde{\mathbb{X}}_*/W)^{H^1(\mathbb{E}, \Gamma)}, \quad (3.1)$$

where $H^1(\mathbb{E}, \Gamma)^\vee$ is the Pontryagin dual of $H^1(\mathbb{E}, \Gamma)$. Note that we can identify $\mathbb{E} \otimes_{\mathbb{Z}} \tilde{\mathbb{X}}_*/W = \mathcal{E}_H \times (\mathbb{E} \otimes_{\mathbb{Z}} Q^\vee/W)$, where Q^\vee is the coroot lattice of G' . We also note that there is an étale locally trivial fibration $\mathbb{E} \otimes_{\mathbb{Z}} \mathbb{X}_*/W \rightarrow \mathcal{E}_{H/\Gamma}$ with fibers isomorphic to $\mathbb{E} \otimes_{\mathbb{Z}} Q^\vee/W$, such that the trivial fibration $\mathcal{E}_H \times (\mathbb{E} \otimes_{\mathbb{Z}} Q^\vee/W)$ is the pullback along the quotient map $\mathcal{E}_H \rightarrow \mathcal{E}_{H/\Gamma}$. The fiber sequence (3.1) is compatible with its counterpart coming from the quotient map between the bases $\mathcal{E}_H \rightarrow \mathcal{E}_{H/\Gamma}$. In other words, we have the commutative diagram of fiber sequences

$$\begin{array}{ccccc} H^1(\mathbb{E}, \Gamma)^\vee & \longrightarrow & \text{Pic}(\mathcal{E}_{H/\Gamma}) & \longrightarrow & \text{Pic}(\mathcal{E}_H)^{H^1(\mathbb{E}, \Gamma)} \\ \parallel & & \downarrow & & \downarrow \\ H^1(\mathbb{E}, \Gamma)^\vee & \longrightarrow & \text{Pic}(\mathbb{E} \otimes_{\mathbb{Z}} \mathbb{X}_*/W) & \longrightarrow & \text{Pic}(\mathbb{E} \otimes_{\mathbb{Z}} \tilde{\mathbb{X}}_*/W)^{H^1(\mathbb{E}, \Gamma)}. \end{array}$$

Now the image of $\Theta(\mathfrak{g})$ in $\text{Pic}(\mathbb{E} \otimes_{\mathbb{Z}} \tilde{\mathbb{X}}_*/W)^{H^1(\mathbb{E}, \Gamma)}$ is the $\mathcal{O}_{\mathcal{E}_H} \boxtimes \Theta(\mathfrak{g}')$, which has a square root $\mathcal{O}_{\mathcal{E}_H} \boxtimes \Theta(\mathfrak{g}')^{\otimes \frac{1}{2}}$ by the previous step. The square root is invariant under the action $H^1(\mathbb{E}, \Gamma)$ since $H^1(\mathbb{E}, \Gamma)$ acts trivially on $\text{Pic}(\mathbb{E} \otimes_{\mathbb{Z}} Q^\vee/W)$. We lift such square root to $\mathcal{L} \in \text{Pic}(\mathbb{E} \otimes_{\mathbb{Z}} \mathbb{X}_*/W)$ so that $\Theta(\mathfrak{g}) \cong \mathcal{L}^{\otimes 2} \otimes \mathcal{K}$ for some $\mathcal{K} \in H^1(\mathbb{E}, \Gamma)^\vee$. Since $H^1(\mathbb{E}, \Gamma)^\vee \in \text{Pic}^0(\mathcal{E}_{H/\Gamma})$, so we can find $\mathcal{K}_1 \in \text{Pic}^0(\mathcal{E}_{H/\Gamma})$ such that $\mathcal{K} \cong \mathcal{K}_1^{\otimes 2}$, hence we have $\Theta(\mathfrak{g}) \cong (\mathcal{L} \otimes \mathcal{K}_1)^{\otimes 2}$. ■

Remark 3.2. When $G = \text{GL}_n$, there is an alternative proof of the above lemma. In this case $\text{adj}(\mathcal{P}) \cong \text{End}(\mathcal{V})$ where \mathcal{V} is a \mathbb{T} -equivariant vector bundle. Notice that $\Theta(\mathcal{V}\mathcal{V}^\vee)$ is isomorphic to pullback $\Delta^* \Theta(\mathcal{V}_1 \mathcal{V}_2^\vee)$ along the diagonal morphism $\text{Ell}_{\mathbb{T}}(X) \hookrightarrow \text{Ell}_{\mathbb{T}}(X) \times \text{Ell}_{\mathbb{T}}(X)$, where $\mathcal{V}_i = \text{pr}_i^* \mathcal{V}$ are pullback of \mathcal{V} along projections to the first and the second component. Moreover, $(12)^* \Theta(\mathcal{V}_1 \mathcal{V}_2^\vee) \cong \Theta(\mathcal{V}_1^\vee \mathcal{V}_2)$ which is isomorphic to $\Theta(\mathcal{V}_1 \mathcal{V}_2^\vee)$, where (12) is the permutation of factors, thus we can apply [100, Lemma 6.1] to conclude that $\Theta(\mathcal{V}\mathcal{V}^\vee)$ admits a square root.

Example 3.1. Suppose that we are in the situation of Section 2.1, namely, $G = G_{\text{ev}} \times G_{\text{odd}}$ acts on a representation \mathcal{R} and $\mu_{\text{ev}}: \mathcal{R} \oplus \mathcal{R}^\vee \rightarrow \mathfrak{g}_{\text{ev}}$ is the moment map for G_{ev} action. Assume moreover that G_{odd} is the reductive group whose derived subgroup is simply connected and every simple constituent is of type A, C, D, E₆, or G₂. Then $X := \mu_{\text{ev}}^{-1}(0)^{\zeta-ss} // G$ is a partially-polarized variety with $\text{Pol}_X = \text{Pol}_X^+ = \mathcal{R} - \text{adj}(\mathcal{P}_b)$, where \mathcal{R} is the bundle associated to the representation \mathcal{R} and $\mathcal{P}_b \times \mathcal{P}_f$ is the principal $G_{\text{ev}} \times G_{\text{odd}} = G$ bundle $\mu_{\text{ev}}^{-1}(0)^{\zeta-ss} \rightarrow X$. We have the decomposition $T_X = \text{Pol}_X + \hbar^{-1} \text{Pol}_X^\vee - \text{adj}(\mathcal{P}_f) \in K_{\mathbb{T}}(X)$.

Example 3.2. In the quiver case, adjoint bundles can be written in terms of tautological bundles. Let Q be a quiver with decomposition of nodes $Q_0 = Q_0^{\text{ev}} \sqcup Q_0^{\text{odd}}$ into even and odd parts. In this case, $\mathbb{T} = \mathbb{A} \times \mathbb{C}_\hbar^\times$, where $\mathbb{A} = T_W$ is the maximal torus of the flavour group. Our choice of \mathbb{C}_\hbar^\times -weight convention is as follows. We split the set of arrows into two parts $Q_1 = Q_1^+ \sqcup Q_1^-$, such that

- (\star) All arrows connected to an even node are from the same group, i.e., either all of them are in Q_1^+ or all of them are in Q_1^- . Denote by $Q_0^{\text{ev}, \pm}$ the subset of Q_0^{ev} consisting of even nodes such that arrows connected to them are all in Q_1^\pm .

Then we have $Q_0^{\text{ev}} = Q_0^{\text{ev}, +} \sqcup Q_0^{\text{ev}, -}$ by our assumption. We define the action of \mathbb{C}_\hbar^\times on the doubled quiver \bar{Q} by assigning the \mathbb{C}_\hbar^\times -weights as follows:

	x_a	$\tilde{x}_a, a \in Q_1^\pm$	α_i	$\tilde{\alpha}_i, i \in Q_0^{\text{odd}}$	$\tilde{\alpha}_i, i \in Q_0^{\text{ev}, \pm}$
\mathbb{C}_\hbar^\times -weight	0	∓ 1	0	-1	∓ 1

Let ζ be generic stability parameter, then $\mathcal{M}^\zeta(\mathbf{v}, \mathbf{w})$ is smooth. Denote by \mathcal{V}_i the tautological bundle on $\mathcal{M}^\zeta(\mathbf{v}, \mathbf{w})$ corresponding to the i -th gauge node, and denote by \mathcal{W}_i the tautological bundle on $\mathcal{M}^\zeta(\mathbf{v}, \mathbf{w})$ corresponding to the i -th framing, then $\text{Pol}_{\mathcal{M}} = \text{Pol}_{\mathcal{M}}^+ + \text{Pol}_{\mathcal{M}}^-$ such that

$$\begin{aligned} \text{Pol}_{\mathcal{M}}^+ &:= \sum_{i \in Q_0 \setminus Q_0^{\text{ev}, -}} \mathcal{W}_i \mathcal{V}_i^\vee + \sum_{a \in Q_1^+} \mathcal{V}_{h(a)} \mathcal{V}_{t(a)}^\vee - \sum_{j \in Q_0^{\text{ev}, +}} \mathcal{V}_j \mathcal{V}_j^\vee, \\ \text{Pol}_{\mathcal{M}}^- &:= \sum_{i \in Q_0^{\text{ev}, -}} \mathcal{W}_i \mathcal{V}_i^\vee + \sum_{a \in Q_1^-} \mathcal{V}_{h(a)} \mathcal{V}_{t(a)}^\vee - \sum_{j \in Q_0^{\text{ev}, -}} \mathcal{V}_j \mathcal{V}_j^\vee \end{aligned} \quad (3.2)$$

is a partial polarization on $\mathcal{M}(\mathbf{v}, \mathbf{w})$. In fact,

$$T_{\mathcal{M}^\zeta(\mathbf{v}, \mathbf{w})} = \text{Pol}_{\mathcal{M}}^+ + \hbar^{-1} (\text{Pol}_{\mathcal{M}}^+)^{\vee} + \text{Pol}_{\mathcal{M}}^- + \hbar (\text{Pol}_{\mathcal{M}}^-)^{\vee} - \sum_{i \in Q_0^{\text{odd}}} \mathcal{V}_i \mathcal{V}_i^\vee,$$

in $K_{\mathbb{T}}(\mathcal{M}^\zeta(\mathbf{v}, \mathbf{w}))$.

Example 3.3. Let G be a semisimple and simply connected algebraic group, P be a parabolic subgroup, and $U \subset P$ be the unipotent radical. Let M be a connected normal subgroup of P such that M contains U , and \mathfrak{m} be its Lie algebra. Denote by $K = P/M$, which is a quotient of the Levi $L \cong P/U$. Take $X = G \times^P \mathfrak{m}$, and $\mathbb{T} = \mathbb{A} \times \mathbb{C}_\hbar^\times$ acts on X such that \mathbb{A} is maximal torus of G and \mathbb{C}_\hbar^\times only acts on the fiber \mathfrak{m} with weight \hbar^{-1} . Then tangent bundle of X decomposes into

$$T_X = T_{G/P} + \hbar^{-1} T_{G/P}^\vee + \mathcal{L} - \mathcal{K} \in K_{\mathbb{T}}(X),$$

where $T_{G/P}$ is the pullback of the tangent bundle of G/P via the canonical projection $G \times^P \mathfrak{m} \rightarrow G/P$, and \mathcal{L} (resp. \mathcal{K}) is the associated adjoint bundle of the principal L -bundle (resp. K -bundle) induced from the principal P -bundle $G \times \mathfrak{m} \rightarrow X$. Assume that every simple constituent of L is of type A, C, D, E₆, or G₂, then the same holds for K . Therefore, $T_{G/P}$ is a partial polarization of X under this assumption.

Example 3.4. A special case of the previous example is when $G = \mathrm{SL}_n$ and P is the subgroup fixing a flag $F_\bullet = (0 = F_0 \subset \mathcal{F}_1 \subset \cdots \subset F_{k-1} \subset F_k = \mathbb{C}^n)$, we choose the Lie algebra \mathfrak{m}_s according to a marking number $s \in \{\pm 1\}^k$ such that

$$\mathfrak{m}_s(F_i) \subset \begin{cases} F_{i-1}, & s_i = +1, \\ F_i, & s_i = -1, \end{cases}$$

then $\mathbf{X} = G \times^P \mathfrak{m}_s$ has a partial polarization for any choice of marking s . These varieties show up in the work of Rimányi and Rozansky [117, Section 2.3.3] as the moduli space of certain quiver-like diagram construction.

3.1 Chambers and attracting sets

Let $\mathrm{Cochar}(\mathbf{A})$ be the cocharacter lattice of \mathbf{A} , and we denote $\mathfrak{a}_{\mathbb{R}} := \mathrm{Cochar}(\mathbf{A}) \otimes_{\mathbb{Z}} \mathbb{R} \subset \mathfrak{a} = \mathrm{Lie}(\mathbf{A})$. We define the *roots* of the pair (\mathbf{X}, \mathbf{A}) to be the set of weights $\{\alpha\}$ appearing in the normal bundle to $\mathbf{X}^{\mathbf{A}}$. A *chamber* is defined to be a connected component of the complement of hyperplanes cut out by roots, i.e.,

$$\mathfrak{a}_{\mathbb{R}} \setminus \bigcup_{\alpha \in \text{roots}} \alpha^\perp = \bigsqcup_i \mathfrak{C}_i. \quad (3.3)$$

For every $\sigma \in \mathfrak{C}_i \cap \mathrm{Cochar}(\mathbf{A})$, we have $\mathbf{X}^\sigma = \mathbf{X}^{\mathbf{A}}$.

Let \mathfrak{C} be a chamber, then we say that a root α is attracting (resp. repelling) if α is positive (resp. negative) on \mathfrak{C} . For a connected component $\mathcal{F} \subset \mathbf{X}^{\mathbf{A}}$, we define the attracting part $N_{\mathbf{X}/\mathcal{F}}^+$ of normal bundle to be the span of attracting root space in $N_{\mathbf{X}/\mathcal{F}}$; similarly, the repelling part $N_{\mathbf{X}/\mathcal{F}}^-$ is defined to be the span of repelling root space in $N_{\mathbf{X}/\mathcal{F}}$. Then we define attracting submanifold $\mathrm{Attr}_{\mathfrak{C}}(\mathcal{F})$ to be the subset

$$\{x \in \mathbf{X} \mid \lim_{t \rightarrow 0} \sigma(t) \cdot x \in \mathcal{F}\}, \quad (3.4)$$

for some $\sigma \in \mathfrak{C} \cap \mathrm{Cochar}(\mathbf{A})$. $\mathrm{Attr}_{\mathfrak{C}}(\mathcal{F})$ does not depend on the choice of σ , in fact, it is the exponential of attracting part of the normal bundle $N_{\mathbf{X}/\mathcal{F}}^+$. Define the union $\mathrm{Attr}_{\mathfrak{C}} := \bigsqcup_{\mathcal{F}} \mathrm{Attr}_{\mathfrak{C}}(\mathcal{F})$, then $\mathrm{Attr}_{\mathfrak{C}}$ admits an immersion $\mathrm{Attr}_{\mathfrak{C}} \hookrightarrow \mathbf{X} \times \mathbf{X}^{\mathbf{A}}$, $x \mapsto (x, \lim_{t \rightarrow 0} \sigma(t) \cdot x)$. In general, $\mathrm{Attr}_{\mathfrak{C}}$ is not closed in $\mathbf{X} \times \mathbf{X}^{\mathbf{A}}$, since $\lim_{t \rightarrow \infty} \sigma(t) \cdot x$, if exists, is not in $\mathrm{Attr}_{\mathfrak{C}}$. We define $\mathrm{Attr}_{\mathfrak{C}}^f$ to be the set of pairs (x, y) that belongs to a chain of closures of attracting \mathbf{A} -orbits. $\mathrm{Attr}_{\mathfrak{C}}^f$ is closed in $\mathbf{X} \times \mathbf{X}^{\mathbf{A}}$.

A partial order \preceq on the set of connected components of $\mathbf{X}^{\mathbf{A}}$ is generated by letting

$$\mathcal{F}_j \subset \overline{\mathrm{Attr}_{\mathfrak{C}}(\mathcal{F}_i)} \Rightarrow \mathcal{F}_j \preceq \mathcal{F}_i.$$

We define closed subvarieties $\mathrm{Attr}_{\mathfrak{C}}^{\leq} \subset \mathrm{Attr}_{\mathfrak{C}}^f \subset \mathbf{X} \times \mathbf{X}^{\mathbf{A}}$ by

$$\mathrm{Attr}_{\mathfrak{C}}^{\leq} := \bigcup_{\mathcal{F}_j \prec \mathcal{F}_i} \mathrm{Attr}_{\mathfrak{C}}(\mathcal{F}_j) \times \mathcal{F}_i, \quad \mathrm{Attr}_{\mathfrak{C}}^f := \bigcup_{\mathcal{F}_j \preceq \mathcal{F}_i} \mathrm{Attr}_{\mathfrak{C}}(\mathcal{F}_j) \times \mathcal{F}_i.$$

Note that $\mathrm{Attr}_{\mathfrak{C}}^f$ is a closed subvariety of $\mathrm{Attr}_{\mathfrak{C}}^{\leq}$ and $\mathrm{Attr}_{\mathfrak{C}}^f \cap (\mathrm{Attr}_{\mathfrak{C}}^{\leq} \setminus \mathrm{Attr}_{\mathfrak{C}}^f) = \mathrm{Attr}_{\mathfrak{C}}$.

Example 3.5. In the case when \mathbf{X} is constructed from quiver representation, there is a nice description of attracting sets, due to Andrei Neguț [96]. The setting in loc. cit. is for a Nakajima quiver variety (i.e., $G_{\mathrm{odd}} = 1$), nevertheless the arguments are applicable to the situation when $G_{\mathrm{odd}} \neq 1$. We record the result here. Consider $\mathbf{X} = \mathcal{M}^{\zeta_+}(\mathbf{v}, \mathbf{w})$ for a quiver Q with dimension vectors (\mathbf{v}, \mathbf{w}) and the stability condition ζ_+ , let $\mathbf{A} = \mathbb{C}_a^\times$ that acts on framing

vector space as $W = aW^{(1)} \oplus W^{(2)}$, so that the framing dimension vector decomposes accordingly $\mathbf{w} = \mathbf{w}^{(1)} + \mathbf{w}^{(2)}$, then

$$\mathcal{X}^A = \bigsqcup_{\mathbf{v}^{(1)} + \mathbf{v}^{(2)} = \mathbf{v}} \mathcal{M}^\zeta(\mathbf{v}^{(1)}, \mathbf{w}^{(1)}) \times \mathcal{M}^\zeta(\mathbf{v}^{(2)}, \mathbf{w}^{(2)}).$$

We choose the chamber \mathfrak{C} such that $a > 0$, then the same argument in [96, Lemma 3.10] can be applied, and $\text{Attr}_{\mathfrak{C}}$ consists of quiver representations (V, W) , $(V^{(1)}, W^{(1)})$, $(V^{(2)}, W^{(2)})$ in $\mathcal{X} \times \mathcal{X}^A$ that fits into exact sequences $0 \rightarrow V^{(1)} \hookrightarrow V \rightarrow V^{(2)} \rightarrow 0$, that commutes with (x_a, \tilde{x}_a) and also commutes with $(\alpha_i, \tilde{\alpha}_i)$ through the splitting exact sequence $0 \rightarrow W^{(1)} \hookrightarrow W \rightarrow W^{(2)} \rightarrow 0$. Moreover, the same argument in [96, Lemma 3.11] can be applied (since the moment map equation is not essential in the proof), and $\text{Attr}_{\mathfrak{C}}^f$ consists of quiver representations (V, W) , $(V^{(1)}, W^{(1)})$, $(V^{(2)}, W^{(2)})$ in $\mathcal{X} \times \mathcal{X}^A$ that fits into the complex $V^{(1)} \xrightarrow{f} V \xrightarrow{g} V^{(2)}$, such that

- the composition $g \circ f = 0$, and
- f and g commutes with (x_a, \tilde{x}_a) and also commutes with $(\alpha_i, \tilde{\alpha}_i)$ through the splitting exact sequence $0 \rightarrow W^{(1)} \hookrightarrow W \rightarrow W^{(2)} \rightarrow 0$.
- there exist filtrations of quiver representations

$$\begin{aligned} V^{(1)} &= E^0 \rightarrow E^1 \rightarrow \dots \rightarrow E^{k-1} \rightarrow E^k = \text{Im}(f), \\ \text{Ker}(g) &= F^k \rightarrow F^{k-1} \rightarrow \dots \rightarrow F^1 \rightarrow F^0 = V^{(2)}, \end{aligned}$$

such that the kernels of $E^l \rightarrow E^{l+1}$ and $F^{l+1} \rightarrow F^l$ are isomorphic quiver representations.

In particular,

$$\mathcal{M}^\zeta(\mathbf{v}^{(1)}, \mathbf{w}^{(1)}) \times \mathcal{M}^\zeta(\mathbf{v}^{(2)}, \mathbf{w}^{(2)}) \preceq \mathcal{M}^\zeta(\mathbf{v}^{(1)} + \mathbf{u}, \mathbf{w}^{(1)}) \times \mathcal{M}^\zeta(\mathbf{v}^{(2)} - \mathbf{u}, \mathbf{w}^{(2)}),$$

for all $\mathbf{u} \in \mathbb{N}^{\mathcal{Q}_0}$.

The restriction of partial polarization $\text{Pol}_{\mathcal{X}}$ to \mathcal{X}^A decomposes according to the chamber \mathfrak{C} as

$$\text{Pol}_{\mathcal{X}}|_{\mathcal{X}^A} = \text{Pol}_{\mathcal{X}}|_{\mathcal{X}^A, >0} + \text{Pol}_{\mathcal{X}}|_{\mathcal{X}^A, \text{fixed}} + \text{Pol}_{\mathcal{X}}|_{\mathcal{X}^A, <0}.$$

Definition 3.2. We define the index bundle $\text{ind} = \text{Pol}_{\mathcal{X}}^+|_{\mathcal{X}^A, >0} - \text{Pol}_{\mathcal{X}}^-|_{\mathcal{X}^A, >0} \in K_{\mathcal{T}}(\mathcal{X}^A)$, the alternating sum of attracting part of partial polarization.

Lemma 3.2. $\text{Pol}_{\mathcal{X}}|_{\mathcal{X}^A, \text{fixed}}$ is a partial polarization on \mathcal{X}^A .

Proof. Let $T_{\mathcal{X}} = \text{Pol}_{\mathcal{X}}^+ + \hbar^{-1}(\text{Pol}_{\mathcal{X}}^+)^{\vee} + \text{Pol}_{\mathcal{X}}^- + \hbar(\text{Pol}_{\mathcal{X}}^-)^{\vee} + \mathcal{E}$, then there exists a decomposition of \mathcal{E} into eigenspaces of \mathbf{A} -actions

$$\mathcal{E}|_{\mathcal{X}^A} = \sum_{\lambda \in \text{Cochar}(\mathbf{A})} \mathcal{E}^\lambda.$$

By the duality $\mathcal{E} \cong \mathcal{E}^{\vee}$, we have $\mathcal{E}^{-\lambda} \cong (\mathcal{E}^\lambda)^{\vee}$, therefore we have

$$\Theta(\mathcal{E}^0) = \Theta(\mathcal{E}|_{\mathcal{X}^A}) \otimes \bigotimes_{\lambda(\mathfrak{C}) > 0} \Theta(\mathcal{E}^\lambda)^{\otimes 2}.$$

In particular, we see that $\Theta(\mathcal{E}^0)$ admits a square root. Moreover, we have the decomposition of the tangent bundle of \mathcal{X}^A in $K_{\mathcal{T}}(\mathcal{X}^A)$

$$T_{\mathcal{X}^A} = \text{Pol}_{\mathcal{X}}^+|_{\mathcal{X}^A, \text{fixed}} + \hbar^{-1}(\text{Pol}_{\mathcal{X}}^+|_{\mathcal{X}^A, \text{fixed}})^{\vee} + \text{Pol}_{\mathcal{X}}^-|_{\mathcal{X}^A, \text{fixed}} + \hbar(\text{Pol}_{\mathcal{X}}^-|_{\mathcal{X}^A, \text{fixed}})^{\vee} + \mathcal{E}^0,$$

which implies that $\text{Pol}_{\mathcal{X}}|_{\mathcal{X}^A, \text{fixed}}$ is a partial polarization on \mathcal{X}^A . ■

We denote by $\text{Pol}_{\mathcal{X}^A}$ the restriction of partial polarization $\text{Pol}_{\mathcal{X}}|_{\mathcal{X}^A, \text{fixed}}$ on \mathcal{X}^A .

3.2 Attractive line bundle

This is the most crucial part of our construction of elliptic stable envelopes for partially-polarized varieties. An important ingredient that we need from [109] is the following.

Definition 3.3. A line bundle \mathcal{L} on $\text{Ell}_\Gamma(\mathbf{X})$ is called attractive for a given chamber \mathfrak{C} if $\deg_{\mathbf{A}} \mathcal{L} = \deg_{\mathbf{A}} \Theta(N_{\mathbf{X}/\mathbf{X}^{\mathbf{A}}}^-)$, where $\deg_{\mathbf{A}} \mathcal{L}$ is the degree of the restriction of \mathcal{L} to the fiber along the projection $\text{Ell}_\Gamma(\mathbf{X}^{\mathbf{A}}) \rightarrow \text{Ell}_{\Gamma/\mathbf{A}}(\mathbf{X}^{\mathbf{A}})$.⁶ The $\deg_{\mathbf{A}}$ takes value in $H^0(\mathbf{X}^{\mathbf{A}}, S^2\text{Char}(\mathbf{A}))$.

Definition 3.4. From now on, we fix a square root for $\Theta(\mathcal{E})$ and define $\mathcal{S}_{\mathbf{X}} := \Theta(\text{Pol}_{\mathbf{X}}) \otimes \Theta(\mathcal{E})^{\otimes \frac{1}{2}}$.

Proposition 3.2. $\mathcal{S}_{\mathbf{X}}$ is an attractive line bundle for every chamber \mathfrak{C} of $\text{Cochar}(\mathbf{A})$.

Proof. Since $S^2\text{Char}(\mathbf{A})$ is a torsion-free abelian group, it suffices to show that $\deg_{\mathbf{A}}(\mathcal{S}_{\mathbf{X}}^{\otimes 2}) = \deg_{\mathbf{A}}(\Theta(N_{\mathbf{X}/\mathbf{X}^{\mathbf{A}}}^- + N_{\mathbf{X}/\mathbf{X}^{\mathbf{A}}}^{-\vee}))$. Since $N_{\mathbf{X}/\mathbf{X}^{\mathbf{A}}}^{-\vee}$ is \mathbf{A} -equivariantly isomorphic to $N_{\mathbf{X}/\mathbf{X}^{\mathbf{A}}}^+$, we have

$$\deg_{\mathbf{A}}(\Theta(N_{\mathbf{X}/\mathbf{X}^{\mathbf{A}}}^- + N_{\mathbf{X}/\mathbf{X}^{\mathbf{A}}}^{-\vee})) = \deg_{\mathbf{A}}(\Theta(N_{\mathbf{X}/\mathbf{X}^{\mathbf{A}}})) = \deg_{\mathbf{A}}(\Theta(T_{\mathbf{X}})).$$

On the other hand, $\mathcal{S}_{\mathbf{X}}^{\otimes 2} \cong \Theta(\text{Pol}_{\mathbf{X}} + \text{Pol}_{\mathbf{X}}^{\vee} + \mathcal{E})$, and its \mathbf{A} -degree equals to that of

$$\Theta(\text{Pol}_{\mathbf{X}}^+ + \hbar^{-1}(\text{Pol}_{\mathbf{X}}^+)^{\vee} + \text{Pol}_{\mathbf{X}}^- + \hbar(\text{Pol}_{\mathbf{X}}^-)^{\vee} + \mathcal{E}) = \Theta(T_{\mathbf{X}}),$$

this proves the lemma. ■

Remark 3.3. For a line bundle \mathcal{L} on $\text{Ell}_\Gamma(\mathbf{X})$, define $\mathcal{L}^{\nabla} := \mathcal{L}^{\vee} \otimes \Theta(T_{\mathbf{X}})$. Then there is an isomorphism $\mathcal{S}_{\mathbf{X}}^{\nabla} \cong \Theta(\text{Pol}_{\mathbf{X}}^{\text{op}}) \otimes \Theta(\mathcal{E})^{\otimes \frac{1}{2}}$, therefore $\mathcal{S}_{\mathbf{X}}^{\nabla}$ is the attractive line bundle associated to the opposite partial polarization.

The attractive line bundle associated to a polarization naturally restricts to \mathbf{A} -fixed points. In fact we can define $\mathcal{S}_{\mathbf{X},\mathbf{A}} := i^* \mathcal{S}_{\mathbf{X}} \otimes \Theta(-N_{\mathbf{X}/\mathbf{X}^{\mathbf{A}}}^-)$ where $i: \text{Ell}_\Gamma(\mathbf{X}^{\mathbf{A}}) \rightarrow \text{Ell}_\Gamma(\mathbf{X})$ is the map induced by the inclusion $\mathbf{X}^{\mathbf{A}} \hookrightarrow \mathbf{X}$. On the other hand, we also have the line bundle $\mathcal{S}_{\mathbf{X}^{\mathbf{A}}}$ defined using the restriction of the partial polarization $\text{Pol}_{\mathbf{X}^{\mathbf{A}}} \in \text{Pic}(\text{Ell}_\Gamma(\mathbf{X}^{\mathbf{A}}))$, i.e., $\mathcal{S}_{\mathbf{X}^{\mathbf{A}}} = \Theta(\text{Pol}_{\mathbf{X}^{\mathbf{A}}}) \otimes \Theta(\mathcal{E}^0)^{\otimes \frac{1}{2}}$. $\mathcal{S}_{\mathbf{X},\mathbf{A}}$ is not isomorphic to $\mathcal{S}_{\mathbf{X}^{\mathbf{A}}}$ in general, and they are related as follows

$$\mathcal{S}_{\mathbf{X},\mathbf{A}} \otimes \mathcal{U} \cong \mathcal{S}_{\mathbf{X}^{\mathbf{A}}} \otimes \Theta(\hbar)^{-\text{rk ind}} \otimes \tau(-\hbar \det \text{ind})^* \mathcal{U}. \quad (3.5)$$

\mathcal{U} is the universal line bundle and $\tau(-\hbar \det \text{ind})$ is the translation [2], whose definitions are recalled below.

3.3 Universal line bundle, Kähler torus and resonant locus

Let \mathcal{A} be an abelian variety and \mathcal{A}^{\vee} its dual abelian variety, then there is a universal line bundle $\mathcal{U}_{\text{Poincaré}}$ on $\mathcal{A}^{\vee} \times \mathcal{A}$. The sections of $\mathcal{U}_{\text{Poincaré}}$ on the universal cover of $\mathcal{A}_s^{\vee} \times \mathcal{A}_z$ has the same factor of automorphy as the function $\frac{\vartheta(sz)}{\vartheta(s)\vartheta(z)}$.

Assumption. We assume that $\text{Pic}(\mathbf{X})$ is finitely generated as an abelian group, and we fix a set of generators $\{\mathcal{L}_i^{\circ}\}_{i=1}^r$, which induces a group homomorphism $\mathbf{K} = (\mathbb{C}^{\times})^r \rightarrow \text{Pic}(\mathbf{X}) \otimes_{\mathbb{Z}} \mathbb{C}^{\times}$. \mathbf{K} is called the Kähler torus.

We choose an equivariant lift $\mathcal{L}_i \in \text{Pic}_\Gamma(\mathbf{X})$ for each \mathcal{L}_i° , then the elliptic Chern class gives a map $c_i: \text{Ell}_\Gamma(\mathbf{X}) \rightarrow \mathbb{E}$. We consider $c_i \times 1: \text{Ell}_\Gamma(\mathbf{X}) \times \mathcal{E}_{z_i} \rightarrow \mathbb{E} \times \mathcal{E}_{z_i}$ and define

$$\mathcal{U}(\mathcal{L}_i, z_i) := (c_i \times 1)^* \mathcal{U}_{\text{Poincaré}},$$

⁶The degree of a line bundle \mathcal{L} on an abelian variety \mathcal{A} is defined as the image of \mathcal{L} in the Néron–Severi group $\text{NS}(\mathcal{A}) = \text{Pic}(\mathcal{A})/\text{Pic}^0(\mathcal{A})$, see Appendix A for details.

where we identify $\mathbb{E} \cong \mathcal{E}_{z_i}^\vee$. Moreover, we define the extended equivariant elliptic cohomology $E_T(X) := \text{Ell}_T(X) \times \mathcal{E}_K$ which is a scheme over $\mathcal{B}_{T,X} := \mathcal{E}_T \times \mathcal{E}_K$. Using the identification $\mathcal{E}_K \cong \prod_{i=1}^r \mathcal{E}_{z_i}$, the line bundle $\mathcal{U}(\mathcal{L}_i, z_i)$ is naturally defined on the extended equivariant elliptic cohomology $E_T(X)$, and we set $\mathcal{U} := \bigotimes_{i=1}^r \mathcal{U}(\mathcal{L}_i, z_i)$.

Let $\mathcal{F}_j \preceq \mathcal{F}_i$ be a pair of connected components of X^A and consider the commutative diagram

$$\begin{array}{ccc} E_T(\mathcal{F}_j \times \mathcal{F}_i) & \xrightarrow{\phi} & E_{T/A}(\mathcal{F}_j \times \mathcal{F}_i) \\ \downarrow p & \searrow \psi & \downarrow p \\ \mathcal{B}_{T,X} & \xrightarrow{\phi} & \mathcal{B}_{T/A,X}, \end{array}$$

where $\mathcal{B}_{T/A,X} := \mathcal{E}_{T/A} \times \mathcal{E}_K$.

Definition 3.5. Let $\mathcal{S}_{X,\mathcal{F}}$ be the restriction of $\mathcal{S}_{X,A}$ to a connected component \mathcal{F} of X^A . The resonant locus Δ is defined as the union of

$$p(\text{supp} R\phi_* (\mathcal{S}_{X,\mathcal{F}_j} \otimes \mathcal{U} \boxtimes (\mathcal{S}_{X,\mathcal{F}_i} \otimes \mathcal{U})^\vee)) \subset \mathcal{B}_{T/A,X},$$

over all pairs $\mathcal{F}_j \preceq \mathcal{F}_i$ of connected components of X^A . We also use the same notation Δ for its preimages.

Verbatim to that of [109, Proposition 2.6], we have the following.

Lemma 3.3. *The complement of Δ in $E_T(X)$ is open and dense.*

For a homomorphism between abelian varieties $g: \mathcal{E}_T \rightarrow \mathcal{E}_K$, we associate an automorphism $\tau(g): \mathcal{B}_{T,X} \cong \mathcal{B}_{T,X}$ by $(t, z) \mapsto (t, z + g(t))$, where t is the coordinate on \mathcal{E}_T . We also use the same notation $\tau(g)$ for its pullback to $E_T(X)$. Such transformation does not affect $\text{Ell}_T(X)$, only the Kähler parameters are shifted. An example is as follows: for a pair

$$\mu \in \text{Char}(T) = \text{Hom}(\mathcal{E}_T, \mathbb{E}), \quad \lambda \in \text{Cochar}(K) = \text{Hom}(\mathbb{E}, \mathcal{E}_K),$$

$\lambda\mu$ is an abelian variety homomorphism, so we have a shift map $\tau(\lambda\mu)$.

The line bundle $\tau(g)^*\mathcal{U} \otimes \mathcal{U}^{-1}$ depends linearly on g , i.e.,

$$\frac{\tau(g_1)^*\mathcal{U}}{\mathcal{U}} \otimes \frac{\tau(g_2)^*\mathcal{U}}{\mathcal{U}} \cong \frac{\tau(g_1 \cdot g_2)^*\mathcal{U}}{\mathcal{U}}.$$

In the case when $\mu \in \text{Char}(T)$ and $\lambda \in \text{Cochar}(K)$, there is a meromorphic section $\frac{\vartheta(\lambda \cdot \mu)}{\vartheta(\lambda)\vartheta(\mu)}$, of the line bundle $\tau(\lambda\mu)^*\mathcal{U} \otimes \mathcal{U}^{-1}$, here $\lambda \cdot \mu$ is the coordinate multiplication [2, Lemma 2.4]. We will write $\mathcal{U}(\lambda, \mu)$ for the line bundle $\tau(\lambda\mu)^*\mathcal{U} \otimes \mathcal{U}^{-1}$.

3.4 Elliptic stable envelope

Now we have all the ingredients, and the main result of this section: the existence and uniqueness of the elliptic stable envelope for partially-polarized varieties, directly follows from the general construction in [109].

Theorem 3.1. *If X is partially polarized, then there exists a unique section*

$$\text{Stab}_{\mathfrak{e}, \mathcal{S}_X} \in \Gamma(E_T(X \times X^A) \setminus \Delta, \mathcal{S}_X \otimes \mathcal{U} \boxtimes (\mathcal{S}_{X,A} \otimes \mathcal{U})^\vee),$$

such that it is supported on $\text{Attr}_{\mathfrak{e}}^f$ and that its restriction to the complement of $\text{Attr}_{\mathfrak{e}}^<$ is given by $(-1)^{\text{rk ind}}[\text{Attr}_{\mathfrak{e}}]$.⁷

⁷For the definition of supports, see Appendix A.

Proof. This is a special case of [109, Theorem 1] applied to the attractive line bundle \mathcal{S}_X . ■

Equivalently, $\mathbf{Stab}_{\mathfrak{C}, \mathcal{S}_X}$ gives rise to a map between sheaves $\mathcal{S}_{X, \mathbf{A}} \otimes \mathcal{U} \rightarrow \mathcal{S}_X \otimes \mathcal{U}$ over the non-resonant locus $\mathcal{B}_{\mathbb{T}, X} \setminus \Delta$, which is uniquely characterized by

- (1) The support of $\mathbf{Stab}_{\mathfrak{C}, \mathcal{S}_X}$ is triangular with respect to \mathfrak{C} , i.e., $\mathbf{Stab}_{\mathfrak{C}, \mathcal{S}_X}([\mathcal{F}_i])|_{\mathcal{F}_j} = 0$ for a pair of connected components of $X^{\mathbf{A}}$ such that $\mathcal{F}_j \not\leq \mathcal{F}_i$.
- (2) The diagonal $\mathbf{Stab}_{\mathfrak{C}, \mathcal{S}_X}([\mathcal{F}_i])|_{\mathcal{F}_i} = (-1)^{\text{rk ind}} \vartheta(N_{X/\mathcal{F}_i}^-)$, where $\vartheta(N_{X/\mathcal{F}_i}^-)$ is the canonical section of the elliptic Thom line bundle $\Theta(N_{X/\mathcal{F}_i}^-)$.

We will use the above point of view on the stable envelope in the following discussions.

According to (3.5), the elliptic stable envelope is a map between sheaves

$$\mathbf{Stab}_{\mathfrak{C}, \mathcal{S}_X}: \mathcal{S}_{X^{\mathbf{A}}} \otimes \Theta(\hbar)^{-\text{rk ind}} \otimes \tau(-\hbar \det \text{ind})^* \mathcal{U} \rightarrow \mathcal{S}_X \otimes \mathcal{U}.$$

This turns out to be more practical since it is usually easier to compute $\text{Pol}_{X^{\mathbf{A}}}$ than $\mathcal{S}_{X, \mathbf{A}}$.

Change of partial polarization. If we change the partial polarization by

$$\text{Pol}'_X = \text{Pol}_X - (\mathcal{F}_+ + \mathcal{F}_-) + \hbar^{-1} \mathcal{F}_+^{\vee} + \hbar \mathcal{F}_-^{\vee},$$

for some $\mathcal{F}_+, \mathcal{F}_- \in K_{\mathbb{T}}(X)$, then the associated attractive line bundle changes by

$$\mathcal{S}'_X \otimes \mathcal{U} \cong \mathcal{S}_X \otimes \tau(\hbar \det(\mathcal{F}_+ - \mathcal{F}_-))^* \mathcal{U} \otimes \Theta(\hbar)^{\text{rk } \mathcal{F}},$$

and the elliptic stable envelope changes accordingly

$$\mathbf{Stab}_{\mathfrak{C}, \mathcal{S}'_X}(z) = \mathbf{Stab}_{\mathfrak{C}, \mathcal{S}_X}(z + \hbar \det(\mathcal{F}_+ - \mathcal{F}_-)) \cdot (-1)^{\text{rk } \mathcal{F}_{\text{moving}}},$$

where z is the Kähler parameter (coordinate on $\mathcal{E}_{\mathbb{K}}$), and $\mathcal{F}_{\text{moving}}$ is the \mathbf{A} -moving part of $\mathcal{F}|_{X^{\mathbf{A}}}$. In particular, if we choose the opposite partial polarization $\text{Pol}_X^{\text{op}} = \hbar^{-1}(\text{Pol}_X^+)^{\vee} + \hbar(\text{Pol}_X^-)^{\vee}$, then

$$\mathbf{Stab}_{\mathfrak{C}, \mathcal{S}'_X}(z) = \mathbf{Stab}_{\mathfrak{C}, \mathcal{S}_X}(z + \hbar \det(\text{Pol}_X^+ - \text{Pol}_X^-)) \cdot (-1)^{\text{rk Pol}_{X, \text{moving}}}.$$

Example 3.6 (abelian Gauge groups). Let us consider the Higgs branch of an Abelian $\mathcal{N} = 2$ gauge theory, i.e., Example 2.1. In this case $X := \mu_{\text{ev}}^{-1}(0)^{\zeta - ss} // G$ is a \mathbb{T} -equivariant smooth deformation of $X_0 := \mu^{-1}(0)^{\zeta - ss} // G$ over the base $\mathfrak{g}_{\text{ev}}^{\perp}$, where $\mathbb{T} = \mathbf{A} \times \mathbb{C}_\hbar^\times = ((\mathbb{C}^\times)^{\text{rk } \mathcal{R}} / G) \times \mathbb{C}_\hbar^\times$. \mathbb{C}_\hbar^\times acts on $\mathcal{R} \oplus \mathcal{R}^\vee$ as $\mathcal{R} \oplus \hbar^{-1} \mathcal{R}^\vee$ and acts on \mathfrak{g} as $\hbar^{-1} \mathfrak{g}$. Then every \mathbf{A} -fixed point of the hypertoric variety X_0 extends to an étale cover of the base $\mathfrak{g}_{\text{ev}}^{\perp}$, which must be trivial cover since every point in this cover is repelled by the \mathbb{C}_\hbar^\times action from the fixed point. Let \mathcal{F}_p be the component of $X^{\mathbf{A}}$ that contains fixed point $p \in X_0^{\mathbf{A}}$, then for any chamber \mathfrak{C} of $\text{Lie}(\mathbf{A})_{\mathbb{R}}$, $\mathcal{F}_p \leq \mathcal{F}_q$ in X if and only if $p \leq q$ in X_0 . In other words, the chamber \mathfrak{C} induces exactly the same partial order on $X^{\mathbf{A}}$ and $X_0^{\mathbf{A}}$. Moreover, the inclusion $X_0 \hookrightarrow X$ induces isomorphism on equivariant elliptic cohomology $\text{Ell}_{\mathbb{T}}(X_0) \cong \text{Ell}_{\mathbb{T}}(X)$. It is straightforward to see that $\mathcal{S}_X \cong \Theta(\mathcal{R}) \cong \mathcal{S}_{X_0}$, i.e., X and X_0 have the same attractive line bundle. The normal bundles are isomorphic $N_{X/\mathcal{F}_p}^{\pm} |_{\mathcal{F}_p} = N_{X_0/\mathcal{F}_p}^{\pm} |_{\mathcal{F}_p}$, so the index bundles are also isomorphic. To summarize, X and X_0 have the same elliptic stable envelope $\mathbf{Stab}_{\mathfrak{C}, \mathcal{S}_X}([\mathcal{F}_p]) = \mathbf{Stab}_{\mathfrak{C}, \mathcal{S}_{X_0}}([p])$. The explicit formula of the elliptic stable envelope for hypertoric varieties can be found in [2], see also [133].

Example 3.7 (resolved determinantal varieties). Consider the quiver Q with one odd node, i.e., $Q_0 = Q_0^{\text{odd}} = \{1\}$ and $\mathbf{v} = N, \mathbf{w} = L$, and $\zeta = \zeta_-$, then $\mathcal{M}^{\zeta}(N, L)$ is nonempty if and only if $N \leq L$, and if nonempty then it is isomorphic to the total space of the vector bundle $\text{Tot}(\mathcal{V}^{\oplus L} \rightarrow \text{Gr}(N, L))$, where \mathcal{V} is the tautological bundle of rank N on the Grassmannian. It is a resolution of the determinantal variety which parameterizes $L \times L$ matrices of rank $\leq N$.

In this case $\mathbf{A} = T_W = (\mathbb{C}^\times)^L$ and $\mathbf{T} = \mathbf{A} \times \mathbb{C}_\hbar^\times$. \mathbf{A} acts on both the base and the fiber of the bundle $\text{Tot}(\mathcal{V}^{\oplus L} \rightarrow \text{Gr}(N, L))$, and \mathbb{C}_\hbar^\times scales the fiber with weight \hbar^{-1} . Let the equivariant parameters of the gauge group GL_N be $\mathbf{s} = \{s_a\}_{a=1}^N$ and the equivariant parameters of \mathbf{A} be $\mathbf{x} = \{x_i\}_{i=1}^L$. Connected components of \mathbf{A} -fixed points are labelled by order-preserving embedding $p: \{1, \dots, N\} \hookrightarrow \{1, \dots, L\}$. Denote by $\{m_1, \dots, m_L\}$ the coordinates on $\text{Lie}(\mathbf{A})$, then we choose the chamber \mathfrak{C} such that $m_1 < \dots < m_L$. We choose the partial polarization as in (3.2). Then we claim that the stable envelope is the following:⁸

$$\mathbf{Stab}_{\mathfrak{C}, \mathcal{S}_X}([\mathcal{F}_p]) = \text{Sym}_{S_N} \left[\left(\prod_{a=1}^N \mathbf{f}_{p(a)}(s_a, \mathbf{x}, \hbar, z) \right) \cdot \left(\prod_{a>b} \frac{1}{\vartheta(s_a s_b^{-1})} \right) \right], \quad (3.6)$$

where Sym_{S_N} means summation over permutations $\{s_a\} \mapsto \{s_{\sigma(a)}\}_{\sigma \in S_N}$, and $\mathbf{f}_m(s, \mathbf{x}, \hbar, z)$ is the following function:

$$\mathbf{f}_m(s, \mathbf{x}, \hbar, z) := \frac{\vartheta(s x_m \hbar^{m-L} z)}{\vartheta(\hbar^{m-L} z)} \prod_{i < m} \vartheta(s x_i) \prod_{j > m} \vartheta(s x_j \hbar),$$

where z is the Kähler parameter corresponding to the ample line bundle $(\det \mathcal{V})^{-1}$.

Let us show that (3.6) is indeed the stable envelope. The first thing to check is that (3.6) is a map between line bundles $\mathcal{S}_{X^A} \otimes \Theta(\hbar)^{-\text{rk ind}} \otimes \tau(-\hbar \det \text{ind})^* \mathcal{U} \rightarrow \mathcal{S}_X \otimes \mathcal{U}$. Note that \mathcal{S}_{X^A} is a trivial line bundle, and ind has equivariant weights $x_i/x_{p(a)}$ such that $i > p(a)$. Every summand in the symmetrization (3.6) can be rewritten as

$$\begin{aligned} & \left(\prod_{i,a} \vartheta(s_a x_i) \right) \left(\prod_a \frac{\vartheta(s_a x_{p(a)} \hbar^{p(a)-L} z)}{\vartheta(s_a x_{p(a)}) \vartheta(\hbar^{p(a)-L} z)} \right) \left(\prod_{i>p(a)} \frac{\vartheta(s_a x_i \hbar)}{\vartheta(s_a x_i) \vartheta(\hbar)} \right) \\ & \times \left(\prod_{a>b} \frac{1}{\vartheta(s_a s_b^{-1})} \right) \vartheta(\hbar)^{\#(i>p(a))}, \end{aligned}$$

which is a meromorphic section of the line bundle

$$\Theta(\text{Pol}_X) \otimes \mathcal{U} \otimes (\tau(-\hbar \det \text{ind})^* \mathcal{U}|_{\mathcal{F}_p})^{-1} \otimes \Theta(\mathcal{V} \mathcal{V}^\vee)^{\otimes \frac{-1}{2}} \otimes \Theta(\hbar)^{\text{rk ind}},$$

which is isomorphic to $\mathcal{S}_X \otimes \mathcal{U} \otimes (\tau(-\hbar \det \text{ind})^* \mathcal{U})^{-1} \otimes \Theta(\hbar)^{\text{rk ind}}$, this shows that (3.6) lives in the correct line bundle.

Next, we verify that (3.6) is triangular with respect to \mathfrak{C} . In fact, if $\mathcal{F}_{p'} \not\leq \mathcal{F}_p$ is another connected component of fixed points, in other words, $p': \{1, \dots, N\} \hookrightarrow \{1, \dots, L\}$ is an order-preserving embedding and there exists $1 \leq a \leq N$ such that $p'(a) < p(a)$, then for every $\sigma \in S_N$ there exists b such that $p'(\sigma(b)) < p(b)$, so every summand in the symmetrization (3.6) vanishes.

Finally, the restriction of (3.6) to diagonal is

$$(-1)^{\#(i>p(a))} \prod_{a=1}^N \prod_{\substack{i < p(a) \\ i \notin \text{Im}(p)}} \vartheta\left(\frac{x_i}{x_{p(a)}}\right) \prod_{i>p(a)} \vartheta\left(\frac{x_{p(a)}}{x_i \hbar}\right),$$

which equals to $(-1)^{\text{rk ind}} \vartheta(N_{X/\mathcal{F}_p}^-)$. This verifies that (3.6) is indeed the stable envelope for $\mathcal{M}^\zeta(N, L)$ with respect to the chamber \mathfrak{C} .

⁸For the gauge-theoretic derivation of (3.6), see Section 6.1. Note that the gauge-theoretic computation uses the normalization of stable envelope such that $\mathbf{Stab}_{\mathfrak{C}, \mathcal{S}_X}([\mathcal{F}])|_{\mathcal{F}} = \vartheta(N_{X/\mathcal{F}}^-)$.

3.5 Triangle lemma and duality

Let $\mathfrak{C}' \subset \mathfrak{C}$ be a face, and let $A' \subset A$ be the subtorus associated to the span of \mathfrak{C}' in $\text{Lie}(A)$.

Proposition 3.3 (triangle lemma). *We have $\mathbf{Stab}_{\mathfrak{C}, \mathcal{S}_X} = \mathbf{Stab}_{\mathfrak{C}', \mathcal{S}_X} \circ \mathbf{Stab}_{\mathfrak{C}/\mathfrak{C}', \mathcal{S}_{X, A'}}$. Moreover,*

$$\mathbf{Stab}_{\mathfrak{C}/\mathfrak{C}', \mathcal{S}_{X, A'}}(z) = \mathbf{Stab}_{\mathfrak{C}/\mathfrak{C}', \mathcal{S}_{X, A'}}(z - \hbar \det \text{ind}_{X, A'}), \quad (3.7)$$

where $\text{ind}_{X, A'} = \text{Pol}_X^+|_{X, A', >0} - \text{Pol}_X^-|_{X, A', >0}$ is the alternating sum of the attracting part of partial polarization with respect to the chamber \mathfrak{C}' .

Proof. This is verbatim to that of [2, Proposition 3.3], replacing polarization with partial polarization. \blacksquare

Assume that X^Γ is proper, then we have the equivariant pushforward $f_\otimes: \Theta(T_X) \rightarrow \mathcal{O}_{\mathcal{E}_T, \text{localized}}$ for the map $f: X \rightarrow \text{pt}$, and accordingly we define $\mathbf{Stab}_{\mathfrak{C}, \mathcal{S}_X}^*(z) := \mathbf{Stab}_{-\mathfrak{C}, \mathcal{S}_X^\nabla}(-z)^\vee$, where $z \in \mathcal{E}_{\text{Pic}(X)}$ is the Kähler parameter and the dual means transpose with respect to the duality

$$\mathcal{S}_X \otimes \mathcal{S}_X^\nabla \xrightarrow{\text{product}} \Theta(T_X) \xrightarrow{f_\otimes} \mathcal{O}_{\mathcal{E}_T, \text{localized}}.$$

Proposition 3.4 (duality). *We have*

$$\mathbf{Stab}_{\mathfrak{C}, \mathcal{S}_X}^* \circ \mathbf{Stab}_{\mathfrak{C}, \mathcal{S}_X} = 1. \quad (3.8)$$

Proof. This is verbatim to that of [2, Proposition 3.4], replacing polarization by partial polarization. \blacksquare

Combine the duality (3.8) with the formula of the stable envelope for opposite partial polarization, and we obtain the following corollary.

Corollary 3.1 (inverse stable envelope for the opposite chamber). *We have*

$$\mathbf{Stab}_{-\mathfrak{C}, \mathcal{S}_X}^{-1}(z) = (-1)^{\text{rk Pol}_{X, \text{moving}}} \mathbf{Stab}_{\mathfrak{C}, \mathcal{S}_X}(-z + \hbar \det(\text{Pol}_X^+ - \text{Pol}_X^-))^\vee.$$

where z is the Kähler parameter (coordinate on \mathcal{E}_K).

From now on, unless specified, we will drop the attractive line bundle in the notation and write $\mathbf{Stab}_{\mathfrak{C}}$ for $\mathbf{Stab}_{\mathfrak{C}, \mathcal{S}_X}$ and $\mathbf{Stab}_{\mathfrak{C}/\mathfrak{C}'}$ for $\mathbf{Stab}_{\mathfrak{C}/\mathfrak{C}', \mathcal{S}_{X, A'}}$, etc.

3.6 Abelianization

A general strategy to compute elliptic stable envelope for GIT quotients by nonabelian gauge group is abelianization. This is described in detail for the symplectic reduction in [2, Section 4.3], and in this subsection, we adapt their technique to the more general setting in Section 2.1. In the following, we point out the necessary modification to [2, Section 4.3].

Let S_b (resp. S_f) be a maximal torus of G_{ev} (resp. G_{odd}) and let $B_b \supset S_b$ (resp. $B_f \supset S_f$) be a Borel subgroup with Lie algebra \mathfrak{b}_b (resp. \mathfrak{b}_f). Denote by $S := S_b \times S_f$ and $B := B_b \times B_f$, then we have a diagram similar to that of [2, equation (59)]

$$\begin{array}{ccc} \text{Fl} & \xrightarrow{j_+} & \mu^{-1}(\mathfrak{b}_b^\perp)^{\zeta^{-ss}} // S \xrightarrow{j_-} X_S \\ \downarrow \pi & & \\ X & & \end{array} \quad (3.9)$$

$\text{Fl} = \mu^{-1}(\mathfrak{g}_{\text{ev}}^\perp)^{\zeta^{-ss}} // B \cong (\mu \times \mu_{\mathbb{R}})^{-1}(\mathfrak{g}_{\text{ev}}^\perp \times \{\zeta\}) / U \cap S$, where $\mu_{\mathbb{R}}: \mathcal{R} \oplus \mathcal{R}^\vee \rightarrow \text{Lie}(U)^\vee$ is the real moment map with respect to the action of a maximal compact subgroup $U \subset G$ and a U -invariant

Hermitian metric on $\mathcal{R} \oplus \mathcal{R}^\vee$, and $\mathsf{X}_S = \mu^{-1}(\mathfrak{s}_b^\perp) //_\zeta S$ is the GIT quotient of $\mu^{-1}(\mathfrak{s}_b^\perp)$ by the torus S . Note that j_+ is a non-holomorphic embedding and j_- is a holomorphic immersion. π is a G/B -bundle, so every statement in [2, Section 4.3.2] still holds for π .

We have decompositions

$$\pi^* \mathcal{G}_{\text{ev}} = \mathcal{N}_b + \mathcal{N}_b^\vee + \text{Lie}(S_b) \otimes \mathcal{O}_{\text{Fl}}, \quad \pi^* \mathcal{G}_{\text{odd}} = \mathcal{N}_f + \mathcal{N}_f^\vee + \text{Lie}(S_f) \otimes \mathcal{O}_{\text{Fl}},$$

where \mathcal{N}_b (resp. \mathcal{N}_b^\vee) is the bundle on X_S associated to the adjoint action of S on $\mathfrak{n}_b = [\mathfrak{b}_b, \mathfrak{b}_b]$ (resp. $\mathfrak{n}_f = [\mathfrak{b}_f, \mathfrak{b}_f]$). The relative tangent bundle of π is $T_\pi = \mathcal{N}^\vee = \mathcal{N}_b^\vee + \mathcal{N}_f^\vee$, and the normal bundle of j_+ is $N_{j_+} = \mathcal{N} + \hbar^{-1} \mathcal{N}_b$, and the normal bundle of j_- is $N_{j_-} = \hbar^{-1} \mathcal{N}_b^\vee$.

For simplicity, let us choose the partial polarizations

$$\text{Pol}_\mathsf{X} = \text{Pol}_\mathsf{X}^+ = \mathcal{R} - \mathcal{G}_{\text{ev}}, \quad \text{Pol}_{\mathsf{X}_S} = \text{Pol}_{\mathsf{X}_S}^+ = \mathcal{R} + \hbar^{-1} \mathcal{N}_b^\vee - \mathcal{N}_b - \text{Lie}(S_b) \otimes \mathcal{O}_{\mathsf{X}_S}.$$

Note that the partial polarization on X_S differs from the choice in Example 3.1 by $\hbar^{-1} \mathcal{N}_b^\vee - \mathcal{N}_b$, which affects the stable envelope by

$$\mathbf{Stab}_\mathfrak{C}(z) \mapsto \mathbf{Stab}_\mathfrak{C}(z + \hbar \det \mathcal{N}_b) \cdot (-1)^{\text{rk } \mathcal{N}_b, \text{moving}},$$

thus (3.24) still holds for such modified choice of partial polarization.

Now we have the modified version of [2, equation (63)]

$$j^*(\text{Pol}_{\mathsf{X}_S} - \hbar^{-1} \mathcal{N}_b^\vee + \mathcal{N}_f^\vee) = \pi^* \text{Pol}_\mathsf{X} + T_\pi,$$

where $j = j_- \circ j_+$. Also notice that $\Theta(\mathcal{N}_f^\vee) \cong \Theta(\mathcal{N}_f) \cong \pi^* \Theta(\mathcal{G}_{\text{odd}})^{\otimes \frac{1}{2}}$. Therefore, we have the modified version of [2, equation (64)]

$$\mathcal{S}_\mathsf{X} \xleftarrow{\pi^\circledast} \pi^* \mathcal{S}_\mathsf{X} \otimes \Theta(T_\pi) \xleftarrow{j_+^*} j_-^* \mathcal{S}_{\mathsf{X}_S} \otimes \Theta(-\hbar^{-1} \mathcal{N}_b^\vee) \xrightarrow{j_-^\circledast} \mathcal{S}_{\mathsf{X}_S}. \quad (3.10)$$

The map j_+^* in (3.10) is surjective since any tautological class extends to X_S .

Next, we choose a connected component $\mathcal{F} \subset \mathsf{X}^A$, and let \mathcal{F}' be the unique connected component of $\pi^{-1}(\mathcal{F})^A$ such that normal weights of \mathcal{F}' in $\pi^{-1}(\mathcal{F})$ are repelling with respect to the chamber \mathfrak{C} , i.e., $(T_\pi|_{\mathcal{F}'})_{>0} = 0$. This implies that $(\mathcal{N}^\vee|_{\mathcal{F}'})_{>0} = 0$ by the same argument with [2, Section 4.3.8]. Consider the following analog of (3.9)

$$\begin{array}{ccc} \mathcal{F}' & \xrightarrow{j_+^*} & \mathcal{F}_S \cap \mu^{-1}(\mathfrak{b}_b^\perp)^{\zeta-ss} // S \xrightarrow{j_-^*} \mathcal{F}_S \\ \downarrow \pi' & & \\ F, & & \end{array}$$

where \mathcal{F}_S is the connected component of X_S^A that contains \mathcal{F}' . Then by the same argument with [2, Section 4.3.9], j_+^* is surjective, and $\pi^*(\text{ind } \mathcal{F}) = j'^*(\text{ind } \mathcal{F}_S)$, where $j' = j_- \circ j_+$.

Now we have all ingredients to state the abelianization result. We claim that the following analog of [2, equation (72)] is well defined and the diagram is commutative

$$\begin{array}{ccc} \mathcal{S}_\mathcal{F} \otimes \mathcal{U}' & \xrightarrow{j_-^\circledast (j_+^*)^{-1} (\pi_\circledast')^{-1}} & \mathcal{S}_{\mathcal{F}_S} \otimes \mathcal{U}' \\ \downarrow \mathbf{Stab} & & \downarrow \mathbf{Stab}_S \\ \mathcal{S}_\mathsf{X} \otimes \mathcal{U} \otimes \Theta(\hbar)^{\text{rk ind}} & \xleftarrow{\pi_\circledast j_+^* (j_-^\circledast)^{-1}} & \mathcal{S}_{\mathsf{X}_S} \otimes \mathcal{U} \otimes \Theta(\hbar)^{\text{rk ind}}, \end{array} \quad (3.11)$$

where $\mathcal{U}' = \tau(-\hbar \det \text{ind})^* \mathcal{U}$. In fact, the argument in [2, Section 4.3.11] works verbatim, so the inverse of the top horizontal arrow is well defined; the normal weights to $\mathfrak{b}_b^\perp \subset \mathfrak{g}^\vee$

are non-attracting then it follows that $\text{Attr}^f(\mu^{-1}(\mathfrak{b}_b^\perp)^\Lambda) \subset \mu^{-1}(\mathfrak{b}_b^\perp)$, thus $\mathbf{Stab}_S \circ j'_{-\otimes}$ factors through $j_{-\otimes}$, so the inverse of the bottom horizontal arrow is also well defined. To show that the diagram commutes, we only need to show that \mathbf{Stab} agrees with the composition of the rest of the arrows in the diagram when restricted to the diagonal. The point is to compare the difference between $\pi^* N_{\mathcal{X}/\mathcal{F}}^-$ and $N_{\mathcal{X}_S/\mathcal{F}'}^-$, and from our previous discussion we learn that the new repelling directions are $(T_\pi|_{\mathcal{F}'})_{\text{moving}}$ and $(N_{j_-}|_{\mathcal{F}'})_{\text{moving}}$. $(T_\pi|_{\mathcal{F}'})_{\text{moving}}$ is cancelled out by π_{\otimes} , and $(N_{j_-}|_{\mathcal{F}'})_{\text{moving}}$ is cancelled with $(j_{-\otimes})^{-1} \circ \mathbf{Stab}_S \circ j'_{-\otimes}$. This concludes the proof of abelianization (3.11).

3.7 R -matrices and dynamical Yang–Baxter equations

Definition 3.6. Let $\mathfrak{C}_1, \mathfrak{C}_2$ be two chambers in $\text{Lie}(\mathbf{A})$, then we define the R -matrix

$$\mathbf{R}_{\mathfrak{C}_2 \leftarrow \mathfrak{C}_1} := \mathbf{Stab}_{\mathfrak{C}_2}^{-1} \circ \mathbf{Stab}_{\mathfrak{C}_1}. \quad (3.12)$$

This is a map from $\mathcal{S}_{\mathcal{X}^{\mathbf{A}}} \otimes \tau(-\hbar \det \text{ind}_1)^* \mathcal{U}$ to $\mathcal{S}_{\mathcal{X}^{\mathbf{A}}} \otimes \tau(-\hbar \det \text{ind}_2)^* \mathcal{U}$, where ind_1 and ind_2 are index bundles for the chambers \mathfrak{C}_1 and \mathfrak{C}_2 respectively.

Obviously it follows from definition that

$$\mathbf{R}_{\mathfrak{C}_3 \leftarrow \mathfrak{C}_2} \mathbf{R}_{\mathfrak{C}_2 \leftarrow \mathfrak{C}_1} = \mathbf{R}_{\mathfrak{C}_3 \leftarrow \mathfrak{C}_1}. \quad (3.13)$$

Moreover, if \mathfrak{C}_1 and \mathfrak{C}_2 share a face \mathfrak{C}' , then (3.7) implies that

$$\mathbf{R}_{\mathfrak{C}_2 \leftarrow \mathfrak{C}_1}(z) = \mathbf{R}_{\mathfrak{C}_2/\mathfrak{C}' \leftarrow \mathfrak{C}_1/\mathfrak{C}'}(z - \hbar \det \text{ind}_{\mathcal{X}^{\mathbf{A}'}, \mathfrak{C}'}), \quad (3.14)$$

where \mathbf{A}' is the subtorus corresponding to the chamber \mathfrak{C}' and $\text{ind}_{\mathcal{X}^{\mathbf{A}'}, \mathfrak{C}'}$ is the index bundle on $\mathcal{X}^{\mathbf{A}'}$ for the chamber \mathfrak{C}' . If \mathfrak{C}' is of codimension one in both \mathfrak{C}_1 and \mathfrak{C}_2 , then we say \mathfrak{C}' is a wall and define $\mathbf{R}_{\text{wall}} := \mathbf{R}_{\mathfrak{C}_2/\mathfrak{C}' \leftarrow \mathfrak{C}_1/\mathfrak{C}'}$. The dynamical Yang–Baxter equation is a direct consequence of (3.13) and (3.14) for $\mathbf{A} = (\mathbb{C}^\times)^3$. To see this, let m_1, m_2, m_3 be coordinates on $\text{Lie}(\mathbf{A})$, then the two paths connecting the chamber $\mathfrak{C}_{123} = \{m_1 < m_2 < m_3\}$ to its opposite $\mathfrak{C}_{321} = \{m_3 < m_2 < m_1\}$, namely $\mathfrak{C}_{123} \rightarrow \mathfrak{C}_{132} \rightarrow \mathfrak{C}_{312} \rightarrow \mathfrak{C}_{321}$ and $\mathfrak{C}_{123} \rightarrow \mathfrak{C}_{213} \rightarrow \mathfrak{C}_{231} \rightarrow \mathfrak{C}_{321}$, gives rise to equation between two sequence of composing R -matrices

$$\begin{aligned} & \mathbf{R}_{12}(z - \hbar \det \text{ind}_{3(12)}) \mathbf{R}_{13}(z - \hbar \det \text{ind}_{(13)2}) \mathbf{R}_{23}(z - \hbar \det \text{ind}_{1(23)}) \\ &= \mathbf{R}_{23}(z - \hbar \det \text{ind}_{(23)1}) \mathbf{R}_{13}(z - \hbar \det \text{ind}_{2(13)}) \mathbf{R}_{12}(z - \hbar \det \text{ind}_{(12)3}), \end{aligned} \quad (3.15)$$

where \mathbf{R}_{ij} is the R -matrix for the ij -wall, and the subscript of ind means the wall and chamber for which the index bundle is taken, for example, $3(12)$ means the chamber $\{m_3 < m_1 = m_2\}$.

The particular case $\mathfrak{C}_3 = \mathfrak{C}_1$ in (3.13) implies the unitarity $\mathbf{R}_{21}(-u, \hbar, z) \mathbf{R}(u, \hbar, z) = 1$, where $\mathbf{R}(u, \hbar, z)$ is the wall R -matrix and subscript 21 means swapping the two tensor components.

3.8 R -matrices for quiver varieties

Let us write down the dynamical Yang–Baxter equation for the quiver variety $\mathcal{M}^{\zeta^-}(\mathbf{w})$ for a quiver Q with decomposition of nodes $Q_0 = Q_0^{\text{ev}} \sqcup Q_0^{\text{odd}}$ into even and odd parts and decomposition of arrows $Q_1 = Q_1^+ \sqcup Q_1^-$ such that the condition (\star) in Example 3.2 is satisfied. We take $\mathbf{A} = (\mathbb{C}^\times)^3 \subset T_W$ such that the framing vector space decomposes into eigenspaces as $W = a_1 W^{(1)} + a_2 W^{(2)} + a_3 W^{(3)}$. Choosing the partial polarization (3.2), we can write down the index bundle for the chamber corresponding to the weight decomposition $W = aW' + bW''$ such that b/a is attracting

$$\text{ind} = \sum_{i,j \in Q_0} D_{ij} \text{Hom}(\mathcal{V}'_i, \mathcal{W}''_j) + (Q_{ij} - P_{ij}) \text{Hom}(\mathcal{V}'_i, \mathcal{V}''_j),$$

where Q_{ij} is the *weighted* adjacency matrix of Q defined by

$$Q_{ij} = \#(a \in Q_1^+ : h(a) = j, t(a) = i) - \#(a \in Q_1^- : h(a) = j, t(a) = i),$$

and D_{ij} is the diagonal matrix such that

$$D_{ii} = \begin{cases} 1, & i \notin Q_0^{\text{ev},-}, \\ -1, & i \in Q_0^{\text{ev},-}, \end{cases}$$

and P_{ij} is the diagonal matrix such that

$$P_{ii} = \begin{cases} 1, & i \in Q_0^{\text{ev},+}, \\ -1, & i \in Q_0^{\text{ev},-}, \\ 0, & i \in Q_0^{\text{odd}}. \end{cases}$$

Define the quiver R -matrix $\mathbf{R}^Q(\mathbf{z}) := \mathbf{R}_{\text{wall}}(\mathbf{z} + \hbar(P - Q^t)\mathbf{v})$, where $\mathbf{z} := \{z_i\}_{i \in Q_0}$ are the equivariant parameters corresponding to the ample line bundles $(\det \mathcal{V}_i)^{-1}$. Let $\boldsymbol{\mu} := D\mathbf{w} - C\mathbf{v}$, where

$$C = 2P - Q - Q^t, \quad (3.16)$$

then (3.15) is specialized to

$$\mathbf{R}_{12}^Q(\mathbf{z})\mathbf{R}_{13}^Q(\mathbf{z} - \hbar\boldsymbol{\mu}^{(2)})\mathbf{R}_{23}^Q(\mathbf{z}) = \mathbf{R}_{23}^Q(\mathbf{z} - \hbar\boldsymbol{\mu}^{(1)})\mathbf{R}_{13}^Q(\mathbf{z})\mathbf{R}_{12}^Q(\mathbf{z} - \hbar\boldsymbol{\mu}^{(3)}). \quad (3.17)$$

This kind of equation is called the dynamical Yang–Baxter equations and was first studied by Felder [51, 52].

Example 3.8 ($\mathfrak{sl}(1|1)$). Consider the resolved determinantal variety case (see Example 3.7), namely a quiver Q with one odd node, i.e., $Q_0 = Q_0^{\text{odd}} = \{1\}$. We choose stability condition $\zeta = \zeta_-$ and partial polarization (3.2). We choose the framing dimensions $\mathbf{w}^{(1)} = \mathbf{w}^{(2)} = \mathbf{w}^{(3)} = 1$, then $\mathcal{M}^\zeta(\mathbf{w}^{(1)})$ is disjoint union of two points, and

$$\mathcal{M}^\zeta(\mathbf{w}^{(1)} + \mathbf{w}^{(2)}) = \text{pt} \sqcup \text{Tot}(\mathcal{O}(-1)^{\oplus 2} \rightarrow \mathbb{P}^1) \sqcup \mathbb{C}^4.$$

Let us label the basis for the elliptic cohomology $E_T(\mathcal{M}^\zeta(\mathbf{w}^{(1)}))$, which is a free module of rank two over $\mathcal{E}_x \times \mathcal{E}_h \times \mathcal{E}_z$, by $v_0 = [\mathcal{M}^\zeta(0, 1)]$ and $v_1 = [\mathcal{M}^\zeta(1, 1)]$, then we can write down the quiver R -matrix in the basis $\{v_0 \otimes v_0, v_1 \otimes v_0, v_0 \otimes v_1, v_1 \otimes v_1\}$

$$\mathbf{R}^Q(u, z) = \left(\begin{array}{c|cc|c} 1 & 0 & 0 & 0 \\ \hline 0 & \frac{\vartheta(z)\vartheta(z\hbar^{-2})\vartheta(u)}{\vartheta(z\hbar^{-1})^2\vartheta(u\hbar^{-1})} & \frac{\vartheta(\hbar)\vartheta(uz\hbar^{-1})}{\vartheta(z\hbar^{-1})\vartheta(u^{-1}\hbar)} & 0 \\ \hline 0 & \frac{\vartheta(\hbar)\vartheta(zu^{-1}\hbar^{-1})}{\vartheta(z\hbar^{-1})\vartheta(u^{-1}\hbar)} & \frac{\vartheta(u)}{\vartheta(u\hbar^{-1})} & 0 \\ \hline 0 & 0 & 0 & \frac{\vartheta(u\hbar)}{\vartheta(u^{-1}\hbar)} \end{array} \right), \quad (3.18)$$

where $u = x_2/x_1$ is the ratio between equivariant weights in the decomposition $W = x_1W' + x_2W''$. In this case $C = 0$, which is the Cartan matrix for $\mathfrak{sl}(1|1)$, and $\boldsymbol{\mu}(v_0) = \boldsymbol{\mu}(v_1) = 1$, so the dynamical Yang–Baxter equation (3.17) reads

$$\mathbf{R}_{12}^Q(u, z)\mathbf{R}_{13}^Q(uw, z/\hbar)\mathbf{R}_{23}^Q(w, z) = \mathbf{R}_{23}^Q(w, z/\hbar)\mathbf{R}_{13}^Q(uw, z)\mathbf{R}_{12}^Q(u, z/\hbar), \quad (3.19)$$

which is the dynamical Yang–Baxter equation for $\mathfrak{sl}(1|1)$ in the fundamental representation.

Example 3.9 ($\mathfrak{sl}(n|1)$). Next, we consider a generalization to the above example, namely the quiver in Example 2.3. Let Q be an A_n quiver with $Q_0^{\text{ev}} = \{1, \dots, n-1\}$ and $Q_0^{\text{odd}} = \{n\}$, and the orientation is such that $h(a) < t(a)$ for all arrows $a \in Q_1$. We take $Q_1 = Q_1^+$. We choose stability condition $\zeta = \zeta_-$ and the partial polarization (3.2). We choose framing dimensions $\mathbf{w}^{(1)} = \mathbf{w}^{(2)} = \mathbf{w}^{(3)} = (1, 0, \dots, 0)$, then $\mathcal{M}^\zeta(\mathbf{w}^{(1)})$ is the disjoint union of $n+1$ points which is labeled by integers $\alpha \in \{0, 1, \dots, n\}$ so that the gauge dimension vector of the corresponding quiver variety is

$$\mathbf{v}_\alpha = \underbrace{(1, \dots, 1)}_\alpha, 0, \dots, 0.$$

Using the above notation, we have

$$\mathcal{M}^\zeta(\mathbf{v}_\alpha + \mathbf{v}_\beta, \mathbf{w}^{(1)} + \mathbf{w}^{(2)}) = \begin{cases} \text{pt}, & \alpha = \beta < n, \\ T^*\mathbb{P}^1, & \alpha < \beta < n \text{ or } \beta < \alpha < n, \\ \text{Tot}(\mathcal{O}(-1)^{\oplus 2} \rightarrow \mathbb{P}^1), & \alpha < \beta = n \text{ or } \beta < \alpha = n, \\ \mathbb{C}^4, & \alpha = \beta = n. \end{cases} \quad (3.20)$$

Let us label the basis for the elliptic cohomology $\mathbf{E}_\Gamma(\mathcal{M}^\zeta(\mathbf{w}^{(1)}))$, which is a free module of rank $n+1$ over $\mathcal{E}_x \times \mathcal{E}_\hbar \times \mathcal{E}_z$, by $v_\alpha = [\mathcal{M}^\zeta(\mathbf{v}_\alpha, \mathbf{w}^{(1)})]$. Here the Kähler parameter $\mathbf{z} = (z_i)_{i=1}^n$ corresponds to the ample line bundles $(\det \mathcal{V}_i^\vee)_{i=1}^n$, where \mathcal{V}_i is the tautological bundle corresponding to the i -th node.

For the quiver Q , we have the Cartan matrix

$$C_{i,j} = 2\delta_{i,j}(1 - \delta_{i,n}) - \delta_{i+1,j} - \delta_{i-1,j}, \quad 1 \leq i, j \leq n,$$

which is the symmetric Cartan matrix of $\mathfrak{sl}(n|1)$ [55]. The weights of v_α are

$$\mu(v_\alpha) = \mathbf{w}^{(1)} - C\mathbf{v}_\alpha = \begin{cases} (1, 0, \dots, 0), & \alpha = 0, \\ \underbrace{(0, \dots, 0)}_{\alpha-1}, -1, 1, 0, \dots, 0, & 0 < \alpha < n, \\ (0, \dots, 0, 1), & \alpha = n, \end{cases} \quad (3.21)$$

which is aligned with the weights for the fundamental representation of $\mathfrak{sl}(n|1)$. Moreover, the Kähler parameter shifts between the wall R -matrix and the quiver R -matrix is

$$(\mathbf{P} - \mathbf{Q}^t)\mathbf{v}_\alpha = \begin{cases} \underbrace{(0, \dots, 0)}_{\alpha-1}, 1, 0, \dots, 0, & 0 < \alpha < n, \\ (0, \dots, 0), & \alpha = 0 \text{ or } \alpha = n. \end{cases}$$

Using the known results on the R -matrices for all four varieties in (3.20), see (3.18) and [2, equation (85)], we can write down the quiver R -matrix in this case. We introduce shorthand notations for some special functions

$$A(u, z) = \frac{\vartheta(z\hbar)\vartheta(z\hbar^{-1})\vartheta(u)}{\vartheta(z)^2\vartheta(u\hbar^{-1})}, \quad B(u, z) = \frac{\vartheta(\hbar)\vartheta(uz)}{\vartheta(z)\vartheta(u^{-1}\hbar)}, \quad C(u) = \frac{\vartheta(u)}{\vartheta(u\hbar^{-1})},$$

$$D(u) = \frac{\vartheta(u\hbar)}{\vartheta(u^{-1}\hbar)},$$

where we have suppressed the \hbar -dependence. Then the quiver R -matrix in this case is the following:

$$\mathbf{R}^Q(u, \mathbf{z})(v_\alpha \otimes v_\beta)$$

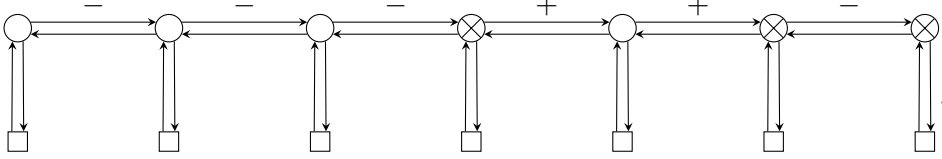
$$= \begin{cases} v_\alpha \otimes v_\beta, & \alpha = \beta < n, \\ D(u)v_\alpha \otimes v_\beta, & \alpha = \beta = n, \\ C(u)v_\alpha \otimes v_\beta + B(u, \hbar^{-\delta_{\beta,n}} \prod_{i=\alpha+1}^{\beta} z_i)v_\beta \otimes v_\alpha, & \alpha < \beta, \\ A(u, \hbar^{-\delta_{\alpha,n}} \prod_{i=\beta+1}^{\alpha} z_i)v_\alpha \otimes v_\beta + B(u, \hbar^{\delta_{\alpha,n}} \prod_{i=\beta+1}^{\alpha} z_i^{-1})v_\beta \otimes v_\alpha, & \beta < \alpha. \end{cases}$$

\mathbf{R}^Q satisfies the dynamical Yang–Baxter equation (3.17) for μ given by weights for the fundamental representation of $\mathfrak{sl}(n|1)$ (3.21).

For quivers of finite or affine type A, we make the following observation. In addition to the condition (\star) in Example 3.2, we also require that

$(\star\star)$ if the valency at an odd node is two then the arrows connected to it are from different groups, i.e., exactly one of the arrows is from Q_1^+ and the other is from Q_1^- .

Here is an example of an allowed quiver (after adding framing and doubling)



For simplicity, we consider an \widehat{A}_{m+n-1} quiver Q and label the nodes by $Q_0 = \mathbb{Z}/(m+n)\mathbb{Z}$. We choose the orientation $Q_1 = \{i-1 \rightarrow i : i \in \mathbb{Z}/(m+n)\mathbb{Z}\}$, and choose a splitting $Q_1 = Q_1^+ \sqcup Q_1^-$ such that $|Q_1^+| = m$ and $|Q_1^-| = n$. Define

$$c_i = \begin{cases} +1, & \text{if } (i-1 \rightarrow i) \in Q_1^+, \\ -1, & \text{if } (i-1 \rightarrow i) \in Q_1^-. \end{cases}$$

Then the condition (\star) is equivalent to $c_i = c_{i+1}$ for all $i \in Q_0^{\text{ev}}$, and the condition $(\star\star)$ is equivalent to $c_i = -c_{i+1}$ for all $i \in Q_0^{\text{odd}}$. The Cartan matrix (3.16) reads

$$C_{i,j} = (c_i + c_{i+1})\delta_{i,j} - c_i\delta_{i,j+1} - c_j\delta_{i+1,j}.$$

$C_{i,j}$ is exactly the symmetric Cartan matrix of the Kac–Dynkin diagram for $\widehat{\mathfrak{sl}}(m|n)$ associated to the quiver Q . Moreover, the quiver R -matrix \mathbf{R}^Q satisfies the dynamical Yang–Baxter equation (3.17) for μ given by weights in a certain highest-weight module of $\widehat{\mathfrak{sl}}(m|n)$.

3.9 K-theory and cohomology limit

In the $q \rightarrow 0$ limit, the elliptic curve $\mathbb{E} = \mathbb{C}^\times/q^{\mathbb{Z}}$ degenerates to a nodal compactification of \mathbb{C}^\times , and the elliptic cohomology $\text{Ell}_\tau(\mathbf{X})$ degenerates to the K-theory $K_\tau(\mathbf{X}) \otimes \mathbb{C}$, such that sections of line bundles on $\text{Ell}_\tau(\mathbf{X})$ becomes sections of line bundles on $K_\tau(\mathbf{X}) \otimes \mathbb{C}$. Strictly speaking, because of half-periodicity property of ϑ -function: $\vartheta(e^{2\pi i}x) = -\vartheta(x)$, their $q \rightarrow 0$ limit is defined on the double cover of \mathbb{C}^\times .

We expect that the elliptic stable envelope degenerates to the K-theoretic stable envelope in the $q \rightarrow 0$ limit, such that the Kähler parameter z degenerates to the slope parameter in the K-theory side. Suppose that

$$\lim_{\substack{\ln z \rightarrow \infty \\ \ln q \rightarrow \infty}} \text{Re} \left(-\frac{\ln z}{\ln q} \right) = \mathfrak{s} \in \text{Pic}_\tau(\mathbf{X}) \otimes_{\mathbb{Z}} \mathbb{R}, \quad (3.22)$$

such that the slope \mathfrak{s} is *generic*, and then define

$$\mathbf{Stab}_{\mathfrak{C}}^{\mathfrak{s}} := \lim_{q \rightarrow 0} [(\det \text{Pol}_{\mathbb{X}})^{-\frac{1}{2}} \circ \mathbf{Stab}_{\mathfrak{C}} \circ (\det \text{Pol}_{\mathbb{X}^{\mathbb{A}}})^{\frac{1}{2}}] \in K_{\mathbb{T}}(\mathbb{X} \times \mathbb{X}^{\mathbb{A}}). \quad (3.23)$$

For every connected component $\mathcal{F} \subset \mathbb{X}^{\mathbb{A}}$,

$$\begin{aligned} \mathbf{Stab}_{\mathfrak{C}}^{\mathfrak{s}}([\mathcal{F}]|_{\mathcal{F}}) &= (-1)^{\text{rk ind}} \left(\frac{\det N_{\mathbb{X}/\mathcal{F}}^-}{\det \text{Pol}_{\mathbb{X}|_{\mathcal{F}, \text{moving}}}^-} \right)^{\frac{1}{2}} \bigwedge (N_{\mathbb{X}/\mathcal{F}}^-)^{\vee} \\ &= (-\sqrt{\hbar})^{\text{rk ind}} (\det \text{Pol}_{\mathbb{X}|_{\mathcal{F}, >0}})^{-1} (\det \mathcal{G}_{\text{odd}}|_{\mathcal{F}, <0})^{-\frac{1}{2}} \bigwedge (N_{\mathbb{X}/\mathcal{F}}^-)^{\vee}. \end{aligned}$$

Moreover, $\mathbf{Stab}_{\mathfrak{C}}^{\mathfrak{s}}$ is supported on $\text{Attr}_{\mathfrak{C}}^f$, i.e., if $\mathcal{F}_2 \not\leq \mathcal{F}_1$ then $\mathbf{Stab}_{\mathfrak{C}}^{\mathfrak{s}}([\mathcal{F}_1])|_{\mathcal{F}_2} = 0$. We expect that $\mathbf{Stab}_{\mathfrak{C}}^{\mathfrak{s}}$ is the K-theoretic stable envelope with slope \mathfrak{s} for the chamber \mathfrak{C} . In other words, $\mathbf{Stab}_{\mathfrak{C}}^{\mathfrak{s}}$ also satisfies the degree condition, which is checked in the following.

For a connected component \mathcal{F} of $\mathbb{X}^{\mathbb{A}}$, define

$$\Delta_{\mathcal{F}} := \text{Convex hull}(\text{Supp}_{\mathbb{A}} \mathbf{Stab}_{\mathfrak{C}}^{\mathfrak{s}}([\mathcal{F}]|_{\mathcal{F}})),$$

where $\text{Supp}_{\mathbb{A}}$ of a $K_{\mathbb{T}}(\mathcal{F})$ class is the set of \mathbb{A} -weights that appears in that class.

Proposition 3.5. *Let \mathbb{X} be a quotient of Hamiltonian reduction (2.1), assume moreover that either*

- *the gauge group is abelian, i.e., the situation in Example 2.1, or*
- *\mathbb{X} is constructed from the quiver representation, i.e., the situation in Section 2.2,*

then for every pair of connected components $\mathcal{F}_2 \preceq \mathcal{F}_1$ in $\mathbb{X}^{\mathbb{A}}$ with respect to the chamber \mathfrak{C} ,

$$\text{Supp}_{\mathbb{A}} \mathbf{Stab}_{\mathfrak{C}}^{\mathfrak{s}}([\mathcal{F}_1])|_{\mathcal{F}_2} \subset \Delta_{\mathcal{F}_2} + \mathbb{A}\text{-weight of } \mathfrak{s}|_{\mathcal{F}_2} - \mathbb{A}\text{-weight of } \mathfrak{s}|_{\mathcal{F}_1}. \quad (3.24)$$

Proof. Suppose that $\mathbb{X} = \mu_{\text{ev}}^{-1}(0)^{\zeta - \text{ss}} // G$, where $G = G_{\text{ev}} \times G_{\text{odd}}$ acts on the representation \mathcal{R} with moment map $\mu: \mathcal{R} \oplus \mathcal{R}^{\vee} \rightarrow \mathfrak{g}^{\vee}$ and μ_{ev} is the composition of μ with projection $\text{pr}_{\text{ev}}: \mathfrak{g}^{\vee} \rightarrow \mathfrak{g}_{\text{ev}}^{\vee}$, and ζ is a generic stability such that $\mu_{\text{ev}}^{-1}(0)^{\zeta - \text{ss}} = \mu_{\text{ev}}^{-1}(0)^{\zeta - \text{s}}$. As we have assumed in the setup of Section 2.1, $\mu_{\text{ev}}^{-1}(0)^{\zeta - \text{ss}}$ is smooth with free G action.

Step 1. Assume that G is abelian, then according to our previous discussions in Example 3.6, the equivariant elliptic cohomology of \mathbb{X} is isomorphic to that of the central fiber $\mathbb{X}_0 = \mu^{-1}(0)^{\zeta - \text{ss}} // G: \text{Ell}_{\mathbb{T}}(\mathbb{X}) \cong \text{Ell}_{\mathbb{T}}(\mathbb{X}_0)$, and elliptic stable envelopes of \mathbb{X} and \mathbb{X}_0 are identified via this isomorphism. \mathbb{X}_0 is a hypertoric variety, thus the result follows from [2, Proposition 4.2].

Step 2. If G is nonabelian, then we shall utilize the abelianization (3.11). It suffices to prove in the special case when $\mathbb{A} = \mathbb{C}^{\times}$, by the same argument of [2, Proposition 4.2]. By step 1, we have

$$\text{Supp}_{\mathbb{A}} \mathbf{Stab}_{\mathfrak{S}}^{\mathfrak{s}}([\mathcal{F}'_1])|_{\mathcal{F}'_2} \subset \Delta_{\mathcal{F}'_2} + \mathbb{A}\text{-weight of } \mathfrak{s}|_{\mathcal{F}'_2} - \mathbb{A}\text{-weight of } \mathfrak{s}|_{\mathcal{F}'_1}.$$

The abelianization gives

$$\begin{aligned} \mathbf{Stab}^{\mathfrak{s}}([\mathcal{F}_1])|_{\mathcal{F}_2} &= \frac{(\det \mathcal{N}_f^{\vee}|_{\mathcal{F}'_2})^{-\frac{1}{2}} \mathbf{Stab}_{\mathfrak{S}}^{\mathfrak{s}}([\mathcal{F}'_1])|_{\mathcal{F}'_2}}{\bigwedge ((N_{j_-}|_{\mathcal{F}'_2})_{\text{moving}})^{\vee} \cdot \bigwedge ((T_{\pi}|_{\mathcal{F}'_2})_{\text{moving}})^{\vee}} (\det \mathcal{N}_f^{\vee}|_{\mathcal{F}'_1, \text{fixed}})^{\frac{1}{2}}, \\ \mathbf{Stab}^{\mathfrak{s}}([\mathcal{F}_2])|_{\mathcal{F}_2} &= \frac{(\det \mathcal{N}_f^{\vee}|_{\mathcal{F}'_2, \text{moving}})^{-\frac{1}{2}} \mathbf{Stab}_{\mathfrak{S}}^{\mathfrak{s}}([\mathcal{F}'_2])|_{\mathcal{F}'_2}}{\bigwedge ((N_{j_-}|_{\mathcal{F}'_2})_{\text{moving}})^{\vee} \cdot \bigwedge ((T_{\pi}|_{\mathcal{F}'_2})_{\text{moving}})^{\vee}}. \end{aligned}$$

The first line in the above equation implies that

$$\begin{aligned} & \text{Supp}_{\mathbf{A}} \text{Stab}^s([\mathcal{F}_1])|_{\mathcal{F}_2} \cdot \bigwedge ((N_{j_-}|_{\mathcal{F}'_2})_{\text{moving}})^\vee \cdot \bigwedge ((T_\pi|_{\mathcal{F}'_2})_{\text{moving}})^\vee \\ & \subset \Delta_{\mathcal{F}'_2} - \frac{1}{2} \mathbf{A}\text{-weight of } \det \mathcal{N}_f^\vee|_{\mathcal{F}'_2} + \mathbf{A}\text{-weight of } \mathfrak{s}|_{\mathcal{F}'_2} - \mathbf{A}\text{-weight of } \mathfrak{s}|_{\mathcal{F}'_1}, \end{aligned}$$

and the second line implies that

$$\Delta_{\mathcal{F}'_2} = \Delta_{\mathcal{F}_2} \cdot \bigwedge ((N_{j_-}|_{\mathcal{F}'_2})_{\text{moving}})^\vee \cdot \bigwedge ((T_\pi|_{\mathcal{F}'_2})_{\text{moving}})^\vee + \frac{1}{2} \mathbf{A}\text{-weight of } \det \mathcal{N}_f^\vee|_{\mathcal{F}'_2},$$

thus we have

$$\text{Supp}_{\mathbf{A}} \text{Stab}^s([\mathcal{F}_1])|_{\mathcal{F}_2} \subset \Delta_{\mathcal{F}_2} + \mathbf{A}\text{-weight of } \mathfrak{s}|_{\mathcal{F}_2} - \mathbf{A}\text{-weight of } \mathfrak{s}|_{\mathcal{F}_1}.$$

This finishes the proof. ■

A further reduction from the K-theory to cohomology can be defined by

$$\text{Stab}_{\mathfrak{C}} := \text{lowest cohomological degree term in } \text{ch}(\text{Stab}_{\mathfrak{C}}^s), \quad (3.25)$$

where $\text{ch}: K_{\mathbb{T}}(\mathbb{X} \times \mathbb{X}^{\mathbf{A}}) \rightarrow H_{\mathbb{T}}(\mathbb{X} \times \mathbb{X}^{\mathbf{A}})$ is the \mathbb{T} -equivariant Chern character map. The composition of reductions $\text{Stab}_{\mathfrak{C}} \rightarrow \text{Stab}_{\mathfrak{C}}^s \rightarrow \text{Stab}_{\mathfrak{C}}$ amounts to replacing $\vartheta(x)$ by x .

For every connected component $\mathcal{F} \subset \mathbb{X}^{\mathbf{A}}$, $\text{Stab}_{\mathfrak{C}}([\mathcal{F}])|_{\mathcal{F}} = (-1)^{\text{rk ind}} e(N_{\mathbb{X}/\mathcal{F}}^-)$, where $e(\cdot)$ is the \mathbb{T} -equivariant Euler class. Moreover, $\text{Stab}_{\mathfrak{C}}$ is supported on $\text{Attr}_{\mathfrak{C}}^f$, i.e., if $\mathcal{F}_2 \not\leq \mathcal{F}_1$ then $\text{Stab}_{\mathfrak{C}}([\mathcal{F}_1])|_{\mathcal{F}_2} = 0$. We expect that $\text{Stab}_{\mathfrak{C}}$ is the cohomological stable envelope for the chamber \mathfrak{C} . In other words, $\text{Stab}_{\mathfrak{C}}$ also satisfies the degree condition, which is checked in the following.

Proposition 3.6. *Under the assumption of Proposition 3.5, then for every pair of connected components $\mathcal{F}_2 \preceq \mathcal{F}_1$ in $\mathbb{X}^{\mathbf{A}}$ with respect to the chamber \mathfrak{C} , $\deg_{\mathbf{A}} \text{Stab}_{\mathfrak{C}}([\mathcal{F}_1])|_{\mathcal{F}_2} < \dim N_{\mathbb{X}/\mathcal{F}_2}^-$, where $\deg_{\mathbf{A}}$ of an element in $H_{\mathbb{T}}(\mathbb{X}^{\mathbf{A}})$ is the degree in $\mathbb{C}[\mathfrak{a}]$ under the non-canonical splitting $H_{\mathbb{T}}(\mathbb{X}^{\mathbf{A}}) \cong H_{\mathbb{T}/\mathbf{A}}(\mathbb{X}^{\mathbf{A}}) \otimes \mathbb{C}[\mathfrak{a}]$.⁹*

Proof. The idea is essentially the same as Proposition 3.5. Namely, the abelian case is the same as hypertoric varieties, which is shown by using the explicit formula [2, equation (56)]. For the nonabelian case, we use the abelianization and get

$$\text{Stab}([\mathcal{F}_1])|_{\mathcal{F}_2} = \frac{\text{Stab}_S([\mathcal{F}'_1])|_{\mathcal{F}'_2}}{e((N_{j_-}|_{\mathcal{F}'_2})_{\text{moving}}) \cdot e((T_\pi|_{\mathcal{F}'_2})_{\text{moving}})}.$$

Since the abelian stable envelope satisfies $\deg_{\mathbf{A}}(\text{Stab}_S([\mathcal{F}'_1])|_{\mathcal{F}'_2}) < \dim N_{\mathbb{X}_S/\mathcal{F}_2,S}^-$, we conclude that

$$\begin{aligned} \deg_{\mathbf{A}} \text{Stab}([\mathcal{F}_1])|_{\mathcal{F}_2} &= \deg_{\mathbf{A}}(\text{Stab}_S([\mathcal{F}'_1])|_{\mathcal{F}'_2}) - \dim(N_{j_-}|_{\mathcal{F}'_2})_{\text{moving}} - \dim(T_\pi|_{\mathcal{F}'_2})_{\text{moving}} \\ &< \dim N_{\mathbb{X}_S/\mathcal{F}_2,S}^- - \dim(N_{j_-}|_{\mathcal{F}'_2})_{\text{moving}} - \dim(T_\pi|_{\mathcal{F}'_2})_{\text{moving}} \\ &= \dim N_{\mathbb{X}/\mathcal{F}_2}^-. \end{aligned}$$

This finishes the proof. ■

Propositions 3.5 and 3.6 are summarized in the following

Corollary 3.2. *Under the assumption of Proposition 3.5, $\text{Stab}_{\mathfrak{C}}^s$ is the K-theoretic stable envelope with slope s for the chamber \mathfrak{C} , and $\text{Stab}_{\mathfrak{C}}$ is the cohomological stable envelope for the chamber \mathfrak{C} .*

⁹Different choice of splitting does not change the degree in $\mathbb{C}[\mathfrak{a}]$.

Example 3.10 (Resolved Determinantal Varieties). We apply (3.23) to (3.6) and obtain the K -theoretic stable envelope for the resolved determinantal variety¹⁰

$$\mathrm{Stab}_{\mathcal{E}}^s([\mathcal{F}_p]) = \hbar^{\frac{\#\{i>p(a)\}}{2}} \mathrm{Sym}_{S_N} \left[\left(\prod_{a=1}^N f_{p(a)}(s_a, \mathbf{x}, \hbar, \mathbf{s}) \right) \cdot \left(\prod_{a>b} \hat{\mathbf{a}}(s_a s_b^{-1}) \right) \right], \quad (3.26)$$

where $f_m(s, \mathbf{x}, \hbar, \mathbf{s})$ is the following function

$$f_m(s, \mathbf{x}, \hbar, \mathbf{s}) := (s x_m)^{|\mathbf{s}|} \prod_{i<m} (1 - s^{-1} x_i^{-1}) \prod_{j>m} (1 - \hbar^{-1} s^{-1} x_j^{-1}),$$

and $\hat{\mathbf{a}}$ is a version of \hat{A} -genus $\hat{\mathbf{a}}(w) = \frac{1}{w^{\frac{1}{2}} - w^{-\frac{1}{2}}}$. A further reduction to cohomology using (3.25) gives the cohomological stable envelope

$$\mathrm{Stab}_{\mathcal{E}}([\mathcal{F}_p]) = \mathrm{Sym}_{S_N} \left[\left(\prod_{a=1}^N f_{p(a)}(s_a, \mathbf{x}, \hbar) \right) \cdot \left(\prod_{a>b} \frac{1}{s_a - s_b} \right) \right], \quad (3.27)$$

where $f_m(s, \mathbf{x}, \hbar)$ is the following function $f_m(s, \mathbf{x}, \hbar) := \prod_{i<m} (s + x_i) \prod_{j>m} (s + x_j + \hbar)$. Note that (3.27) equals to $(-1)^{\#\{i>p(a)\}} W_{w_0, I}^{(10)}(-\mathbf{s}, \mathbf{x}, -\hbar)$, where $W_{w_0, I}^{(10)}$ is the super weight function in [117] for the longest element $w_0 \in S_N$ and $I = w_0(\mathrm{image}(p))$. The sign $(-1)^{\#\{i>p(a)\}}$, which equals to $(-1)^{\mathrm{rk\,ind}}$, reflects the choice of normalization of the stable envelope in the *loc. cit.* being $\mathrm{Stab}_{\mathcal{E}}([\mathcal{F}])|_{\mathcal{F}} = e(N_{\mathcal{X}/\mathcal{F}}^-)$ as opposed to $(-1)^{\mathrm{rk\,ind}} e(N_{\mathcal{X}/\mathcal{F}}^-)$, see Axiom A1 in Definition 4.1 of *loc. cit.*

Part II

The solution to dYBE for $\mathfrak{sl}(1|1)$ from 3d $\mathcal{N} = 2$ SQCD

4 Stable envelopes and R -matrix from gauge theory: the setup

In the previous sections, we have defined the elliptic stable envelope for certain non symplectic varieties. Examples of such varieties include Higgs branches of 3d $\mathcal{N} = 2$ theories. We have characterized the stable envelopes by their formal properties and using these stable envelopes, we defined the R -matrix satisfying the dynamical Yang–Baxter equations for superspin chains. In practice, it is rather difficult to construct stable envelopes from their definitions alone. To write down explicit formulas for these elliptic functions, we turn to the 3d $\mathcal{N} = 2$ theories. In this section, we explain the motivation behind computing certain partition functions of these theories that we argue should give us stable envelopes. Finally, in Section 6 when we actually compute these partition functions, we shall give an a posteriori justification for the computation by showing that the partition functions satisfy the criteria for elliptic stable envelopes set up earlier and that the R -matrix satisfies the dynamical Yang–Baxter equations. In fact, results from these computations were already used to define the functional forms of the elliptic $\mathfrak{sl}(1|1)$ stable envelopes (3.6) and the R -matrix (3.18) and we checked that they indeed fulfill all the necessary criteria. Let us now go through the details of how we actually computed them.

¹⁰Here we use the identities $\lim_{q \rightarrow 0} \vartheta(w) = w^{\frac{1}{2}} - w^{-\frac{1}{2}}$ and $\lim_{q \rightarrow 0} \frac{\vartheta(wz)}{\vartheta(z)} = w^{|\mathbf{s}| + \frac{1}{2}}$.

4.1 The Bethe/Gauge correspondence

The correspondence relates integrable spin chains to supersymmetric quiver gauge theories, for both purely bosonic [102, 103] and superspin [99] chains. One way to introduce this is from a correspondence between Kac–Dynkin diagrams and quivers.

An integrable spin chain is based on a Lie algebra, say $\mathfrak{sl}(m|n)$, which can be characterized by its Kac–Dynkin diagram. For example, the following is a Kac–Dynkin diagram for $\mathfrak{sl}(1|3)$



A hollow circle represents an even simple root and a crossed circle represents an odd simple root. In order to fully characterize a spin chain, in addition to the symmetry, we also need to specify the representations or the spins that appear in the chain and the anisotropy. For simplicity, let us only consider spin chains with highest-weight representations, with the highest-weight being the fundamental weight, then we just need to specify the number of sites in the spin chain. For the anisotropy, we have three choices (see Table 1), fully isotropic (XXX/rational), partially anisotropic (XXZ/trigonometric/K-theoretic), and fully anisotropic (XYZ/elliptic).

The Bethe/Gauge correspondence relates this data to a quiver such as the following:¹¹



Each circular/gauge node corresponds to a node in the Kac–Dynkin diagram – even nodes are distinguished from the odd ones by the presence of a self-loop. The presence of a single framing node attached to the left-most gauge node reflects our choice regarding the spin chain only having the fundamental highest weights. The value L of the framing node is the size of the spin chain. The gauge dimensions N_1 , N_2 and N_3 are not part of the data defining the spin chain. Rather, they correspond to a sector of the spin chain with definite magnon numbers. To find the gauge theory dual of the full spin chain we need to consider multiple quivers, varying the gauge dimensions over all possible magnon numbers. The choice of anisotropy is not part of the quiver itself, but it will be reflected in the choice of gauge theory that we shall assign to the quiver. The quiver (4.1) is a special case of Example 2.4 with $n = 3$.

To quivers like (4.1), we can assign supersymmetric gauge theories with 4 supercharges in various dimensions. In the version of the Bethe/Gauge correspondence in which we are interested in this paper, the relation between anisotropy and gauge theory is as follows:

$$\begin{array}{ll}
 \text{Rational} & 1\text{d } \mathcal{N} = 4 \text{ (quantum mechanics),} \\
 \text{Trigonometric} & 2\text{d } \mathcal{N} = (2, 2), \\
 \text{Elliptic} & 3\text{d } \mathcal{N} = 2.
 \end{array} \tag{4.2}$$

In the absence of any odd node, the amount of supersymmetry is doubled in all cases. All these theories possess the same classical Higgs branch \mathcal{M}_H , which is the quiver variety $\mathcal{M}^\zeta(\mathbf{v}, \mathbf{w})$ (2.3) with $\mathbf{v} = (N_1, N_2, N_3)$ and $\mathbf{w} = (L, 0, 0)$. The stability condition ζ corresponds to the real FI

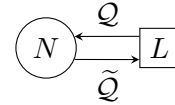
¹¹In the hierarchy of quivers, the first one is just a Kac–Dynkin diagram (e.g., (1.8)), then there’s the framed and doubled quiver (e.g., (1.7)), and finally the one in (4.1) is a tripled quiver that is traditionally used in physics to define gauge theories with 4 supercharges.

parameter in gauge theories. Supersymmetric ground states of these theories correspond to different cohomologies of the Higgs branch. For quantum mechanics, Witten's Morse theory arguments [144] describe the space of ground states as the ordinary equivariant cohomology $H_{\mathbf{A}}(\mathcal{M}_H)$ where \mathbf{A} is the maximal torus of the $SU(L)$ flavor symmetry of the quiver. In the 2d and 3d cases the cohomology is replaced by K-theory $K_{\mathbf{A}}(\mathcal{M}_H)$ and elliptic cohomology $\text{Ell}_{\mathbf{A}}(\mathcal{M}_H)$ respectively [145, 146]. On the spin chain side, for the rational, trigonometric, and elliptic cases, we have the Yangian $Y_{\hbar}(\mathfrak{g})$, the quantum affine algebra $\mathcal{U}_{\hbar}(\widehat{\mathfrak{g}})$, and the elliptic dynamical quantum group $E_{\tau, \hbar}(\mathfrak{g})$ as the spectrum generating algebras. The Bethe/Gauge correspondence identifies the Bethe eigenstates of the spin chain with the space of supersymmetric vacua of the corresponding gauge theories. Consequently, it conjectures an action of the spectrum generating algebra on the corresponding cohomology [53, 88, 106].¹²

There are more choices of gauge theories that we can assign to quivers such as (4.1) and find dual gauge theoretic interpretations of spin chain quantities. For example, in the original formulation of the Bethe/Gauge correspondence [99, 102, 103], for both bosonic and supersymmetric spin chains, the gauge dual of the rational spin chain was presented as a 2d $\mathcal{N} = (4, 4)$ (bosonic spin chain) or an $\mathcal{N} = (2, 2)$ (superspin chain) theory. The quantum mechanics from (4.2) is simply the dimensional reduction of these 2d theories. The supersymmetric vacua of these 2d theories correspond to the quantum deformation of the equivariant cohomology. In principle, there should be 3d and 4d theories with 4 supercharges that quantize K-theory [72, 73, 75, 106] and elliptic cohomology, and whose dimensional reductions correspond to the 2d and 3d theories in (4.2). In this paper we shall not be concerned with quantum cohomology, regardless, it is of some interest to look at the 2d dual of the rational spin chains. We consider the example of a $\mathfrak{sl}(1|1)$ spin chain in the following.



(a) Kac–Dynkin diagram.



(b) Gauge theory quiver.

Figure 3. $\mathfrak{sl}(1|1)$ Kac–Dynkin diagram and the corresponding quiver.

Example 4.1 ($\mathfrak{sl}(1|1)$ spin chains and 2d $\mathcal{N} = (2, 2)$ Gauge theories). Consider a length L spin chain whose sites are labeled by $i = 1, \dots, L$. At the i th site we assign the $\mathfrak{sl}(1|1)$ representation with the highest-weight $(\lambda_i^{(1)}, \lambda_i^{(2)})$. We are using the $\mathfrak{gl}(1|1)$ notation to write down the weights, i.e., $\lambda_i^{(1)} + \lambda_i^{(2)}$ is the $\mathfrak{sl}(1|1)$ weight of the i th site. We also consider complex inhomogeneities v_i . Excitations containing N magnons with rapidities $\sigma_a^{\mathbb{C}}$ for $a = 1, \dots, N$ correspond to Bethe eigenstates if the rapidities satisfy the Bethe ansatz equations (BAE) [114] (also see [143])

$$\prod_{i=1}^L \frac{+\sigma_a^{\mathbb{C}} + \frac{i}{2}\epsilon - v_i + i\epsilon\lambda_i^{(1)}}{-\sigma_a^{\mathbb{C}} - \frac{i}{2}\epsilon + v_i + i\epsilon\lambda_i^{(2)}} = e^{i\varphi}, \quad a = 1, \dots, N, \quad (4.3)$$

The inhomogeneities and the phase on the right-hand side are arbitrary auxiliary parameters of the spin chain. ϵ is the quantization parameter of the spin chain.¹³ Note that the value of $\lambda_i^{(1)} - \lambda_i^{(2)}$ is somewhat ambiguous since it can be changed by shifting the (arbitrary) inhomogeneity v_i . The $\mathfrak{sl}(1|1)$ charge $\lambda_i^{(1)} + \lambda_i^{(2)}$ is invariant under such shifts.

¹²Since a single gauge theory corresponds to a specific magnon sector, we need to take direct sum of the cohomologies over gauge dimensions to get the full spin chain Hilbert space and the algebra action (cf. the sums over \mathbf{v} in Conjecture 1.1).

¹³Usually the quantization parameter in a quantum mechanical system is written as \hbar , however in this paper, we are using \hbar for the elliptic deformation parameter which is roughly related to ϵ by $\epsilon \sim \ln \hbar$.

The BAE is the condition of extremizing a potential function called the Yang–Yang function

$$\begin{aligned}
Y(\boldsymbol{\sigma}^{\mathbb{C}}) &= \sum_{a=1}^N \sum_{i=1}^L (\sigma_a^{\mathbb{C}} - m_i^{\mathbb{C}}) (\ln(\sigma_a^{\mathbb{C}} - m_i^{\mathbb{C}}) - 1) \\
&\quad + \sum_{a=1}^N \sum_{i=1}^L (-\sigma_a^{\mathbb{C}} - \tilde{m}_i^{\mathbb{C}}) (\ln(-\sigma_a^{\mathbb{C}} - \tilde{m}_i^{\mathbb{C}}) - 1) - \mathrm{i}\varphi \sum_{a=1}^N \sigma_a^{\mathbb{C}},
\end{aligned} \tag{4.4}$$

where we have defined the new parameters

$$m_i^{\mathbb{C}} := v_i - \mathrm{i}\epsilon\lambda_i^{(1)} - \frac{\mathrm{i}}{2}\epsilon, \quad \tilde{m}_i^{\mathbb{C}} := -v_i - \mathrm{i}\epsilon\lambda_i^{(2)} + \frac{\mathrm{i}}{2}\epsilon. \tag{4.5}$$

In terms of this function, the BAE becomes: $\exp(2\pi\mathrm{i}\frac{\partial Y(\boldsymbol{\sigma}^{\mathbb{C}})}{\partial \sigma_a^{\mathbb{C}}}) = 1$, $a=1, \dots, N$. The Bethe/Gauge correspondence identifies the Yang–Yang function (4.3) as the effective twisted superpotential of the 2d $\mathcal{N} = (2, 2)$ theory defined by the quiver Figure 3b in the presence of twisted masses [99]. We identify the rapidities $\sigma_a^{\mathbb{C}}$ with the adjoint scalars in the Cartan part of the $\mathcal{N} = (2, 2)$ $U(N)$ vector multiplet. Then we recognize the 1-loop contributions of the fundamental and the anti-fundamental chirals \mathcal{Q}_a^i , $\tilde{\mathcal{Q}}_i^a$ in the first and the second line of (4.4) [65, 147]. Masses of these chirals are precisely the parameters defined in (4.5). We also notice the microscopic linear twisted superpotential as the last term in the second line.

For positive real FI parameter, the 2d theory flows in the infrared to a phase where it is described by a sigma model into its Higgs branch [147]. The Higgs branch is acted on by $SU(L)$ and the fixed points of the $U(1)^{L-1}$ action correspond to the vacua of the theory. When put on an interval, such as $I \times S^1$ for some interval I , the theory has brane-type boundary conditions supported on the attracting and repelling sets of these fixed points. All this Higgs branch structure can just as well be studied by looking at the dimensionally reduced $\mathcal{N} = 4$ quantum mechanics (from (4.2)) on I using Morse theory [144].

4.2 3d $\mathcal{N} = 2$ SQCD and its parameters

The 3d theory assigned to the quiver Figure 3b is the $\mathcal{N} = 2$ SQCD. The theory has a gauge group $G := U(N)$ with Lie algebra $\mathfrak{g} := \mathfrak{u}(N)$. The flavor symmetry visible in the quiver is $F_{\mathbb{A}} := SU(L)$ with Lie algebra $\mathfrak{f}_{\mathbb{A}} := \mathfrak{su}(L)$. There is an additional $U(1)_{\hbar}$ symmetry with Lie algebra \mathbb{R}_{\hbar} . The total flavor symmetry we consider is $F := SU(L) \times U(1)_{\hbar}$, whose Lie algebra we denote by \mathfrak{f} .

Let $\mathrm{tr} A$ be the component of the gauge field valued in the center of $\mathfrak{u}(N)$. From this abelian gauge field, we can form a current $\star \mathrm{tr} F$ in 3d which is conserved by Bianchi identity. This gives rise to an abelian topological symmetry $U(1)_{\mathrm{top}}$. It is topological in the sense that only the monopoles and none of the fields in the Lagrangian are coupled to this current. The Lie algebra of $U(1)_{\mathrm{top}}$ is written as $\mathbb{R}_{\mathrm{top}}$. This global symmetry can be weakly gauged by introducing a background vector multiplet for it. Let A_{top} , σ_{top} and D_{top} be the gauge field, the scalar, and the auxiliary scalar in this vector multiplet. The purely bosonic part of the Lagrangian coupling this multiplet to the topological current is

$$A_{\mathrm{top}} \wedge \mathrm{tr} F + \sigma_{\mathrm{top}} \mathrm{tr} D + D_{\mathrm{top}} \mathrm{tr} \sigma. \tag{4.6}$$

The theory has a \mathfrak{g} -valued vector multiplet σ and chiral multiplets¹⁴

$$(\mathcal{Q}, \tilde{\mathcal{Q}}) \in \mathcal{R} \oplus \mathcal{R}^{\vee}, \quad \text{where} \quad \mathcal{R} = \mathrm{Hom}(\mathbb{C}^L, \mathbb{C}^N), \quad \mathcal{R}^{\vee} = \mathrm{Hom}(\mathbb{C}^N, \mathbb{C}^L). \tag{4.7}$$

¹⁴We refer to a multiplet by its scalar component. Their full field content and our conventions about supersymmetry are presented in Section B.2.

We also define $\mathbf{M} := \mathcal{R} \oplus \mathcal{R}^\vee$ to refer to the space of all the chirals. The chiral multiplets are accompanied by their conjugate anti-chiral multiplets $(\widetilde{\mathcal{Q}}, \overline{\widetilde{\mathcal{Q}}}) \in \mathcal{R}^\vee \oplus \mathcal{R}$. Charges of these multiplets under the groups defined above are as follows

	$U(N)$	$SU(L)$	$U(1)_{\hbar}$	$U(1)_{\text{top}}$
σ	\mathbf{ad}	$\mathbf{1}$	0	0
\mathcal{Q}	\mathbf{N}	\mathbf{L}^\vee	$\frac{1}{2}$	0
$\widetilde{\mathcal{Q}}$	\mathbf{N}^\vee	\mathbf{L}	$\frac{1}{2}$	0

Table 4. Charges of 3d $\mathcal{N} = 2$ multiplets in the quiver gauge theory of Figure 3b. \mathbf{N} and \mathbf{L} refer to the fundamental representations of $U(N)$ and $SU(L)$ respectively.

There is also a $U(1)_R$ R -symmetry but charges of various multiplets under this symmetry are a bit ambiguous since the symmetry can be mixed with any other $U(1)$ global symmetry. Also, we do not turn on any background for the R -symmetry.

We put the 3d theory on a Euclidean space-time manifold $[y_-, y_+] \times \mathbb{E}_\tau$. Here \mathbb{E}_τ is the elliptic curve with complex structure τ , defined as the quotient $\mathbb{E}_\tau = \mathbb{C}^\times / q^{\mathbb{Z}}$, $q := e^{2\pi i \tau}$. For the global symmetries $SU(L) \times U(1)_{\hbar} \times U(1)_{\text{top}}$ we turn on background vector multiplets. The multiplets contain, among other fields, flat connections.¹⁵ These flat connections are parameterized by their holonomies around the non-contractible cycles, S_A^1 and S_B^1 , of \mathbb{E}_τ . For example, if $A_{\mathbf{a}}$ is the background connection for the $F_{\mathbf{A}} = SU(L)$ flavor symmetry, then we can compute the $\mathfrak{a}_{\mathbb{C}} = \mathfrak{su}(L) \otimes \mathbb{C}$ valued periods $a_{\mathbf{a}} := \oint_{S_B^1} A_{\mathbf{a}} - \tau \oint_{S_A^1} A_{\mathbf{a}}$. We can always take the background fields to be Cartan valued

$$a_{\mathbf{a}} = \text{diag}((a_{\mathbf{a}})_1, \dots, (a_{\mathbf{a}})_L), \quad (a_{\mathbf{a}})_1 + \dots + (a_{\mathbf{a}})_L = 0.$$

Once we quotient by equivalences due to gauge transformations, these periods are really valued in the torus $a_{\mathbf{a}} \in \frac{\mathfrak{a}_{\mathbb{C}}}{\Lambda_{\mathfrak{a}}^\vee \oplus \tau \Lambda_{\mathfrak{a}}^\vee}$, where $\Lambda_{\mathfrak{a}}^\vee$ is the coroot lattice of \mathfrak{a} . Consequently, the holonomies of this connection around the cycles of the elliptic curve, also called the elliptic equivariant parameters, are valued in the elliptic curve as well

$$(x_i, \dots, x_L) := (e^{2\pi i \tau (a_{\mathbf{a}})_1}, \dots, e^{2\pi i \tau (a_{\mathbf{a}})_L}) \in \mathbb{E}_\tau^L, \quad i = 1, \dots, L.$$

The tracelessness condition translates to $x_1 \cdots x_L = 1$ for the equivariant parameters. Another common term for these parameters is fugacity, which we shall also use.

In a completely analogous fashion, we turn on elliptic fugacities for the flavor symmetries $U(1)_{\hbar}$, $U(1)_{\text{top}}$, as well as for the gauge symmetry $U(N)$

Symmetry	Fugacities
$U(1)_{\hbar}$	$\hbar \in \mathbb{E}_\tau$
$U(1)_{\text{top}}$	$z \in \mathbb{E}_\tau$
$U(N)$	$(s_1, \dots, s_N) \in \mathbb{E}_\tau^N$

The values of the flavor fugacities are constant along the interval $[y_-, y_+]$ due to BPS equations (5.6a). The values of the gauge fugacities along the interval are unconstrained and only their values at the boundaries with fixed holonomy are parameters of the theory. These fugacities are the same equivariant parameters as in Example 3.7.

¹⁵The connections need to be flat to preserve supersymmetry (cf. (5.5a)). The full constraints on the background fields follow from the BPS equations of Section 5.1.2.

In addition to the holonomies, we also turn on $SU(L)$ twisted masses, by giving nonzero VEV to the adjoint scalar in the background $SU(L)$ vector multiplet $\langle \sigma_a \rangle = m = \text{diag}(m_1, \dots, m_L)$, as well as $U(1)_{\text{top}}$ twisted mass $\langle \sigma_{\text{top}} \rangle = \zeta$. Note that, due to the coupling (4.6) between the topological and the dynamical gauge multiplets, the ζ is nothing but the real FI parameter. The absolute values of these mass parameters do not play any role in this paper. However, the relative ordering of their values matter. We fix the sign of the FI parameter once and for all localization computations $\zeta > 0$. For the $SU(L)$ -masses, we shall refer to different chambers at different times. A chamber simply refers to a particular ordering of the masses. For any permutation $\varsigma: \{1, \dots, L\} \hookrightarrow \{1, \dots, L\}$ we have a chamber

$$\mathfrak{C}_\varsigma = \{(m_1, \dots, m_L) \mid m_{\varsigma(1)} < m_{\varsigma(2)} < \dots < m_{\varsigma(L)}\} \subseteq \mathfrak{a}. \quad (4.8)$$

These are the same chambers as in (3.3). Two arbitrary but specific chambers defined by $i \mapsto i$ and $i \mapsto L + 1 - i$, reintroduced in (5.41), will be used for explicit computations of stable envelopes and the R -matrix in Section 6.

Reduction to 2d. In Section 7, we shall consider dimensional reduction of the 3d partition functions down to 2d and 1d. The 3d \rightarrow 2d reduction is done by compactifying the S_B^1 cycle of the elliptic curve. Then the period of the connections along the S_B^1 cycle becomes a real adjoint scalar field in the 2d theory. Combined with the real adjoint scalar already in the 3d vector multiplet, this becomes the complex adjoint scalar of the 2d $\mathcal{N} = (2, 2)$ vector multiplet. For example, for the F_A vector multiplet we define the following complex scalar

$$\sigma_a^{\mathbb{C}} := \sigma_a + i \oint_{S_B^1} A_a. \quad (4.9)$$

We use $(\sigma_{\mathfrak{g}})^{\mathbb{C}}$, $(\sigma_{\mathfrak{h}})^{\mathbb{C}}$, and $\zeta^{\mathbb{C}}$ for the similarly defined complex adjoint scalars in the dynamical vector multiplet, in the $U(1)_{\mathfrak{h}}$ vector multiplet, and the complex FI parameter in 2d respectively. A further reduction on S_A^1 gives an $\mathcal{N} = 4$ quantum mechanics.

4.3 Branes and the Bethe/Gauge correspondence

In this section, we are going to look at the brane construction of the 3d quiver theory and relations between certain string dualities and the Bethe/Gauge correspondence. The goal is to find hints regarding which gauge theoretic quantities may correspond to stable envelopes and the R -matrix. The brane constructions will serve merely as a motivation to set up the purely gauge theoretic computations of the future sections, we will be rather schematic in the present section.

Gauge theories with 4 supercharges have standard brane constructions in terms of D-branes suspended between rotated NS5s [37, 48, 49]. We consider a 10d space-time $\mathbb{R}_y \times \mathbb{R}_1 \times \mathbb{E}_\tau \times \mathbb{R}_2 \times \mathbb{R}_3 \times \mathbb{R}_{+\epsilon}^2 \times \mathbb{R}_{-\epsilon}^2$. Here \mathbb{R}_y is a real direction parameterized by the coordinate y and $\mathbb{R}_1, \mathbb{R}_2, \mathbb{R}_3$ are three real directions parameterized by some unspecified coordinates. $\mathbb{R}_{\pm\epsilon}^2$ are two Ω -deformed planes, ϵ being the deformation parameter. The 3d $\mathcal{N} = 2$ SQCD can be seen as the world-volume theory of a stack of N D3 branes in this background. The complete brane configuration is in Table 5, and we draw a cartoon of the branes in Figure 4.

T-dualizing the elliptic curve and the \mathbb{R}_1 direction takes us to a type IIA configuration with N D2 branes suspended between rotated NS5 branes, with L D4 branes about. This is precisely the configuration studied in [66]. In this duality frame, the world-volume theory of the D2 branes is the 2d $\mathcal{N} = (2, 2)$ theory from Example 4.1. If we do not T-dualize \mathbb{R}_1 , then we find D1 branes suspended between NS5s and additional D3 branes. The D1 branes are described the quantum mechanics from (4.2) for which the D3 branes provide flavor symmetry. The main result of [66] was to relate this setup to 4d Chern–Simons theory.¹⁶ By a further S-duality we can

¹⁶Also see [33] for relations between Ω -deformed D5s and bosonic 4d Chern–Simons theory.

No.		\mathbb{R}_y	\mathbb{R}_1	\mathbb{E}_τ	\mathbb{R}_2	\mathbb{R}_3	$\mathbb{R}_{+\epsilon}^2$	$\mathbb{R}_{-\epsilon}^2$
1	NS5	×	×	×			×	
1	NS5'	×	×	×				×
N	D3	×		×	×			
L	D5	×		×		×	×	

Table 5. Brane configuration for 3d $\mathcal{N} = 2$ SQCD with $U(N)$ gauge group and $SU(L)$ flavor (see Figure 3b). Simultaneous rotation of the $\mathbb{R}_{+\epsilon}^2$ and $\mathbb{R}_{-\epsilon}^2$ planes is also a symmetry of this configuration. This corresponds to the $U(1)_\hbar$ symmetry of the 3d theory (see Table 4), the two parameters being roughly related by $\epsilon \sim \ln \hbar$.

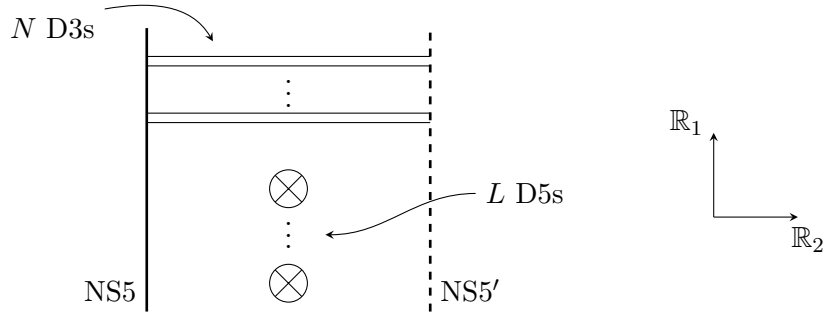


Figure 4. A cross-section of the brane configuration from Table 5.

convert the D1s, NS5s, and D3s to F1s, D5s, and D3s respectively. The Ω -deformed D5s give rise to a $\mathfrak{gl}(1|1)$ 4d holomorphic-topological Chern–Simons theory on $\mathbb{R}_y \times \mathbb{R}_1 \times \mathbb{E}_\tau$ where $\mathbb{R}_y \times \mathbb{R}_1$ is the topological plane and the elliptic curve provides the holomorphic direction. The D3 branes create line operators in this theory [66, 67]. From [31] we know that crossing of line operators in the topological plane of 4d Chern–Simons creates R -matrix for certain spin chains. In the starting configuration of Table 5 this crossing corresponds to changing the \mathbb{R}_1 -positions of two D5 branes as we move along in the \mathbb{R}_y direction, ultimately swapping the two (see Figure 5). The

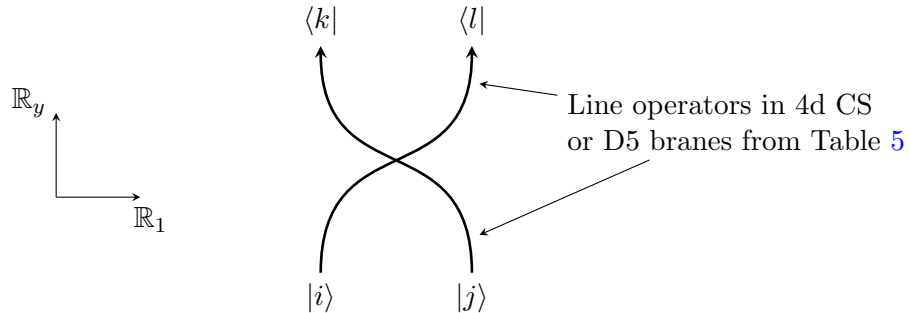


Figure 5. Line operators crossing in the topological plane of 4d Chern–Simons, computing the matrix element $\langle k| \otimes \langle l| R |i\rangle \otimes |j\rangle$ of an R -matrix. This translates to bending D5 branes (from Table 5) in the $\mathbb{R}_y \times \mathbb{R}_1$ plane.

incoming and outgoing asymptotic states correspond to states in the spin chain, or according to the Bethe/Gauge philosophy, vacua of the D3 brane theory. Without further analysis of the string background, it is unclear how exactly to bend branes while preserving supersymmetry or how to specify incoming and outgoing states. However, there is a natural supersymmetric

process in the field theory language that accomplishes something analogous. The locations of the D5 branes in the \mathbb{R}_1 direction are real twisted masses for the chiral multiplets of the SQCD that lives on the D3 branes. Swapping the \mathbb{R}_1 -locations, m_1 and m_2 , of two D5 branes then correspond to giving y -dependent masses $m_1(y)$ and $m_2(y)$ such that

$$\lim_{y \rightarrow -\infty} (m_1(y), m_2(y)) = (m_1, m_2), \quad \lim_{y \rightarrow +\infty} (m_1(y), m_2(y)) = (m_2, m_1). \quad (4.10)$$

Such position-dependent masses can be made supersymmetric, creating a mass Janus interface [4, 19, 42, 61]. The interface interpolates between two distinct chambers, as defined in (4.8). In Section 5, we shall be more specific and identify suitable boundary conditions in the 3d theory, in any chamber, that mimics supersymmetric vacua, and define mass Janus interpolating between them.

4.4 Gauge-theoretic definition of elliptic stable envelope

The gauge-theoretic definition of elliptic stable envelopes as Janus partition functions first appeared in [24, 40]. They define the Janus interfaces for 3d $\mathcal{N} = 2$ supersymmetry but provide boundary conditions for 3d $\mathcal{N} = 4$ vacua, which give elliptic stable envelopes of 3d $\mathcal{N} = 4$ Higgs branches. We take our previous discussion on the brane construction and in particular the position dependent masses (4.10) as motivations to study Janus interfaces. In this section, we briefly review the definition of the elliptic stable envelopes as Janus partition functions, and in Section 5, we generalize the boundary conditions to correspond to the vacua of 3d $\mathcal{N} = 2$ theories computing the elliptic stable envelopes for the 3d $\mathcal{N} = 2$ Higgs branches.

Our starting point here is the 3d $\mathcal{N} = 2$ SQCD on $[y_-, y_+] \times \mathbb{E}_\tau$. For the moment, replace the interval $[y_-, y_+]$ with the infinite line $(-\infty, +\infty)$. Suppose, for any chamber \mathfrak{C} , we have a set of asymptotic boundary conditions $\{p\}$ for the 3d $\mathcal{N} = 2$ SQCD corresponding to its Higgs branch vacua.¹⁷ We can construct a Janus interface interpolating between a vacuum p_- in a chamber \mathfrak{C}_- at infinite past and a vacuum p_+ in a chamber \mathfrak{C}_+ at infinite future. Then, our previous discussion on branes suggests that the expectation value of this interface with p_- and p_+ boundary conditions at infinite past and future should give us a matrix element of an R -matrix

$$\mathbf{R}_{\mathfrak{C}_+ \leftarrow \mathfrak{C}_-}(p_+, p_-) = \langle p_+ | \mathcal{J}(m_{\mathfrak{C}_-}, m_{\mathfrak{C}_+}) | p_- \rangle,$$

where $m_{\mathfrak{C}_-}$ and $m_{\mathfrak{C}_+}$ are two generic masses from the chambers \mathfrak{C}_- and \mathfrak{C}_+ respectively, and \mathcal{J} is the Janus interface interpolating between these two masses. A similar expression for the R -matrix in terms of Janus interfaces was also used in [23] in the context of bosonic rational spin chains.

As already mentioned in the introduction (equations (1.3), (1.4) and (1.5)), a more fundamental quantity than the R -matrix is the stable envelope. While the R -matrix corresponds to changing masses across chambers, the stable envelope corresponds to changing masses from some nonzero value to zero. The full R -matrix is then a composition of a stable envelope and an inverse stable envelope (3.12), the first one taking the nonzero masses from one chamber to zero, the second one taking the zero masses to a nonzero value in a different chamber. In the gauge theory setup, a stable envelope then corresponds to a Janus interface interpolating between nonzero and zero masses. We expect that due to the presence of nonzero equivariant

¹⁷In the presence of twisted masses, the classical Higgs branch of 3d $\mathcal{N} = 2$ SQCD retracts down to the fixed point set of the flavor torus action. This set is a disjoint union of connected components. Here p labels a connected component, and by the boundary condition labeled by p we roughly refer to a brane-type boundary supported on the attracting/repelling set of p . We shall describe in detail the fixed points, their attracting sets and the corresponding boundary conditions in Sections 5.2.2, 5.2.3 and 5.3, respectively.

elliptic parameters, even in the massless side of the interface we can choose a basis of boundary conditions labeled by the same p that labels the massive boundaries.¹⁸ Then the stable envelope corresponds to the Janus partition function

$$\mathbf{Stab}_{\mathfrak{E}}(p_+, p_-) = \langle p_+ | \mathcal{J}(m_{\mathfrak{E}}, 0) | p_- \rangle.$$

We want to compute these partition functions using supersymmetric localization. In practice, localization on non-compact manifolds are technically subtle and it is more convenient to replace the infinite line with the finite interval $[y_-, y_+]$. Then we need to find boundary conditions for the boundaries at finite distances that are cohomologous to the asymptotic vacuum boundaries with respect to our choice of localizing supercharge. We describe our choice of supercharge in Appendix B.3 and in Section 5, we shall identify the vacua and various boundary conditions equivalent to these vacua. For the massive boundary, we shall construct two types of boundary conditions, the so-called thimbles $\mathcal{D}_{\mathfrak{E}}(p)$ (see Section 5.3.1) and the enriched Neumann $\mathcal{N}_{\mathfrak{E}}(p)$ (see Section 5.3.2). For the massless boundary, we shall construct one type of boundary conditions depending on a choice of polarization $\mathbb{L}(p)$ of the space of chirals $\mathcal{B}_{\mathbb{L}(p)}(p)$ (see Section 5.3.3). All these boundary conditions are labeled by the connected components of the (flavor) torus fixed-point set of the Higgs branch. We shall compute the matrix elements of the stable envelopes as the partition function

$$\mathbf{Stab}_{\mathfrak{E}}(p_+, p_-) = \langle \mathcal{B}_{\mathbb{L}(p_+)}(p_+) | \mathcal{J}(m_{\mathfrak{E}}, 0) | \mathcal{N}_{\mathfrak{E}}(p_-) \rangle. \quad (4.11)$$

Here we have the boundary conditions $\mathcal{N}_{\mathfrak{E}}(p_-)$ and $\mathcal{B}_{\mathbb{L}(p_+)}(p_+)$ at the two endpoints of the finite interval $[y_-, y_+]$ and a Janus interface interpolating between them. By keeping the masses constant throughout most of the interval and varying them rapidly near $y = 0$ we can think of the Janus interface as being located at $y = 0$. The setup of this computation is depicted in Figure 6.

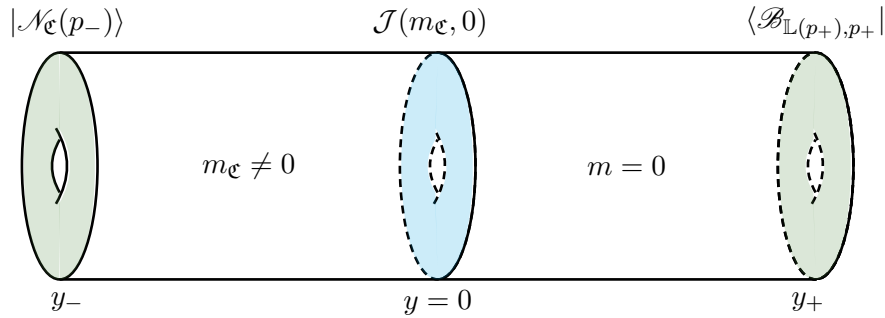


Figure 6. The physical setup for the computation of elliptic stable envelopes. We start with an enriched Neumann boundary condition $\mathcal{N}_{\mathfrak{E}}(p_-)$ at the past boundary y_- . This boundary imitates the choice of the vacuum p_- at $-\infty$. We insert a mass-Janus interface $\mathcal{J}(m_{\mathfrak{E}}, 0)$ at $y = 0$ whose role is to change the real masses from nonzero to zero as we cross over to $y > 0$. We then choose an exceptional Dirichlet boundary condition $\mathcal{B}_{\mathbb{L}(p_+), p_+}$ at y_+ . This future state is associated with the choice of a vacuum p_+ at $+\infty$ and a polarization $\mathbb{L}(p_+)$ of the space of the chirals. The interval partition function of this configuration gives the matrix elements (4.11) of elliptic stable envelopes.

Our choice of localizing supercharge makes the 3d theory topological in the y -direction (B.10), and therefore the length of the interval $[y_-, y_+]$ will not be a parameter of the theory. In fact, we shall find an effective description of the theory as a 2d $\mathcal{N} = (0, 2)$ theory on \mathbb{E}_τ – the amount of supersymmetry reduced to half by the half BPS boundaries and interfaces. The 3d interval partition function will be given by the elliptic genus of this 2d theory.

¹⁸Note that, if we go down in dimension then the periods of the background connections become part of the twisted masses (4.9).

Due to the presence of a mass-Janus interface along which real masses change, we have two different effective descriptions of the theory on the two sides of the interface. The theory living on $[0, y_+]$, where $m = 0$, is the full gauge theory with the gauge Lie algebra $\mathfrak{g} = \mathfrak{u}(N)$. On the other hand, the theory living on $[y_-, 0]$, where $m \neq 0$, is the gauge theory describing the effective theory of low energy (relative to m) excitations in the vacuum p_- . This theory depends on the choice of a chamber \mathfrak{C} for the mass m , which in turn determines the phase of the theory. For generic masses, the adjoint scalar σ from the vector multiplet takes a generic VEV at this boundary (according to (5.25)), and as a result gauge symmetry is broken down to its commutant, the maximal abelian subgroup $\mathfrak{h} = \mathbb{R}^N$.

5 Boundaries and interfaces in 3d $\mathcal{N} = 2$ SQCD

In this section, we look at the classical Higgs branch vacua of 3d $\mathcal{N} = 2$ SQCDs and $\mathcal{N} = (0, 2)$ boundary conditions that are cohomologous to these vacua. Our goal is to compute Janus partition functions (4.11) of the 3d theory on space-time manifolds of the form $[y_-, y_+] \times \mathbb{E}_\tau$. We do so using supersymmetric localization using the supercharge (B.8). The supercharge belongs to an $\mathcal{N} = (0, 2)$ subalgebra of the 3d $\mathcal{N} = 2$ supersymmetry algebra, which is why we shall look at boundary conditions preserving this subalgebra. We follow [40] to find such boundary conditions that mimic the Higgs branch vacua of the theory. The main difference in our case is that, unlike in the 3d $\mathcal{N} = 4$ cases, we do not have a discrete set of completely massive vacua even in the presence of generic twisted masses. This requires us to slightly modify the notion of the so-called thimble boundaries so that they can be attached to extended subspaces of the Higgs branch.

5.1 Vacua and BPS equations in 3d $\mathcal{N} = 2$ Gauge theories

5.1.1 Vacuum equations

Consider a 3d $\mathcal{N} = 2$ Lagrangian gauge theory with gauge group G and flavor symmetry group F . We minimize its potential (consisting of bosonic fields only) to find the classical moduli of vacua.

All terms of the action containing dynamical and background vector multiplet fields can be packaged together in terms of a single vector multiplet for the group $G' := G \times F$. We denote the fields of the G' -multiplet as $(A, \sigma, \lambda, \bar{\lambda}, D)$. Part of the Lagrangian containing A , σ and D only is

$$\frac{1}{2} \text{tr} \left(\frac{1}{2} F^{\mu\nu} F_{\mu\nu} + D^\mu \sigma D_\mu \sigma + D^2 \right) - i\zeta \text{tr} D. \quad (5.1)$$

The first trace is from the Yang–Mills action and the second is the FI term. We are assuming that G has a $U(1)$ center and so there is a single real FI parameter ζ . Note that ζ corresponds to the background value of the adjoint scalar in the vector multiplet for the $U(1)_{\text{top}}$ global symmetry (4.6).

We consider chiral multiplets in the representation \mathbf{M} of the gauge group G and anti-chiral multiplets in the contragradient representation \mathbf{M}^\vee . Let us denote the gauge invariant pairing between \mathbf{M} and \mathbf{M}^\vee as $\langle \cdot, \cdot \rangle$. Then, the part of the chiral multiplet Lagrangian containing only bosonic fields from the chiral and vector multiplets is

$$-\langle \bar{\phi}, D^\mu D_\mu \phi \rangle + \langle \sigma \cdot \bar{\phi}, \sigma \cdot \phi \rangle + i \langle \bar{\phi}, D\phi \rangle + \langle \bar{F}, F \rangle, \quad (5.2)$$

The equations of motion for the auxiliary fields D and F are

$$D = i\zeta - i\mu_{\mathfrak{g}}, \quad F = \bar{F} = 0, \quad (5.3)$$

where $\mu_{\mathfrak{g}} = \phi\bar{\phi}$ is the real moment map of the G -action on \mathbf{M} . Eliminating the auxiliary fields from the Lagrangians (5.1) and (5.2) using the equations of motion we find the on-shell Lagrangian containing bosonic fields only

$$\text{tr} \left(\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} D^\mu \sigma D_\mu \sigma \right) - \langle \bar{\phi}, D^\mu D_\mu \phi \rangle + \text{tr}(-\zeta + \mu_{\mathfrak{g}})^2 + \langle \sigma \cdot \bar{\phi}, \sigma \cdot \phi \rangle.$$

The derivative terms are the kinetic terms and the non-derivative terms constitute the potential that we want to minimize. The potential, being a sum of squares, is minimized when the individual terms are zero

$$\mu_{\mathfrak{g}} - \zeta = 0, \tag{5.4a}$$

$$(\sigma_{\mathfrak{g}} + \sigma_{\mathfrak{f}}) \cdot \phi = 0. \tag{5.4b}$$

Here $\sigma_{\mathfrak{g}} + \sigma_{\mathfrak{f}} = \sigma$ is the decomposition of σ into its dynamical and background components.

The classical Higgs branch of the theory is defined as the real symplectic quotient $\mu_{\mathfrak{g}}^{-1}(\zeta)/G$. The first vacuum equation (5.4a) is precisely this real moment map equation, and after implementing the G -quotient, the second equation (5.4b) becomes the criteria for fixed points under the infinitesimal action generated by $\sigma_{\mathfrak{f}}$. $\sigma_{\mathfrak{f}}$ will be some element from the Cartan subalgebra \mathfrak{t} of the flavor symmetry group. Nonzero $\sigma = \sigma_{\mathfrak{g}} + \sigma_{\mathfrak{f}}$ generates mass terms for various chiral multiplets and W-bosons. Thus we come to the following characterization of the (classical) massive vacua of a 3d $\mathcal{N} = 2$ gauge theory: they correspond to the fixed point set in the Higgs branch under the action of the maximal torus of the flavor symmetry group.

5.1.2 BPS equations

We look for field configurations left invariant by the localizing supercharge \mathcal{Q} (B.8). We consider the 3d $\mathcal{N} = 2$ theory on the space-time $I \times \mathbb{E}_\tau$, where I is an interval with real coordinate y and \mathbb{E}_τ is an elliptic curve with holomorphic coordinate z . A supercharge of the theory is parameterized by 2 Dirac spinors ϵ and $\bar{\epsilon}$, as in (B.4), and our choice of \mathcal{Q} is determined by

$$\mathcal{Q} = \mathcal{Q}_{\epsilon, \bar{\epsilon}}, \quad \text{where } \epsilon = \bar{\epsilon} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

We thus find the BPS equations by setting the supersymmetry variations for the fermions from the vector (B.5) and the chiral (B.6) multiplets to zero for this specific choice of parameters. Doing so, we find the BPS equations – the ones containing derivatives along the elliptic curve \mathbb{E}_τ

$$F_{z\bar{z}} = 0, \tag{5.5a}$$

$$D_{\bar{z}}\sigma_{\mathfrak{g}} = D_{\bar{z}}\sigma_{\mathfrak{f}} = 0, \tag{5.5b}$$

$$D_{\bar{z}}\phi = D_{\bar{z}}\bar{\phi} = 0, \tag{5.5c}$$

and the ones containing derivatives along I

$$D_y A_{\bar{z}} = 0, \tag{5.6a}$$

$$D_y \sigma_{\mathfrak{g}} + \zeta + \mu_{\mathfrak{g}} = 0, \tag{5.6b}$$

$$D_y \sigma_{\mathfrak{f}} + iD_F = 0, \tag{5.6c}$$

$$D_y \phi + (\sigma_{\mathfrak{g}} + \sigma_{\mathfrak{f}}) \cdot \phi = 0, \tag{5.6d}$$

along with the conjugate equation for $\bar{\phi}$. We have used the equations of motion (5.3) to eliminate the auxiliary fields $D_{\mathfrak{g}}$, F and \bar{F} . The only constraint on the background auxiliary field D_F is (5.6c) and we use this equation as the definition of this particular background. The equations (5.5)

determine the fields along the elliptic curve \mathbb{E}_τ and the equations (5.6) determine their flow in the y -direction.

The connections in the above equations all contain both dynamical and background fields for the gauge and the flavor symmetry groups respectively. According to (5.5a) all these connections are flat and will be characterized by their holonomies around the cycles S_A^1 and S_B^1 of \mathbb{E}_τ . Equation (5.6a) further implies that these holonomies are constant along the flow in the y -direction. $\sigma_{\mathfrak{g}}$ and $\sigma_{\mathfrak{f}}$ are holomorphic according to (5.5b) but since they are elements of real Lie algebras, they have to be constant on \mathbb{E}_τ . Similarly, ϕ and $\bar{\phi}$ are holomorphic according to (5.5c). However, in the path integral we would like to impose the reality condition that the anti-chiral field $\bar{\phi}$ is the complex conjugate of the chiral field ϕ . Then both ϕ and $\bar{\phi}$ have to be constant on \mathbb{E}_τ as well. Having nonzero holonomies for the flat connections is only compatible with constant VEVs for ϕ if the holonomies act trivially on ϕ

$$sx^{-1}\hbar^{\frac{1}{2}}z\phi = \phi. \quad (5.7)$$

Here s , x , \hbar and z are the group valued holonomies for the Cartan valued connections for the gauge group G , flavor groups F_A and $U(1)_{\hbar}$, and the topological symmetry group $U(1)_{\text{top}}$. None of the chirals are charged under $U(1)_{\text{top}}$ so z acts as identity on all chirals. The equation (5.7) is saying that the combined holonomies are invisible to the chirals, which is also referred to as a screening condition.

There is no equation for A_y , which is gauge equivalent to the zero connection since the y -direction is contractible. We are now left with the flow equations for $\sigma_{\mathfrak{g}}$ (5.6b) and ϕ (5.6d), which we shall address in Section 5.2.3.

5.2 Classical Higgs branch

We consider the classical Higgs branch, where the chiral multiplet scalars have VEVs in the absence of twisted masses. The vacuum equation (5.4b) is then automatically satisfied because $\sigma_{\mathfrak{g}} = \sigma_{\mathfrak{f}} = 0$ and we find the classical Higgs branch to be (cf. (2.3))

$$\mathcal{M}_H = \mu_{\mathfrak{g}}^{-1}(\zeta)/G, \quad (5.8)$$

where G is the gauge group, $\mu_{\mathfrak{g}}$ is the real moment map of the G -action on the representation space \mathbf{M} of the chirals and ζ is a positive central element in the gauge Lie algebra. The construction (5.8) is the real symplectic reduction of \mathbf{M} by G $\mathcal{M}_H = \mathbf{M} // G$. In 3d $\mathcal{N} = 2$, the space of chirals is always Kähler and so is its symplectic quotient by a real Lie group. If we are only interested in the complex structure of the Higgs branch, we can equivalently define it as a GIT quotient by the complexified gauge group instead¹⁹

$$\mathcal{M}_H = \mathbf{M}^s / G_{\mathbb{C}}. \quad (5.9)$$

Here \mathbf{M}^s is the stable locus of \mathbf{M} , which consists of all points of \mathbf{M} whose $G_{\mathbb{C}}$ orbits intersect $\mu_{\mathfrak{g}}^{-1}(\zeta)$.

Twisted masses are turned on by giving t -valued nonzero VEV to the adjoint scalar $\sigma_{\mathfrak{f}}$ in the flavor vector multiplet. We then have the additional vacuum equation $(\sigma_{\mathfrak{g}} + \sigma_{\mathfrak{f}}) \cdot \phi = 0$ (5.4b). This tells us that most of the Higgs branch is now lifted and only the fixed points of the T -action in the Higgs branch remain as vacua of the theory.²⁰

We work out some details of the construction of the classical Higgs branches and their flavor fixed points for the specific theories defined by the quiver in Figure 3b.

¹⁹This is how the Higgs branch was defined in (2.1).

²⁰Later in computations, we only turn on A -twisted masses and look for A -fixed points where $T = A \times U(1)_{\hbar}$.

5.2.1 Classical Higgs branches of 3d $\mathcal{N} = 2$ SQCDs

The real moment map of the G -action on the space of chirals is dictated by the D-term in the chiral multiplet Lagrangian (5.2). For the theory defined by Figure 3b with the representations of the chirals as in (4.7), the moment map is²¹ $\mu = QQ^\dagger - \tilde{Q}^\dagger \tilde{Q}$. And so the real moment map equation becomes:

$$QQ^\dagger - \tilde{Q}^\dagger \tilde{Q} - \zeta = 0, \quad (5.10)$$

where $\zeta \in \mathbb{R}_+$ is the positive FI parameter. The Higgs branch is the quotient $\mathcal{M}_H(N, L) := \mu^{-1}(\zeta)/U(N)$, From this, we get the dimension of the Higgs branch

$$\dim_{\mathbb{C}} \mathcal{M}_H(N, L) = 2NL - N^2. \quad (5.11)$$

Furthermore, since $Q, \tilde{Q} \in \text{Hom}(\mathbb{C}^L, \mathbb{C}^N)$, we have $\text{rk } \mu \leq L$. On the other hand, the moment map equation sets $\mu_{\mathfrak{g}}$ to ζ which has rank N . We thus get the constraint

$$N \leq L. \quad (5.12)$$

Notice from (5.11) that for odd values of N the complex dimension is odd and the resulting space can not have a holomorphic symplectic form. This is in sharp contrast with the case of 3d $\mathcal{N} = 4$ supersymmetry where the hyper-Kähler structure of Higgs branches equips them with a holomorphic symplectic form.

For our purposes, we mainly need the complex structure of the Higgs branch instead of the full Kähler structure. Let us look at an example.

Example 5.1 ($N = 1, L = 2$). For gauge group $G = U(1)$ and flavor group $F_A = SU(2)$, the Higgs branch is a 3-dimensional complex manifold. The space of the chirals is, $\mathbf{M} = \text{Hom}(\mathbb{C}^2, \mathbb{C}) \oplus \text{Hom}(\mathbb{C}, \mathbb{C}^2)$ with holomorphic coordinates Q^i and \tilde{Q}_i for $i = 1, 2$. The moment-map equation is given by

$$|Q^1|^2 + |Q^2|^2 - |\tilde{Q}_1|^2 - |\tilde{Q}_2|^2 - \zeta = 0. \quad (5.13)$$

Under a complexified gauge transformation by $\lambda \in \mathbb{C}^\times$ the coordinates transform as

$$(Q, \tilde{Q}) \mapsto (\lambda Q, \lambda^{-1} \tilde{Q}).$$

So a set of gauge invariant holomorphic functions on the Higgs branch is given by the matrix elements of $\tilde{Q}Q$

$$\tilde{Q}Q = \begin{pmatrix} \tilde{Q}_1 Q^1 & \tilde{Q}_1 Q^2 \\ \tilde{Q}_2 Q^1 & \tilde{Q}_2 Q^2 \end{pmatrix}.$$

Any gauge invariant polynomial function on the Higgs branch can be written in terms of these. Since $\tilde{Q}Q: \mathbb{C}^2 \rightarrow \mathbb{C}^2$ factors through \mathbb{C} , it has rank at most 1, consequently

$$\det \tilde{Q}Q = 0. \quad (5.14)$$

The equation (5.14) defines the singular conifold as a subspace of $\text{Hom}(\mathbb{C}^2, \mathbb{C}^2)$ with singularity at $Q = \tilde{Q} = 0$ where the rank of $\tilde{Q}Q$ is strictly less than 1. The Higgs branch $\mathcal{M}_H(1, 2)$ is a resolution of this singularity by the moment map equation (5.13). The moment map equation requires a nonzero $|Q^1|^2 + |Q^2|^2$, which means that with $\zeta > 0$, at $\det \tilde{Q}Q = 0$ we have $\tilde{Q} = 0$.

²¹ Q and \tilde{Q} are a priori independent fields in Euclidean signature, but we have to choose some reality condition on them to do our path integrals and we choose $\tilde{Q} = Q^\dagger$.

Then the moment map equation becomes $|\mathcal{Q}^1|^2 + |\mathcal{Q}^2|^2 = \zeta$. This defines an S^3 with radius $\sqrt{\zeta}$, which together with the $U(1)$ -gauge identification implies that the singularity of the conifold is replaced with $S^3/U(1) \simeq \mathbb{P}^1$ of size determined by ζ .

The resolved complex geometry can be described as follows. For $\zeta > 0$, the requirement $(\mathcal{Q}^1, \mathcal{Q}^2) \neq (0, 0)$ coming from the moment map equation says that the map $\mathcal{Q}: \mathbb{C}^2 \rightarrow \mathbb{C}$ is surjective. Due to the quotient by complex gauge transformations, only the kernel of this map is physical. Different choices of $\ker \mathcal{Q} \simeq \mathbb{C} \subseteq \mathbb{C}^2$ are parameterized by a \mathbb{P}^1 . \mathcal{Q}^1 and \mathcal{Q}^2 are the homogeneous coordinates on this \mathbb{P}^1 . We can define the two local coordinates $z := \frac{\mathcal{Q}^1}{\mathcal{Q}^2}$, $z' := \frac{\mathcal{Q}^2}{\mathcal{Q}^1}$, to cover this \mathbb{P}^1 . There is no constraint on $\tilde{\mathcal{Q}}: \mathbb{C} \rightarrow \mathbb{C}^2$ and since we have already used up the gauge transformations to classify \mathcal{Q} , all choices of $\tilde{\mathcal{Q}}: \mathbb{C} \rightarrow \mathbb{C}^2$ are physically significant. A map $\mathbb{C} \rightarrow \mathbb{C}^2$ is characterized by the image of 1 which is unconstrained. Over the chart of \mathbb{P}^1 with local coordinate z we can parameterize these images of 1 by $(\frac{\tilde{\mathcal{Q}}_1}{\tilde{\mathcal{Q}}^2}, \frac{\tilde{\mathcal{Q}}_2}{\tilde{\mathcal{Q}}^2})$ and over the chart with coordinate z' , we can parameterize the same images by

$$\left(\frac{\tilde{\mathcal{Q}}_1}{\tilde{\mathcal{Q}}^1}, \frac{\tilde{\mathcal{Q}}_2}{\tilde{\mathcal{Q}}^1} \right) = z' \left(\frac{\tilde{\mathcal{Q}}_1}{\tilde{\mathcal{Q}}^2}, \frac{\tilde{\mathcal{Q}}_2}{\tilde{\mathcal{Q}}^2} \right).$$

Overall, the maps $(\mathcal{Q}, \tilde{\mathcal{Q}})$ modulo complex gauge transformations form a rank 2 vector bundle over \mathbb{P}^1 with transition function z' as we change coordinate $z \mapsto z'$. Dualizing this transition function we get the underlying space as the Higgs branch

$$\mathcal{M}_H(1, 2) = \mathcal{O}(-1) \oplus \mathcal{O}(-1) \rightarrow \mathbb{P}^1. \quad (5.15)$$

It is easy to generalize this example to arbitrary N and $L \geq N$. First, as a consequence of the moment map equation we have

Proposition 5.1. *For $\zeta > 0$, on the Higgs branch $\text{rk } \mathcal{Q} = N$.*

Proof. Suppose \mathcal{Q} has rank less than N . Then $\mathcal{Q}\mathcal{Q}^\dagger$ has rank less than N and we can choose a nonzero $v \in \ker \mathcal{Q}\mathcal{Q}^\dagger$. The moment map equation (5.10) then implies $-\tilde{\mathcal{Q}}^\dagger \tilde{\mathcal{Q}}v = \zeta v$. However, the operator on the left-hand side is negative semi-definite and can not have a positive eigenvalue – we have a contradiction. ■

Now let us consider the Higgs branch. Once again, a generating set of holomorphic functions on the Higgs branch are given by the matrix elements of $\tilde{\mathcal{Q}}\mathcal{Q}$. Since $\tilde{\mathcal{Q}}\mathcal{Q}$ factors through \mathbb{C}^N we have the constraint $\text{rk } \tilde{\mathcal{Q}}\mathcal{Q} \leq N$. This defines the determinantal variety, denoted by $\text{Det}(N, L)$, which has singularities where the rank of $\tilde{\mathcal{Q}}\mathcal{Q}$ is strictly less than N . These singularities are resolved by the real moment map equation.

Since $\mathcal{Q}: \mathbb{C}^L \rightarrow \mathbb{C}^N$ is surjective (see Proposition 5.1), modulo complexified gauge transformation by GL_N , \mathcal{Q} is completely fixed by its kernel. Once \mathcal{Q} is fixed, $\tilde{\mathcal{Q}}: \mathbb{C}^N \rightarrow \mathbb{C}^L$ is unconstrained, though the choices of $\tilde{\mathcal{Q}}$ are fibered nontrivially over the choices of \mathcal{Q} due to the GL_N quotient that acts on both the choice of \mathcal{Q} and that of $\tilde{\mathcal{Q}}$. The quotient of \mathbb{C}^L by $\ker \mathcal{Q}$ is parameterized by the Grassmannian $\text{Gr}(N, L)$. We can remember the GL_N action on \mathcal{Q} by the tautological principal GL_N bundle \mathcal{K}_P over $\text{Gr}(N, L)$. Then the choices of $(\mathcal{Q}, \tilde{\mathcal{Q}})$ such that \mathcal{Q} is surjective are parameterized by $\mathcal{K}_P \times \text{Hom}(\mathbb{C}^N, \mathbb{C}^L)$. Taking the gauge quotient, we find the Higgs branch

$$\mathcal{M}_H(N, L) = \mathcal{K}_P \times^{\text{GL}_N} \text{Hom}(\mathbb{C}^N, \mathbb{C}^L) = \text{Hom}(\mathcal{K}, \mathbb{C}^L),$$

where \mathcal{K} and \mathbb{C}^L refer to the tautological vector bundle and a trivial bundle over $\text{Gr}(N, L)$. By construction, this is a vector bundle over $\text{Gr}(N, L)$.²² The above space is also called the desingularization of the determinantal variety $\text{Det}(N, L)$ [83], and can be denoted by $\widehat{\text{Det}}(N, L)$.

²²Had we used the opposite sign $\zeta < 0$ of the real FI parameter in the moment map equation (5.10), the Higgs branch would be the dual bundle $\text{Hom}(\mathbb{C}^L, \mathcal{K})$.

The singular conifold (5.14) is the special case of a determinantal variety for $N = 1$ and $L = 2$, and the resolved conifold (5.15) is the aforementioned resolution thereof. In all abelian cases, we get

$$\mathcal{M}_H(1, L) = \widehat{\text{Det}}(1, L) = \mathcal{O}(-1)^{\oplus L} \rightarrow \mathbb{P}^{L-1}.$$

5.2.2 Fixed points

Flavor fixed points in the Higgs branch are points of \mathbf{M} satisfying the real moment map equation where the flavor torus action can be undone by a gauge transformation. We are considering the gauge and flavor symmetry group $G = \text{U}(N)$ and $\mathbf{A} = \text{U}(1)^{L-1} \subseteq \text{SU}(L)$. Under an infinitesimal $G \times \mathbf{A}$ action, the chiral multiplets transform as $\mathcal{Q} \rightarrow \mathcal{Q} + \sigma\mathcal{Q} - \mathcal{Q}m$, $\tilde{\mathcal{Q}} \rightarrow \tilde{\mathcal{Q}} - \tilde{\mathcal{Q}}\sigma + m\tilde{\mathcal{Q}}$ where $\sigma \in \mathfrak{u}(N)$, and

$$m = \begin{pmatrix} m_1 & & & \\ & m_2 & & \\ & & \ddots & \\ & & & m_L \end{pmatrix} \in \mathfrak{a} = \mathbb{R}^L, \quad (5.16)$$

with the constraint $m_1 + \dots + m_L = 0$. Therefore, $(\mathcal{Q}, \tilde{\mathcal{Q}})$ is a fixed point if the following equations admit solutions for σ

$$\sigma\mathcal{Q} - \mathcal{Q}m = 0, \quad \tilde{\mathcal{Q}}\sigma - m\tilde{\mathcal{Q}} = 0. \quad (5.17)$$

And $(\mathcal{Q}, \tilde{\mathcal{Q}})$ must satisfy the moment map equations (5.10) to be in the Higgs branch.

Let us denote by \mathcal{Q}^i and $\tilde{\mathcal{Q}}_i$ for $i = 1, \dots, L$ the columns and rows of \mathcal{Q} and $\tilde{\mathcal{Q}}$, respectively. Then we can write (5.17) as

$$\sigma\mathcal{Q}^i = m_i\mathcal{Q}^i, \quad \tilde{\mathcal{Q}}_i\sigma = m_i\tilde{\mathcal{Q}}_i. \quad (5.18)$$

These are eigenvalue equations for the $2L$ vectors \mathcal{Q}^i , $\tilde{\mathcal{Q}}_i$. We assume that the masses m_i are all distinct. There are then L different eigenvalue equations since \mathcal{Q}^i and $\tilde{\mathcal{Q}}_i$ share an eigenvalue. Since σ is an $N \times N$ matrix with $N \leq L$, at most N out of these L eigenvalue equations can produce nonzero eigenvectors. On the other hand, for $\zeta > 0$, the moment map equation (5.10) implies that $\text{rk } \mathcal{Q} = N$ (see Proposition 5.1). Consequently, a solution to the eigenvalue equations (5.18) contains exactly N nonzero eigenvectors. The nonzero eigenvectors are labeled by an inclusion

$$p: \{1, \dots, N\} \hookrightarrow \{1, \dots, L\}, \quad (5.19)$$

such that the nonzero eigenvectors are $\mathcal{Q}^{p(a)}$. Eigenvectors of hermitian operators with distinct eigenvalues are mutually orthogonal. So, as an orthonormal basis for the N -dimensional vector space spanned by $\mathcal{Q}^{p(a)}$ we can choose $\left\{ \frac{\mathcal{Q}^{p(1)}}{\|\mathcal{Q}^{p(1)}\|}, \dots, \frac{\mathcal{Q}^{p(N)}}{\|\mathcal{Q}^{p(N)}\|} \right\}$. In this basis, the matrix \mathcal{Q} satisfies

$$\mathcal{Q}_a^i = \delta_{p(a)}^i \|\mathcal{Q}^{p(a)}\| \quad (\text{no sum over } a). \quad (5.20)$$

$\tilde{\mathcal{Q}}_i$ must be a multiple of \mathcal{Q}^i since they have the same eigenvalue

$$\tilde{\mathcal{Q}}_i^a = \delta_i^{p(a)} e^{i\varphi} \|\tilde{\mathcal{Q}}_{p(a)}\|. \quad (5.21)$$

The phase is not fixed. Substituting (5.20) and (5.21) in the moment map equation (5.10), we get

$$\mathcal{Q}\mathcal{Q}^\dagger - \tilde{\mathcal{Q}}^\dagger\tilde{\mathcal{Q}} = \text{diag}(\|\mathcal{Q}^{p(1)}\|^2 - \|\tilde{\mathcal{Q}}_{p(1)}\|^2, \dots, \|\mathcal{Q}^{p(N)}\|^2 - \|\tilde{\mathcal{Q}}_{p(N)}\|^2) = \zeta\mathbb{1}_N.$$

Using gauge symmetry, we can fix all components of \mathcal{Q} to be real and positive, then there is a one-parameter family of solutions to the above moment map equation:

$$\mathcal{Q}_a^i = \delta_{p(a)}^i \sqrt{\zeta + |\lambda|^2}, \quad \tilde{\mathcal{Q}}_i^a = \delta_i^{p(a)} \lambda, \quad \lambda \in \mathbb{C}. \quad (5.22)$$

To summarize, we have the following characterization of \mathbf{A} -fixed points of the Higgs branch $\mathcal{M}_H(N, L)$. There are $\binom{L}{N}$ mutually disjoint connected components of the fixed point set, labeled by a choice p of N numbers out of L (5.19).²³ For such a choice p , the corresponding component of the fixed point set in the Higgs branch is given by the gauge orbit of a reference N -dimensional subspace $\widehat{\mathcal{F}}_p \subseteq \mathbf{M}$ which we take to be (no sum over a, b)

$$\widehat{\mathcal{F}}_p := \{(\mathcal{Q}, \tilde{\mathcal{Q}}) \in \mathbf{M} \mid (\mathcal{Q}_a^i, \tilde{\mathcal{Q}}_j^b) = (\delta_{p(a)}^i \sqrt{\zeta + |\lambda_a|^2}, \delta_j^{p(b)} \lambda_b), (\lambda_1, \dots, \lambda_N) \in \mathbb{C}^N\}. \quad (5.23)$$

We denote the image of $\widehat{\mathcal{F}}_p$ in the complex Higgs branch (5.9) under the bundle projection

$$\mathbf{M}^s \xrightarrow{\pi} \mathbf{M}^s / \mathrm{GL}_N = \mathcal{M}_H \quad (5.24)$$

by $\mathcal{F}_p := \pi(\widehat{\mathcal{F}}_p)$. The projection π is one-to-one when restricted to $\widehat{\mathcal{F}}_p$.

Observe that the fixed points also satisfy the vacuum equations (5.4) in the case $\sigma_{\mathfrak{f}} = \sigma_{\mathbf{A}}$, in other words, when we turn on twisted masses only with respect to the flavor symmetry $\mathrm{SU}(L)$ (the framing node of the quiver Figure 3b). At these vacua, the value of the vector multiplet scalar σ is determined by (5.4b) (or equivalently (5.18)), which is

$$\sigma_p(m) := \begin{pmatrix} m_{p(1)} & & & \\ & m_{p(2)} & & \\ & & \ddots & \\ & & & m_{p(N)} \end{pmatrix}. \quad (5.25)$$

Thus, the \mathbf{A} -fixed point set corresponds to the residual classical Higgs branch in the presence of the \mathbf{A} -twisted masses

$$\mathcal{M}_H^{\mathbf{A}} = \bigsqcup_p \mathcal{F}_p = \text{classical Higgs branch with } \mathbf{A}\text{-twisted masses}. \quad (5.26)$$

At the vacuum p , we also get constraints on the holonomies for the flat connections around the elliptic curve from the screening conditions (5.7). The nontrivial screening conditions come from the chirals (5.22) that take nonzero VEVs, the conditions being

$$s_a x_{p(a)}^{-1} \hbar^{1/2} = 1. \quad (5.27)$$

We shall refer to the gauge holonomies satisfying this condition by $s_a^{(p)}$.

5.2.3 Attracting sets and Morse flow

Consider the definition of the attracting set of a fixed point $x_0 \in \mathcal{F}_p$ (3.4)

$$\mathrm{Attr}_{\mathfrak{C}}(x_0) = \{x \in \mathcal{M}_H \mid \lim_{t \rightarrow 0} f(t) \cdot x = x_0 \text{ for all } f \in \mathfrak{C}\}.$$

Choose some generic $m = \mathrm{diag}(m_1, \dots, m_L) \in \mathfrak{a}$ such that $t \mapsto t^m$ is a cocharacter in the chamber \mathfrak{C} . Now, introduce the variable $y = -\ln t$ such that the limit $t \rightarrow 0$ corresponds to

²³Only maps (5.19) modulo permutations of $\{1, \dots, N\}$ lead to physically inequivalent vacua since the permutations coincide with the action of the Weyl subgroup of the gauge group $\mathrm{U}(N)$.

the limit $y \rightarrow \infty$. We take the limit $t \rightarrow 0$ along the positive real axis so that y becomes a non-negative real number. Then we can rewrite the definition of the above attracting set as

$$\text{Attr}_{\mathfrak{C}}(x_0) = \{x \in \mathcal{M}_H \mid \lim_{y \rightarrow \infty} e^{-ym} \cdot x = x_0\}.$$

Define the trajectory of x in the Higgs branch, as we take the limit $y \rightarrow \infty$, as a path

$$x: [0, \infty) \rightarrow \mathcal{M}_H, \quad x(y) := e^{-ym} \cdot x_0.$$

This path satisfies the flow equation

$$\partial_y x + m \cdot x = 0, \tag{5.28}$$

with the asymptotic boundary condition $\lim_{y \rightarrow \infty} x = x_0$. The definition of the above attracting set can now be rewritten as

$$\text{Attr}_{\mathfrak{C}}(x_0) = \{x(0) \in \mathcal{M}_H \mid \partial_y x + m \cdot x = 0, \lim_{y \rightarrow \infty} x = x_0\}. \tag{5.29}$$

Any smooth path $x(y)$ in the Higgs branch can be lifted to a smooth path in the stable locus \mathbf{M}^s by undoing the projection (5.24) and non-uniquely choosing some point from each fiber over $x(y)$. Let $\hat{x}(y)$ be such a lift of $x(y)$. The flow equation (5.28) for x lifts to the following flow equation for \hat{x} $D_y \hat{x} + (\sigma + m) \cdot \hat{x} = 0$. Here D_y involves some \mathfrak{gl}_N -connection and $\sigma(y)$ is some \mathfrak{gl}_N -valued function of y . These are the extra ingredients that will be lost once we take the projection by the GL_N -quotient. Let $\hat{x}_0 \in \hat{\mathcal{F}}_p$ be a lift of x_0 . We can constraint the lift \hat{x} by requiring that its asymptotic value, instead of being something arbitrary from the GL_N -orbit of \hat{x}_0 , be exactly equal to \hat{x}_0 $\lim_{y \rightarrow \infty} \hat{x}(y) = \hat{x}_0$. This completely breaks gauge symmetry at $y = \infty$ and therefore the value of σ must also be constrained asymptotically to take the value (5.25) $\lim_{y \rightarrow \infty} \sigma(y) = \sigma_p(m)$. We can rewrite the definition of the attracting set (5.29) once more by projecting \hat{x} as follows

$$\text{Attr}_{\mathfrak{C}}(x_0) = \pi(\text{Flow}_{\mathfrak{C}}^+(\hat{x}_0)), \tag{5.30}$$

where we have defined

$$\text{Flow}_{\mathfrak{C}}^{\pm}(\hat{x}_0) := \{\hat{x}(0) \in \mathbf{M}^s \mid D_y \hat{x} + (\sigma + m) \cdot \hat{x} = 0, \lim_{y \rightarrow \pm\infty} (\hat{x}, \sigma) = (\hat{x}_0, \sigma_p(m))\}. \tag{5.31}$$

This relates Morse flow in \mathbf{M} to the attracting sets in the Higgs branch. The flows in \mathbf{M} are not completely fixed because both the connection D_y and σ can be changed by GL_N gauge transformations as long as we satisfy the limit condition. We can choose some convenient representatives of these flows as follows.

Let us zoom in on the relation (5.30) at x_0 , i.e., we look at the tangent space to the attracting set at x_0

$$T_{x_0} \text{Attr}_{\mathfrak{C}}(x_0) = \pi_* \left(\{\hat{x}(0) \in \mathbf{M}^s \mid D_y \hat{x} + (\sigma_p(m) + y D_y \sigma|_{x_0} + m) \cdot \hat{x} = 0, \lim_{y \rightarrow \infty} \hat{x} = \hat{x}_0\} \right). \tag{5.32}$$

We further use our gauge freedom to set $D_y = \partial_y$ and $D_y \sigma|_{x_0} = 0$. Small displacements around $x_0 \in \mathbf{M}^s$ can be parameterized by the chiral scalars \mathcal{Q}_a^i and $\tilde{\mathcal{Q}}_i^a$. These displacements are scaled by the action of $(\sigma_p(m) + m)$ as

$$(\sigma_p(m) + m) \cdot \mathcal{Q}_a^i = (m_{p(a)} - m_i) \mathcal{Q}_a^i, \quad (\sigma_p(m) + m) \cdot \tilde{\mathcal{Q}}_i^a = -(m_{p(a)} - m_i) \tilde{\mathcal{Q}}_i^a. \tag{5.33}$$

Solutions to the flow equation $\partial_y \hat{x} + (\sigma_p(m) + m) \cdot \hat{x} = 0$ are therefore given by

$$\mathcal{Q}_a^i(y) = e^{-y(m_{p(a)} - m_i)} \mathcal{Q}_a^i(0), \quad \tilde{\mathcal{Q}}_i^a(y) = e^{y(m_{p(a)} - m_i)} \tilde{\mathcal{Q}}_i^a(0).$$

The linear combination $\widehat{x}(y) = \widehat{x}_0 + \kappa_i^a \mathcal{Q}_a^i(y) + \widetilde{\kappa}_i^a \widetilde{\mathcal{Q}}_a^i(y)$ then satisfies the same flow equation as well as the limit condition $\lim_{y \rightarrow \infty} \widehat{x} = \widehat{x}_0$ if and only if the coefficients κ and $\widetilde{\kappa}$ vanish for all chirals with non-positive weights. Let us decompose the space of chirals into subspaces according to the sign of their weights under the action (5.33)

$$\mathbf{M} = \mathbf{M}_{\mathfrak{C};+}(p) \oplus \mathbf{M}_0(p) \oplus \mathbf{M}_{\mathfrak{C};-}(p), \quad (5.34a)$$

where

$$\begin{aligned} \mathbf{M}_{\mathfrak{C};+}(p) &:= \text{Span}_{\mathbb{C}}\{\mathcal{Q}_a^i, \widetilde{\mathcal{Q}}_j^b \mid m_{p(a)} - m_i > 0, m_{p(b)} - m_j < 0\}, \\ \mathbf{M}_0(p) &:= \text{Span}_{\mathbb{C}}\{\mathcal{Q}_a^i, \widetilde{\mathcal{Q}}_j^b \mid m_{p(a)} - m_i = m_{p(b)} - m_j = 0\}, \\ \mathbf{M}_{\mathfrak{C};-}(p) &:= \text{Span}_{\mathbb{C}}\{\mathcal{Q}_a^i, \widetilde{\mathcal{Q}}_j^b \mid m_{p(a)} - m_i < 0, m_{p(b)} - m_j > 0\}. \end{aligned}$$

We can now rewrite (5.32) as

$$T_{x_0} \text{Attr}_{\mathfrak{C}}(x_0) \simeq \pi_* (\mathbf{M}_{\mathfrak{C};+}(p)). \quad (5.35)$$

Here the vector space inside the projection on the right-hand side has its origin at \widehat{x}_0 . The attracting set $\text{Attr}_{\mathfrak{C}}(x_0)$ is generated by the action of $f(t) = t^m$ for some generic cocharacter f in the chamber \mathfrak{C} . This action can be lifted to an action \widehat{f} on \mathbf{M} via an embedding $\mathbf{A}_{\mathbb{C}} \hookrightarrow \mathbf{H}_{\mathbb{C}} \times \mathbf{A}_{\mathbb{C}}$, such that for $\widehat{x} \in \mathbf{M}$ we have $\widehat{f}(t) \cdot \widehat{x} = t^{\sigma_p(m)+m} \cdot \widehat{x}$. We can then define the attracting set of \widehat{x}_0 in \mathbf{M}^s with respect to this action:

$$\widehat{\text{Attr}}_{\mathfrak{C}}(\widehat{x}_0) := \{\widehat{x} \in \mathbf{M} \mid \lim_{t \rightarrow 0} t^{\sigma_p(m)+m} \cdot \widehat{x} = \widehat{x}_0\}.$$

The tangent space to this affine space at \widehat{x}_0 is $\mathbf{M}_{\mathfrak{C};+}(p)$. The projection π is $\mathbf{A}_{\mathbb{C}}$ equivariant and it is a surjection (5.35) on tangent spaces. Therefore, it is a surjection globally, i.e., $\widehat{\text{Attr}}_{\mathfrak{C}}(\widehat{x}_0)$ projects onto the full attracting set $\text{Attr}_{\mathfrak{C}}(x_0)$. We can also immediately see that the attracting set $\widehat{\text{Attr}}_{\mathfrak{C}}(\widehat{x}_0)$ is nothing but $\{\widehat{x}_0\} \times \mathbf{M}_{\mathfrak{C};+}(p)$. Therefore,

$$\text{Attr}_{\mathfrak{C}}(x_0) = \pi(\{\widehat{x}_0\} \times \mathbf{M}_{\mathfrak{C};+}(p)).$$

Comparing with (5.30) we find a gauge equivalence between positive weight spaces in \mathbf{M} and Morse flow in the stable locus \mathbf{M}^s that asymptotes some fixed point in $\widehat{\mathcal{F}}_p$. More specifically, by varying x_0 over \mathcal{F}_p , we find

$$\pi(\text{Flow}_{\mathfrak{C}}^+(\widehat{\mathcal{F}}_p)) = \text{Attr}_{\mathfrak{C}}(\mathcal{F}_p) = \pi(\widehat{\mathcal{F}}_p \times \mathbf{M}_{\mathfrak{C};+}(p)). \quad (5.36)$$

Replacing the limit $y \rightarrow \infty$ with $y \rightarrow -\infty$, and following the same arguments we find

$$\pi(\text{Flow}_{\mathfrak{C}}^-(\widehat{\mathcal{F}}_p)) = \text{Rep}_{\mathfrak{C}}(\mathcal{F}_p) = \pi(\widehat{\mathcal{F}}_p \times \mathbf{M}_{\mathfrak{C};-}(p)). \quad (5.37)$$

Here Rep refers to the repelling set defined similarly to the attracting set (3.4) but with the opposite limit $t \rightarrow \infty$.

5.3 Boundaries and interfaces

We discuss half-BPS boundary conditions that are equivalent, in localization computations, to forcing the fields to reach any particular connected component of the solutions to the vacuum equation (5.4). The choice of localizing supercharge is discussed in Appendix B.3, (B.8) in particular. We discuss half-BPS boundaries preserving \mathcal{Q}_- , $\overline{\mathcal{Q}}_-$. So these are $\mathcal{N} = (0, 2)$ boundary conditions in the 3d $\mathcal{N} = 2$ theory. We also discuss the mass Janus interface preserving the same supersymmetry. Our discussion mostly follows similar boundary conditions for the 3d $\mathcal{N} = 4$ case considered in [40]. Our description of the thimble boundaries (see Section 5.3.1) differs slightly from this reference due to the non-isolated nature of 3d $\mathcal{N} = 2$ Higgs branch vacua.

5.3.1 Thimble (exceptional Dirichlet), $\mathcal{D}_{\mathfrak{C}}(p)$

To find a boundary condition at $y = y_-$ that is cohomologous to a choice of vacuum, we extend our space-time to negative infinity and consider the theory on $(-\infty \times y_-] \times \mathbb{E}_\tau$. We impose the constraint that the theory must reach the vacuum p at infinity. Then we observe the supersymmetric evolution of the fields along the interval $(-\infty, y_-]$ and see what values the fields take at $y = y_-$. More specifically, we solve the BPS equations (5.6) with the boundary condition that at $y = -\infty$ the fields satisfy the vacuum equation (5.4). Then the values of these solutions at $y = y_-$ provide a past boundary condition for the theory at $y \geq y_-$ that mimics the chosen vacuum in BPS computations. Similar boundary conditions were constructed and called thimble or exceptional Dirichlet boundary conditions in [22].

In this process, we observe that the BPS equation (5.6d) with the condition that at past infinity the chiral and vector multiplet scalars satisfy the vacuum equations (5.4) is precisely what defines the set $\text{Flow}_{\mathfrak{C}}^-(\widehat{\mathcal{F}}_p)$ (5.31). The choice of chamber and vacuum determine the Morse function and the connected component of the vacuum configurations defining the boundary condition. The BPS equation (5.6b) puts constraints on the evolution of $\sigma_{\mathfrak{g}}$ but $\text{Flow}_{\mathfrak{C}}^-(\widehat{\mathcal{F}}_p)$ is agnostic to such constraints. According to (5.37), the set $\text{Flow}_{\mathfrak{C}}^-(\widehat{\mathcal{F}}_p)$ is gauge equivalent to the subspace $\widehat{\mathcal{F}}_p \times \mathbf{M}_{\mathfrak{C};-}(p)$ of \mathbf{M} . So, we formulate a $(0, 2)$ boundary condition of brane type with support for the chiral multiplet scalars on $\widehat{\mathcal{F}}_p \times \mathbf{M}_{\mathfrak{C};-}(p)$.

Remark 5.1 (thimbles in $\mathcal{N} = 4$ vs $\mathcal{N} = 2$). Support of this brane (at the past boundary) on the Higgs branch is the projection of $\widehat{\mathcal{F}}_p \times \mathbf{M}_{\mathfrak{C};-}(p)$, which is the repelling set (5.37). In [22], the thimble boundaries are supported on holomorphic Lagrangians in the hyperkähler Higgs branches of 3d $\mathcal{N} = 4$ theories. For 3d $\mathcal{N} = 2$ theories, the Higgs branches are generally only Kähler and the attracting/repelling sets (5.36), (5.37) are not supported on holomorphic Lagrangians, they are just holomorphic.

Let us decompose the space of chirals further from (5.34a) based on where the \mathcal{Q} s and the $\widetilde{\mathcal{Q}}$ s belong

$$\mathbf{M}_{\mathfrak{C};\pm}(p) = \mathcal{R}_{\mathfrak{C};\pm}(p) \oplus \widetilde{\mathcal{R}}_{\mathfrak{C};\pm}(p), \quad \mathbf{M}_0(p) = \mathcal{R}_0(p) \oplus \widetilde{\mathcal{R}}_0(p).$$

Since \mathcal{Q}_a^i and $\widetilde{\mathcal{Q}}_i^a$ have weights with opposite signs (see (5.33)), we have the following relations among these subspaces

$$\mathcal{R}_{\mathfrak{C};\pm}(p) = \widetilde{\mathcal{R}}_{\mathfrak{C};\mp}^\vee(p), \quad \mathcal{R}_0(p) = \widetilde{\mathcal{R}}_0^\vee(p).$$

We thus have a Lagrangian splitting of the chirals

$$\begin{aligned} \mathbf{M} &= \mathbb{L}_{\mathfrak{C}}(p) \oplus \mathbb{L}_{\mathfrak{C}}^\perp(p), & \mathbb{L}_{\mathfrak{C}}(p) &:= \mathbf{M}_{\mathfrak{C};+}(p) \oplus \mathcal{R}_0(p), \\ \mathbb{L}_{\mathfrak{C}}^\perp(p) &:= \mathbf{M}_{\mathfrak{C};-}(p) \oplus \widetilde{\mathcal{R}}_0(p). \end{aligned} \tag{5.38}$$

$\widehat{\mathcal{F}}_p$ is parameterized by arbitrary VEVs of fields from $\widetilde{\mathcal{R}}_0(p)$ (see (5.23)), therefore, chirals from $\mathbb{L}_{\mathfrak{C}}^\perp(p)$ parameterize the brane $\widehat{\mathcal{F}}_p \times \mathbf{M}_{\mathfrak{C};-}(p)$. So we shall put Dirichlet and Neumann boundary conditions on the scalars from the chirals in $\mathbb{L}_{\mathfrak{C}}(p)$ and $\mathbb{L}_{\mathfrak{C}}^\perp(p)$ respectively. The boundary conditions will be extended to the rest of the fields of these multiplets by the requirement of preserving $(0, 2)$ supersymmetry.

Gauge symmetry is broken at the vacuum by the VEVs of the chirals and as such the gauge field receives Dirichlet boundary condition, the rest of the vector multiplet receiving boundary conditions accordingly. The holonomy of the dynamical gauge field at the vacuum is given in terms of the holonomy of the background gauge fields via the screening condition (5.27).

Altogether, the thimble boundary condition $\mathcal{D}_{\mathfrak{C}}(p)$ involves:

$$\begin{aligned} \text{Chirals in } \mathbb{L}_{\mathfrak{C}}(p): & \text{ Dirichlet with VEVs according to vacuum } p, \\ \text{Chirals in } \mathbb{L}_{\mathfrak{C}}^{\perp}(p): & \text{ Neumann,} \\ \text{Vector multiplet:} & \text{ Dirichlet with screening (5.27) of gauge holonomy.} \end{aligned} \tag{5.39}$$

Boundary 't Hooft anomaly. Let $\mathbf{f} \in \mathfrak{h}$, $\mathbf{f}_x \in \mathfrak{a}$ and $\mathbf{f}_h \in \mathbb{R}_h$ be the curvatures of the gauge, $SU(L)$ flavor, and $U(1)_h$ flavor symmetry at the boundary. The gauge symmetry is completely broken at the boundary and \mathbf{f} is a function of the flavor curvatures. Their dependence is fixed by demanding that the nonzero VEVs at the boundary do not observe any flux at the boundary $\mathbf{f} = f^{(p)}(\mathbf{f}_x, \mathbf{f}_h)$ such that, $w(\mathbf{f} + \mathbf{f}_x + \mathbf{f}_h) = 0$ for all $w \in \mathcal{R}_0(p)$. Contributions to the anomaly polynomial come from two sources: 1) Chern–Simons terms created by integrating out massive fermions and the Chern–Simons coupling to background multiplet associated with $U(1)_{\text{top}}$ symmetry, 2) purely from boundary conditions. In the present case, these contributions are [40, 43]²⁴

$$P_{-}^{\text{CS}}(\mathcal{D}_{\mathfrak{C}}(p)) = 2 \text{tr}_{\mathbf{M}_{\mathfrak{C},+}(p)}(\mathbf{f}_h(\mathbf{f} + \mathbf{f}_x)) - 2 \text{tr}(\mathbf{f}\mathbf{f}_{\text{top}}), \quad P^{\partial}(\mathcal{D}_{\mathfrak{C}}(p)) = 2 \text{tr}_{\mathcal{R}_0(p)}(\mathbf{f}_h(\mathbf{f} + \mathbf{f}_x)).$$

The subscript in P_{-}^{CS} refers to the past boundary. The Chern–Simons anomaly depends on the orientation of the boundary and therefore $P_{-}^{\text{CS}} = -P_{+}^{\text{CS}}$. P^{∂} is independent of the choice of past or future. If we define

$$P(\mathcal{D}_{\mathfrak{C}}(p)) := P_{-}^{\text{CS}}(\mathcal{D}_{\mathfrak{C}}(p)) + P^{\partial}(\mathcal{D}_{\mathfrak{C}}(p)) = 2 \text{tr}_{\mathbb{L}_{\mathfrak{C}}(p)}(\mathbf{f}_h(\mathbf{f} + \mathbf{f}_x)) - 2 \text{tr}(\mathbf{f}\mathbf{f}_{\text{top}}),$$

then the total anomaly polynomial associated to the thimble boundary is $P(\mathcal{D}_{\mathfrak{C}}(p))$ evaluated at $\mathbf{f} = \mathbf{f}^{(p)}(\mathbf{f}_x, \mathbf{f}_h)$.

5.3.2 Enriched Neumann, $\mathcal{N}_{\mathfrak{C}}(p)$

Swap the boundary conditions on the chiral multiplets from (5.39). Meaning, the chirals in $\mathbb{L}_{\mathfrak{C}}(p)$ now receive Neumann, and those in $\mathbb{L}_{\mathfrak{C}}^{\perp}(p)$ receive Dirichlet boundary conditions with zero VEV. The gauge field in the vector multiplet receives the Neumann boundary condition. This implies that the adjoint scalar $\sigma_{\mathfrak{g}}$ from the vector multiplet gets Dirichlet [43, Section 2.5.2]. We set the value of $\sigma_{\mathfrak{g}}$ equal to the VEV $\sigma_p(m)$ (5.25) corresponding to the vacuum p . This breaks the gauge symmetry down to the maximal torus $\mathbf{H} = U(1)^N$. Additionally, we couple a 2d $\mathcal{N} = (0, 2)$ theory $\Upsilon_{\mathfrak{C}}(p)$ at the boundary by gauging some of its global symmetries with the boundary gauge symmetry \mathbf{H} . We shall characterize the theory $\Upsilon_{\mathfrak{C}}(p)$ momentarily. Let us first summarize this enriched Neumann boundary condition $\mathcal{N}_{\mathfrak{C}}(p)$

$$\begin{aligned} \text{Chirals in } \mathbb{L}_{\mathfrak{C}}(p): & \text{ Neumann,} \\ \text{Chirals in } \mathbb{L}_{\mathfrak{C}}^{\perp}(p): & \text{ Dirichlet with zero VEVs,} \\ \text{Vector multiplet:} & \text{ Neumann,} \\ \text{Coupled to the boundary:} & \text{ 2d } \mathcal{N} = (0, 2) \text{ theory } \Upsilon_{\mathfrak{C}}(p). \end{aligned}$$

The relation between the thimble boundary conditions (5.39) and vacua are relatively straightforward. However, in computation, we shall use this new boundary condition $\mathcal{N}_{\mathfrak{C}}(p)$ to represent the vacuum p . In [40, Section 6.3], Dedushenko–Nekrasov argue that both $|\mathcal{D}_{\mathfrak{C}}(p)\rangle \langle \mathcal{N}_{\mathfrak{C}}(p)|$ and $|\mathcal{N}_{\mathfrak{C}}(p)\rangle \langle \mathcal{D}_{\mathfrak{C}}(p)|$ are projections onto the vacuum p in cohomology, where $\langle \cdot |$ and $|\cdot \rangle$ refer to

²⁴The trace without explicit representation is taken over the fundamental representation.

future and past boundary condition, respectively. A necessary condition for this to be true is that these projectors are free of 't Hooft anomaly

$$P_{\pm}(\mathcal{D}_{\mathfrak{C}}(p))\Big|_{\mathbf{f}=\mathbf{f}^{(p)}(\mathbf{f}_x, \mathbf{f}_h)} + P_{\mp}(\mathcal{N}_{\mathfrak{C}}(p)) = 0. \quad (5.40)$$

Here P_+ and P_- refer to anomaly polynomials of the future and past boundary conditions, respectively. In this paper, we only describe the $\mathcal{N}_{\mathfrak{C}}(p)$ boundary condition and show that this anomaly condition is satisfied, beyond that, we take their equivalence as an ansatz. For more details, see [40].

Boundary theory $\Upsilon_{\mathfrak{C}}(p)$. At the boundary preserving $\mathcal{N} = (0, 2)$ supersymmetry, we have $U(1)^N$ gauge symmetry. So we can couple a 2d $\mathcal{N} = (0, 2)$ theory with $U(1)^N$ global symmetry by gauging it as long as the anomaly matching condition (5.40) is satisfied. It is straightforward to write down a boundary theory for any chamber, but for the sake of explicit formulas we shall focus here on the following two chambers

$$\mathfrak{C}_1 := m_1 < m_2 < \cdots < m_L, \quad \mathfrak{C}_2 := m_1 > m_2 > \cdots > m_L. \quad (5.41)$$

For \mathfrak{C}_1 , we propose the following field content and charges for the theory under various symmetry groups applying the procedure of field content of $\Upsilon_{\mathfrak{C}_1}(p)$:

	$U(1)_a$	$U(1)_h$	$U(1)_{\text{top}}$
Chiral $_{1,a}$	-1	1	0
Chiral $_{2,a}$	0	$\frac{L}{2} - p(a)$	1
Fermi $_{1,a}$	1	$\frac{L}{2} - p(a)$	1
Fermi $_{2,a}$	0	1	0

We have one set of these fields for each $a = 1, \dots, N$, totaling in $2N$ Fermi and $2N$ chiral multiplets. The $U(1)_a$ global symmetry is identified with a $U(1)$ subgroup of the bulk $U(1)^N \times U(1)^{L-1} \times U(1)_h$ symmetry such that the 2d $U(1)_a$ -fugacity is identified with the bulk fugacity $s_a x_{p(a)}^{-1} \hbar^{1/2}$. The remaining symmetries are identified straightforwardly. The elliptic genus of this theory is [17, 18]

$$\mathbb{W}(\Upsilon_{\mathfrak{C}_1}(p)) = \prod_{a=1}^N \frac{\vartheta(s_a x_{p(a)}^{-1} \hbar^{\frac{1}{2} + \frac{L}{2} - p(a)} z) \vartheta(\hbar)}{\vartheta(s_a^{-1} x_{p(a)} \hbar^{\frac{1}{2}}) \vartheta(z \hbar^{\frac{L}{2} - p(a)}}. \quad (5.42)$$

Here we are using the theta functions defined in the notation of [2]

$$\vartheta(t; q) := (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \prod_{n=1}^{\infty} (1 - tq^n)(1 - t^{-1}q^n). \quad (5.43)$$

Some of its identities are

$$\begin{aligned} \vartheta(1; q) &= 0, & \vartheta(t^{-1}; q) &= -\vartheta(t; q), & \vartheta(e^{2\pi i} t; q) &= -\vartheta(t; q), \\ \vartheta(q^k t; q) &= q^{-\frac{k^2}{2}} (-t)^{-k} \vartheta(t; q), & k &\in \mathbb{Z}. \end{aligned}$$

This theta function can be related to Jacobi's theta function and the Dedekind eta function as follows:

$$q^{\frac{1}{12}} \vartheta(t; q) = \frac{i\theta_1\left(\frac{\ln t}{2\pi i}, \frac{\ln q}{2\pi i}\right)}{\eta\left(\frac{\ln q}{2\pi i}\right)}.$$

We omit the argument q in the theta function when it is clear from the context.

For the chamber \mathfrak{C}_2 , we simply write down the elliptic genus

$$\mathbb{W}(\Upsilon_{\mathfrak{C}_2}(p)) = \prod_{a=1}^N \frac{\vartheta(s_a x_{p(a)}^{-1} \hbar^{-\frac{1}{2} - \frac{L}{2} + p(a)} z) \vartheta(\hbar)}{\vartheta(s_a^{-1} x_{p(a)} \hbar^{\frac{1}{2}}) \vartheta(z \hbar^{-\frac{L}{2} + p(a) - 1})}. \quad (5.44)$$

The field content of the theory and their charges can be read off from this if needed. Notice that when the screening conditions (5.27) are imposed the elliptic genera reduce to 1.

The choice of the boundary theory is defined only up to the addition of anomaly-free 2d $\mathcal{N} = (0, 2)$ theories. This leads to some ambiguities regarding the overall partition function, and consequently regarding the stable envelope. We can fix some of this ambiguity by imposing certain normalization conditions on the stable envelopes (as was done in their definition in Theorem 3.1). The normalization condition restricts our choice of boundary theory to those without anomaly and whose elliptic genus reduces to 1 when the screening conditions (5.27) are imposed. Even then the choice is not unique. Our choice differs minimally from the closely related example of the $T^*\text{Gr}(N, L)$ Grassmannian theory with 3d $\mathcal{N} = 4$ symmetry from [40, Section 6.5.3].

Boundary 't Hooft anomaly. The enriched Neumann boundary condition (5.3.2) receives contributions to its anomaly polynomial from Chern–Simons terms, boundary conditions on chirals and vectors, and the boundary theory

$$P(\mathcal{N}_{\mathfrak{C}}(p)) := P_-^{\text{CS}}(\mathcal{N}_{\mathfrak{C}}(p)) + P^\partial(\mathcal{N}_{\mathfrak{C}}(p)) + P(\Upsilon_{\mathfrak{C}}(p)).$$

The Chern–Simons anomalies only depend on the orientation of the boundary, $P_-^{\text{CS}}(\mathcal{N}_{\mathfrak{C}}(p)) = P_-^{\text{CS}}(\mathcal{D}_{\mathfrak{C}}(p))$. Whereas, the boundary conditions on the chirals and the vector multiplet fields are swapped between the thimble (5.39) and the enriched Neumann (5.3.2). So they contribute with opposite signs: $P_-^\partial(\mathcal{N}_{\mathfrak{C}}(p)) = -P_-^\partial(\mathcal{D}_{\mathfrak{C}}(p))$. Since we have a gauge theory at the boundary, the gauge anomalies from all the different sources must neatly cancel. In fact, the boundary theory is defined to satisfy the constraint that only 't Hooft anomalies survive and they match the negative of the future thimble boundary anomaly (5.40) define $\Upsilon_{\mathfrak{C}}(p)$ such that

$$P_-(\mathcal{N}_{\mathfrak{C}}(p)) = -P_+(\mathcal{D}_{\mathfrak{C}}(p)) \Big|_{\mathbf{f}=\mathbf{f}^{(p)}(\mathbf{f}_r, \mathbf{f}_\hbar)}.$$

The anomaly of the elliptic genera (5.42) and (5.44) satisfy this condition for the chambers \mathfrak{C}_1 and \mathfrak{C}_2 (5.41) respectively. For reference, we note that in any chamber \mathfrak{C} , the anomaly of the boundary theory must be

$$P(\Upsilon_{\mathfrak{C}}(p)) = -2(\text{tr}_{\mathbb{L}_{\mathfrak{C}}(p)} - 2 \text{tr}_{\mathcal{R}_0(p)})((\mathbf{f} - \mathbf{f}^{(p)})\mathbf{f}_\hbar) + 2 \text{tr}((\mathbf{f} - \mathbf{f}^{(p)})\mathbf{f}_{\text{top}}).$$

It can be checked that the elliptic genera (5.42) and (5.44) contribute the appropriate anomalies for their respective chambers.

5.3.3 Massless boundary condition, $\mathcal{B}_{\mathbb{L}(p)}(p)$

Broadly speaking, stable envelope is a map from the cohomology of the flavor fixed points of the Higgs branch to the cohomology of the Higgs branch itself (1.3). To compute this map, we compute the partition function of the 3d theory on $[y_-, y_+] \times \mathbb{E}_\tau$ such that at the past boundary we have twisted masses which eventually vanish at the future boundary (4.11). At the past boundary, we can, in principle, use either the thimble (see Section 5.3.1) or the enriched Neumann (see Section 5.3.2) boundary conditions both of which are labeled by connected components of the fixed point set and provide a basis for the cohomology of the fixed point set. As vector spaces, the cohomology of the Higgs branch is isomorphic to that of its fixed points. We, therefore, look for a set of boundary conditions to impose at the future massless boundary

parameterized by connected components of the fixed point set as well, such that we can write down the matrix elements of the stable envelope in terms of pairs of connected components. Such a set of boundary conditions was proposed by Dedushenko–Nekrasov in [40]. These boundary conditions are very similar to the thimble boundary conditions. However, since there are no masses at this boundary, there is no choice of chamber and we can not choose a polarization, as we did for the thimbles in (5.38), based on positive and negative masses. More specifically, there are no obvious candidates for $\mathbf{M}_{\mathfrak{C};\pm}(p)$. The massless parts of the polarization, namely $\mathcal{R}_0(p)$ and $\tilde{\mathcal{R}}_0(p)$ still make sense as they are chamber independent. The proposal of Dedushenko–Nekrasov in this case is to arbitrarily choose a polarization of the chirals $\mathbf{M} = \mathbb{L}(p) \oplus \mathbb{L}^\perp(p)$, as long as they satisfy (cf. (5.38))

$$\mathcal{R}_0(p) \subseteq \mathbb{L}(p), \quad \tilde{\mathcal{R}}_0(p) \subseteq \mathbb{L}^\perp(p). \quad (5.45)$$

In all our computations, we make the following choice $\mathbb{L}(p) := \mathcal{R}$, $\mathbb{L}^\perp(p) := \tilde{\mathcal{R}}$, which obviously satisfies (5.45).

Once the choice of polarization is made, this boundary condition, which we call $\mathcal{B}_{\mathbb{L}(p)}(p)$, can be described just as in (5.39):

Chirals in $\mathbb{L}(p)$: Dirichlet with VEVs according to vacuum p ,

Chirals in $\mathbb{L}^\perp(p)$: Neumann,

Vector multiplet: Dirichlet with screening (5.27) of gauge holonomy.

5.3.4 Janus interface, $\mathcal{J}(m_{\mathfrak{C}}, \mathbf{0})$

The transition from a massive boundary in the past labeled by a fixed point p to a massless boundary in the future is achieved via a mass Janus interface. We assume that the masses in the past belong to the chamber \mathfrak{C} . Take $y_- < 0$ and $y_+ > 0$ and put the Janus interface at $y = 0$ wrapping the elliptic curve \mathbb{E}_τ .

We only vary the $F_A = \mathrm{SU}(L)$ twisted masses in the theory across the Janus interface. We do not turn on any twisted masses for the $\mathrm{U}(1)_h$ flavor symmetry. In principle, the mass Janus is implemented by giving a y -dependent background value to the adjoint scalar $\sigma_{\mathfrak{f}_A}$ in the flavor vector multiplet. The background value interpolates between some generic mass m (5.16) in the chamber \mathfrak{C} and zero. This is supersymmetric as long as we also turn on the auxiliary field in the multiplet satisfying the BPS equation (5.6c) (with $\sigma_h = \mathbf{D}_h = 0$ such that $\sigma_{\mathfrak{f}} = \sigma_{\mathfrak{f}_A}$ and $\mathbf{D}_F = \mathbf{D}_{F_A}$). In particular, we can choose a background profile for the mass that is constant on the intervals $[y_-, 0)$ and $(0, y_+]$ and changes rapidly near $y = 0$. Similar Janus interfaces in 3d $\mathcal{N} = 4$ theories were previously used to compute stable envelopes [40] and R -matrices [23, 24] for bosonic spin chains.

The supersymmetric quantities we compute are supposed to be renormalization group invariants and capture the infrared dynamics. We, therefore, do not lose anything by considering the nonzero masses to be arbitrarily large. In this limit, we can integrate out the massive fields from the interval $[y_-, 0)$. But these fields are all present on the other side $(0, y_+]$. Thus, the fields that acquire masses in the massive vacuum p , including chirals and W-bosons, terminate at the interface when approaching from the massless side. This is implemented by a set of half-BPS boundary conditions at $y = 0$ that were worked out in [40]. We are simply going to use their prescription.

The chirals with positive and negative masses in the vacuum p belong to $\mathbf{M}_{\mathfrak{C};+}$ and $\mathbf{M}_{\mathfrak{C};-}$ respectively, according to our earlier definition of these spaces (5.34a). To characterize the massive W-bosons, let us decompose the set Δ of roots of the gauge Lie algebra into subsets corresponding to the sign of their masses at the vacuum p . At p , the adjoint scalar in the vector multiplet takes the VEV $\sigma_p(m)$ given by (5.25). Therefore, the vector field associated to the

root $\alpha \in \Delta$ picks up a mass $\alpha(\sigma_p(m))$. The decomposition of the roots can then be written as $\Delta = \Delta_+(p) \sqcup \Delta_0 \sqcup \Delta_-(p)$, where

$$\begin{aligned} \Delta_+(p) &:= \{\alpha \in \Delta | \alpha(\sigma_p(m)) > 0\}, & \Delta_0 &:= \{\alpha \in \Delta | \alpha(\sigma_p(m)) = 0\}, \\ \Delta_-(p) &:= \{\alpha \in \Delta | \alpha(\sigma_p(m)) < 0\}. \end{aligned}$$

For generic masses, always only the Cartan valued fields remain massless, so $\Delta_0 = \{0\}$ in any vacuum.

The interface is then implemented by the following boundary conditions on fields [40]:

$$\begin{aligned} \text{Chirals in } \mathbf{M}_{\mathfrak{E},+}(p) &: \text{ Dirichlet,} \\ \text{Chirals in } \mathbf{M}_{\mathfrak{E},-}(p) &: \text{ Neumann,} \\ \text{Vectors in } \Delta_+(p) &: \text{ Dirichlet,} \\ \text{Vectors in } \Delta_-(p) &: \text{ Neumann.} \end{aligned} \tag{5.46}$$

From the definition of the Janus interface as a supersymmetric position-dependent background profile for the masses, it is clear that it does not introduce any anomaly. However, in practice, we implement the Janus interface effectively using the above-mentioned boundary conditions and it is not obvious that they are anomaly free. After all, both the boundary conditions and the Chern–Simons terms created by integrating out massive fermions contribute to anomaly at the boundary. It is however a simple exercise to check that these two anomalies precisely cancel each other, as was checked explicitly in [40].

6 Stable envelopes and R -matrix from gauge theory: the computation

6.1 Elliptic stable envelope

We are now in a position to compute the matrix elements of the stable envelope of $\mathfrak{sl}(1|1)$ elliptic spin chains. For the purpose of BPS computations, we can shrink the interval $[y_-, y_+]$ to a point, resulting in an effective 2d $\mathcal{N} = (0, 2)$ theory on \mathbb{E}_τ . Then the 3d $\mathcal{N} = 2$ partition function on $[y_-, y_+] \times \mathbb{E}_\tau$ is given by the elliptic genus of this effective 2d theory. To derive the field content of this effective 2d theory, we need to look at the intervals $[y_-, 0]$ and $[0, y_+]$, in addition to their union, and determine which 3d multiplets have compatible boundary conditions at both ends of all intervals where it is defined. If any field living on $[y_{\text{start}}, y_{\text{end}}]$ has opposite boundary conditions at y_{start} and y_{end} then it vanishes in the effective theory. A 3d multiplet that has the same boundary condition at both ends of an interval survives the collision of boundaries and decomposes into 2d $\mathcal{N} = (0, 2)$ multiplets. Of these $(0, 2)$ multiplets, the ones with Dirichlet boundary conditions are simply frozen to their boundary values. The remaining $(0, 2)$ multiplets with Neumann boundary conditions, along with the theory $\Upsilon_{\mathfrak{E}_1}(p_1)$ at the past massive boundary, comprise the field content of the effective 2d theory. Note that throughout the computation of the stable envelope, we use the specific chamber (5.41) to write down formulas.

The vector fields that remain massless in the vacuum p_1 are part of the theory along the entire interval $[y_-, y_+]$ and they have opposite boundary conditions at $|\mathcal{B}_{\mathbb{L}(p_2)}(p_2)\rangle$ (Dirichlet) and $\langle \mathcal{N}_{\mathfrak{E}_1}(p_1) |$ (Neumann). These vector fields, therefore, do not contribute to the elliptic genus. All the vector fields receive Dirichlet boundary condition at $|\mathcal{B}_{\mathbb{L}(p_2)}(p_2)\rangle$ and all the vector fields with positive mass in the vacuum p_1 receive Dirichlet at the Janus interface. In the IR, these vector multiplets with positive masses in p_1 are only part of the massless theory along $y > 0$ and therefore they survive. They decompose into 2d $\mathcal{N} = (0, 2)$ vector and adjoint chiral multiplets. Dirichlet for 3d $\mathcal{N} = 2$ vector multiplet imposes Dirichlet and Neumann on its constituent 2d

vector and 2d adjoint chiral multiplets respectively [43]. Contributions from these adjoint chirals to the elliptic genus are

$$\mathbb{V}(p_1) := \prod_{\substack{a,b=1 \\ p_1(a) > p_1(b)}}^N \frac{q^{-1/12}}{\vartheta(s_a s_b^{-1})}. \quad (6.1)$$

These 2d chirals are not charged under any global symmetry group.

Next, let us look at all the 3d $\mathcal{N} = 2$ chiral multiplets that survive with Dirichlet boundary conditions. Multiplets receiving Dirichlet at $|\mathcal{N}_{\mathfrak{e}_1}(p_1)\rangle$ are in $\mathbb{L}_{\mathfrak{e}_1}^\perp(p_1) = \mathbf{M}_{\mathfrak{e}_1;-}(p_1) \oplus \tilde{\mathcal{R}}_0(p_1)$ ((5.3.2) and (5.38)). But everything in $\mathbf{M}_{\mathfrak{e}_1;-}(p_1)$ receives Neumann at the Janus (5.46) and therefore vanishes in the effective 2d theory. Multiplets in $\tilde{\mathcal{R}}_0(p_1)$ extends unconstrained across the Janus and intersects with the multiplets $\mathbb{L}(p_2)$ receiving Dirichlet at $\langle \mathcal{B}_{\mathbb{L}(p_2)}(p_2) |$. Similarly, everything in $\mathbf{M}_{\mathfrak{e}_1;+}(p_1)$ receives Dirichlet at the Janus and intersects with multiplets receiving Dirichlet at $\langle \mathcal{B}_{\mathbb{L}(p_2)}(p_2) |$. Therefore, 3d chiral multiplets surviving the collision of boundaries with Dirichlet are from

$$\mathbb{L}(p_2) \cap (\tilde{\mathcal{R}}_0(p_1) \oplus \mathbf{M}_{\mathfrak{e}_1;+}(p_1)) = \mathcal{R} \cap (\tilde{\mathcal{R}}_0(p_1) \oplus \mathcal{R}_{\mathfrak{e}_1;+}(p_1) \oplus \tilde{\mathcal{R}}_{\mathfrak{e}_1;+}(p_1)) = \mathcal{R}_{\mathfrak{e}_1;+}(p_1).$$

Dirichlet on a 3d chiral multiplet means Dirichlet and Neumann on its constituent 2d $\mathcal{N} = (0, 2)$ chiral and fermi multiplets respectively. Fields with Dirichlet are frozen at their boundary values in the effective 2d theory and the remaining fermi multiplets contribute to the elliptic genus

$$\mathbb{M}_{\text{fe}}(p_1) := \prod_{a=1}^N \prod_{\substack{i=1 \\ i < p_1(a)}}^L q^{1/12} \vartheta(s_a x_i^{-1} \hbar^{1/2}).$$

Now we look for the 3d $\mathcal{N} = 2$ chiral multiplets with Neumann boundary condition contributing to the effective 2d theory. At $|\mathcal{N}_{\mathfrak{e}_1}(p_1)\rangle$ all fields in $\mathbb{L}_{\mathfrak{e}_1}(p_1) = \mathbf{M}_{\mathfrak{e}_1;+} \oplus \mathcal{R}_0(p_1)$ get Neumann. Only $\mathcal{R}_0(p_1)$ passes through the Janus unconstrained. On the interval $[0, y_+]$ all fields with negative masses in the vacuum p_1 receive Neumann boundary condition at the Janus interface. Taking the intersection of all these fields with the fields receiving Neumann at $\langle \mathcal{B}_{\mathbb{L}(p_2)} |$, we find

$$\mathbb{L}^\perp(p_2) \cap (\mathcal{R}_0(p_1) \oplus \mathbf{M}_{\mathfrak{e}_1;-}(p_1)) = \tilde{\mathcal{R}} \cap (\mathcal{R}_0(p_1) \oplus \mathcal{R}_{\mathfrak{e}_1;-}(p_1) \oplus \tilde{\mathcal{R}}_{\mathfrak{e}_1;-}(p_1)) = \tilde{\mathcal{R}}_{\mathfrak{e}_1;-}(p_1).$$

Neumann on a 3d chiral multiplet means Neumann and Dirichlet on its constituent 2d $\mathcal{N} = (0, 2)$ chiral and fermi multiplets respectively [43]. These 2d chirals contribute to the elliptic genus

$$\mathbb{M}_{\text{ch}}(p_1) := \prod_{a=1}^N \prod_{\substack{i=1 \\ i < p_1(a)}}^L \frac{q^{-1/12}}{\vartheta(s_a^{-1} x_i \hbar^{1/2})}.$$

Lastly, we have the contribution (5.42) of the boundary theory $\Upsilon_{\mathfrak{e}_1}(p_1)$ to the elliptic genus. Putting all these together, we find the (unnormalized) elliptic stable envelope

$$\widetilde{\text{Stab}}_{\mathfrak{e}_1}(p) = \text{Sym}_{S_N}(\mathbb{V}(p) \mathbb{M}_{\text{fe}}(p) \mathbb{M}_{\text{ch}}(p) \mathbb{W}(\Upsilon_{\mathfrak{e}_1}(p))). \quad (6.2)$$

The symmetrization is over the gauge fugacities. In addition to the maximal torus $U(1)^N$, the Weyl group S_N of $U(N)$ also remains as gauge symmetry at the massive vacuum p_1 and it acts on the gauge fugacities by permuting them.

Let us define

$$\mathbf{f}_{\mathfrak{e}_1,m}(s, \mathbf{x}, \hbar, z) := \left(\prod_{i < m} \vartheta(s x_i) \right) \frac{\vartheta(s x_m z \hbar^{m-L})}{\vartheta(z \hbar^{m-L})} \left(\prod_{i > m} \vartheta(s x_i \hbar) \right). \quad (6.3)$$

Then the stable envelope (6.2) becomes

$$\widetilde{\mathbf{Stab}}_{\mathfrak{C}_1}(p) = \frac{q^{-\binom{N}{2}/12} \vartheta(\tilde{h})^N}{\prod_{a,i} \vartheta(s_a \tilde{x}_i \tilde{h})} \mathbf{Stab}_{\mathfrak{C}_1}(p),$$

where

$$\begin{aligned} \mathbf{Stab}_{\mathfrak{C}_1}(p) &:= (-1)^{\sum_a \#(i < p(a))} \\ &\times \text{Sym}_{S_N} \left[\left(\prod_a \mathbf{f}_{\mathfrak{C}_1, p(a)}(s_a, \tilde{\mathbf{x}}, \tilde{h}, \tilde{z}) \right) \left(\prod_{p(a) > p(b)} \frac{1}{\vartheta(s_a s_b^{-1})} \right) \right], \end{aligned} \quad (6.4)$$

and we have also redefined the fugacities as

$$\tilde{x}_i := x_i^{-1} \tilde{h}^{\frac{1}{2}}, \quad \tilde{h} := \tilde{h}^{-1}, \quad \tilde{z} := z \tilde{h}^{-\frac{L}{2}}. \quad (6.5)$$

These redefinitions and the specific form of the stable envelope in terms of the function (6.3) are merely to facilitate comparison with analogous known formulas coming from BPS computations in 3d $\mathcal{N} = 4$ SQCDs [40] or from the theory of elliptic stable envelopes on complex symplectic varieties [2] that correspond to bosonic XYZ spin chains.

Remark 6.1 (normalization of the stable envelopes). The overall factor in $\widetilde{\mathbf{Stab}}_{\mathfrak{C}_1}(p)$ is independent of the choice of the chamber \mathfrak{C} and the vacuum p . This makes it ambiguous in the context of defining stable envelopes since we can change it by normalizing all the stable envelopes simultaneously. Only the relative normalization of the stable envelopes associated to different vacua and different chambers is meaningful. We have therefore defined the normalized stable envelope $\mathbf{Stab}_{\mathfrak{C}_1}(p)$ without any leading factor. This normalized version is used in Section 3.4 to check that it satisfies all formal properties of elliptic stable envelopes for $\mathfrak{sl}(1|1)$ spin chains. Note however that, the overall sign from (6.4) is missing in (3.6) due to a slightly different choice of normalization for the stable envelopes, see footnote 8.

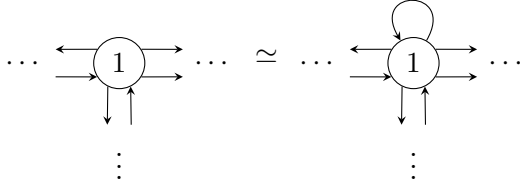
In the expression (6.4), we have not yet fully imposed the boundary condition $\mathcal{B}_{\mathbb{L}(p_2)}(p_2)$. We still need to impose the screening condition (5.27) associated to the fixed point p_2 . Doing so completes the computation of the 3d $\mathcal{N} = 2$ interval partition function, providing us with the matrix elements of the stable envelope

$$\mathbf{Stab}_{\mathfrak{C}_1}(p_2, p_1) = \langle \mathcal{B}_{\mathbb{L}(p_2)}(p_2) | \mathcal{J}(m, 0) | \mathcal{N}_{\mathfrak{C}_1}(p_1) \rangle = \mathbf{Stab}_{\mathfrak{C}_1}(p_1) \Big|_{s=s(p_2)}.$$

The specific values of the mass m is not important as long as it belongs to the chamber \mathfrak{C}_1 .

Remark 6.2 (SQED and comparison with 3d $\mathcal{N} = 4$ /bosonic stable envelopes). The difference between the quiver (3b) we used to compute elliptic $\mathfrak{sl}(1|1)$ stable envelope and the quiver used in [40] to compute the elliptic $\mathfrak{sl}(2)$ stable envelope is the lack of a self-loop at the gauge node in the $\mathfrak{sl}(1|1)$ -quiver. In field theory, this simply means the absence of the 3d $\mathcal{N} = 2$ adjoint chiral multiplet, which is necessarily a part of the 3d $\mathcal{N} = 4$ vector multiplet. In the computation of the last section, the 3d $\mathcal{N} = 4$ vector multiplet contribution to the elliptic genus would involve $\mathbb{V}(p)$ from (6.1) along with the contributions from this adjoint chiral. This is the only difference between the $\mathfrak{sl}(1|1)$ and $\mathfrak{sl}(2)$ cases. Notice then, in the 1-magnon sector, i.e., for SQEDs, there is no distinction between the two cases. Because the vector multiplet, as well as any adjoint chiral multiplet, do not contribute anything to the elliptic genus in abelian theories. Thus, the $\mathfrak{sl}(1|1)$ stable envelopes and the R -matrix in the 1-magnon sector must coincide with

those of the $\mathfrak{sl}(2)$ spin chains. More generally, the following two quivers will always lead to the same stable envelopes, assuming that the hidden parts of the quivers coincide:



Another way of saying this is that in a magnon sector with magnon number 1 assigned to a simple root, the stable envelopes (and consequently the R -matrix) will remain invariant if we change the parity of the simple root. The equivalence between even and odd gauge nodes in the case of abelian gauge groups was also pointed out in Example 3.6 from a more mathematical perspective.

6.2 R -matrix for elliptic $\mathfrak{sl}(1|1)$ spin chains

From the perspective of integrable systems, we are interested in computing the R -matrix for supersymmetric XYZ spin chains, not just the stable envelope. As stated in (3.12), R -matrix is a composition of a stable envelope and an inverse stable envelope.

The action of an R -matrix on a spin chain preserves magnon numbers. As a result, an R -matrix takes a block diagonal form, one block for each sector with fixed magnon numbers. We focus on the $\mathfrak{sl}(1|1)$ spin chain with two fundamental spins.²⁵ The fundamental representation of $\mathfrak{sl}(1|1)$ is $\mathbb{C}^{1|1}$, where the lowest weight state is bosonic, and the only excited state is fermionic. Thus, the tensor product of two such representations is four complex dimensional, consisting of a bosonic one-dimensional subspace of magnon number 0, a fermionic two-dimensional subspace of magnon number 1, and a bosonic one-dimensional subspace of magnon number 2. The R -matrix also decomposes as

$$\mathbf{R} = \begin{pmatrix} \mathbf{R}^{(0)} & & \\ & \mathbf{R}^{(1)} & \\ & & \mathbf{R}^{(2)} \end{pmatrix}, \quad (6.6)$$

where the superscript refers to the magnon number.

The gauge theory dual of a spin chain with two fundamental spins for $\mathfrak{sl}(1|1)$ corresponds to a 3d $\mathcal{N} = 2$ SQCD with $SU(2)$ flavor symmetry. Each magnon sector corresponds to a different unitary gauge group, the rank of the gauge group is equal to the magnon number. The zero magnon sector thus corresponds to a trivial gauge theory and accordingly, the R -matrix in the 0-magnon sector is always trivial

$$\mathbf{R}_{\mathfrak{C}_2 \leftarrow \mathfrak{C}_1}^{(0)} = 1. \quad (6.7)$$

In the above formula, \mathfrak{C}_1 and \mathfrak{C}_2 are the two chambers for the Cartan of $\mathfrak{su}(2)$. To compute the R -matrix, we always need stable envelopes in two different chambers. We use two real masses m_1 and m_2 to parameterize the Cartan subalgebra of $\mathfrak{su}(2)$ with the tracelessness condition $m_1 + m_2 = 0$. There are then two chambers for the masses $\mathfrak{C}_1 = m_1 < m_2$, $\mathfrak{C}_2 = m_1 > m_2$. We computed the stable envelope for \mathfrak{C}_1 in (6.4). The stable envelope in the other chamber can

²⁵The spin chain can be arbitrarily large, but the R -matrix always acts on the tensor product of two representations. So for the purpose of computing the R -matrix it suffices to consider just two sites. While testing Yang–Baxter we will have to extend the R -matrix to act trivially on a third site as well.

be computed following identical steps as in the previous section. Here we simply write down the answer

$$\begin{aligned} \mathbf{Stab}_{\mathfrak{c}_2}(p) &:= (-1)^{\sum_a \#(i>p(a))} \\ &\times \text{Sym}_{S_N} \left[\left(\prod_a \mathbf{f}_{\mathfrak{c}_2, p(a)}(s_a, \mathbf{x}, \hbar, z) \right) \left(\prod_{p(a)<p(b)} \frac{1}{\vartheta(s_a s_b^{-1})} \right) \right], \end{aligned} \quad (6.8)$$

where we have defined

$$\mathbf{f}_{\mathfrak{c}_2, m}(s, \mathbf{x}, \hbar, z) := \left(\prod_{i>m} \vartheta(sx_i) \right) \frac{\vartheta(sx_m z \hbar^{1-m})}{\vartheta(z \hbar^{1-m})} \left(\prod_{i<m} \vartheta(sx_i \hbar) \right).$$

Note that, in this section, all fugacities are the newly defined ones from (6.5), but to avoid clutter we omit the tildes.

Let us now specialize these formulas to the cases of gauge group $U(1)$ and $U(2)$ to compute the stable envelopes and the R -matrix in the 1 and 2-magnon sectors.

1-magnon. The Higgs branch (see Example 5.1) has two fixed points p_1 and p_2 , corresponding to the two possible maps $\{1\} \hookrightarrow \{1, 2\}$ mapping 1 to either 1 or 2 (cf. (5.19)). There is a single gauge fugacity and two screening conditions (5.27) for the two fixed points. We can now compute the four matrix elements of each of the two stable envelopes (6.4) and (6.8) in the two chambers, we find

$$\begin{aligned} \mathbf{Stab}_{\mathfrak{c}_1}^{(1)} &= \begin{pmatrix} \vartheta(x_1^{-1} x_2 \hbar) & 0 \\ \frac{\vartheta(x_1 x_2^{-1} z \hbar^{-1}) \vartheta(\hbar)}{\vartheta(z \hbar^{-1})} & -\vartheta(x_1 x_2^{-1}) \end{pmatrix}, \\ \mathbf{Stab}_{\mathfrak{c}_2}^{(1)} &= \begin{pmatrix} -\vartheta(x_1^{-1} x_2) & \frac{\vartheta(x_1^{-1} x_2 z \hbar^{-1}) \vartheta(\hbar)}{\vartheta(z \hbar^{-1})} \\ 0 & \vartheta(x_1 x_2^{-1} \hbar) \end{pmatrix}. \end{aligned}$$

The R -matrix is a simple matrix product

$$\begin{aligned} \mathbf{R}_{\mathfrak{c}_2 \leftarrow \mathfrak{c}_1}^{(1)} &= \mathbf{Stab}_{\mathfrak{c}_2}^{-1} \cdot \mathbf{Stab}_{\mathfrak{c}_1} \\ &= \frac{-1}{\vartheta(x_1^{-1} x_2 \hbar^{-1})} \begin{pmatrix} \frac{\vartheta(x_1^{-1} x_2) \vartheta(z) \vartheta(z \hbar^{-2})}{\vartheta(z \hbar^{-1})^2} & \frac{\vartheta(\hbar) \vartheta(x_1^{-1} x_2 z \hbar^{-1})}{\vartheta(z \hbar^{-1})} \\ \frac{\vartheta(\hbar) \vartheta(x_1 x_2^{-1} z \hbar^{-1})}{\vartheta(z \hbar^{-1})} & -\vartheta(x_1 x_2^{-1}) \end{pmatrix}. \end{aligned} \quad (6.9)$$

To simplify the upper left component, we have used the identity

$$\sum_{\text{cyclic}} \vartheta(AB) \vartheta(AB^{-1}) \vartheta(C)^2 = 0.$$

2-magnon. The gauge theory dual of the 2-magnon sector has gauge group $U(2)$. There are two gauge fugacities and a unique vacuum corresponding to the unique (modulo permutation) map $\{1, 2\} \hookrightarrow \{1, 2\}$. The stable envelopes are just functions

$$\mathbf{Stab}_{\mathfrak{c}_1}^{(2)} = -\vartheta(x_1^{-1} x_2 \hbar), \quad \mathbf{Stab}_{\mathfrak{c}_2}^{(2)} = -\vartheta(x_1 x_2^{-1} \hbar).$$

As is the R -matrix,

$$\mathbf{R}_{\mathfrak{c}_2 \leftarrow \mathfrak{c}_1}^{(2)} = \mathbf{Stab}_{\mathfrak{c}_2}^{-1} \mathbf{Stab}_{\mathfrak{c}_1} = \frac{\vartheta(x_1^{-1} x_2 \hbar)}{\vartheta(x_1 x_2^{-1} \hbar)}. \quad (6.10)$$

Putting together (6.7), (6.9), and (6.10) as in (6.6), we find the complete R -matrix for the elliptic $\mathfrak{sl}(1|1)$ spin chain

$$\mathbf{R}_{\mathfrak{c}_2 \leftarrow \mathfrak{c}_1} = \left(\begin{array}{c|cc|c} 1 & 0 & 0 & 0 \\ \hline 0 & -\frac{\vartheta(x_1^{-1}x_2)\vartheta(z)\vartheta(z\hbar^{-2})}{\vartheta(x_1^{-1}x_2\hbar^{-1})\vartheta(z\hbar^{-1})^2} & \frac{\vartheta(\hbar)\vartheta(x_1^{-1}x_2z\hbar^{-1})}{\vartheta(x_1x_2^{-1}\hbar)\vartheta(z\hbar^{-1})} & 0 \\ 0 & \frac{\vartheta(\hbar)\vartheta(x_1x_2^{-1}z\hbar^{-1})}{\vartheta(x_1x_2^{-1}\hbar)\vartheta(z\hbar^{-1})} & -\frac{\vartheta(x_1^{-1}x_2)}{\vartheta(x_1^{-1}x_2\hbar^{-1})} & 0 \\ \hline 0 & 0 & 0 & \frac{\vartheta(x_1^{-1}x_2\hbar)}{\vartheta(x_1x_2^{-1}\hbar)} \end{array} \right). \quad (6.11)$$

Here x_1 and x_2 are referred to as the spectral parameters associated to the two sites, z as the dynamical parameter, and \hbar as the quantization parameter. This R -matrix solves the dynamical Yang–Baxter equation for $\mathfrak{sl}(1|1)$ spin chains (3.19). Note that, relative to the R -matrix presented in (3.18), the signs in front of the two diagonal entries in the 1-magnon sector are different in the above formula. This is due to the slightly different choice of normalization of the stable envelopes, as pointed out in Remark 6.1.

7 2d and 1d avatars of elliptic stable envelopes and the R -matrix

This section is devoted to taking the 3d \rightarrow 2d and 2d \rightarrow 1d reduction of elliptic stable envelopes. In Section 7.1, we take the 3d \rightarrow 2d reduction, which produces the K-theoretic stable envelopes and the trigonometric R -matrix for $\mathfrak{sl}(1|1)$. The 2d \rightarrow 1d reduction, which produces cohomological stable envelopes and the rational R -matrix, is the subject of Section 7.2. In the case of cohomological stable envelopes and the rational R -matrix for $\mathfrak{sl}(1|1)$, we compare our result with that of Rimányi and Rozansky in [117] and find perfect agreement. The discussion in this section is based on Section 3.9 and also the physical description in [40, Section 2.1].

7.1 The K-theory limit

In this section, we reduce the formulas for elliptic stable envelopes to two dimensions and give a prediction for K-theoretic stable envelopes associated with Lie superalgebra $\mathfrak{sl}(1|1)$.

Procedure for 3d \rightarrow 2d reduction. In Section 4.2, we explained the parameters in three-dimensional theory on $I \times \mathbb{E}_\tau$. One could shrink one of the cycles of \mathbb{E}_τ , corresponding to $\tau \rightarrow 0$, to get a two-dimensional theory $S_A^1 \times I$. The procedure of going from three to two dimensions is that we shrink the B-cycles of \mathbb{E}_τ . This limit could be better handled if we take our parameters to take value in the S -transformed elliptic curve $\mathbb{E}_{-\frac{1}{\tau}}$. Then, the two-dimensional limit is given by $\tau \rightarrow 0$, $q^{-\frac{2\pi i}{\tau}} \rightarrow 0$. In this limit, the elliptic curve becomes nodal and the smooth locus is isomorphic to \mathbb{C}^\times . It turns out that the K-theoretic stable envelope depends on the choice of a slope. This makes the behavior of elliptic Kähler parameters in the $q \rightarrow 0$ limit more subtle. Defining the slope parameter as (3.22) such that \mathfrak{s} is generic (i.e., it belongs to an alcove). The precise procedure to take the K-theory limit is explained in Section 3.9.

In the case of $\mathcal{N} = 2$ SQCD, we need two results for deriving the K-theoretic stable envelopes:

- (1) the behavior of this function in the limit $q \rightarrow 0$, that can be deduced from its very definition (5.43) $\lim_{q \rightarrow 0} \vartheta(t; q) = t^{\frac{1}{2}} - t^{-\frac{1}{2}}$.
- (2) and that [2, equation (2)]

$$\lim_{q \rightarrow 0} \left(\frac{\vartheta(t_1 z; q)}{\vartheta(t_2 z; q)} \right) \Big|_{z=q^{\mathfrak{s}}} = \left(\frac{t_1}{t_2} \right)^{|\mathfrak{s}| + \frac{1}{2}}. \quad (7.1)$$

Finally, we note that all the holonomies introduced in Section 4.2 are still multiplicative but are now valued in \mathbb{C}^\times .

K-theoretic stable envelopes for $\mathfrak{sl}(1|1)$. We now perform a 3d \rightarrow 2d reduction of the elliptic stable envelope, which gives us the K-theoretic stable envelopes.

From the definition of the function $\mathbf{f}_{\mathfrak{C}_1, m}$ (6.3) and (7.1), we find that

$$\lim_{q \rightarrow 0} \left[\frac{\vartheta(s_a x_{p(a)} \hbar^{-(L-p(a))} z)}{\vartheta(\hbar^{-(L-p(a))} z)} \Big|_{z=q^s} \right] = (s_a x_{p(a)})^{[s] + \frac{1}{2}}.$$

Hence,

$$\begin{aligned} \lim_{q \rightarrow 0} \mathbf{f}_{\mathfrak{C}_1, m}(s, \mathbf{x}, \hbar, z) &= \left(\hbar^{\frac{\#(i > p(a))}{2}} \prod_i (s x_i)^{\frac{1}{2}} \right) (s x_m)^{[s]} \prod_{i < m} (1 - (s x_i)^{-1}) \prod_{i > m} (1 - (s x_i \hbar)^{-1}) \\ &= \left(\hbar^{\frac{\#(i > p(a))}{2}} \prod_i (s x_i)^{\frac{1}{2}} \right) \mathbf{f}_{\mathfrak{C}_1, s, m}(s, \mathbf{x}, \hbar), \end{aligned}$$

where we have defined

$$\mathbf{f}_{\mathfrak{C}_1, s, m}(s, \mathbf{x}, \hbar) := (s x_m)^{[s]} \prod_{i < m} (1 - (s x_i)^{-1}) \prod_{i > m} (1 - (s x_i \hbar)^{-1}).$$

Furthermore,

$$\lim_{q \rightarrow 0} \left(\prod_{a > b} \frac{1}{\vartheta(s_a s_b^{-1})} \right) = \prod_{a > b} \frac{1}{(s_a s_b^{-1})^{\frac{1}{2}} - (s_a s_b^{-1})^{-\frac{1}{2}}}.$$

Finally, we know from (3.23) that we need to multiply the result by two factors: (1) the inverse of the square root of the determinant of the partial polarization $(\det \text{Pol}_X)^{-\frac{1}{2}}$ for $X = \mathcal{M}_H(N, L)$, which is given by the product of fugacities of the quarks Q

$$(\det \text{Pol}_X)^{-\frac{1}{2}} = \prod_{a, i} \frac{1}{(s_a x_i^{-1} \hbar^{1/2})^{\frac{1}{2}}}.$$

However, we need to implement the redefinition of fugacities according to (6.5). Therefore,

$$(\det \text{Pol}_X)^{-\frac{1}{2}} = \prod_{a, i} \frac{1}{(s_a x_i)^{\frac{1}{2}}},$$

(2) the square root of the determinant of the polarization of the fixed loci $(\det \text{Pol}_{X_A})^{\frac{1}{2}}$, which in our case is trivial $(\det \text{Pol}_{X_A})^{+\frac{1}{2}} = 1$. Putting these results together, and using (3.23), we see that²⁶

$$\begin{aligned} \text{Stab}_{\mathfrak{C}_1, s}(p) &= (-1)^{\sum_a \#(i < p(a))} \hbar^{\frac{1}{2} \sum_a \#(i > p(a))} \\ &\quad \times \text{Sym}_{S_N} \left[\left(\prod_{a=1}^N \mathbf{f}_{\mathfrak{C}_1, s, p(a)}(s_a, \mathbf{x}, \hbar) \right) \cdot \left(\prod_{a > b} \frac{1}{(s_a s_b^{-1})^{\frac{1}{2}} - (s_a s_b^{-1})^{-\frac{1}{2}}} \right) \right]. \end{aligned} \quad (7.2)$$

²⁶The K-theory limit of elliptic stable envelope for the resolved determinantal variety $\mathcal{M}_H(N, L)$ has also been computed in (3.26) using a different polarization as explained there. Compared to (3.26), the expression (7.2) contains the extra factor $(-1)^{\sum_a \#(i < p(a))}$, which is the result of a choice of different polarization in the BPS computation of elliptic stable envelope.

By Lemma 3.2, this is the K-theoretic stable envelope for the Lie superalgebra $\mathfrak{sl}(1|1)$. Notice that it is manifestly a locally-constant function of \mathfrak{s} , which is a characteristic feature of K-theoretic stable envelopes as a function of \mathfrak{s} .

Trigonometric R -matrix for $\mathfrak{sl}(1|1)$. We can now perform the reduction of the $\mathfrak{sl}(1|1)$ elliptic R -matrix (6.11) to construct the $\mathfrak{sl}(1|1)$ trigonometric R -matrix. Let us define the function $G(x) := x^{\frac{1}{2}} - x^{-\frac{1}{2}}$. The result of the reduction is

$$R_{\mathfrak{e}_2 - \mathfrak{e}_1}(u) = \left(\begin{array}{c|cc|c} 1 & 0 & 0 & 0 \\ \hline 0 & \frac{G(u)}{G(u^{-1}\hbar)} & u^{|\mathfrak{s}| + \frac{1}{2}} \frac{G(\hbar)}{G(u^{-1}\hbar)} & 0 \\ 0 & u^{-|\mathfrak{s}| - \frac{1}{2}} \frac{G(\hbar)}{G(u^{-1}\hbar)} & \frac{G(u)}{G(u^{-1}\hbar)} & 0 \\ \hline 0 & 0 & 0 & \frac{G(u\hbar)}{G(u^{-1}\hbar)} \end{array} \right). \quad (7.3)$$

One can check that it satisfies the trigonometric version of Yang–Baxter equations given by

$$R_{12}(u/v)R_{13}(u/w)R_{23}(v/w) = R_{23}(v/w)R_{13}(u/w)R_{12}(u/v).$$

7.2 The cohomology limit

In the previous section, we have constructed the K-theoretic stable envelopes and the trigonometric R -matrix for Lie superalgebra $\mathfrak{sl}(1|1)$ by performing a $3d \rightarrow 2d$ reduction. It is then natural to consider a further $2d \rightarrow 1d$ reduction to construct the cohomological stable envelopes and the rational R -matrix for $\mathfrak{sl}(1|1)$. Fortunately, the mathematical construction of cohomological stable envelope for Lie superalgebra $\mathfrak{sl}(1|1)$ is available in the literature [117], and we can compare our result. In this section, we perform this $2d \rightarrow 1d$ reduction, and we recover the cohomological stable envelopes and the geometrical R -matrix of Rimányi and Rozansky.

Procedure for $2d \rightarrow 1d$ reduction. We need to know what happens by going from $2d$ to $1d$. In the $3d \rightarrow 2d$, reduction, one considers the reduction of the theory on one of the circles of the elliptic curve. Similarly, the $2d \rightarrow 1d$ result is obtained by the reduction of the theory on the remaining circle. This amounts to the following substitution of gauge and flavor holonomies²⁷ (see Section 3.9 and [40, Section 2.1])

$$(s_a, x_i, \hbar) \mapsto \lim_{\epsilon \rightarrow 0} (e^{\epsilon s_a}, e^{\epsilon x_i}, e^{\epsilon \hbar}), \quad (7.4)$$

On the other hand, the holonomy associated with the topological symmetry defines Kähler parameter in three dimensions. Upon $3d \rightarrow 2d$ reduction, this becomes a theta angle, i.e., the slope parameter, as explained in Section 7.1. By $2d \rightarrow 1d$ reduction, it turns out that this parameter completely disappears [40, p. 16]. This is consistent with the fact that cohomological stable envelopes are not dependent on extra choices like a slope. Therefore, one should (1) use (7.4) to replace multiplicative holonomies with additive parameters in the $\epsilon \rightarrow 0$ limit by throwing away terms of order $\mathcal{O}(\epsilon^2)$, and (2) discard all \mathfrak{s} -dependent factors. In this process, all gauge and flavor holonomies become additive and \mathbb{C} -valued.

Cohomological stable envelope for $\mathfrak{sl}(1|1)$. We can now perform the reduction explicitly. Different factors of (7.2) are reduced as follows

$$f_{\mathfrak{e}_1, \mathfrak{s}, m} \rightarrow f_{\mathfrak{e}_1, m}(s, \mathbf{x}, \hbar) := \prod_{i < m} (s + x_i) \prod_{i > m} (s + x_i + \hbar),$$

²⁷The precise statement in going from K-theoretic stable envelope to the cohomological one is given in (3.25).

and

$$\prod_{a>b} \frac{1}{(s_a s_b^{-1})^{\frac{1}{2}} - (s_a s_b^{-1})^{-\frac{1}{2}}} \rightarrow \prod_{a>b} \frac{1}{s_a - s_b}.$$

Therefore, according to Lemma 3.2, we end up with the cohomological stable envelope²⁸

$$\text{Stab}_{\mathfrak{e}_1}(p) = (-1)^{\sum_a \#(i < p(a))} \text{Sym}_{S_N} \left[\left(\prod_{a=1}^N f_{\mathfrak{e}_1, m}(s, \mathbf{x}, \hbar) \right) \cdot \left(\prod_{a>b} \frac{1}{s_a - s_b} \right) \right]. \quad (7.5)$$

Rational R -matrix for $\mathfrak{sl}(1|1)$. We can now perform the reduction of the $\mathfrak{sl}(1|1)$ trigonometric R -matrix (7.3) to construct the $\mathfrak{sl}(1|1)$ rational R -matrix. The result is

$$\mathbf{R}_{\mathfrak{e}_2 \leftarrow \mathfrak{e}_1}(u) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{u}{\hbar-u} & \frac{\hbar}{\hbar-u} & 0 \\ 0 & \frac{\hbar}{\hbar-u} & \frac{u}{\hbar-u} & 0 \\ 0 & 0 & 0 & \frac{\hbar+u}{\hbar-u} \end{pmatrix}. \quad (7.6)$$

This satisfies the rational Yang–Baxter equations given by

$$\mathbf{R}_{12}(u-v)\mathbf{R}_{13}(u-w)\mathbf{R}_{23}(v-w) = \mathbf{R}_{23}(v-w)\mathbf{R}_{13}(u-w)\mathbf{R}_{12}(u-v).$$

Let us finally compare our result with the construction in the mathematical literature [117]. For this matter, we first give a synopsis of [117].

Brief recap of the work of Rimányi and Rozanski. Consider an index with values (00), (01), (10), or (11). Depending on the choice of r , we have

$$\begin{aligned} r = 00: & \mathbb{C}_{\text{even}} \oplus \mathbb{C}_{\text{even}}, & r = 10: & \mathbb{C}_{\text{even}} \oplus \mathbb{C}_{\text{odd}}, & r = 01: & \mathbb{C}_{\text{odd}} \oplus \mathbb{C}_{\text{even}}, \\ r = 11: & \mathbb{C}_{\text{odd}} \oplus \mathbb{C}_{\text{odd}}, \end{aligned}$$

which corresponds to $\mathfrak{sl}(2, 0)$, $\mathfrak{sl}(1, 1)$, $\mathfrak{sl}(1|1)$, and $\mathfrak{sl}(0|2)$, respectively. We are only interested in $r = (10)$ (or equivalently $r = (01)$).²⁹ Analog to the classical case, there is a super-stable map $\text{Stab}_{\pi}^{(r)}$, which depends on a permutation $\pi \in S_L$. The definition of this map involves the triple (N, L, π) and are defined if certain classes $\kappa_{p, \pi}^{(r)} := \text{Stab}_{\pi}^{(r)}(1_p)$ ³⁰ exist and satisfy certain axioms with p denoting the injective map (5.19), that uniquely determine $\text{Stab}_{\pi}^{(r)}$ if it exists [117, Definition 4.1]. The explicit forms of the stable envelopes are then given in terms of certain rational functions $W_{l; N, L}^{(r)}(\mathbf{s}, \mathbf{x}, \hbar)$ that are called super-weight functions and are the super-analog of weight functions defined by Tarasov and Varchenko [121, 142]. Here, $\mathbf{x} := \{s_1, \dots, s_N\}$ and $\mathbf{x} := \{x_1, \dots, x_L\}$ are Chern roots and are equivariant parameters, respectively. The super-weight functions are defined as [117, Section 5.2]

$$W_p^{(10)}(\mathbf{s}, \mathbf{x}, \hbar) := \text{Sym}_{S_N}(U_p^{(10)}(\mathbf{s}, \mathbf{x}; \hbar)), \quad (7.7)$$

²⁸The cohomology limit of K-theoretic stable envelope for the resolved determinantal variety $\mathcal{M}_H(N, L)$ has also been computed in (3.27) using a different polarization as explained there. Compared to (3.27), the expression (7.5) contains the extra factor $(-1)^{\sum_a \#(i < p(a))}$, which is the result of a choice of different polarization in the BPS computation of elliptic stable envelope.

²⁹The case $r = (00)$ corresponds to $\mathfrak{sl}(2)$ spin chains, which is explored in [23, 88].

³⁰In [117], these classes are denoted as $\kappa_{\mathbf{I}, \pi}^{(r)}$, where $\mathbf{I} = \{I_1, \dots, I_N\} \subseteq \{1, \dots, L\}$. \mathbf{I} is determined by the injective map $p: \{1, \dots, N\} \rightarrow \{1, \dots, L\}$. We prefer to use the map p to have a uniform notation throughout the paper.

where³¹

$$U_p^{(10)}(\mathbf{s}, \mathbf{x}, \hbar) = \prod_{a=1}^N \left\{ \prod_{i=1}^{p(a)-1} (s_a - x_i + \hbar) \prod_{i=p(a)+1}^L (-s_a + x_i) \right\} \cdot \left(\prod_{a>b} \frac{1}{(s_a - s_b)} \right). \quad (7.8)$$

More generally, one can define the superweight functions for any permutation $\pi \in S_L$ through

$$W_{p,\pi}^{(10)}(\mathbf{s}, \mathbf{x}; \hbar) := W_{\pi^{-1}(p)}^{(10)}(\mathbf{s}, \pi(\mathbf{x}), \hbar), \quad (7.9)$$

where $\pi(\mathbf{x}) = \{x_{\pi(1)}, \dots, x_{\pi(L)}\}$. Consider the set of all subsets of $\{1, \dots, L\}$ consisting of N elements. Each such subset is determined by an injective map $p: \{1, \dots, N\} \rightarrow \{1, \dots, L\}$. We denote the set of all such injective maps as \mathbf{p}_N . For any $p, p' \in \mathbf{p}_N$, we denote the polynomial function³² obtained by substituting $s_a = x_{p'(a)}$ ³³ by $W_{p,\pi}^{(10)}(\mathbf{x}_J, \mathbf{x}, \hbar)$. Then, the tuple $(W_{p,\pi}^{(10)}(\mathbf{x}_{p'}, \mathbf{x}, \hbar))_{p' \in \mathbf{p}_N}$, is a class in the equivariant cohomology $H_T(\text{Gr}(N, L))$, which we denote as $[W_{p,\pi}^{(10)}]$ [117, Proposition 7.1], and satisfies the axioms for the class $\kappa_{p,\pi}^{(r)}$ [117, Theorem 7.3]. Hence by [117, Definition 4.1], $\kappa_{p,\pi}^{(10)} = \text{Stab}_{\pi}^{(10)}(1_p) = [W_{p,\pi}^{(10)}]$. Therefore, the stable envelope maps the identity class to the class $[W_{p,\pi}^{(10)}]$, and as a result is given by the superweight functions $W_{p,\pi}^{(10)}(\mathbf{s}, \mathbf{m}, \hbar)$. Furthermore, the geometric R -matrix for $r = (10)$ is given by [117, Section 8.2]

$$\mathbf{R}^{\text{RR}}(u) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{u}{\hbar-u} & \frac{\hbar}{\hbar-u} & 0 \\ 0 & \frac{\hbar}{\hbar-u} & \frac{u}{\hbar-u} & 0 \\ 0 & 0 & 0 & \frac{\hbar+u}{\hbar-u} \end{pmatrix}. \quad (7.10)$$

Recovering the result of Rimányi and Rozansky. We can now compare our results to that of [117]. First consider a permutation π such that $\{x_1, \dots, x_{p(a)}, \dots, x_L\} \rightarrow \{x_L, \dots, x_{p(a)}, \dots, x_1\}$. Then,

$$W_{p,\pi}^{(10)}(\mathbf{s}, -\mathbf{x}, \hbar) = (-1)^{\sum_a \#(i < p(a))} \times \text{Sym}_{S_N} \left[\prod_{a=1}^N \left(\prod_{i < p(a)} (s_a + x_i) \prod_{i > p(a)} (s_a + x_i + \hbar) \right) \cdot \left(\prod_{a > b} \frac{1}{(s_a - s_b)} \right) \right].$$

Comparing this expression to (7.5), we find that $\text{Stab}_{\mathfrak{e}_1}(p) = W_{p,\pi}^{(10)}(\mathbf{s}, -\mathbf{x}, \hbar)$. Furthermore, by comparing (7.10) to (7.6), we find that $\mathbf{R}_{\mathfrak{e}_2 \leftarrow \mathfrak{e}_1}(u) = \mathbf{R}^{\text{RR}}(u)$. This verifies our construction of cohomological stable envelopes for $\mathfrak{sl}(1|1)$.

A Equivariant elliptic cohomology

In this appendix, we give a brief overview of equivariant elliptic cohomology, we only list the essential ingredients that are used in this paper. We basically follow the presentation in [109, Section 1.4, Appendices A and B]. The general theory on equivariant elliptic cohomology can be found in [58, 59, 60, 62, 87, 126],

³¹The explicit form of $U_l^{(r)}(\mathbf{s}; \mathbf{x}; \hbar)$ for other values of r can be found in [117, Section 5].

³²The fact that $W_{p,\pi}(\mathbf{x}_{p'}; \mathbf{x}; \hbar)$ is a polynomial function is proven in [117, Proposition 6.1].

³³This is the additive analog of the screening condition $s_a \rightarrow s_a^{(p)}$ that we used to construct matrix elements of elliptic stable envelopes.

Let q be a nonzero complex number such that $|q| < 1$, then take the elliptic curve $\mathbb{E} = \mathbb{C}^\times/q\mathbb{Z}$. For simplicity, we choose the elliptic curve \mathbb{E} to be general enough such that it has no complex multiplication, i.e., $\text{End}(\mathbb{E}) = \mathbb{Z}$.

Fix a compact Lie group G and denote its complexification by $G_{\mathbb{C}}$, then the G -equivariant elliptic cohomology is a functor

$$\{\text{pairs of } G\text{-spaces}\} \rightarrow \{\text{graded super schemes over } \mathcal{E}_G\} \quad (X, \partial X) \mapsto \text{Ell}_G(X, \partial X),$$

where \mathcal{E}_G is the moduli scheme of semistable principal $G_{\mathbb{C}}$ -bundles of trivial topological type on the dual elliptic curve \mathbb{E}^\vee . For a torus T , $\mathcal{E}_T = \mathbb{E} \otimes_{\mathbb{Z}} \text{Cochar}(T)$, and for a compact Lie group G with maximal torus T , it is known that [56] $\mathcal{E}_G \cong \mathcal{E}_T/W$, where W is the Weyl group of G which acts on $\text{Cochar}(T)$ naturally. When G is simple and simply-connected, it is known that [86] \mathcal{E}_G is isomorphic to the weighted projective space $\mathbb{P}(1, g_1, \dots, g_r)$, where g_i are coefficients in the decomposition $\theta^\vee = \sum_i g_i \alpha_i^\vee$, of the dual of highest root into simple coroots. The map Ell_G is functorial with respect to the change of groups, i.e., for a group homomorphism $G \rightarrow G'$, there is a natural transformation $\text{Ell}_G \rightarrow \text{Ell}_{G'}$. In the following, we focus on the case when $\partial X = \emptyset$ and X is a complex algebraic variety.

The graded super scheme means the structure sheaf $\mathcal{O}_{\text{Ell}_G(X)}^\bullet = \bigoplus_{d \in \mathbb{Z}} \mathcal{O}_{\text{Ell}_G(X)}^d$ is graded and graded-commutative. It is known that each homogeneous piece $\mathcal{O}_{\text{Ell}_G(X)}^d$ is a coherent sheaf on \mathcal{E}_G . Moreover, Ell_G is two-periodic $\mathcal{O}_{\text{Ell}_G(X)}^d = \mathcal{O}_{\text{Ell}_G(X)}^{d+2} \otimes \omega$, $\omega = T_0^* E$. Since the stable envelope is in degree zero, we will focus on $\mathcal{O}_{\text{Ell}_G(X)}^0$, which is the structure sheaf of a scheme, and it is finite over \mathcal{E}_G [60]. In the later discussions, we use the same notation $\text{Ell}_G(X)$ to denote the degree zero piece.

A.1 Chern class

If H is another compact Lie group, and $P \rightarrow X$ is a G -equivariant principal H -bundle, then P induces a map $c: \text{Ell}_G(X) \rightarrow \text{Ell}_H(\text{pt}) = \mathcal{E}_H$, called the *Chern class* map. For a vector bundle V of rank r , which is equivalent to a principal GL_r -bundle, the *Thom line bundle* associated to V is defined as

$$\Theta(V) := c^* \mathcal{O}(D_\Theta), \quad D_\Theta = \{0\} + S^{r-1}\mathbb{E} \subset S^r\mathbb{E} = \mathcal{E}_{\text{GL}_r}.$$

$\Theta(V)$ inherits a canonical section $\vartheta(V)$ from the effective divisor D_Θ . $\Theta: V \rightarrow \Theta(V)$ descends to a group homomorphism $K_G(X) \rightarrow \text{Pic}(\text{Ell}_G(X))$, this follows from

$$\Theta(V) = \Theta(V_1) \otimes \Theta(V_2) \quad \text{for short exact sequence } 0 \rightarrow V_1 \rightarrow V \rightarrow V_2 \rightarrow 0.$$

Note that the canonical section simply multiplies: $\vartheta(V) = \vartheta(V_1)\vartheta(V_2)$. We also have $\Theta(V^\vee) \cong \Theta(V)$, such that the canonical section picks up a sign $\vartheta(V^\vee) = (-1)^{\text{rk } V} \vartheta(V)$.

For vector bundles V_1, V_2 of ranks r_1, r_2 , $\Theta(V_1 \otimes V_2)$ is the pullback of $\mathcal{O}(D_\Theta)$ on $S^{r_1 r_2} \mathbb{E}$ via the composition of maps

$$\text{Ell}_G(X) \xrightarrow{c_1 \times c_2} S^{r_1} \mathbb{E} \times S^{r_2} \mathbb{E} \xrightarrow{m} S^{r_1 r_2} \mathbb{E},$$

where m is induced from the multiplication of elliptic curve

$$m: (\{x_1, \dots, x_{r_1}\}, \{y_1, \dots, y_{r_2}\}) \mapsto \{x_i + y_j\}.$$

In particular, for $G = 1$ and considering the tensor product of line bundles, the above gives rise to the formal group law of non-equivariant elliptic cohomology.

A.2 Gysin map

For a proper G -equivariant map $f: X \rightarrow Y$, assume that f factors as a regular embedding $i: X \hookrightarrow Z$ and a smooth projection $p: Z \rightarrow Y$, and that both i and p are G -equivariant, then there exists a distinguished element

$$f_{\otimes} \in \mathrm{Hom}_{\mathcal{O}_{\mathrm{Ell}_G(Y)}}(f_*\Theta(T_f), \mathcal{O}_{\mathrm{Ell}_G(Y)}),$$

where $f_*: \mathrm{Ell}_G(X) \rightarrow \mathrm{Ell}_G(Y)$ is the induced map between elliptic cohomologies, and T_f is the relative tangent bundle. We call f_{\otimes} the Gysin map.

If T_f equals to f^*V in $K_G(X)$ for some $V \in K_G(Y)$, then we denote by $[X]$ the section $\Gamma(\mathrm{Ell}_G(Y), \Theta(-V))$ induced by the Gysin map $f_{\otimes}: f_*\mathcal{O}_{\mathrm{Ell}_G(X)} \rightarrow \Theta(-V)$ precomposed with the canonical map $\mathcal{O}_{\mathrm{Ell}_G(Y)} \rightarrow f_*\mathcal{O}_{\mathrm{Ell}_G(X)}$.

For example, let $N \rightarrow X$ be a G -equivariant vector bundle and let $i: X \hookrightarrow N$ be the zero section, then $f_*: \mathrm{Ell}_G(X) \rightarrow \mathrm{Ell}_G(N)$ is an isomorphism of schemes due to the homotopy invariance of elliptic cohomology, and $i_{\otimes} \in \Gamma(\mathrm{Ell}_G(X), \Theta(N))$ is the section $\vartheta(N)$.

A.3 Supports

For a section α of a coherent sheaf \mathcal{F} on $\mathrm{Ell}_G(X)$, and a G -invariant open subset $j: U \hookrightarrow X$, we say that α is supported on $X \setminus U$ if $j^*(\alpha) = 0$ in $\mathrm{Ell}_G(U)$. Define the support $\mathrm{supp}(\alpha)$ to be the intersection of G -invariant closed subset that α is supported on.

The Gysin map can be defined for compactly supported sections. Namely for a G -equivariant map $f: X \rightarrow Y$ such that it factors as a regular embedding followed by a smooth projection, then there exists a distinguished element

$$f_{\otimes} \in \mathrm{Hom}_{\mathcal{O}_{\mathrm{Ell}_G(Y)}}(f_*\Theta(T_f)_c, \mathcal{O}_{\mathrm{Ell}_G(Y)}),$$

where $\Theta(T_f)_c \subset \Theta(T_f)$ is the subsheaf of sections α such that $f|_{\mathrm{supp}(\alpha)}$ is proper.

A.4 Correspondences

Consider the diagram

$$\begin{array}{ccc}
 & X_2 \times X_1 & \\
 p_2 \swarrow & & \searrow p_1 \\
 X_2 & & X_1 \\
 q_2 \searrow & & \swarrow q_1 \\
 & \mathrm{pt} &
 \end{array}$$

Assume that X_1 is smooth, then for a pair of line bundles $\mathcal{L}_i \in \mathrm{Pic}(\mathrm{Ell}_G(X_i))$, and for any section $\alpha \in \Gamma(\mathrm{Ell}_G(X_2 \times X_1), \mathcal{L}_2 \boxtimes (\mathcal{L}_1^\vee \otimes \Theta(T_{X_1})))$, such that $\mathrm{supp}(\alpha)$ is proper over X_2 , α induces a map

$$q_{1*}\mathcal{L}_1 \xrightarrow{p_{2\otimes}(\alpha p_1^*(\cdot))} q_{2*}\mathcal{L}_2,$$

in $\mathrm{Coh}(\mathcal{E}_G)$.

A.5 Degree of a line bundle

For a line bundle \mathcal{L} on an abelian variety \mathcal{A} , we define its degree $\deg \mathcal{L}$ to be its image in the Néron–Severi group $\text{NS}(\mathcal{A}) = \text{Pic}(\mathcal{A})/\text{Pic}^0(\mathcal{A})$. It is known that $\text{NS}(\mathcal{A})$ is isomorphic to the subgroup of homomorphisms $f \in \text{Hom}(\mathcal{A}, \mathcal{A}^\vee)$ such that $f = f^\vee$, and the isomorphism is given by³⁴ $\mathcal{L} \mapsto (\phi_{\mathcal{L}}: x \mapsto x^* \mathcal{L} \otimes \mathcal{L}^{-1})$. Let T be a torus, then $\text{Hom}(\mathcal{E}_T, \mathcal{E}_T^\vee)$ is isomorphic to $\text{Char}(T)^{\otimes 2} \otimes_{\mathbb{Z}} \text{End}(\mathbb{E})$. As we have assumed in the beginning, \mathbb{E} is chosen such that $\text{End}(\mathbb{E}) = \mathbb{Z}$, so the Néron–Severi group $\text{NS}(\mathcal{E}_T)$ is isomorphic to $S^2 \text{Char}(T)$. Explicitly, any $\mu \in \text{Char}(T)$ gives rise to a map $\phi_\mu: \mathcal{E}_T \rightarrow \mathbb{E}$, then $\deg \phi_\mu^* \mathcal{O}(D_\Theta) = \mu \otimes \mu \in S^2 \text{Char}(T)$. So for $V = \sum_\mu V_\mu \cdot \mu \in K_T(\text{pt})$, we have $\deg \Theta(V) = \sum_\mu (\dim V_\mu) \mu \otimes \mu$.

B Notations and conventions for supersymmetry

In this appendix, we collect the notations and conventions regarding supersymmetry used in the computation of elliptic stable envelopes in Section 6.

B.1 Conventions for spinors

For a two component $\text{SU}(2)$ spinor ϵ , we write its components as ϵ^α where $\alpha = 1, 2$. We use α, β, \dots as spinor indices. Our spinors will be anti-commuting. Let $\varepsilon^{\alpha\beta}$ and $\varepsilon_{\alpha\beta}$ be the totally antisymmetric tensors satisfying $\varepsilon^{12} = \varepsilon_{21} = 1$. These tensors are used to raise and lower spinor indices $\epsilon_\alpha = \varepsilon_{\alpha\beta} \epsilon^\beta$, $\epsilon^\alpha = \varepsilon^{\alpha\beta} \epsilon_\beta$. Let γ^μ for $\mu = 1, 2, 3$ be the Pauli matrices

$$[(\gamma^1)_\alpha^\beta] = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \quad [(\gamma^2)_\alpha^\beta] = \begin{pmatrix} & -i \\ i & \end{pmatrix}, \quad [(\gamma^3)_\alpha^\beta] = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}.$$

We also define the charge conjugation matrix

$$C = [C^{\alpha\beta}] := -i\gamma^2 = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}.$$

In other words, $C^{\alpha\beta} = -\varepsilon^{\alpha\beta}$. Some properties of the charge conjugation matrix are

$$C^2 = -1, \quad C^T = C^{-1} = -C, \quad C\gamma^\mu = -(\gamma^\mu)^T C. \quad (\text{B.1})$$

Contraction of spinors ϵ and λ are defined as

$$\epsilon\lambda := \epsilon^\alpha \lambda_\alpha, \quad (\text{B.2})$$

i.e., we use the NE–SW convention for contracting indices on anti-commuting spinors. We can think of two-component spinors as column vectors, with lowered indices, e.g., $\epsilon = \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix}$. Then we can write the spinor bilinear (B.2) in matrix notation as³⁵

$$\epsilon\lambda = \epsilon^T C \lambda. \quad (\text{B.3})$$

Using the properties (B.1) of the charge conjugation matrix, we can derive some symmetry properties of various spinor bilinears:

$$\epsilon \gamma^{\mu_1} \dots \gamma^{\mu_n} \lambda = (-1)^{\frac{n(n+1)}{2}} \lambda \gamma^{\mu_1} \dots \gamma^{\mu_n} \epsilon.$$

³⁴This fact can be derived from Theorem 2 together with the remark after that in [92, Section 20].

³⁵There should not be any confusion between when the matrix notation is being used in spinor bilinears, since in the matrix notation there will always be the transpose of a spinor to the left (as in the right-hand side of (B.3)), otherwise there will not be any transpose (as in the left-hand side of (B.3)).

B.2 3d $\mathcal{N} = 2$ supersymmetry

The supersymmetry algebra has four supercharges which we can denote as $\mathcal{Q}_\alpha, \overline{\mathcal{Q}}_\alpha$ with α being a spinor index. A generic supercharge is a linear combination of these

$$\mathcal{Q}_{\epsilon, \bar{\epsilon}} := \epsilon^\alpha \mathcal{Q}_\alpha + \bar{\epsilon}^\alpha \overline{\mathcal{Q}}_\alpha, \quad (\text{B.4})$$

where the coefficients $\epsilon, \bar{\epsilon}$ are two components spinors. The variation of a field in the theory caused by the supercharge $\mathcal{Q}_{\epsilon, \bar{\epsilon}}$ will be denoted $\delta_{\epsilon, \bar{\epsilon}}$.

For a Lie group G , the corresponding 3d $\mathcal{N} = 2$ vector multiplet consists of a vector field A , two scalars σ and D and two spinors λ and $\bar{\lambda}$. All these fields are valued in the Lie algebra \mathfrak{g} of G . We adopt the convention that elements of real Lie algebras are hermitian. Their supersymmetry variations are

$$\begin{aligned} \delta_{\epsilon, \bar{\epsilon}} A_\mu &= \frac{i}{2} (\bar{\epsilon} \gamma_\mu \lambda - \bar{\lambda} \gamma_\mu \epsilon), & \delta_{\epsilon, \bar{\epsilon}} \sigma &= \frac{1}{2} (\bar{\epsilon} \lambda - \bar{\lambda} \epsilon), \\ \delta_{\epsilon, \bar{\epsilon}} \lambda &= -\frac{1}{2} \gamma^{\mu\nu} F_{\mu\nu} \epsilon - D \epsilon + i \gamma^\mu D_\mu \sigma \epsilon, & \delta_{\epsilon, \bar{\epsilon}} \bar{\lambda} &= -\frac{1}{2} \gamma^{\mu\nu} F_{\mu\nu} \bar{\epsilon} + D \bar{\epsilon} - i \gamma^\mu D_\mu \sigma \bar{\epsilon}, \\ \delta_{\epsilon, \bar{\epsilon}} D &= -\frac{i}{2} \bar{\epsilon} \gamma^\mu D_\mu \lambda - \frac{i}{2} D_\mu \bar{\lambda} \gamma^\mu \epsilon + \frac{i}{2} [\bar{\epsilon} \lambda, \sigma] + \frac{i}{2} [\bar{\lambda} \epsilon, \sigma]. \end{aligned} \quad (\text{B.5})$$

Given a representation R of G we can add an R -valued chiral multiplet containing two scalars ϕ and F , and a spinor ψ . We also have the anti-chiral multiplet containing scalars $\bar{\phi}, \bar{F}$, and spinor $\bar{\psi}$ valued in the dual representation R^\vee . These fields transform under supersymmetry as

$$\begin{aligned} \delta_{\epsilon, \bar{\epsilon}} \phi &= \bar{\epsilon} \psi, & \delta_{\epsilon, \bar{\epsilon}} \bar{\phi} &= \epsilon \bar{\psi}, & \delta_{\epsilon, \bar{\epsilon}} \psi &= i \gamma^\mu \epsilon D_\mu \phi + i \epsilon \sigma \cdot \phi + \bar{\epsilon} F, \\ \delta_{\epsilon, \bar{\epsilon}} \bar{\psi} &= i \gamma^\mu \bar{\epsilon} D_\mu \bar{\phi} - i \bar{\epsilon} \sigma \cdot \bar{\phi} + \epsilon \bar{F}, & \delta_{\epsilon, \bar{\epsilon}} F &= \epsilon (i \gamma^\mu D_\mu \psi - i \sigma \cdot \psi - i \lambda \cdot \phi), \\ \delta_{\epsilon, \bar{\epsilon}} \bar{F} &= \bar{\epsilon} (i \gamma^\mu D_\mu \bar{\psi} + i \sigma \cdot \bar{\psi} - i \bar{\lambda} \cdot \bar{\phi}). \end{aligned} \quad (\text{B.6})$$

Here $\sigma \cdot$ is the action of σ on whatever follows in the corresponding representation.

By computing the commutator $[\delta_{\epsilon, \bar{\epsilon}}, \delta_{\eta, \bar{\eta}}]$ on fields, we find the (anti)commutation relations between the supercharges

$$\{\mathcal{Q}_\alpha, \overline{\mathcal{Q}}_\beta\} = -i \gamma_{\alpha\beta}^\mu D_\mu - i \varepsilon_{\alpha\beta} \sigma \cdot. \quad (\text{B.7})$$

In a gauge theory, we always have a dynamical vector multiplet for the gauge group. So the last term above is an infinitesimal gauge transformation by the adjoint scalar σ of this vector multiplet. But we can also introduce background vector multiplets for flavor symmetry groups. If G is the gauge group and F the flavor group with Lie algebras \mathfrak{g} and \mathfrak{f} , respectively, then we can treat the vector multiplet from (B.5) as $(\mathfrak{g} \oplus \mathfrak{f})$ -valued. Thus all the fields of the multiplet become sums of \mathfrak{g} and \mathfrak{f} -valued fields. In particular, we write $\sigma = \sigma_G + \sigma_F$ according to this decomposition and the supersymmetry algebra (B.7) contains a gauge and a flavor symmetry transformation by the dynamical σ_G and the background σ_F respectively. We only turn on background values for scalar fields, namely σ_F and D_F . Furthermore, these values are constrained by the requirement to preserve some amount of supersymmetry, i.e., to solve the BPS equations. More on BPS equations in Section 5.1.2.

B.3 Localizing supercharge

Suppose the elliptic curve \mathbb{E}_τ in our 3d space-time $I \times \mathbb{E}_\tau$ has a holomorphic volume form dz and two 1-cycles, S_A^1 and S_B^1 defined by: $\oint_{S_A^1} dz = 1$, $\oint_{S_B^1} dz = \tau$. Let y, θ_A and θ_B be the real coordinates on I, S_A^1 and S_B^1 , respectively. We take the orientation defining volume form on M to be $dy \wedge d\theta_A \wedge d\theta_B$. This induces the volume forms $dy \wedge d\theta_A$ on $\Sigma := \mathbb{R} \times S_A^1$ and $d\theta_A \wedge d\theta_B$

on \mathbb{E}_τ . We use holomorphic coordinates w and z on Σ and \mathbb{E}_τ respectively such that the volume forms in real and complex coordinates are related by $dy \wedge d\theta_A = \frac{i}{2} dw \wedge d\bar{w}$, $d\theta_A \wedge d\theta_B = \frac{i}{2} dz \wedge d\bar{z}$. The real and complex coordinates are related by $w = y + i\theta_A$, $z = \theta_A + i\theta_B$. We denote the derivatives with respect to y , θ_A and θ_B by ∂_y , ∂_A and ∂_B , respectively. They are related to the derivatives with respect to the complex coordinates as $\partial_w = \frac{1}{2}(\partial_y - i\partial_A)$, $\partial_z = \frac{1}{2}(\partial_A - i\partial_B)$. Similarly for the covariant derivatives D_w , D_z , D_y , D_A and D_B .

We can relabel our 3d $\mathcal{N} = 2$ supercharges \mathcal{Q}_α , $\overline{\mathcal{Q}}_\alpha$ to adjust to the complex structures of either Σ or \mathbb{E}_τ . If we define

$$\begin{aligned}\mathcal{Q}_+ &:= \frac{1}{\sqrt{2}}(\mathcal{Q}_1 + \mathcal{Q}_2), & \mathcal{Q}_- &:= \frac{1}{\sqrt{2}}(\mathcal{Q}_1 - \mathcal{Q}_2), & \overline{\mathcal{Q}}_+ &:= \frac{1}{\sqrt{2}}(\overline{\mathcal{Q}}_1 + \overline{\mathcal{Q}}_2), \\ \overline{\mathcal{Q}}_- &:= \frac{1}{\sqrt{2}}(\overline{\mathcal{Q}}_1 - \overline{\mathcal{Q}}_2),\end{aligned}$$

then the subscripts \pm refer to the chirality on \mathbb{E}_τ and the algebra (B.7) reflects the complex structure of \mathbb{E}_τ

$$\begin{aligned}\{\mathcal{Q}_+, \overline{\mathcal{Q}}_+\} &= 2D_z, & \{\mathcal{Q}_-, \overline{\mathcal{Q}}_-\} &= 2D_{\bar{z}}, & \{\mathcal{Q}_+, \overline{\mathcal{Q}}_-\} &= i(D_y - \sigma), \\ \{\mathcal{Q}_-, \overline{\mathcal{Q}}_+\} &= i(D_y + \sigma).\end{aligned}$$

Alternatively, if we define

$$q_+ := \frac{\mathcal{Q}_1}{\sqrt{2}}, \quad q_- := -\frac{\mathcal{Q}_2}{\sqrt{2}}, \quad \bar{q}_+ := \frac{\overline{\mathcal{Q}}_1}{\sqrt{2}}, \quad \bar{q}_- := -\frac{\overline{\mathcal{Q}}_2}{\sqrt{2}},$$

then the subscripts \pm refer to the chirality on Σ and the algebra (B.7) reflects the complex structure of Σ

$$\begin{aligned}\{q_+, \bar{q}_+\} &= iD_w, & \{q_-, \bar{q}_-\} &= -iD_{\bar{w}}, & \{q_+, \bar{q}_-\} &= \frac{i}{2}(D_B - \sigma), \\ \{q_-, \bar{q}_+\} &= \frac{i}{2}(D_B + \sigma).\end{aligned}$$

The localization results used in this note are with respect to the supercharge

$$\mathcal{Q} := \mathcal{Q}_- + \overline{\mathcal{Q}}_- = q_+ + q_- + \bar{q}_+ + \bar{q}_-, \tag{B.8}$$

which satisfies

$$\mathcal{Q}^2 = 2D_{\bar{z}}. \tag{B.9}$$

Furthermore, translation in the y -direction is \mathcal{Q} -exact

$$\{\mathcal{Q}, \mathcal{Q}_+ + \overline{\mathcal{Q}}_+\} = 2iD_y. \tag{B.10}$$

From the point of view of both Σ and \mathbb{E}_τ , the 3d $\mathcal{N} = 2$ algebra appears as the 2d $\mathcal{N} = (2, 2)$ algebra. On the elliptic curve, the supercharge \mathcal{Q} belongs to the $\mathcal{N} = (0, 2)$ subalgebra. On Σ , the supercharge \mathcal{Q} does not belong to any proper subalgebra of $(2, 2)$ but rather it can be related to certain deformations of both the A-model and the B-model BRST operators, as we find below.

Dimensional reduction. By compactifying the S_B^1 direction of the 3d space-time, we land on a 2d $\mathcal{N} = (2, 2)$ theory on Σ with four supercharges Q_\pm , \overline{Q}_\pm that descend from the 3d supercharges q_\pm , \bar{q}_\pm , respectively. The 2d $\mathcal{N} = (2, 2)$ supersymmetry has the usual A-model

and B-model supercharges $Q_A := \bar{Q}_+ + Q_-$, $Q_B := \bar{Q}_+ + \bar{Q}_-$, that are nilpotent up to gauge transformations.³⁶ The remaining supersymmetry in the A and B-model can be labeled as

$$G_+^A := -iQ_+, \quad G_-^A := i\bar{Q}_-, \quad G_+^B := -iQ_+, \quad G_-^B := iQ_-.$$

For any vector field $V = V_+\partial_w + V_-\partial_{\bar{w}}$, we define the linear combination $\iota_V G := V_+G_+ + V_-G_-$ and we have the anti-commutation relations

$$\{Q_A, \iota_V G^A\} = \{Q_B, \iota_V G^B\} = V_+D_w + V_-D_{\bar{w}} = \mathcal{L}_V + \text{gauge transformation.}$$

Ω -deforming the A and B-model supercharges with respect to a vector field V means changing the BRST operator from Q_A and Q_B to $Q_A + \iota_V G^A$ and $Q_B + \iota_V G^B$ such that the new BRST operator squares to the space-time transformation generated by V

$$(Q_A + \iota_V G^A)^2 = (Q_B + \iota_V G^B)^2 = \mathcal{L}_V + \text{gauge transformation.}$$

We observe that the localizing supercharge (B.8), upon reduction to 2d, descends to an Ω -deformed supercharge both in the A and in the B-model

$$\mathcal{Q} \xrightarrow{3d \rightarrow 2d} Q_+ + Q_- + \bar{Q}_+ + \bar{Q}_- = Q_A + iG_+^A - iG_-^A = Q_B + iG_+^B - iG_-^B.$$

These Ω -deformations are defined with respect to the vector field $i\partial_w - i\partial_{\bar{w}} = \partial_A$, which rotates the S_A^1 circle. So, the 3d localizing supercharge can be thought of as a lift of the S_A^1 -rotating Ω -deformed supercharge from either the A or the B-model on Σ .

If we further compactify the S_A^1 direction and reduce to a 1d theory on I , we get an $\mathcal{N} = 4$ supersymmetric gauged quantum mechanics. The localizing supercharge \mathcal{Q} descends to a nilpotent (up to gauge transformation) supercharge. Nilpotency in 1d follows from the fact that there is no D_y on the right-hand side of (B.9), which is the only remaining translation in 1d.

C $\mathfrak{sl}(1|1)$ rational R -matrix from the A-model localization

The aim of this appendix is to compute the geometric rational R -matrix for $\mathfrak{sl}(1|1)$ superspin chain from its natural theory from the perspective of the Bethe/Gauge correspondence, i.e., a 2d gauge theory corresponding to the $\mathfrak{sl}(1|1)$ spin chains with fundamental representations, and using the A-model localization formula of Closset, Cremonesi and Park [28]. The computation here is inspired by insights and observations in [23, 97, 98], and its $\mathfrak{sl}(2)$ counterpart has been performed in [23]. Unlike the construction in the main body of this work, the computation in this appendix should be considered an interesting observation rather than being based on a systematic framework. Furthermore, in this section, we assume that the explicit form of cohomological stable envelopes is known. Of course, by knowing the cohomological stable envelopes, the computation of R -matrix is a simple matrix manipulation. However, viewing the R -matrix elements as the two-point functions of certain (normalized) A-model observables would provide an alternative route and may lead to a better understanding of the Bethe/Gauge correspondence.

C.1 Construction of geometric rational R -matrix

In Section 7, we have successfully constructed cohomological stable envelopes and the geometric R -matrix associated with $\mathfrak{sl}(1|1)$ by starting from their 3d counterparts and performing a $3d \rightarrow 2d \rightarrow 1d$ reduction, corresponding to reducing the three-dimensional gauge theory, living on $I \times \mathbb{E}_\tau$, on the two cycles of the elliptic curve \mathbb{E}_τ . This is expected since the interval partition

³⁶After compactifying S_B^1 the component of a gauge field in this direction becomes an adjoint scalar field in the 2d theory and the derivative D_B becomes a gauge transformation generated by this field.

function of the 3d theory reduces to the interval partition function of the corresponding 1d quantum mechanical system after discarding all instanton corrections coming from particles wrapping S_A^1 and S_B^1 . Before presenting the setup and the computation of the R -matrix, let us make a series of remarks.

Remark C.1 (2d Gauge theory). The first remark is the theory we use for localization computation. This is the gauge theory that Bethe/Gauge correspondence naturally associates to an $\mathfrak{sl}(1|1)$ spin chain in a fundamental representation. This gauge theory is constructed in Example 4.1. For the N -magnon sector of a spin chain with L sites, the 2d gauge theory is a $U(N)$ gauge theory with $\mathcal{N} = (2, 2)$ supersymmetry coupled to a pair of $SU(L) \times U(1)_\hbar$ fundamental/anti-fundamental chiral multiplets.

Remark C.2 (relation to the dimensional reduction of 3d Gauge theory of Section 4.2). A natural question is the relation between the gauge theory we use for the localization computation of this section and the 3d \rightarrow 2d dimensional reduction of the 3d theory of Section 4.2 used for the computation of elliptic stable envelopes. The dimensional reduction of the 3d theory of Section 4.2 is again a 2d gauge theory with an $\mathcal{N} = (2, 2)$ supersymmetry. However, this is a *different* 2d theory, although with the same gauge and matter content. To see the relation, it is expected [40] that we should start from a 4d theory³⁷ with $\mathcal{N} = 1$ supersymmetry on $I \times S_A^1 \times S_B^1 \times S_C^1$,³⁸ where S_C^1 is an extra circle. The dimensional reduction of the 4d theory on S_C^1 would be the 3d theory of Section 4.2.³⁹ On the other hand, the dimensional reduction of the 4d theory on $S_A^1 \times S_B^1$ gives a 2d theory on $I \times S_C^1$, which is the gauge theory that the 2d gauge theory we use for the localization of this section. This is precisely the gauge theory that the Bethe/Gauge correspondence assigns to an $\mathfrak{sl}(1|1)$ spin chain with fundamental representations. Furthermore, the 2d gauge theory on $I \times S_C^1$ can be thought of as a theory living on the sphere from which two cigars have been capped off as shown in Figure 7.

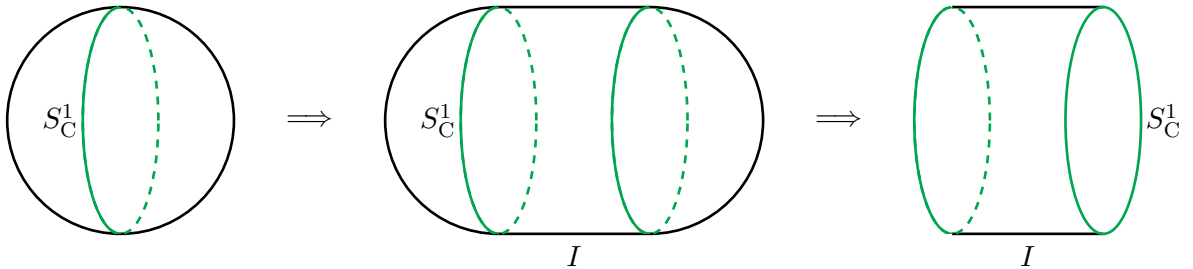


Figure 7. The cylinder $I \times S_C^1$ can be thought as starting from the sphere (left), stretching (middle), and capping off two cigars (right).

The hierarchy of theories we have considered in this work is shown in Figure 8.

Remark C.3 (identification of parameters). As we have seen in Section 6.1, the elliptic stable envelopes depend on the holonomies for the gauge \mathfrak{s} , flavor (\mathbf{x}, \hbar) and the topological symmetry z along the cycle of the elliptic curve. Furthermore, we constructed cohomological stable envelopes by reducing the elliptic ones to 1d upon which the holonomy z disappears and all

³⁷The reason we need to start in 4d is that such a theory is a natural arena for the quantum version of equivariant elliptic cohomology. The BRST cohomology of a 4d analog of the A-model supercharge is expected to be equivalent to quantum equivariant elliptic cohomology of the Higgs branch of the 4d theory.

³⁸Alternatively, we can think of the 4d theory on $S^2 \times S_A^1 \times S_B^1$. We can then think of S^2 as $I \times S_C^1$ to which two cigars are capped. Reduction of the theory on $S_A^1 \times S_B^1$ gives rise to a theory on S^2 .

³⁹The reason that this 3d theory is enough for the purpose of the computation of the R -matrix is that the latter is insensitive to the deformed non-perturbative corrections coming from the extra circle S_C^1 or equivalently, the deformed ring structure of the quantum equivariant elliptic cohomology.

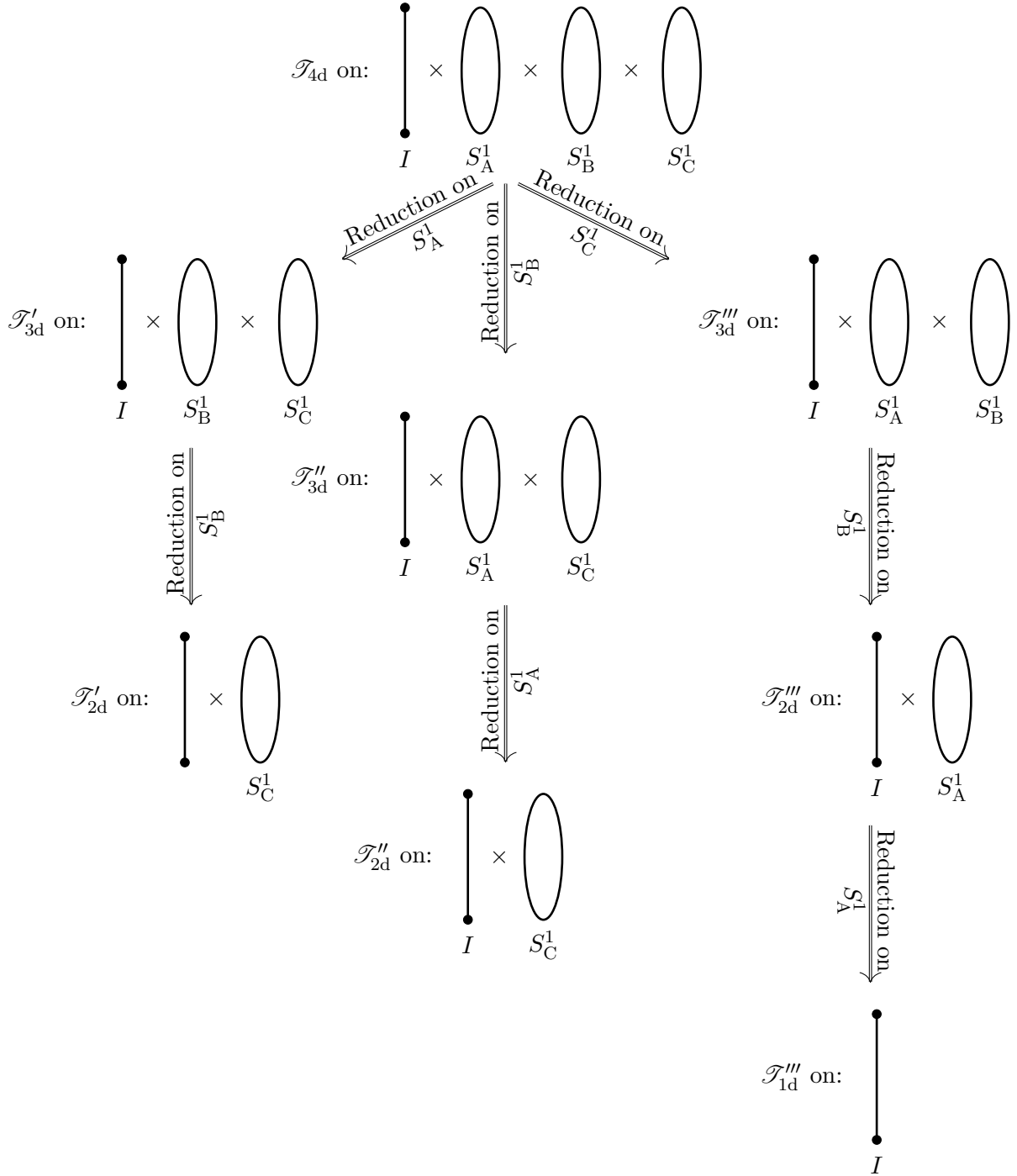


Figure 8. Since the order of dimensional reduction should not matter, one can identify \mathcal{T}'_{2d} and \mathcal{T}''_{2d} and it is the theory we use in this appendix. \mathcal{T}'''_{3d} is the theory we used for the computation of elliptic stable envelopes in Section 6.1.

other holonomies become complex twisted masses of the 2d theory on $I \times S_C^1$. We thus need to compare the quantities in the theory \mathcal{T}'''_{1d} (the theory used in deriving (7.5)) and \mathcal{T}'_{2d} (the theory we use in this appendix) to be able to write the formulas in terms of parameters of the latter (see Figure 8). First note that in the theory \mathcal{T}'''_{1d} ,

$$\mathbf{s} := \oint_A A_{4d}^b + i \oint_B A_{4d}^b,$$

where $A_{4d}^{\mathfrak{h}}$ is the four-dimensional gauge field for the Cartan of the gauge Lie algebra $\mathfrak{h} = \mathbb{R}^{\oplus N}$. Hence, they will be identified with $\boldsymbol{\sigma}^{\mathbb{C}} := \text{diag}(\sigma_1^{\mathbb{C}}, \dots, \sigma_N^{\mathbb{C}})$, the complex scalar in the $\mathfrak{u}(N)$ vector multiplet, in the theory \mathcal{T}'_{2d} . Furthermore, in the theory \mathcal{T}'''_{1d} ,

$$\boldsymbol{x} := \oint_A A_{4d}^{\mathfrak{a}} + \mathfrak{i} \oint_B A_{4d}^{\mathfrak{a}}, \quad \boldsymbol{h} := \oint_A A_{4d}^{\mathfrak{h}} + \mathfrak{i} \oint_B A_{4d}^{\mathfrak{h}},$$

where $A_{4d}^{\mathfrak{a}}$ is the four-dimensional gauge field for the Cartan of the flavor symmetry $\mathfrak{a} = \mathfrak{u}(1)^{\oplus L-1}$,⁴⁰ and $A_{4d}^{\mathfrak{h}}$ is the four-dimensional gauge field for the $\mathbb{R}_{\mathfrak{h}}$ flavor symmetry. Therefore, they are identified with $\boldsymbol{m}^{\mathbb{C}} := (m_1^{\mathbb{C}}, \dots, m_L^{\mathbb{C}})$ and $\boldsymbol{h}^{\mathbb{C}}$, the complex twisted masses, in the theory \mathcal{T}'_{2d} . Hence, we identify the parameters as follows:

$$s_a \longleftrightarrow \sigma_a^{\mathbb{C}}, \quad a = 1, \dots, N, \quad x_i \longleftrightarrow m_i^{\mathbb{C}}, \quad i = 1, \dots, L, \quad \boldsymbol{h} \longleftrightarrow \boldsymbol{h}^{\mathbb{C}}. \quad (\text{C.1})$$

Let us emphasize that say $m_i^{\mathbb{C}}$ has nothing to do with the (expectation value of the) quantity defined in (4.9), which is the complex scalar in the vector multiplet of the theory \mathcal{T}'''_{2d} (see the right portion of Figure 8), while $(\sigma_a^{\mathbb{C}}, m_i^{\mathbb{C}}, \boldsymbol{h}^{\mathbb{C}})$ in (C.1) are parameters of the theory \mathcal{T}'_{2d} (or equivalently \mathcal{T}''_{2d}). On the other hand, there are real masses in the theory \mathcal{T}'''_{1d} whose order determines a chamber \mathfrak{C} in the theory \mathcal{T}'''_{1d} ,

$$m_i = \oint_{\mathfrak{C}} A_{4d}^{\mathfrak{a}}, \quad i = 1, \dots, L. \quad (\text{C.2})$$

From the perspective of \mathcal{T}'_{2d} , these are holonomies of the 2d gauge field⁴¹ along $S^1_{\mathfrak{C}}$. This completes the identification of parameters. Furthermore, for convenience in the computation, we consider a shift in the parameters⁴²

$$s_a \rightarrow s_a, \quad a = 1, \dots, N, \quad x_i \rightarrow x_i + \frac{\boldsymbol{h}}{2}, \quad i = 1, \dots, L. \quad (\text{C.3})$$

Remark C.4 (using the A-model localization). Note that the stable envelope we have constructed is written in the stable basis, which is a basis for the equivariant quantum cohomology of the Higgs branch of the gauge theory. The quantum equivariant cohomology consists of the vector space of the underlying classical equivariant cohomology together with a deformed ring structure [140, 148]. On the other hand, for an A-twisted theory, the equivariant quantum cohomology of the target space is the cohomology of the ordinary A-model supercharge, which by definition is the twisted chiral ring of the gauge theory [65, 140, 148]. Therefore, the full Bethe–Gauge correspondence would have to invoke the cohomology of A-model supercharge, as has been argued in [40, 102, 103]. Since the stable envelopes provide a map between quantum equivariant cohomology⁴³ of the fixed-point loci⁴⁴ and the target, one would like to construct a basis for the twisted chiral ring which coincides with the stable basis for the equivariant quantum cohomology. Therefore, cohomological stable envelopes (7.5) are the corresponding A-model observables. As it is known, these generators are functions of the complex scalar in the vector multiplet of the 2d theory [28].

⁴⁰Note that $x_1 + \dots + x_L = 0$.

⁴¹Note that the gauge field in the theory \mathcal{T}'_{2d} comes from the components of the gauge field in 4d along $I \times S^1_{\mathfrak{C}}$.

⁴²These shifts are not essential and as we mentioned are done just for the convenience in the computations and coping with existing literature.

⁴³Stable envelope map preserves the ring structure of the quantum equivariant cohomology in some weak sense [88, Theorem 7.2.1].

⁴⁴This is the fixed-point loci of the flavor-symmetry action on the corresponding Higgs branch.

Remark C.5 (identifying the A-model observables). The A-model observables $\mathcal{O}(\sigma^{\mathbb{C}})$ are functions of complex scalars $\sigma^{\mathbb{C}}$ in the vector multiplet associated with the gauge invariance [28]. In the computation of the R -matrix, we use particular A-model observables obtained from the stable envelopes using the identification (C.1). Considering the identification (C.1) and the shifts (C.3), and using (7.7) and (7.8), the cohomological stable envelopes take the following form:

$$\begin{aligned} \text{Stab}_{\mathfrak{e}}(p) = & \text{Sym}_{S_N} \left[\prod_{a=1}^N \left(\prod_{i < p(a)} \left(+\sigma_a^{\mathbb{C}} - m_i^{\mathbb{C}} + \frac{\hbar^{\mathbb{C}}}{2} \right) \prod_{i > p(a)} \left(-\sigma_a^{\mathbb{C}} + m_i^{\mathbb{C}} + \frac{\hbar^{\mathbb{C}}}{2} \right) \right) \right. \\ & \left. \times \left(\prod_{a > b} \frac{1}{\sigma_a^{\mathbb{C}} - \sigma_b^{\mathbb{C}}} \right) \right]. \end{aligned} \quad (\text{C.4})$$

This is the form of the cohomological stable envelopes for $\mathfrak{sl}(1|1)$ that we use in the following. Note that these observables can be inserted anywhere on S^2 and the final result is independent of the location of insertions [28, Section 3.3].

Remark C.6 (Coulomb vs. Higgs branch localization schemes). For GLSMs, there are two different localization schemes depending on the choice of the localizing action: the Higgs-branch localization and the Coulomb-branch localization [16, 28]. These localization schemes are expected to give rise to the same final results but the explicit details and the localization procedure could be very different. The A-localization computation of this section is an example of the Coulomb-branch localization.

Remark C.7 (R -matrix from the A-model localization). From the mathematical definition, it is expected that the R -matrix is related to the two-point function of stable envelopes [23]. As we briefly recall in Section C.2, the A-model localization computations involve a summation over all instanton sectors. However, R -matrix is insensitive to these non-perturbative corrections. As such, the computation of R -matrix reduces to the computation of a two-point function of stable envelopes in the zero-flux sector. We will further elaborate on the construction below.

Remark C.8 (interpretation in terms of $\mathcal{N} = 4$ quantum mechanics). Since the computation of R -matrix is the restriction of certain two-point functions to zero-flux section, we can interpret the result in the corresponding quantum mechanics. This theory is obtained by the dimensional reduction of our 2d theory on $I \times S^1_{\mathbb{C}}$. The insertion of A-model observables on S^2 leads to certain boundary conditions in quantum mechanics, as illustrated in Figure 9. Therefore, the two-point function of A-model observables in the zero-flux sector is the interval partition function of the corresponding $\mathcal{N} = 4$ quantum mechanics in the presence of these boundary conditions.

Having summarized all the necessary ingredients, we can now compute the R -matrix. In this construction, we use the cohomological stable envelopes (C.4).

Rational R -matrix from Gauge theory. The geometric definition of R -matrix (3.12) (and its trigonometric and rational versions) depends on two pieces of data: (1) the cohomological stable envelope and its inverse, and (2) a choice of the chamber for the cohomological stable envelope and another (in general different) chamber for its inverse. Therefore, it is natural to consider the configuration of Figure 10 of the 2d gauge theories on S^2 .

$\mathcal{J}(m_{\mathfrak{e}}, 0)$ and $\mathcal{J}(0, m'_{\mathfrak{e}'})$ are certain supersymmetric Janus interfaces.⁴⁵ Colliding these interfaces gives rise to a single interface: $\mathcal{J}(m_{\mathfrak{e}}, 0)\mathcal{J}(0, m'_{\mathfrak{e}'}) = \mathcal{J}(m_{\mathfrak{e}}, m'_{\mathfrak{e}'})$. Setting $m'_{\mathfrak{e}'} = m_{\mathfrak{e}}$ and noting that $\mathcal{J}(m_{\mathfrak{e}}, m_{\mathfrak{e}})$ is equivalent to no interface, we see that $\mathcal{J}^{-1}(m_{\mathfrak{e}}, 0) = \mathcal{J}(0, m_{\mathfrak{e}})$. Therefore, we consider the configuration of Figure 11 for the computation of R -matrix.

⁴⁵Through (C.2) and going to 1d by compactification on $S^1_{\mathbb{C}}$, we can think of these interfaces as real-mass Janus interfaces in the corresponding 4d $\mathcal{N} = 4$ quantum mechanics. The presence of these types of interfaces leads to an exact deformation of the theory [40, Section 4.2]. Therefore, the precise shape of the profile of the parameters does not matter and only their asymptotic values affect the computations. This is the only fact we need.

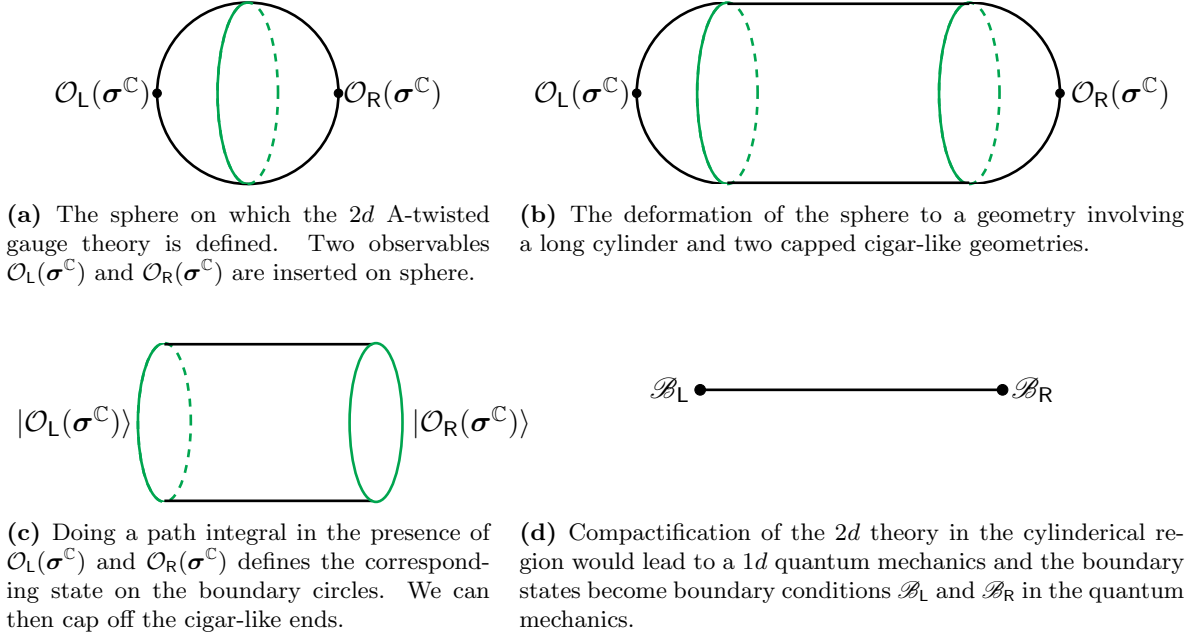


Figure 9. The $2d \rightarrow 1d$ reduction of the $2d$ $\mathcal{N} = (2, 2)$ theory to a quantum mechanical system with boundary conditions prescribed by the observables $\mathcal{O}_L(\sigma)$ and $\mathcal{O}_R(\sigma)$. The subscripts just mean left and right. Restricting to the zero-flux sector, the two-point function on the sphere becomes the partition function of the quantum mechanics with certain boundary conditions.

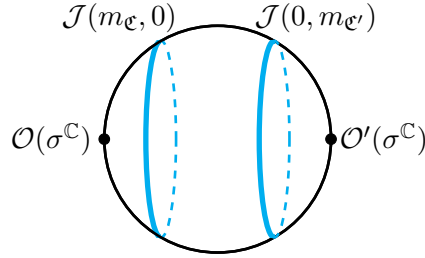


Figure 10. The configuration of two Janus interfaces for computing the rational R -matrix. $\mathcal{O}(\sigma^{\mathbb{C}})$ and $\mathcal{O}'(\sigma^{\mathbb{C}})$ are certain A-model observables.

The effect of the presence of the Janus interface $\mathcal{J}(m_{\mathfrak{C}}, m_{-\mathfrak{C}})$ is that the parameters belong to the two opposite chambers \mathfrak{C} and $-\mathfrak{C}$ on the two sides of the interface.

The next ingredient is the formula for the computation of the two-point functions of GLSM observables on sphere S^2 in the zero-flux sector. In the case of theory \mathcal{T}'_{2d} , it is given by [28] (see Appendix C.2.2 for the derivation)

$$\langle \mathcal{O}\mathcal{O}' \rangle_{S^2, 0} = \frac{1}{N!} \oint_{\text{JK}} d\sigma^{\mathbb{C}} \mathcal{Z}[\mathcal{O}, \mathcal{O}'](\sigma^{\mathbb{C}}), \quad (\text{C.5})$$

where the subscript 0 denotes the zero-flux sector, $\sigma^{\mathbb{C}} = (\sigma_1^{\mathbb{C}}, \dots, \sigma_N^{\mathbb{C}})$,

$$d\sigma^{\mathbb{C}} = d\sigma_1^{\mathbb{C}} \wedge \dots \wedge d\sigma_N^{\mathbb{C}}, \quad \mathcal{Z}[\mathcal{O}, \mathcal{O}'](\sigma^{\mathbb{C}}) = \mathcal{Z}(\sigma^{\mathbb{C}}) \mathcal{O}(\sigma^{\mathbb{C}}) \mathcal{O}'(\sigma^{\mathbb{C}}),$$

and

$$\mathcal{Z}(\sigma^{\mathbb{C}}) \equiv \prod_{a=1}^N \prod_{i=1}^L \frac{1}{(\sigma_a^{\mathbb{C}} - m_i^{\mathbb{C}} + \frac{1}{2}\hbar^{\mathbb{C}})(-\sigma_a^{\mathbb{C}} + m_i^{\mathbb{C}} + \frac{1}{2}\hbar^{\mathbb{C}})} \cdot \prod_{a \neq b}^N (\sigma_a^{\mathbb{C}} - \sigma_b^{\mathbb{C}}). \quad (\text{C.6})$$

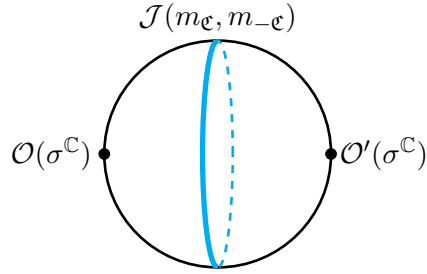


Figure 11. The configuration for the computation of R -matrix obtained by colliding the two interfaces in Figure 10.

The contribution to the integral is determined by the JK-pole prescription and comes from the contribution of chiral multiplets. The location of the poles is given by (C.26).

The A-model observables relevant for the computation of R -matrix are constructed from (C.4) as follows. First of all these observables depend on the pair (N, L) and also the injective map p . Furthermore, we can consider a permutation $\pi: \{1, \dots, L\} \rightarrow \{1, \dots, L\}$ which leads to a permutation of complex masses and the real parameters (C.2), and hence determines a choice of chamber. We denote these observables as $\mathcal{O}_{p,\pi,N,L}$. It turns out that the relevant two-point functions are [23]⁴⁶

$$\langle \mathcal{O}_{p,\pi,N,L}(\sigma^{\mathcal{C}}) \mathcal{O}_{p',\pi';N,L}^*(\sigma^{\mathcal{C}}) \rangle_{S^2,0}, \quad (\text{C.7})$$

where

$$\mathcal{O}_{p,\pi,N,L} \propto \text{Stab}_{\mathfrak{C}}(p), \quad (\text{C.8})$$

where the chamber \mathfrak{C} is determined through the permutation π , and \star operation corresponds to replacing $\sigma_a^{\mathcal{C}} \rightarrow -\sigma_a^{\mathcal{C}}$ and $m_i^{\mathcal{C}} \rightarrow -m_i^{\mathcal{C}}$. This operation corresponds to the longest permutation $\pi: \{1, \dots, L\} \rightarrow \{L, \dots, 1\}$ in (7.9). We comment below (see Remark C.9) on how to determine the proportionality constant in (C.8).

We are now in a position to compute the rational R -matrix for $\mathfrak{sl}(1|1)$. For this purpose, it suffices to restrict to $L = 2$, which through (5.12), implies $N = 0, 1, 2$, corresponding to the vacuum, one-magnon, and two-magnon sectors of the spin chain. The R -matrix of the vacuum sector is trivially one. We can compute the R -matrix in the one- and two-magnon sectors as follows.

Remark C.9 (normalization of observables). Let us make a curious observation before proceeding to details. In the following computations, we demand that the observables $\mathcal{O}_{p,\pi,N,L}$ are orthonormal, meaning $\langle \mathcal{O}_{p,\pi,N,L} \mathcal{O}_{p,\pi,N,L}^* \rangle = 1$ and $\langle \mathcal{O}_{p,\pi,N,L} \mathcal{O}_{p',\pi',N,L}^* \rangle = 0$. If we do not stress on orthonormality of these observables, then we end up with a 4×4 matrix that all the components of the middle 2×2 diagonal block⁴⁷ are multiplied by $(\hbar^{\mathcal{C}})^{-1}$ and the bottom 1×1 diagonal block⁴⁸ is multiplied by $(\hbar^{\mathcal{C}})^{-2}$. Such a matrix *does not* satisfy the Yang–Baxter equation and as such is not an R -matrix at all. In the case of $\mathfrak{sl}(2)$ spin chains considered in [23, Section 6.1], the analogous observables are already normalized and there is no need for a rescaling. The reason is a certain contribution of the adjoint chiral multiplet that takes care of the normalization. In the present case, there is no adjoint chiral multiplet and as such we need this rescaling.

⁴⁶In [23, Section 6.2], this type of two-point functions has been interpreted as an inner product in the corresponding $\mathcal{N} = 4$ quantum mechanics, where the \star operation provides a notion of “complex conjugation”.

⁴⁷For an R -matrix, this 2×2 block is the one-magnon sector R -matrix.

⁴⁸For an R -matrix, this 1×1 block is the two-magnon sector R -matrix.

R-matrix in the one-magnon sector. The one-magnon sector is described by setting $N = 1$, hence there are two possible choices of the injective map, which we call p and p' : $p(1) = 1$ or $p'(1) = 2$. There are also two possible choices of permutation, which we denote by π and π' : $\pi(\{1, 2\}) = \{1, 2\}$ and $\pi'(\{1, 2\}) = \{2, 1\}$. There are two complex twisted masses: $m^{\mathbb{C}} = (m_1^{\mathbb{C}}, m_2^{\mathbb{C}})$ with $m_1^{\mathbb{C}} + m_2^{\mathbb{C}} = 0$, and there is only one effective complex twisted mass $m^{\mathbb{C}} \equiv m_1^{\mathbb{C}} = -m_2^{\mathbb{C}}$. Similarly, there are two real parameters, defined in (C.2), with $m_1 + m_2 = 0$, and a single effective parameter $m \equiv m_1 = -m_2$. Correspondingly, there are two chambers $\mathfrak{C} = \{m > 0\}$ and $-\mathfrak{C} = \{m < 0\}$. These chambers correspond to the permutations π and π' . We then have

$$\begin{aligned}\mathcal{O}_{p,\pi;1,2}(\sigma^{\mathbb{C}}) &= (\hbar^{\mathbb{C}})^{\frac{1}{2}} \left(-\sigma^{\mathbb{C}} + m_2^{\mathbb{C}} + \frac{\hbar^{\mathbb{C}}}{2} \right), \\ \mathcal{O}_{p,\pi';1,2}(\sigma^{\mathbb{C}}) &= (\hbar^{\mathbb{C}})^{\frac{1}{2}} \left(+\sigma^{\mathbb{C}} - m_2^{\mathbb{C}} + \frac{\hbar^{\mathbb{C}}}{2} \right) = \mathcal{O}_{p,\pi;1,2}^*(\sigma^{\mathbb{C}}), \\ \mathcal{O}_{p',\pi;1,2}(\sigma^{\mathbb{C}}) &= (\hbar^{\mathbb{C}})^{\frac{1}{2}} \left(+\sigma^{\mathbb{C}} - m_1^{\mathbb{C}} + \frac{\hbar^{\mathbb{C}}}{2} \right), \\ \mathcal{O}_{p',\pi';1,2}(\sigma^{\mathbb{C}}) &= (\hbar^{\mathbb{C}})^{\frac{1}{2}} \left(-\sigma^{\mathbb{C}} + m_1^{\mathbb{C}} + \frac{\hbar^{\mathbb{C}}}{2} \right) = \mathcal{O}_{p',\pi;1,2}^*(\sigma^{\mathbb{C}}).\end{aligned}$$

The prefactor $(\hbar^{\mathbb{C}})^{\frac{1}{2}}$ is chosen to guarantee the orthonormality of the observables $\mathcal{O}_{p,\pi;N,L}$, which can be seen using (C.5) as follows

$$\begin{aligned}\langle \mathcal{O}_{p,\pi;1,2}(\sigma^{\mathbb{C}}) \mathcal{O}_{p,\pi;1,2}^*(\sigma^{\mathbb{C}}) \rangle_{S^2,0} &= 1, & \langle \mathcal{O}_{p,\pi;1,2}(\sigma^{\mathbb{C}}) \mathcal{O}_{p',\pi;1,2}^*(\sigma^{\mathbb{C}}) \rangle_{S^2,0} &= 0, \\ \langle \mathcal{O}_{p',\pi;1,2}(\sigma^{\mathbb{C}}) \mathcal{O}_{p',\pi;1,2}^*(\sigma^{\mathbb{C}}) \rangle_{S^2,0} &= 1, & \langle \mathcal{O}_{p',\pi;1,2}(\sigma^{\mathbb{C}}) \mathcal{O}_{p,\pi;1,2}^*(\sigma^{\mathbb{C}}) \rangle_{S^2,0} &= 0.\end{aligned}$$

All these observables only depend on a single permutation π , which means that real masses are chosen in a single chamber, say \mathfrak{C} . Physically this means that there is no Janus interface, and these are just orthonormality conditions on the observables. Furthermore,

$$\begin{aligned}\langle \mathcal{O}_{p,\pi';1,2}(\sigma^{\mathbb{C}}) \mathcal{O}_{p,\pi;1,2}^*(\sigma^{\mathbb{C}}) \rangle_{S^2,0} &= -\frac{m_2^{\mathbb{C}} - m_1^{\mathbb{C}}}{m_2^{\mathbb{C}} - m_1^{\mathbb{C}} + \hbar^{\mathbb{C}}}, \\ \langle \mathcal{O}_{p,\pi';1,2}(\sigma^{\mathbb{C}}) \mathcal{O}_{p',\pi;1,2}^*(\sigma^{\mathbb{C}}) \rangle_{S^2,0} &= \frac{\hbar^{\mathbb{C}}}{m_2^{\mathbb{C}} - m_1^{\mathbb{C}} + \hbar^{\mathbb{C}}}, \\ \langle \mathcal{O}_{p',\pi;1,2}(\sigma^{\mathbb{C}}) \mathcal{O}_{p,\pi;1,2}^*(\sigma^{\mathbb{C}}) \rangle_{S^2,0} &= \frac{\hbar^{\mathbb{C}}}{m_2^{\mathbb{C}} - m_1^{\mathbb{C}} + \hbar^{\mathbb{C}}}, \\ \langle \mathcal{O}_{p',\pi';1,2}(\sigma^{\mathbb{C}}) \mathcal{O}_{p',\pi;1,2}^*(\sigma^{\mathbb{C}}) \rangle_{S^2,0} &= -\frac{m_2^{\mathbb{C}} - m_1^{\mathbb{C}}}{m_2^{\mathbb{C}} - m_1^{\mathbb{C}} + \hbar^{\mathbb{C}}}.\end{aligned}$$

These correlation functions involve two observables and in the presence of a Janus interface $\mathcal{J}(m_{\mathfrak{C}}, m_{-\mathfrak{C}})$, the observables in the two sides of the interval depend on two different permutations π and π' . Hence, we find the one-magnon sector R -matrix constructed from the quantum mechanics

$$\text{R}^{\text{gt},(1)} \equiv \begin{pmatrix} -\frac{m_2^{\mathbb{C}} - m_1^{\mathbb{C}}}{m_2^{\mathbb{C}} - m_1^{\mathbb{C}} + \hbar^{\mathbb{C}}} & \frac{\hbar^{\mathbb{C}}}{m_2^{\mathbb{C}} - m_1^{\mathbb{C}} + \hbar^{\mathbb{C}}} \\ \frac{\hbar^{\mathbb{C}}}{m_2^{\mathbb{C}} - m_1^{\mathbb{C}} + \hbar^{\mathbb{C}}} & -\frac{m_2^{\mathbb{C}} - m_1^{\mathbb{C}}}{m_2^{\mathbb{C}} - m_1^{\mathbb{C}} + \hbar^{\mathbb{C}}} \end{pmatrix}. \quad (\text{C.9})$$

Remark C.10. By comparing this with [23, equation (6.7)], we see that the one-magnon sector R -matrix is the same for $\mathfrak{sl}(2)$ and $\mathfrak{sl}(1|1)$. This is expected since the statistics of the excitation do not play any role in the one-magnon. We have seen this above but in a different disguise: the stable envelopes have been the same as we have explained in Remark 6.2.

Finally,

$$\begin{aligned} \langle \mathcal{O}_{p,\pi;1,2}(\sigma^{\mathbb{C}}) \mathcal{O}_{p,\pi';1,2}^*(\sigma^{\mathbb{C}}) \rangle_{S^2,0} &= -\frac{m_2^{\mathbb{C}} - m_1^{\mathbb{C}}}{m_2^{\mathbb{C}} - m_1^{\mathbb{C}} - \hbar^{\mathbb{C}}}, \\ \langle \mathcal{O}_{p,\pi;1,2}(\sigma^{\mathbb{C}}) \mathcal{O}_{p',\pi';1,2}^*(\sigma^{\mathbb{C}}) \rangle_{S^2,0} &= -\frac{\hbar^{\mathbb{C}}}{m_2^{\mathbb{C}} - m_1^{\mathbb{C}} - \hbar^{\mathbb{C}}}, \\ \langle \mathcal{O}_{p',\pi;1,2}(\sigma^{\mathbb{C}}) \mathcal{O}_{p,\pi';1,2}^*(\sigma^{\mathbb{C}}) \rangle_{S^2,0} &= -\frac{\hbar^{\mathbb{C}}}{m_2^{\mathbb{C}} - m_1^{\mathbb{C}} - \hbar^{\mathbb{C}}}, \\ \langle \mathcal{O}_{p',\pi;1,2}(\sigma^{\mathbb{C}}) \mathcal{O}_{p',\pi';1,2}^*(\sigma^{\mathbb{C}}) \rangle_{S^2,0} &= -\frac{m_2^{\mathbb{C}} - m_1^{\mathbb{C}}}{m_2^{\mathbb{C}} - m_1^{\mathbb{C}} - \hbar^{\mathbb{C}}}, \end{aligned}$$

that regarding (C.9) is just the inverse $(R^{\text{gt},(1)})^{-1}$ of $R^{\text{gt},(1)}$, as it is expected

$$(R^{\text{gt},(1)})^{-1}(m_1^{\mathbb{C}}, m_2^{\mathbb{C}}) = \begin{pmatrix} -\frac{m_2^{\mathbb{C}} - m_1^{\mathbb{C}}}{m_2^{\mathbb{C}} - m_1^{\mathbb{C}} - \hbar^{\mathbb{C}}} & -\frac{\hbar^{\mathbb{C}}}{m_2^{\mathbb{C}} - m_1^{\mathbb{C}} - \hbar^{\mathbb{C}}} \\ -\frac{\hbar^{\mathbb{C}}}{m_2^{\mathbb{C}} - m_1^{\mathbb{C}} - \hbar^{\mathbb{C}}} & -\frac{m_2^{\mathbb{C}} - m_1^{\mathbb{C}}}{m_2^{\mathbb{C}} - m_1^{\mathbb{C}} - \hbar^{\mathbb{C}}} \end{pmatrix}.$$

Let us now compute the correlation functions (C.7) for the gauge theory corresponding to the two-magnon sector.

R -matrix in the two-magnon sector. The two-magnon sector is described by setting $N = 2$. There is only one possible choice of the injective map $p(\{1, 2\}) = \{1, 2\}$. As the Abelian case, there are two possible choices of permutation, which we denote by π and π' : $\pi(\{1, 2\}) = \{1, 2\}$ and $\pi'(\{1, 2\}) = \{2, 1\}$. The complex masses and the real parameters are as in the Abelian case. We have $(\sigma^{\mathbb{C}} = (\sigma_1^{\mathbb{C}}, \sigma_2^{\mathbb{C}}))$

$$\mathcal{O}_{p,\pi;2,2}(\sigma^{\mathbb{C}}) = \hbar^{\mathbb{C}} \left(\frac{(-\sigma_1^{\mathbb{C}} + m_2^{\mathbb{C}} + \frac{\hbar^{\mathbb{C}}}{2})(\sigma_2^{\mathbb{C}} - m_1^{\mathbb{C}} + \frac{\hbar^{\mathbb{C}}}{2})}{\sigma_2^{\mathbb{C}} - \sigma_1^{\mathbb{C}}} \right) + (\sigma_1^{\mathbb{C}} \longleftrightarrow \sigma_2^{\mathbb{C}}),$$

and $\mathcal{O}_{p,\pi';2,2}(\sigma^{\mathbb{C}}) = \mathcal{O}_{p,\pi;2,2}^*(\sigma^{\mathbb{C}})$. Again, the prefactor $\hbar^{\mathbb{C}}$ is determined by demanding orthogonality. Using (C.5), we see that

$$\langle \mathcal{O}_{p,\pi;2,2}(\sigma^{\mathbb{C}}) \mathcal{O}_{p,\pi;2,2}^*(\sigma^{\mathbb{C}}) \rangle_{S^2,0} = 1,$$

which means that $\mathcal{O}_{p,\pi;2,2}(\sigma^{\mathbb{C}})$ is normalized. Furthermore,

$$\langle \mathcal{O}_{p,\pi';2,2}(\sigma^{\mathbb{C}}) \mathcal{O}_{p,\pi;2,2}^*(\sigma^{\mathbb{C}}) \rangle_{S^2,0} = -\frac{m_2^{\mathbb{C}} - m_1^{\mathbb{C}} - \hbar^{\mathbb{C}}}{m_2^{\mathbb{C}} - m_1^{\mathbb{C}} + \hbar^{\mathbb{C}}}.$$

This gives the geometric R -matrix in the two-magnon sector

$$R^{\text{gt},(2)} = -\frac{m_2^{\mathbb{C}} - m_1^{\mathbb{C}} - \hbar^{\mathbb{C}}}{m_2^{\mathbb{C}} - m_1^{\mathbb{C}} + \hbar^{\mathbb{C}}}. \quad (\text{C.10})$$

Finally,

$$\langle \mathcal{O}_{p,\pi;2,2}(\sigma^{\mathbb{C}}) \mathcal{O}_{p,\pi';2,2}^*(\sigma^{\mathbb{C}}) \rangle_{S^2,0} = -\frac{m_2^{\mathbb{C}} - m_1^{\mathbb{C}} + \hbar^{\mathbb{C}}}{m_2^{\mathbb{C}} - m_1^{\mathbb{C}} - \hbar^{\mathbb{C}}},$$

which is just the inverse of (C.10).

Putting (C.9) and (C.10) together, the R -matrix of $\mathfrak{sl}(1|1)$ superspin chain constructed through the Bethe–Gauge correspondence is

$$R^{\text{gt}} = \left(\begin{array}{c|cc|c} 1 & 0 & 0 & 0 \\ \hline 0 & -\frac{m_2^{\mathbb{C}} - m_1^{\mathbb{C}}}{m_2^{\mathbb{C}} - m_1^{\mathbb{C}} + \hbar^{\mathbb{C}}} & \frac{\hbar^{\mathbb{C}}}{m_2^{\mathbb{C}} - m_1^{\mathbb{C}} + \hbar^{\mathbb{C}}} & 0 \\ 0 & \frac{\hbar^{\mathbb{C}}}{m_2^{\mathbb{C}} - m_1^{\mathbb{C}} + \hbar^{\mathbb{C}}} & -\frac{m_2^{\mathbb{C}} - m_1^{\mathbb{C}}}{m_2^{\mathbb{C}} - m_1^{\mathbb{C}} + \hbar^{\mathbb{C}}} & 0 \\ \hline 0 & 0 & 0 & -\frac{m_2^{\mathbb{C}} - m_1^{\mathbb{C}} - \hbar^{\mathbb{C}}}{m_2^{\mathbb{C}} - m_1^{\mathbb{C}} + \hbar^{\mathbb{C}}} \end{array} \right).$$

If we identify $u \rightarrow m_1^{\mathbb{C}} - m_2^{\mathbb{C}}$ and $\hbar \rightarrow \hbar^{\mathbb{C}}$, we see from (7.10) that $R^{\text{gt}}(u) = R^{\text{RR}}(u)$. Therefore, our A-model localization successfully reproduces the geometric R -matrix of $\mathfrak{sl}(1|1)$ superspin chain, as it is expected from the Bethe/Gauge correspondence.

C.2 Further details on two-point functions of 2d GLSM observables

Here, we will collect concepts and formulas for the computation of two-point function of Ω -deformed A-model observables. We follow [28] to which we refer for all the details.

C.2.1 Brief recap of Closset–Cremonesi–Park localization formula

Consider a 2d $\mathcal{N}(2, 2)$ GLSM with $\text{rank}(\mathfrak{g}) = r$ gauge Lie algebra \mathfrak{g} , whose Cartan subalgebra we denote by \mathfrak{h} , in the Ω -deformed sphere background S_Ω^2 .⁴⁹ We denote the corresponding groups by G and H and the complexified Lie algebras by $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{h}_{\mathbb{C}}$, respectively. Different topological sectors are labeled by magnetic flux $\mathbf{n} = (n_1, \dots, n_r) \in \Gamma_{\mathfrak{g}^{\vee}} \simeq \mathbb{Z}^r \subset i\mathfrak{h}$, where

$$\Gamma_{\mathfrak{g}^{\vee}} \equiv \{\mathbf{n} \mid \rho(\mathbf{n}), \forall \rho \in \Gamma_{\mathfrak{g}}\},$$

and $\rho(\mathbf{n})$ is given by the canonical pairing of dual vector spaces and electric charges take value in $\Gamma_{\mathfrak{g}}$, the weight lattice. Supersymmetric observables belong to the cohomology of left- and right-moving supercharges Q and \tilde{Q} . In the presence of Ω -deformation, we have $\delta_Q \sigma = \delta_{\tilde{Q}} \sigma = 0$ when the vector field V , that generates the rotation isometry, vanishes. Hence such operators are functions of σ , the complex scalar field in the vector multiplet of the theory. Furthermore, they can only be inserted at the north and south poles of S_Ω^2 . When we turn off the Ω background, they can be inserted anywhere on the sphere. The two-point function of such operators, which we denote by $\mathcal{O}^N(\sigma_N)$ and $\mathcal{O}^S(\sigma_S)$ for N and S denoting the north and south poles, are given by [28]

$$\begin{aligned} & \langle \mathcal{O}^N(\sigma_N) \mathcal{O}^R(\sigma_R) \rangle_{S_\Omega^2} \\ &= \frac{(-1)^{\#}}{|W|} \sum_{\mathbf{n} \in \Gamma_{\mathfrak{g}^{\vee}}} \mathfrak{q}^{\mathbf{n}} \oint_{\text{JK}} \mathcal{Z}_{\mathbf{n}}^{1\text{-loop}}(\sigma, \epsilon) \mathcal{O}^N\left(\sigma - \frac{\epsilon \mathbf{n}}{2}\right) \mathcal{O}^S\left(\sigma + \frac{\epsilon \mathbf{n}}{2}\right), \end{aligned} \quad (\text{C.11})$$

where $|W|$ denotes the order of the Weyl group W of \mathfrak{g} , $\#$ denotes the number of chiral multiplets with R -charge 2,⁵⁰ σ is the coordinate on $\tilde{\mathcal{M}} \equiv \mathfrak{h}_{\mathbb{C}} \simeq \mathbb{C}^r$, which is the cover of $\mathcal{M} \equiv \tilde{\mathcal{M}}/W$, the

⁴⁹We do not need the details of this background, which can be found in [28, Section 2.1].

⁵⁰The prefactor $(-1)^{\#}$ is a sign ambiguity which is irrelevant for the rest of our discussion since we do not have a chiral multiplet of R -charge +2. The prescription to fix this sign ambiguity, which we have stated here, is given in [28, Section 4.5].

Coulomb-branch moduli space, and ϵ is the Ω -deformation parameter. $\mathcal{Z}_{\mathbf{n}}^{1\text{-loop}}(\sigma)$ is the 1-loop contribution to the partition function from the \mathbf{n}^{th} topological sector, which is give by

$$\mathcal{Z}_{\mathbf{n}}^{1\text{-loop}}(\sigma) = \mathcal{Z}_{\mathbf{n}}^{\text{vector}}(\sigma; \epsilon) \prod_i \mathcal{Z}_{\mathbf{n}}^{\text{chiral}_i}(\sigma; \epsilon),$$

where chiral_i denotes the i^{th} chiral multiplet. These contributions could be written explicitly using

$$Z_p(x, \epsilon) \equiv \begin{cases} \prod_{q=-\frac{p}{2}+1}^{\frac{p}{2}-1} (x + \epsilon q), & p > 1, \\ 1, & p = 1, \\ \prod_{q=-\frac{|p|}{2}}^{\frac{|p|}{2}} (x + \epsilon q)^{-1}, & p < 1. \end{cases} \quad (\text{C.12})$$

For our purpose, we only need to know $Z_0(x, \epsilon = 0)$ and $Z_2(x, \epsilon = 0)$ for $\mathbf{n} = \mathbf{0}$. From (C.12), we have

$$Z_2(x, \epsilon = 0) = x, \quad Z_0(x, \epsilon = 0) = \frac{1}{x}. \quad (\text{C.13})$$

We then have

$$\begin{aligned} \mathcal{Z}_{\mathbf{n}}^{\text{vector}}(\sigma, \epsilon) &= \prod_{\alpha \in \Delta} Z_{2-\alpha(\mathbf{n})}(\alpha(\sigma), \epsilon), \\ \mathcal{Z}_{\mathbf{n}}^{\text{chiral}_i}(\sigma, \epsilon) &= \prod_{\rho_i \in \mathcal{R}_i} Z_{q_i^R - \rho_i(\mathbf{n})}(\rho_i(\sigma) + m_i, \epsilon). \end{aligned} \quad (\text{C.14})$$

In this formula, Δ denotes the set of roots, ρ_i labels the weights of representation \mathcal{R}_i of G in which the i^{th} chiral multiplet transforms, and m_i and q_i^R denote the twisted mass and $U(1)_V$ R -symmetry charge of the i^{th} chiral multiplet. Defining the notation

$$(\mathcal{W}_{\mathcal{I}}, m_{\mathcal{I}}, q_{\mathcal{I}}^R) = \begin{cases} (\rho_i, m_i, q_i^R), & \mathcal{I} = (i, \rho_i), \\ (\alpha, 0, 2), & \mathcal{I} = \alpha. \end{cases} \quad (\text{C.15})$$

We can write the 1-loop partition function as follows:⁵¹

$$\mathcal{Z}_{\mathbf{n}}^{1\text{-loop}}(\sigma, \epsilon) = \prod_{\mathcal{I}} \mathcal{Z}_{\mathcal{W}_{\mathcal{I}}(\mathbf{n}) - q_{\mathcal{I}}^R}(\mathcal{W}_{\mathcal{I}}(\sigma) + m_{\mathcal{I}}, \epsilon). \quad (\text{C.16})$$

The parameter \mathbf{q} in (C.11) is related to the complexified FI parameters, defined as follows:

$$\eta^I \equiv \frac{\theta^I}{2\pi} + i\zeta^I \in \mathbb{C}, \quad 1 \leq I \leq \dim(\mathfrak{c}_{\mathbb{C}}^*),$$

where θ^I and ζ^I are the theta angle and real FI parameters. They belong to the dual of the complexified center $\mathfrak{c}_{\mathbb{C}}^* \subset \mathfrak{g}_{\mathbb{C}}^*$ and could be thought of as elements of $\mathfrak{h}_{\mathbb{C}}^*$ through the embedding $\mathfrak{c}_{\mathbb{C}}^* \hookrightarrow \mathfrak{h}_{\mathbb{C}}^*$. We denote the image of the physical FI parameters ζ^I under this embedding by ζ^a , $a = 1, \dots, r$. Then, $\mathbf{q}^n \equiv \exp(2\pi i \sum_I \eta \cdot \mathbf{n})$, where n_I denotes the flux in the free I^{th} $U(1)$

⁵¹Considering Figure 9, we only need the zero-flux sector of the GLSM observables in the computation of interval partition function of quantum mechanics, as we will see momentarily.

part of the center of G . Note that $\mathbf{n} \in \mathfrak{ih}$ and hence $\boldsymbol{\eta} \cdot \mathbf{n}$ is given by the canonical pairing between $\mathfrak{h}_{\mathbb{C}}$ and $\mathfrak{h}_{\mathbb{C}}^*$.

Finally, we need to specify the contour of integration, which is given by the JK contour [21, 68, 137]. The poles of $\mathcal{Z}_{\mathbf{n}}^{1\text{-loop}}(\sigma, \epsilon)$ are the loci of intersections of $s \geq r$ hyperplanes $\{\mathcal{H}_{\mathcal{I}_1}^{k_1}, \dots, \mathcal{H}_{\mathcal{I}_s}^{k_s}\}$, where

$$\mathcal{H}_{\mathcal{I}}^k \equiv \left\{ \sigma \left| \mathcal{W}_{\mathcal{I}}(\sigma) = -m_{\mathcal{I}} - \left(k + \frac{q_{\mathcal{I}}^R - \mathcal{W}_{\mathcal{I}}(\mathbf{n})}{2} \right) \epsilon \right. \right\}, \quad k \in [0, \mathcal{W}_{\mathcal{I}}(\mathbf{n}) - q_{\mathcal{I}}^R]_{\text{int}}, \quad (\text{C.17})$$

and $[a, b]_{\text{int}}$ denotes the set of integers between a and b . There are two types of singularities: the non-degenerate singularities corresponding to the intersection of exactly r hyperplanes and the degenerate ones corresponding to the intersection of $s > r$ hyperplanes. This complex-codimension- r singular loci is denoted as $\widetilde{\mathcal{M}}_{\mathbf{n}}^{\text{sing}} \subset \widetilde{\mathcal{M}}$. Note that $\mathcal{H}_{\mathcal{I}}^k$ is a singular locus of $\mathcal{Z}_{\mathbf{n}}^{1\text{-loop}}$ only when \mathcal{I} labels a chiral multiplet field. \mathcal{H}_{α}^k labels the codimension-1 poles with zero residue. The definition of JK residues at $\sigma_* \in \widetilde{\mathcal{M}}_{\mathbf{n}}^{\text{sing}}$ depends on an additional parameter $\boldsymbol{\eta} \in \mathfrak{ih}^*$, which is defined as follows. The charge $\mathcal{W}_{\mathcal{I}}$, defined in (C.15), belongs to \mathfrak{ih}^* and hence any set of charges $\{\mathcal{W}_{\mathcal{I}_1}, \dots, \mathcal{W}_{\mathcal{I}_r}\}$ generates a cone

$$\text{Con}[\mathcal{W}_{\mathcal{I}_1}, \dots, \mathcal{W}_{\mathcal{I}_r}] \subset \mathfrak{ih}^*. \quad (\text{C.18})$$

The set of all such cones can be divided into dimension- r chambers that are separated by codimension-1 walls. The chamber to FI parameters $\zeta^a \in \mathfrak{ih}^*$, $a = 1, \dots, r$ belong to determines the phase of the GLSM [91, 147]. The effective FI parameter $\zeta_{\text{eff}}^{\text{UV}, I}$ at the infinity (i.e., when $|\sigma| \rightarrow \infty$) of Coulomb branch \mathcal{M} is defined to be

$$\zeta_{\text{eff}}^{\text{UV}, a} \equiv \zeta^a + \frac{b^a}{2\pi} \lim_{R \rightarrow \infty} \ln R, \quad b^a \equiv \sum_i \text{Tr}_{\mathcal{R}_i}(t^a) = \sum_{\mathcal{I}} \mathcal{W}_{\mathcal{I}}.$$

t^a denotes the generator of the a^{th} $\mathfrak{u}(1) \subset \mathfrak{g}$,⁵² and the last equality follows from embedding $\mathfrak{c}^* \subset \mathfrak{ih}^*$. Hence, phases of the theory on the Coulomb branch depend on the chamber $\mathfrak{C} \subset \mathfrak{ih}^*$ that the parameter $\zeta_{\text{eff}}^{\text{UV}}$ belong to. This can be stated as follows: $\zeta_{\text{eff}}^{\text{UV}}$ belongs to a chamber $\mathcal{C} \subset \mathfrak{ih}^*$ if $\zeta + \frac{b}{2\pi} \ln R \in \mathcal{C}$, $\forall R \geq R_*$, for some R_* , which always exists for practical purposes [28]. When $b \neq 0$, there could be two cases: 1) either b belongs to a definite chamber \mathcal{C} ; In this case, ζ is irrelevant and the theory has only one phase determined by the chamber \mathcal{C} , or 2) b belongs to a wall where chambers $\mathcal{C}_1, \dots, \mathcal{C}_p$ meet; In this case, one can turn on ζ and depending on its value, $\zeta_{\text{eff}}^{\text{UV}}$ could belong to any of the chambers \mathfrak{C}_i , $i = 1, \dots, p$, which in turn determines the phase of the theory. When $b = 0$, and hence the IR theory is a CFT, then $\zeta_{\text{eff}}^{\text{UV}} = \zeta$ could be taken to be in any chamber $\mathcal{C} \subset \mathfrak{ih}^*$. We then set

$$\xi \equiv \zeta_{\text{eff}}^{\text{FI}}. \quad (\text{C.19})$$

The JK residues can now be defined as follows. If a singular point σ_* comes from the intersection of $s \geq r$ hyperplanes $\{\mathcal{H}_{\mathcal{I}_1}^{k_1}, \dots, \mathcal{H}_{\mathcal{I}_s}^{k_s}\}$, we denote the set of \mathcal{H} -charges as $\mathcal{W}(\sigma_*) \equiv \{\mathcal{W}_{\mathcal{I}_1}, \dots, \mathcal{W}_{\mathcal{I}_s}\}$. The JK residues at $\sigma_* = \{\sigma_{*,1}, \dots, \sigma_{*,r}\}$ is then give by

$$\begin{aligned} \text{JK-Res}[\mathcal{W}(\sigma), \xi] & \prod_{\mathcal{W}_j \in \mathcal{W}_r} \frac{d\sigma_1 \wedge \dots \wedge d\sigma_r}{\mathcal{W}_1(\sigma) \dots \mathcal{W}_r(\sigma)} \Bigg|_{\sigma=\sigma_*} \\ & = \begin{cases} \frac{1}{|\det(\mathcal{W}_i)|}, & \xi \in \text{Con}[\mathcal{W}_1, \dots, \mathcal{W}_r], \\ 0, & \xi \notin \text{Con}[\mathcal{W}_1, \dots, \mathcal{W}_r], \end{cases} \end{aligned} \quad (\text{C.20})$$

where \mathcal{W}_r is any set among $\binom{s}{r}$ sets of r \mathfrak{h} -charges in $\mathcal{W}(\sigma_*)$. This concludes our discussion of the necessary results from [28].

⁵² t^a is the image of the I^{th} $\mathfrak{u}(1)$ generator of the center of \mathfrak{g}^* under the embedding $\mathfrak{c}^* \hookrightarrow \mathfrak{ih}^*$.

C.2.2 Explicit computation of two-point functions

We would like to compute the contribution of vector and chiral multiplets to the correlation functions (C.7), given in (C.11) or more compactly in (C.16), explicit. For the fields in the vector multiplet, $\mathcal{I} = (i, \rho_i) \equiv (ab, \rho_{ab})$, as (C.15) $\rho_{ab} = \alpha_{ab} = e_a - e_b$, $a \neq b$, where α_{ab} denotes the ab^{th} root of $\mathfrak{u}(N)$, associated to the generator E_{ab} , and $\{e_1, \dots, e_N\}$ is the standard basis for \mathbb{R}^N .⁵³ Therefore,

$$\begin{aligned} \mathcal{W}_{\mathcal{I}}(\sigma^{\mathbb{C}}) &= \rho_{ab}(\sigma^{\mathbb{C}}) = \sigma_a^{\mathbb{C}} - \sigma_b^{\mathbb{C}}, & a \neq b, \\ \mathcal{W}_{\mathcal{I}}(\mathbf{n}) &= \rho_{ab}(\mathbf{n}) = n_a - n_b, & a \neq b. \end{aligned} \quad (\text{C.21})$$

Similarly, for chiral fields in the fundamental (+) and antifundamental (-) representations, the weights are $(\mathcal{I} = (i, \rho_i) \equiv (a, \rho_{a,\pm})$ as in (C.15))

$$\rho_{a,\pm} = \pm e_a, \quad a = 1, \dots, N, \quad (\text{C.22})$$

and hence for the chiral multiplets⁵⁴ $(\mathcal{Q}_a^i, \tilde{\mathcal{Q}}_i^a)$, we have

$$\begin{aligned} \mathcal{Q}_a^i: \mathcal{W}_{\mathcal{I}}(\sigma^{\mathbb{C}}) + m_{\mathcal{I}} &= \rho_{a,+}(\sigma^{\mathbb{C}}) + m_{\mathcal{Q}_a^i} = \sigma_a^{\mathbb{C}} - m_i^{\mathbb{C}} + \hbar^{\mathbb{C}}, \\ \tilde{\mathcal{Q}}_i^a: \mathcal{W}_{\mathcal{I}}(\sigma^{\mathbb{C}}) + m_{\mathcal{I}} &= \rho_{a,-}(\sigma^{\mathbb{C}}) + m_{\tilde{\mathcal{Q}}_i^a} = -\sigma_a^{\mathbb{C}} + m_i^{\mathbb{C}} + \hbar^{\mathbb{C}}. \end{aligned}$$

Using (C.13) and (C.14), the contribution of vector multiplet is⁵⁵

$$\mathcal{Z}_2^{\text{vector}}(\sigma, \epsilon = 0) = \prod_{\substack{a,b=1 \\ a \neq b}}^N (\sigma_a^{\mathbb{C}} - \sigma_b^{\mathbb{C}}). \quad (\text{C.23})$$

Similarly, setting $q_{\tilde{\mathcal{Q}}}^R = q_{\mathcal{Q}}^R = 0$, we obtain the contribution of chiral multiplets

$$\begin{aligned} \mathcal{Z}_0^{\mathcal{Q}}(\sigma^{\mathbb{C}}; \epsilon = 0) &= \prod_{a=1}^N \prod_{i=1}^L \frac{1}{(\sigma_a^{\mathbb{C}} - m_i^{\mathbb{C}} + \frac{\hbar^{\mathbb{C}}}{2})}, \\ \mathcal{Z}_0^{\tilde{\mathcal{Q}}}(\sigma^{\mathbb{C}}; \epsilon = 0) &= \prod_{a=1}^N \prod_{i=1}^L \frac{1}{(-\sigma_a^{\mathbb{C}} + m_i^{\mathbb{C}} + \frac{\hbar^{\mathbb{C}}}{2})}. \end{aligned} \quad (\text{C.24})$$

Putting together (C.23) and (C.24) and using (C.11), we thus see that the zero-flux sector contribution to the sphere partition function can be written explicitly as⁵⁶ ($\mathbf{n} = 0$ in (C.11))

$$\mathcal{Z} \equiv \langle 1 \rangle_{S^2, 0} = \frac{1}{N!} \oint_{\text{JK}} d\sigma_1^{\mathbb{C}} \wedge \dots \wedge d\sigma_N^{\mathbb{C}} \mathcal{Z}(\sigma_1^{\mathbb{C}}, \dots, \sigma_N^{\mathbb{C}}),$$

where

$$\mathcal{Z}(\sigma_1^{\mathbb{C}}, \dots, \sigma_N^{\mathbb{C}}) \equiv \prod_{a=1}^N \prod_{i=1}^L \frac{1}{(\sigma_a^{\mathbb{C}} - m_i^{\mathbb{C}} + \frac{1}{2}\hbar^{\mathbb{C}})(-\sigma_a^{\mathbb{C}} + m_i^{\mathbb{C}} + \frac{1}{2}\hbar^{\mathbb{C}})} \cdot \prod_{a \neq b}^N (\sigma_a^{\mathbb{C}} - \sigma_b^{\mathbb{C}}). \quad (\text{C.25})$$

⁵³For $\alpha_{ab} = e_a - e_b$ for $a \neq b$, we consider the one-dimensional vector space generated by E_{ab} . This space is an eigenspace of the Cartan subalgebra with eigenvalue α_{ab} . These eigenspaces together with the Cartan subalgebra generate $\mathfrak{u}(N)$.

⁵⁴We use the same notation for chiral multiplets of the theory \mathcal{T}_{2d}'' in Figure 8 as the ones for \mathcal{T}_{3d}''' and its dimensional reductions to 2d and 1d.

⁵⁵Note that the contribution of vector multiplet is the same as an adjoint chiral with R -charge +2 [28]. Hence, we have taken $q_{\text{vector}}^R = 2$.

⁵⁶As we mentioned above, there is no overall $(-1)^{\#}$ sign since there is no chiral multiplet of R -charge +2 in our construction.

The factor $1/N!$ comes from the order of the Weyl-group factor in (C.11). This proves (C.6). Substituting (C.25) in (C.11) gives (C.5). The poles of JK prescription for the positive (effective) FI parameter ($\xi > 0$, where ξ is defined in (C.19)) are given by

$$\text{JK Poles: } \sigma_a^{\mathbb{C}} = m_i^{\mathbb{C}} - \frac{\hbar^{\mathbb{C}}}{2}, \quad a = 1, \dots, N, \quad i = 1, \dots, L, \quad (\text{C.26})$$

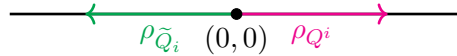
coming from the intersection of hyperplanes $\mathcal{H}_{\mathcal{I}}^k$ in (C.17) associated with fundamental chiral multiplets \mathcal{Q}_a^i . Note that we need to take the contribution from all poles, i.e., all pairs (a, i) . In the following, we provide further details on this choice of JK poles in the computation of R -matrix.

Jeffrey–Kirwan poles for the computation of R -matrix. Let us give the details of the choice (C.26) of JK poles necessary for the computation of R -matrix in C.1. Although we have stated the poles in (C.26), we give more details on this choice using the formalism of [28], as reviewed in Section C.2. As we have explained there, the choice of JK poles depends on the choice of the (effective) FI parameter ξ , defined in (C.19).

For the 1-magnon sector, we have a U(1) gauge theory with SU(2) fundamental \mathcal{Q}^i and antifundamental $\tilde{\mathcal{Q}}_i$ flavors. Using (C.22) and (C.17), the equations of hyperplanes (C.17) are given by

$$\begin{aligned} \mathcal{H}_{\mathcal{I}_{\mathcal{Q}^1}}^{k_1^{\mathcal{Q}}} &= \left\{ \sigma^{\mathbb{C}} \mid \sigma^{\mathbb{C}} = m_1^{\mathbb{C}} - \frac{\hbar^{\mathbb{C}}}{2} - \left(k_1^{\mathcal{Q}} - \frac{n}{2} \right) \epsilon \right\}, & k_1^{\mathcal{Q}} \in [0, n]_{\text{int}}, \\ \mathcal{H}_{\mathcal{I}_{\mathcal{Q}^2}}^{k_2^{\mathcal{Q}}} &= \left\{ \sigma^{\mathbb{C}} \mid \sigma^{\mathbb{C}} = m_2^{\mathbb{C}} - \frac{\hbar^{\mathbb{C}}}{2} - \left(k_2^{\mathcal{Q}} - \frac{n}{2} \right) \epsilon \right\}, & k_2^{\mathcal{Q}} \in [0, n]_{\text{int}}, \\ \mathcal{H}_{\mathcal{I}_{\tilde{\mathcal{Q}}_1}}^{k_1^{\tilde{\mathcal{Q}}}} &= \left\{ \sigma^{\mathbb{C}} \mid -\sigma^{\mathbb{C}} = -m_1^{\mathbb{C}} - \frac{\hbar^{\mathbb{C}}}{2} - \left(k_1^{\tilde{\mathcal{Q}}} + \frac{n}{2} \right) \epsilon \right\}, & k_1^{\tilde{\mathcal{Q}}} \in [0, n]_{\text{int}}, \\ \mathcal{H}_{\mathcal{I}_{\tilde{\mathcal{Q}}_2}}^{k_2^{\tilde{\mathcal{Q}}}} &= \left\{ \sigma^{\mathbb{C}} \mid -\sigma^{\mathbb{C}} = -m_2^{\mathbb{C}} - \frac{\hbar^{\mathbb{C}}}{2} - \left(k_2^{\tilde{\mathcal{Q}}} + \frac{n}{2} \right) \epsilon \right\}, & k_2^{\tilde{\mathcal{Q}}} \in [0, n]_{\text{int}}, \end{aligned}$$

with $m_1^{\mathbb{C}} + m_2^{\mathbb{C}} = 0$. The weight diagram is



We only need to consider the zero-flux sector $n = 0$. From (C.18), we see that there are two “cones”

$$\text{Con}(\mathcal{W}_{\mathcal{I}_{\mathcal{Q}^1}}, \mathcal{W}_{\mathcal{I}_{\mathcal{Q}^2}}) = \text{Con}(+1, +1), \quad \text{Con}(\mathcal{W}_{\mathcal{I}_{\tilde{\mathcal{Q}}_1}}, \mathcal{W}_{\mathcal{I}_{\tilde{\mathcal{Q}}_2}}) = \text{Con}(-1, -1).$$

Using (C.20) and taking $\xi > 0$ and using (C.20), we see that only the poles coming from $H_{\mathcal{Q}^1}^{k_1^{\mathcal{Q}}}$ and $H_{\mathcal{Q}^2}^{k_2^{\mathcal{Q}}}$, which is the same as (C.26).

For the 2-magnon sector, the gauge group is U(2) and the $\text{SU}(2) \times \text{U}(1)_{\hbar}$ matter multiplet consists of \mathcal{Q}_a^i and $\tilde{\mathcal{Q}}_i^a$ with $q_{\mathcal{Q}}^R = q_{\tilde{\mathcal{Q}}}^R = 0$. (C.22), and (C.21), we can write the equations for singular hyperplanes (C.17) associated to the fundamental chiral multiplets

$$\begin{aligned} \mathcal{H}_{\mathcal{I}_{\mathcal{Q}^1}^1}^{k_{11}^{\mathcal{Q}}} &= \left\{ (\sigma_1^{\mathbb{C}}, \sigma_2^{\mathbb{C}}) \mid \sigma_1^{\mathbb{C}} = m_1^{\mathbb{C}} - \frac{\hbar^{\mathbb{C}}}{2} - \left(k_{11}^{\mathcal{Q}} - \frac{n_1}{2} \right) \epsilon \right\}, & k_{11}^{\mathcal{Q}} \in [0, n_1]_{\text{int}}, \\ \mathcal{H}_{\mathcal{I}_{\mathcal{Q}^2}^2}^{k_{12}^{\mathcal{Q}}} &= \left\{ (\sigma_1^{\mathbb{C}}, \sigma_2^{\mathbb{C}}) \mid \sigma_1^{\mathbb{C}} = m_2^{\mathbb{C}} - \frac{\hbar^{\mathbb{C}}}{2} - \left(k_{12}^{\mathcal{Q}} - \frac{n_1}{2} \right) \epsilon \right\}, & k_{12}^{\mathcal{Q}} \in [0, n_1]_{\text{int}}, \\ \mathcal{H}_{\mathcal{I}_{\mathcal{Q}^2}^1}^{k_{21}^{\mathcal{Q}}} &= \left\{ (\sigma_1^{\mathbb{C}}, \sigma_2^{\mathbb{C}}) \mid \sigma_2^{\mathbb{C}} = m_1^{\mathbb{C}} - \frac{\hbar^{\mathbb{C}}}{2} - \left(k_{21}^{\mathcal{Q}} - \frac{n_2}{2} \right) \epsilon \right\}, & k_{21}^{\mathcal{Q}} \in [0, n_2]_{\text{int}}, \end{aligned}$$

$$\mathcal{H}_{\mathcal{I}_{\mathcal{Q}_2^2}}^{k_{22}^{\mathcal{Q}}} = \left\{ (\sigma_1^{\mathbb{C}}, \sigma_2^{\mathbb{C}}) \mid \sigma_2^{\mathbb{C}} = m_2^{\mathbb{C}} - \frac{\hbar^{\mathbb{C}}}{2} - \left(k_{22}^{\mathcal{Q}} - \frac{n_2}{2} \right) \epsilon \right\}, \quad k_{22}^{\mathcal{Q}} \in [0, n_2]_{\text{int}}.$$

Similarly, the hyperplanes associated to the antifundamental chiral multiplets are

$$\begin{aligned} H_{\mathcal{I}_{\tilde{\mathcal{Q}}_1^1}}^{k_{11}^{\tilde{\mathcal{Q}}}} &= \left\{ (\sigma_1^{\mathbb{C}}, \sigma_2^{\mathbb{C}}) \mid -\sigma_1^{\mathbb{C}} = -m_1^{\mathbb{C}} - \frac{\hbar^{\mathbb{C}}}{2} - \left(k_{11}^{\tilde{\mathcal{Q}}} + \frac{n_1}{2} \right) \epsilon \right\}, \quad k_{11}^{\tilde{\mathcal{Q}}} \in [0, n_1]_{\text{int}}, \\ H_{\mathcal{I}_{\tilde{\mathcal{Q}}_1^2}}^{k_{12}^{\tilde{\mathcal{Q}}}} &= \left\{ (\sigma_1^{\mathbb{C}}, \sigma_2^{\mathbb{C}}) \mid -\sigma_1^{\mathbb{C}} = -m_2^{\mathbb{C}} - \frac{\hbar^{\mathbb{C}}}{2} - \left(k_{12}^{\tilde{\mathcal{Q}}} + \frac{n_1}{2} \right) \epsilon \right\}, \quad k_{12}^{\tilde{\mathcal{Q}}} \in [0, n_1]_{\text{int}}, \\ H_{\mathcal{I}_{\tilde{\mathcal{Q}}_2^1}}^{k_{21}^{\tilde{\mathcal{Q}}}} &= \left\{ (\sigma_1^{\mathbb{C}}, \sigma_2^{\mathbb{C}}) \mid -\sigma_2^{\mathbb{C}} = -m_1^{\mathbb{C}} - \frac{\hbar^{\mathbb{C}}}{2} - \left(k_{21}^{\tilde{\mathcal{Q}}} + \frac{n_2}{2} \right) \epsilon \right\}, \quad k_{21}^{\tilde{\mathcal{Q}}} \in [0, n_2]_{\text{int}}, \\ H_{\mathcal{I}_{\tilde{\mathcal{Q}}_2^2}}^{k_{22}^{\tilde{\mathcal{Q}}}} &= \left\{ (\sigma_1^{\mathbb{C}}, \sigma_2^{\mathbb{C}}) \mid -\sigma_2^{\mathbb{C}} = -m_2^{\mathbb{C}} - \frac{\hbar^{\mathbb{C}}}{2} - \left(k_{22}^{\tilde{\mathcal{Q}}} + \frac{n_2}{2} \right) \epsilon \right\}, \quad k_{22}^{\tilde{\mathcal{Q}}} \in [0, n_2]_{\text{int}}. \end{aligned}$$

We only need the zero-flux sector $n_1 = n_2 = 0$. To specify the JK poles, we take ξ to be in the first quadrant, and using (C.18) and (C.20), we see that we need to take the contribution from the hyperplanes forming the following cones associated to $\mathcal{Q}_a^i \text{Con}(\mathcal{W}_{\mathcal{I}_{\mathcal{Q}_1^i}}, \mathcal{W}_{\mathcal{I}_{\mathcal{Q}_2^i}})$, $i = 1, 2$, which contain ξ . These poles are as given in (C.26).

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