Some Differential Equations for the Riemann θ-Function on Jacobians

Robert WILMS

Laboratoire de Mathématiques Nicolas Oresme, Université de Caen–Normandie, BP 5186, 14032 Caen Cedex, France
E-mail: *robert.wilms@unicaen.fr*
URL: https://www.robert-wilms.de/

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Abstract. We prove some differential equations for the Riemann theta function associated to the Jacobian of a Riemann surface. The proof is based on some variants of a formula by Fay for the theta function, which are motivated by their analogues in Arakelov theory of Riemann surfaces.

Key words: θ-functions; Riemann surfaces; Jacobians; differential equations

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1 Introduction

The Riemann θ-function in dimension $g \geq 1$ is given by

$$\theta \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} (\tau, z) = \sum_{m \in \mathbb{Z}^g} \exp\bigl(\pi \mathrm{i}\,{}^{\mathrm{t}}(m+\alpha_1)\tau(m+\alpha_1) + 2\pi \mathrm{i}\,{}^{\mathrm{t}}(m+\alpha_1)(z+\alpha_2)\bigr),$$

where $\alpha = (\alpha_1, \alpha_2) \in \left(\frac{1}{2}\mathbb{Z}\right)^g \times \left(\frac{1}{2}\mathbb{Z}\right)^g$ is called a theta characteristic, $z \in \mathbb{C}^g$ and τ is a complex symmetric $g \times g$ matrix with positive definite imaginary part. Fay [5] studied intensively the case, where τ is the period matrix of a Riemann surface. In particular, he obtained his famous trisecant identity, which can be applied to give solutions of certain differential equations occurring in mathematical physics in terms of θ-functions as in [8, Chapter IIIb, Section 4]. Building on Fay's studies, we will derive some new differential equations for the Riemann θ-function associated to Riemann surfaces.

To give the precise statement, let X be a compact and connected Riemann surface of genus $g \geq 1$ and τ a period matrix for X. Let us shortly write $\theta(z) = \theta[0](\tau, z)$ and $\theta_j = \frac{\partial \theta}{\partial z_j}$ and $\theta_{jk} = \frac{\partial^2 \theta}{\partial z_j \partial z_k}$ for the partial derivatives of θ. Further, we define as in [6]

$$J(w_1, \dots, w_g) = \det(\theta_j(w_k)), \qquad w_1, \dots, w_g \in \mathbb{C}^g$$

and as in [2]

$$\eta = \det \begin{pmatrix} \theta_{jk} & \theta_j \\ \theta_k & 0 \end{pmatrix}.$$

After fixing a symplectic basis of homology $H_1(X, \mathbb{Z})$ and representing curves, we can canonically identify $\mathrm{Pic}_{g-1}(X) \cong \mathrm{Pic}_0(X)$ and fix canonical representatives in $\mathbb{C}^g$ for divisors of degree $g-1$. See Section 2 for details.

Theorem 1.1. *Let X be a compact and connected Riemann surface of genus $g \geq 1$ and $p_1, \dots, p_g, q \in X$ arbitrary points on X in general position. We denote the degree $(g-1)$ divisor $D = \sum_{j=1}^{g} p_j - q$ and the effective degree $(g-1)$ divisors $D_k = \sum_{j=1}^{g} p_j - p_k$ for $1 \leq k \leq g$. Then the following equations hold:*

$$(i) \qquad \prod_{k=1}^{g} \eta(D_k) = (-1)^g \left(\frac{J(D_1, \dots, D_g)}{\theta(D)^{g-1}}\right)^{2g} \prod_{j \neq k}^{g} \frac{\theta(D_j + p_k - q)^2}{\theta(D_j + p_k - p_j)},$$

$$(ii) \qquad \prod_{k=1}^{g} \eta((g-1)p_k) = (-1)^g \left(\frac{J(D_1, \dots, D_g)}{\theta(D)^{g-1}}\right)^{2g} \prod_{j \neq k}^{g} \frac{\theta(gp_j - q)^2}{\theta(gp_j - p_k)},$$

$$(iii) \qquad \eta(D_g)^{g-1} = \prod_{k=1}^{g-1} \left(\eta((g-1)p_k) \left(\frac{\theta(D_g + p_k - q)}{\theta(gp_k - q)}\right)^{g-1} \right).$$

We will prove the theorem in Section 2 by comparing two derived variants of a formula by Fay on θ, the Schottky–Klein prime form $E(\cdot, \cdot)$ and the Brill–Noether matrix. The most difficult problem is to connect the determinant of the Brill–Noether matrix to the determinants J and η. Especially for η this involves ambitious combinatorics. The proof of the theorem is motivated by analogous formulas in Arakelov theory on the normed versions $\|\theta\|$, $\|J\|$ and $\|\eta\|$ of θ, J and η. We will discuss them in Section 3.

2 Variations of Fay's formula

In this section, we prove Theorem 1.1. Let X be a compact and connected Riemann surface of genus $g \geq 1$ and $A_1, \dots, A_g, B_1, \dots, B_g \in H_1(X, \mathbb{Z})$ a basis of homology such that for all j, k we have $(A_j, A_k) = (B_j, B_k) = 0$ and $(A_j, B_k) = \delta_{jk}$ for the intersection pairings. Further, let $v_1, \dots, v_g \in H^0(X, \Omega_X^1)$ be the unique basis of one-forms such that $\int_{A_k} v_j = \delta_{jk}$ and write $v = {}^{\mathrm{t}}(v_1, \dots, v_g)$ for the vector of them. The period matrix τ of X is given by the entries $\tau_{jk} = \int_{B_k} v_j$. It is a complex symmetric $g \times g$ matrix with positive definite imaginary part. Note that τ is not an invariant of X as it also depends on the choice of the basis of homology $A_1, \dots, A_g, B_1, \dots, B_g \in H_1(X, \mathbb{Z})$. Two different bases of homology as above can be transformed in each other by a symplectic matrix $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}(2g, \mathbb{Z})$. This basis transformation acts on the period matrix by $M \cdot \tau = (A\tau + B)(C\tau + D)^{-1}$.

As in the introduction, we associate to a theta characteristic $\alpha = (\alpha_1, \alpha_2) \in \left(\frac{1}{2}\mathbb{Z}\right)^g \times \left(\frac{1}{2}\mathbb{Z}\right)^g$ and to τ the theta function

$$\theta[\alpha](z) = \sum_{m \in \mathbb{Z}^g} \exp\left(\pi \mathrm{i}\, {}^{\mathrm{t}}(m + \alpha_1)\tau(m + \alpha_1) + 2\pi \mathrm{i}\, {}^{\mathrm{t}}(m + \alpha_1)(z + \alpha_2)\right)$$

on $\mathbb{C}^g$. We shortly write $\theta(z) = \theta[0](z)$ and $\theta_j = \frac{\partial \theta}{\partial z_j}$ and $\theta_{jk} = \frac{\partial^2 \theta}{\partial z_j \partial z_k}$ for the partial derivatives. A theta characteristic α is called even if $\theta[\alpha](z)$ is an even function or equivalently if $4\,{}^{\mathrm{t}}\alpha_1 \alpha_2$ is even. Otherwise, it is called odd.

Let $\mathrm{Jac}(X) = \mathbb{C}^g / (\mathbb{Z}^g + \tau \mathbb{Z}^g)$ be the Jacobian of X. Any divisor $D = \sum_{j=1}^{n} p_j - \sum_{j=1}^{n} q_j$ of degree 0 on X defines an element $\sum_{j=1}^{n} \int_{q_j}^{p_j} v$ in $\mathrm{Jac}(X)$. We fix curves $\gamma_1, \dots, \gamma_{2g}$ representing the homology classes $A_1, \dots, A_g$, $B_1, \dots, B_g$. Further, we make the convention that the paths of integration are taken in X cut along these curves. If we assume that the support of D does not intersect $\bigcup_{j=1}^{2g} \gamma_j$ or, more generally, that the points $p_1, \dots, p_n$, $q_1, \dots, q_n$ are in general position, then we obtain a well-defined representative of D in $\mathbb{C}^g$, which we also denote by D. There is an isomorphism

$$\varphi\colon\ \mathrm{Pic}_{g-1}(X) \xrightarrow{\cong} \mathrm{Jac}(X), \tag{2.1}$$

which sends the divisor $\Theta \subseteq \operatorname{Pic}_{g-1}(X)$ of effective line bundles of degree $g-1$ to the zero divisor of θ, see for example [7, Corollary II.3.6]. This can be made explicit by the vector of Riemann constants

$$\varphi\left(\sum_{j=1}^{d} n_j p_j\right) = \left(\sum_{j=1}^{d} n_j \int_{p_0}^{p_j} v_k - \frac{\tau_{kk}-1}{2} - \sum_{\substack{j=1\\ j\neq k}}^{g} \int_{A_j} v_j(x) \int_{p_0}^{x} v_k\right)_{1\leq k\leq g},$$

where $\sum_{j=1}^{d} n_j = g-1$ and $p_0 \in X$ is some fixed base point not lying in $\bigcup_{j=1}^{2g} \gamma_j$. By our convention to take paths of integration in X cut along $\gamma_1, \ldots, \gamma_{2g}$, the above vector gives for any degree $g-1$ divisor D with support outside of $\bigcup_{j=1}^{2g} \gamma_j$ a well-defined representative in $\mathbb{C}^g$, which we also denote by D.

Note that φ sends line bundles $\mathscr{L}$ with $\mathscr{L} \otimes \mathscr{L} = K_X$, where K_X denotes the canonical bundle of X, to theta characteristics in $\frac{1}{2}\mathbb{Z}^g + \frac{1}{2}\tau\mathbb{Z}^g \subseteq \mathbb{C}^g$. Furthermore, $\varphi(\mathscr{L})$ is even if and only if $\dim H^0(X, \mathscr{L})$ is even. We call a theta characteristic α non-singular if the corresponding line bundle $\mathscr{L}_\alpha = \varphi^{-1}(\alpha)$ satisfies $\dim H^0(X, \mathscr{L}_\alpha) \leq 1$. For an odd and non-singular theta characteristic $\alpha \in \left(\frac{1}{2}\mathbb{Z}\right)^{2g}$ and $x \in X$, we define the half-order differential $h_\alpha \in H^0(X, \mathscr{L}_\alpha)$ by

$$h_\alpha(x)^2 = \sum_{j=1}^{g} \frac{\partial \theta[\alpha]}{\partial z_j}(0) v_j(x).$$

Then the Schottky–Klein prime-form is defined by

$$E(x,y) = \frac{\theta[\alpha]\left(\int_x^y v\right)}{h_\alpha(x) h_\alpha(y)}$$

for any $x, y \in X$. It satisfies $E(x,y) = -E(y,x)$. Moreover, it has a simple zero on the diagonal in $X \times X$ and it is non-zero for $x \neq y$. We fix a generic effective divisor $\mathcal{A} = \sum_{j=1}^{g} a_j$ with $a_j \in X$ and we define for any $x \in X$ in general position

$$\sigma_{\mathcal{A}}(x) = \frac{\theta(\mathcal{A} - x)}{\prod_{j=1}^{g} E(a_j, x)}.$$

Now let $p_1, \ldots, p_g, q \in X$ be arbitrary points of X and set $D = \sum_{j=1}^{g} p_j - q$ as well as $D_k = \sum_{j=1}^{g} p_j - p_k$. We will often assume without loss of generality and without mentioning it that these points are in general position. There exists a constant $c(\mathcal{A}) \in \mathbb{C}$ independent of q, $p_1, \ldots, p_g$ such that

$$\theta(D) = c(\mathcal{A}) \cdot \frac{\det(v_j(p_k)) \cdot \sigma_{\mathcal{A}}(q) \cdot \prod_{j=1}^{g} E(p_j, q)}{\prod_{j<k}^{g} E(p_j, p_k) \cdot \prod_{j=1}^{g} \sigma_{\mathcal{A}}(p_j)}. \tag{2.2}$$

This was shown by Fay in the proof of [5, Corollary 2.17] if one sets $\mathcal{A} = \mathcal{B}$ in the notation there. Fay also constructed a multiple σ of $\sigma_{\mathcal{A}}$, which is independent of the choice of $\mathcal{A}$, and he proved a version of equation (2.2) for $\sigma_{\mathcal{A}}$ replaced by σ and a constant c independent of $\mathcal{A}$. But for our purposes, the above equation is sufficient, so we will not discuss the more involved definition of σ here. We will give two alternative equations for the invariant $c(\mathcal{A})$ by differentiating the theta function. For this purpose, we define as in the introduction

$$J(w_1, \ldots, w_g) = \det(\theta_j(w_k))$$

for any $w_1, \ldots, w_g \in \mathbb{C}^g$.

Proposition 2.1. *The following equation holds:*

$$J(D_1,\dots,D_g)=(-1)^{\binom{g+1}{2}}c(\mathcal{A})\cdot\frac{\theta(D)^{g-1}\prod_{j<k}^{g}E(p_j,p_k)}{\sigma_{\mathcal{A}}(q)^{g-1}\prod_{j=1}^{g}E(p_j,q)^{g-1}}.$$

Proof. The proof is motivated by Guàrdia's proof [6] of the analogous result in Arakelov theory, see also (3.1). For any $1\le k\le g$, let U_k be an open neighbourhood of p_k and $t_k\colon U_k\to\mathbb{C}$ a local coordinate such that $v_j=f_{jk}\mathrm{d}t_k$ for some functions f_{jk} on U_k. By the chain rule, we obtain

$$\lim_{q\to p_k}\frac{\theta(D)}{t_k(q)-t_k(p_k)}=-\sum_{j=1}^{g}\theta_j(D_k)f_{jk}(p_k).$$

The derivative of the prime form can be directly computed to be

$$\lim_{q\to p_k}\frac{E(p_k,q)}{t_k(q)-t_k(p_k)}=\frac{1}{\mathrm{d}t_k}. \tag{2.3}$$

Putting these equations together, we obtain

$$\lim_{q\to p_k}\frac{\theta(D)}{E(p_k,q)}=-\sum_{j=1}^{g}\theta_j(D_k)v_j(p_k).$$

Applying this to Fay's identity (2.2) and taking the product over all k gives

$$(-1)^g\prod_{k=1}^{g}\sum_{j=1}^{g}\theta_j(D_k)v_j(p_k)=\frac{(-1)^{\binom{g}{2}}\cdot c(\mathcal{A})^g\cdot\det(v_j(p_k))^g}{\prod_{j<k}^{g}E(p_j,p_k)^{g-2}\cdot\prod_{j=1}^{g}\sigma_{\mathcal{A}}(p_j)^{g-1}}.$$

If the function $q\mapsto\theta\big(\sum_{j=1}^{g}x_j-q\big)$ is not identically 0, it has the zero divisor $\sum_{j=1}^{g}x_j$. Thus, $\theta(D_k+p_l-q)$ has a zero of order 2 in $q=p_l$ as a function of q if $l\neq k$. Hence, we get $\sum_{j=1}^{g}\theta_j(D_k)v_j(p_l)=0$ whenever $k\neq l$. Thus, we can rewrite the left-hand side by

$$\prod_{k=1}^{g}\sum_{j=1}^{g}\theta_j(D_k)v_j(p_k)=J(D_1,\dots,D_g)\cdot\det v_j(p_k).$$

Therefore, we conclude

$$J(D_1,\dots,D_g)=\frac{(-1)^{\binom{g+1}{2}}c(\mathcal{A})^g\det(v_j(p_k))^{g-1}}{\prod_{j<k}^{g}E(p_j,p_k)^{g-2}\cdot\prod_{j=1}^{g}\sigma_{\mathcal{A}}(p_j)^{g-1}}.$$

Now, the proposition follows by combining this formula with Fay's identity (2.2). ∎

To obtain the second equation for $c(\mathcal{A})$, we denote as in the introduction

$$\eta=\det\begin{pmatrix}\theta_{jk}&\theta_j\\ \theta_k&0\end{pmatrix}.$$

Proposition 2.2. *For any $1\le l\le g$, it holds*

$$\eta(D_l)=(-1)^{\binom{g+1}{2}}c(\mathcal{A})^2\prod_{\substack{j=1\\ j\neq l}}^{g}\frac{\theta(D_l+p_j-q)}{\sigma_{\mathcal{A}}(q)\sigma_{\mathcal{A}}(p_j)E(p_j,q)^g}.$$

Proof. The proof is similar to the proof of Proposition 2.1, but more involved. As one can deduce the case $g=1$ directly from Proposition 2.1, we assume $g \geq 2$. By symmetry, we may also assume $l=g$. We will first prove the proposition for the special case $p_1 = \cdots = p_{g-1}$.

Let $t\colon U \to \mathbb{C}$ be a local coordinate for an open neighbourhood U of p_1 such that $v_j = f_j \mathrm{d}t$ for some functions f_j on U. The Wronskian determinant is locally given by

$$W(p_1) = \det\left(\frac{1}{(k-1)!}\frac{\mathrm{d}^{k-1} f_j}{\mathrm{d}t^{k-1}}\right)_{1\leq j,k\leq g}(p_1)$$

and defines a non-zero global section $\widetilde{v} = W \cdot (\mathrm{d}t)^{\otimes g(g+1)/2}$ of $\Omega_X^{g(g+1)/2}$. It can be directly computed that we have

$$\lim_{p_g\to p_1}\lim_{p_{g-1}\to p_1}\cdots\lim_{p_2\to p_1}\frac{\det(v_j(p_k))}{\prod_{j<k}^{g} E(p_j,p_k)} = \widetilde{v}(p_1).$$

Applying this to Fay's identity (2.2), we obtain

$$\theta(gp_1 - q) = \frac{c(\mathcal{A})\sigma_{\mathcal{A}}(q)\widetilde{v}(p_1)E(p_1,q)^g}{\sigma_{\mathcal{A}}(p_1)^g}. \tag{2.4}$$

Note that $\theta(gp_1-q)$ vanishes of order g at p_1 as a function in q. By L'Hôpital's rule and (2.3), we deduce

$$F(p_1) := \lim_{q\to p_1}\frac{\theta(gp_1-q)}{E(p_1,q)^g} = \frac{1}{g!}\left.\frac{\partial^g\theta(gp_1-q)}{\partial q^g}\right|_{q=p_1}\mathrm{d}t^{\otimes g}.$$

Lemma 2.3. *It holds* $F(p_1)^{g+1} = (-1)^{\binom{g+1}{2}}\eta((g-1)p_1)\cdot\widetilde{v}(p_1)^2$.

Proof. The idea of the proof is based on [3, Section 5]. Let us first shorten notations by setting

$$\theta_{j_1\dots j_n} = \frac{\partial^n\theta}{\partial z_{j_1}\cdots\partial z_{j_n}}((g-1)p_1), \qquad f_j^{(k)} = \frac{\mathrm{d}^k f_j}{\mathrm{d}t^k}(p_1).$$

Note that $\eta((g-1)p_1)\cdot\widetilde{v}(p_1)^2$ is given by $\mathrm{d}t^{\otimes g(g+1)}$ multiplied by the determinant of the following symmetric matrix:

$$\begin{pmatrix} \sum_{j,k=1}^g \theta_{jk} f_j f_k & \dots & \sum_{j,k=1}^g \theta_{jk} f_j \frac{f_k^{(g-1)}}{(g-1)!} & \sum_{j=1}^g \theta_j f_j \\ \vdots & \ddots & \vdots & \vdots \\ \sum_{j,k=1}^g \theta_{jk}\frac{f_j^{(g-1)}}{(g-1)!} f_k & \dots & \sum_{j,k=1}^g \theta_{jk}\frac{f_j^{(g-1)}}{(g-1)!}\frac{f_k^{(g-1)}}{(g-1)!} & \sum_{j=1}^g \theta_j \frac{f_j^{(g-1)}}{(g-1)!} \\ \sum_{j=1}^g \theta_j f_j & \dots & \sum_{j=1}^g \theta_j \frac{f_j^{(g-1)}}{(g-1)!} & 0 \end{pmatrix}.$$

Let s be any positive integer. For any vectors $a,b \in \mathbb{Z}^s$ we write $a \leq b$ if $a_i \leq b_i$ for all $1 \leq i \leq s$ and $a < b$ if $a \leq b$ and $a \neq b$. Further, we denote $|a| = \sum_{i=1}^s a_i$. For two vectors $m \in \mathbb{Z}^s$ and $n \in \mathbb{N}_0^s$, we define

$$h_{m,n} = \left.\frac{\partial^{|n|}\theta((g-1)p_1 + m_1(q_1-p_1)+\cdots+m_s(q_s-p_1))}{\partial q_1^{n_1}\cdots\partial q_s^{n_s}}\right|_{q_1=\cdots=q_s=p_1}.$$

Then we have $g!\cdot F(p_1) = h_{-1,g}\mathrm{d}t^{\otimes g}$ and $h_{m,n}=0$ for all n if $m \geq 0$ and $|m| \leq g-1$, since the involved theta function is constantly zero as a function in $(q_1,\dots,q_s)$ in these cases. By Faà di Bruno's formula, we can explicitly write

$$h_{m,n} = \sum_{0\leq l\leq n} m^l \left(\prod_{i=1}^s \frac{n_i!}{l_i!}\right)\sum_{k\in[1,\dots,g]^{|l|}}\theta_k \sum_{r_j}\prod_{j=1}^{|l|}\frac{f_{k_j}^{(r_j-1)}}{r_j!}, \tag{2.5}$$

where the last sum runs over all $|l|$-tuples of positive integers $r_1,\dots,r_{|l|}$ which sum to $r_1+\dots+r_{l_1}=n_1$, $r_{l_1+1}+\dots+r_{l_1+l_2}=n_2$ and so on. Further, m^l should be read as $\prod_{j=1}^s m_j^{l_j}$. Considering $h_{m,n}$ as a multi-degree n polynomial in m, we know that this polynomial has to be identically zero for $|n|\le g-1$. In particular, for all $|n|\le g-1$ the coefficients of the monomials of degree 1 and 2 in m vanish, which implies

$$\sum_{j=1}^g \theta_j f_j^{(a)} = \sum_{j,k=1}^g \theta_{jk} f_j^{(b)} f_k^{(c)} = 0$$

for non-negative integers a, b, c with $a\le g-2$ and $b+c\le g-3$ by choosing $n=a+1$ respectively $n=(b+1,c+1)$. In particular, the determinant of the matrix above is just a product of $g+1$ factors.

Next, we consider the case $s=1$ and $n=g$. We recall that the signed Stirling numbers $s(g,k)=(-1)^{g-k}\left[{g \atop k}\right]$ are defined as the coefficients of $\prod_{k=0}^{g-1}(X-k)=\sum_{k=0}^g s(g,k)X^k$. Since $h_{m,g}$ is a degree g polynomial vanishing for $m=0,\dots,g-1$, its coefficients have to be given by the signed Stirling numbers multiplied by a common non-zero factor. The factor can be obtained by computing the coefficients of m or m^2 in equation (2.5)

$$\begin{aligned} h_{m,g} &= -\frac{1}{(g-1)!}\sum_{j=1}^g \theta_j f_j^{(g-1)} \sum_{k=1}^g (-1)^k \left[{g \atop k}\right] m^k \\ &= \frac{1}{2(g-1)!H_{g-1}} \sum_{r=1}^{g-1}\binom{g}{r} \sum_{j,k=1}^g \theta_{jk} f_j^{(r-1)} f_k^{(g-r-1)} \sum_{k=1}^g (-1)^k \left[{g \atop k}\right] m^k, \end{aligned}$$

for $H_k=\sum_{j=1}^k \frac{1}{j}$ the k-th harmonic number. In particular, we get

$$\frac{h_{-1,g}}{g!} = -\sum_{j=1}^g \theta_j \frac{f_j^{(g-1)}}{(g-1)!} = \frac{g}{2H_{g-1}}\sum_{r=1}^{g-1}\frac{1}{r(g-r)}\sum_{j,k=1}^g \theta_{jk}\frac{f_j^{(r-1)}}{(r-1)!}\frac{f_k^{(g-r-1)}}{(g-r-1)!}.$$

We would like to show that the last sum does not depend on the choice of r. For this purpose, we consider the case $s=2$ and $n=(r,g-r)$ for some $1\le r\le g-1$. The polynomial $h_{m,n}$ is of multi-degree n and vanishes for all $0\le m<n$. Hence, its coefficients have to be given by the products of signed Stirling numbers $s(g,j)\cdot s(g,k)$ multiplied by a common non-zero factor. The factor can be obtained by considering the coefficient of m_1m_2 in equation (2.5)

$$h_{m,(r,g-r)} = \frac{\sum_{j,k=1}^g \theta_{jk} f_j^{(r-1)} f_k^{(g-r-1)}}{(r-1)!(g-r-1)!}\sum_{j,k\ge 1}(-1)^{j+k}\left[{r \atop j}\right]\left[{g-r \atop k}\right] m_1^j m_2^k.$$

Note that the top degree coefficient of $h_{m,(r,g-r)}$ is given by

$$\sum_{k\in[1,\dots,g]^g}\theta_k \prod_{j=1}^g f_{k_j},$$

which does not depend on the choice of r. Thus,

$$\sum_{j,k=1}^g \theta_{jk}\frac{f_j^{(r-1)}}{(r-1)!}\frac{f_k^{(g-r-1)}}{(g-r-1)!}$$

does not depend on the choice of r, either. Therefore, the determinant of the matrix above is given by

$$(-1)^{\binom{g-1}{2}+1}\left(\sum_{j=1}^{g}\theta_j\frac{f_j^{(g-1)}}{(g-1)!}\right)^2\left(\sum_{j,k=1}^{g}\theta_{jk}\frac{f_j^{(g-2)}}{(g-2)!}f_k\right)^{g-1}.$$

Since both expressions in the brackets can be identified with $h_{-1,g}/g!$, we obtain

$$F(p_1)^{g+1}=\left(\frac{h_{-1,g}}{g!}\right)^{g+1}\mathrm{d}t^{\otimes g(g+1)}=(-1)^{\binom{g+1}{2}}\eta((g-1)p_1)\cdot\widetilde{v}(p_1)^2,$$

which proves the lemma. ■

We continue with the proof of the proposition. Equation (2.4) implies

$$F(p_1)=\frac{c(\mathcal{A})\widetilde{v}(p_1)}{\sigma_{\mathcal{A}}(p_1)^{g-1}}.$$

Applying the lemma, we obtain

$$\eta((g-1)p_1)=(-1)^{\binom{g+1}{2}}\frac{c(\mathcal{A})^{g+1}\widetilde{v}(p_1)^{g-1}}{\sigma_{\mathcal{A}}(p_1)^{(g-1)(g+1)}}.$$

Combining this with equation (2.4), we obtain

$$\eta((g-1)p_1)=(-1)^{\binom{g+1}{2}}c(\mathcal{A})^2\left(\frac{\theta(gp_1-q)}{E(p_1,q)^g\sigma_{\mathcal{A}}(q)\sigma_{\mathcal{A}}(p_1)}\right)^{g-1},$$

This proves the proposition in the case $p_1=\cdots=p_{g-1}$.

Next we prove the general case. We write $v_j'\in H^0\big(X,\Omega_X^{\otimes 2}\big)$ for the two-fold holomorphic differential given locally by $\frac{\mathrm{d}f_j}{\mathrm{d}t}(\mathrm{d}t\otimes\mathrm{d}t)$ for some local coordinate t and f_j such that $v_j=f_j\mathrm{d}t$. Note that v_j' is independent of the choice of the local coordinate t and hence, it defines a global two-fold holomorphic differential. Further, we define for any integer $1\le e\le g-1$ the $g\times g$ matrix

$$\mathscr{D}_e(p_1,\dots,p_{g-1})=\det\begin{pmatrix}v_1(p_1)&\dots&v_1(p_{g-1})&v_1'(p_e)\\\vdots&\ddots&\vdots&\vdots\\v_g(p_1)&\dots&v_g(p_{g-1})&v_g'(p_e)\end{pmatrix}.$$

We get the following limit:

$$\lim_{p_g\to p_e}\frac{\det(v_j(p_k))}{E(p_e,p_g)}=\mathscr{D}_e(p_1,\dots,p_{g-1}).$$

Applying this to Fay's identity (2.2), we obtain

$$\theta(D_g+p_e-q)=\frac{c(\mathcal{A})\mathscr{D}_e(p_1,\dots,p_{g-1})\sigma_{\mathcal{A}}(q)E(p_e,q)\prod_{j=1}^{g-1}E(p_j,q)}{\sigma_{\mathcal{A}}(p_e)\prod_{j\neq e}^{g-1}E(p_j,p_e)\prod_{j<k}^{g-1}E(p_j,p_k)\prod_{j=1}^{g-1}\sigma_{\mathcal{A}}(p_j)},\tag{2.6}$$

and hence for the product over all e

$$\prod_{e=1}^{g-1}\theta(D_g+p_e-q)=\frac{c(\mathcal{A})^{g-1}\prod_{e=1}^{g-1}\mathscr{D}_e(p_1,\dots,p_{g-1})\sigma_{\mathcal{A}}(q)^{g-1}\prod_{j=1}^{g-1}E(p_j,q)^g}{(-1)^{\binom{g-1}{2}}\prod_{j<k}^{g-1}E(p_j,p_k)^{g+1}\prod_{j=1}^{g-1}\sigma_{\mathcal{A}}(p_j)^g}.\tag{2.7}$$

Since $\theta(D_g+p_e-q)$ vanishes of at least second order at p_e as a function in q, we can define

$$T_e(p_1,\dots,p_{g-1}) := \lim_{q\to p_e}\frac{\theta(D_g+p_e-q)}{E(p_e,q)^2}$$
$$=\frac{1}{2}\sum_{j,k=1}^{g}\theta_{jk}(D_g)v_j(p_e)v_k(p_e)-\frac{1}{2}\sum_{j=1}^{g}\theta_j(D_g)v_j'(p_e), \tag{2.8}$$

where the second equality follows by L'Hôpital's rule and Faà di Bruno's formula.

Lemma 2.4. *It holds*

$$\prod_{e=1}^{g-1}T_e(p_1,\dots,p_{g-1})^{g+1}=(-\eta(D_g))^{g-1}\prod_{e=1}^{g-1}\mathscr{D}_e(p_1,\dots,p_{g-1})^2.$$

Proof. The proof is very similar to the proof of Lemma 2.3. We write $v_k=f_{j,k}\mathrm{d}t$ locally around the point p_j and shorten notations by setting

$$\theta_{j_1\dots j_n}=\frac{\partial^n\theta}{\partial z_{j_1}\cdots\partial z_{j_n}}(D_g),\qquad f_{j,k}=f_{j,k}(p_j),\qquad f_{j,k}^{(r)}=\frac{\mathrm{d}^r f_{j,k}}{\mathrm{d}t^r}(p_j),\qquad f'_{j,k}=f^{(1)}_{j,k}.$$

We obtain $\eta(D_g)\mathscr{D}_e(p_1,\dots,p_{g-1})^2$ as $\mathrm{d}t^{\otimes(2g+2)}$ multiplied by the determinant of the symmetric matrix

$$M=\sum_{j,k=1}^{g}\begin{pmatrix}\theta_{jk}f_{1,j}f_{1,k} & \dots & \theta_{jk}f_{1,j}f_{g-1,k} & \theta_{jk}f_{1,j}f'_{e,k} & \frac{1}{g}\theta_j f_{1,j}\\ \vdots & \ddots & \vdots & \vdots & \vdots\\ \theta_{jk}f_{g-1,j}f_{1,k} & \dots & \theta_{jk}f_{g-1,j}f_{g-1,k} & \theta_{jk}f_{g-1,j}f'_{e,k} & \frac{1}{g}\theta_j f_{g-1,j}\\ \theta_{jk}f'_{e,j}f_{1,k} & \dots & \theta_{jk}f'_{e,j}f_{g-1,k} & \theta_{jk}f'_{e,j}f'_{e,k} & \frac{1}{g}\theta_j f'_{e,j}\\ \frac{1}{g}\theta_j f_{1,j} & \dots & \frac{1}{g}\theta_j f_{g-1,j} & \frac{1}{g}\theta_j f'_{e,j} & 0\end{pmatrix}.$$

For any $m\in\mathbb{Z}^{g-1}$ and $n\in\mathbb{N}_0^{g-1}$, we define

$$h'_{m,n}=\frac{\partial^{|n|}\theta(D_g+m_1(q_1-p_1)+\dots+m_{g-1}(q_{g-1}-p_{g-1}))}{\partial q_1^{n_1}\cdots\partial q_{g-1}^{n_{g-1}}}\Big|_{q_1=p_1,\dots,q_{g-1}=p_{g-1}}.$$

We have $h'_{m,n}=0$ for $0\le m\le(1,\dots,1)$ and all n, since the involved theta functions are constantly zero as functions in $q_1,\dots,q_{g-1}$. Faà di Bruno's formula gives

$$h'_{m,n}=\sum_{0\le l\le n}m^l\left(\prod_{i=1}^{g-1}\frac{n_i!}{l_i!}\right)\sum_{k\in[1,\dots,g]^l}\theta_k\sum_{r_{i,j}}\prod_{i=1}^{g-1}\prod_{j=1}^{l_i}\frac{f_{i,k_{i,j}}^{(r_{i,j}-1)}(p_i)}{r_{i,j}!},$$

where $[1,\dots,g]^l$ stands for $\prod_{i=1}^{g-1}[1,\dots,g]^{l_i}$ and the last sum runs over all $|l|$-tuples of positive integers $r_{1,1},\dots,r_{g-1,l_{g-1}}$, which sum to $r_{i,1}+\dots+r_{i,l_i}=n_i$ for every i. Considering $h'_{m,n}$ as a multi-degree n polynomial in m, it has to be identically zero for $n\le(1,\dots,1)$. In particular, the coefficients of the monomials of degree 1 and 2 in m

$$\sum_{j=1}^{g}\theta_j f_{a,j}=\sum_{j,k=1}^{g}\theta_{jk}f_{a,j}f_{b,k}=0$$

vanish for all $a\ne b$. Hence, the determinant of the matrix M is just a product of $g+1$ factors.

If we set $n(e) = (0, \dots, 0, 2, 0 \dots, 0)$, where the 2 occurs at the e-th position, the degree 2 polynomial

$$h'_{m,n(e)} = m_e^2 \sum_{j,k=1}^{g} \theta_{jk} f_{e,j} f_{e,k} + m_e \sum_j \theta_j f'_{e,j}$$

has to vanish for $m_e = 0, 1$, which implies $\sum_{j,k=1}^{g} \theta_{jk} f_{e,j} f_{e,k} = -\sum_{j=1}^{g} \theta_j f'_{e,j}$ for all $e < g$. Applying this to equation (2.8), we obtain

$$T_e(p_1, \dots, p_{g-1}) = \sum_{j,k=1}^{g} \theta_{jk} f_{e,j} f_{e,k} \mathrm{d}t^{\otimes 2}.$$

On the other hand, we can compute the determinant of the matrix M as

$$\eta(D_g) \mathscr{D}_e(p_1, \dots, p_{g-1})^2 = -\left(\sum_{j,k=1}^{g} \theta_{jk} f_{e,j} f_{e,k} \right)^2 \left(\prod_{i=1}^{g-1} \sum_{j,k=1}^{g} \theta_{jk} f_{i,j} f_{i,k} \right) \mathrm{d}t^{\otimes(2g+2)}.$$

Now the lemma follows by comparing the last two equations after taking the products over all $e < g$. ■

We continue the proof of the proposition. Applying the definition in equation (2.8) to equation (2.6), we get

$$T_e(p_1, \dots, p_{g-1}) = \frac{c(\mathcal{A}) \mathscr{D}_e(p_1, \dots, p_{g-1})}{\prod_{j<k}^{g-1} E(p_j, p_k) \prod_{j=1}^{g-1} \sigma_{\mathcal{A}}(p_j)}.$$

If we multiply over all $1 \le e \le g-1$ and apply the lemma, we get

$$\eta(D_g)^{g-1} = (-1)^{(g-1)} \frac{c(\mathcal{A})^{(g-1)(g+1)} \prod_{e=1}^{g-1} \mathscr{D}_e(p_1, \dots, p_{g-1})^{g-1}}{\prod_{j<k}^{g-1} E(p_j, p_k)^{(g-1)(g+1)} \prod_{j=1}^{g-1} \sigma_{\mathcal{A}}(p_j)^{(g-1)(g+1)}}.$$

Hence, we can conclude by combining with equation (2.7)

$$\eta(D_g)^{g-1} = \left((-1)^{\binom{g+1}{2}} c(\mathcal{A})^2 \prod_{j=1}^{g-1} \frac{\theta(D_g + p_j - q)}{\sigma_{\mathcal{A}}(q) \sigma_{\mathcal{A}}(p_j) E(p_j, q)^g} \right)^{g-1}.$$

This gives the proposition up to a $(g-1)$-th root of unity. But by the special case $p_1 = \dots = p_{g-1}$ we know that this root of unity must be 1. ■

Corollary 2.5. *We have the following equalities of meromorphic sections:*

$$\prod_{j \ne k}^{g} \frac{\theta(g p_j - q)}{\theta(g p_j - p_k)} = (-1)^{g\binom{g}{2}} \frac{\sigma_{\mathcal{A}}(q)^{g(g-1)} \prod_{j=1}^{g} E(p_j, q)^{g(g-1)}}{\prod_{j=1}^{g} \sigma_{\mathcal{A}}(p_j)^{g-1} \prod_{j<k} E(p_j, p_k)^{2g}} = \prod_{j \ne k}^{g} \frac{\theta(D_j + p_k - q)}{\theta(D_j + p_k - p_j)}.$$

Proof. The first equality follows from Proposition 2.2 applied to divisors of the form $(g-1)p_j$ by comparing it for different choices of q, namely $q = p_k$ and $q = q$, and multiplying over all $j \ne k$. The second equality follows similar, but using the proposition for the divisors D_j instead of $(g-1)p_j$. Note that the $(g-1)$-th root of unity, which may occur, does not depend on the points p_j by continuity. Hence, it cancels out in the quotient. ■

Proof of Theorem 1.1. The first two formulas in Theorem 1.1 are now obtained as combinations of the formulas in Propositions 2.1 and 2.2 and Corollary 2.5. The third formula results by comparing the formula in Proposition 2.2 for the divisors $D = \sum_{j=1}^{g} p_j - q$ and $D = g p_k - q$. ■

3 Analogous results in Arakelov theory

In this section, we will discuss normed variants of the formulas in Theorem 1.1 and Section 2 in Arakelov theory of Riemann surfaces. We continue the notation from the previous section. In Arakelov theory, one is interested in canonical norms for sections of line bundles. For the sections θ, J and η, the norms

$$\|\theta\|(z) = \det(Y)^{1/4} \exp\big(-\pi {}^{\mathrm{t}}y Y^{-1} y\big) \cdot |\theta|(z),$$
$$\|J\|(w_1, \ldots, w_g) = \det(Y)^{(g+2)/4} \exp\left(-\pi \sum_{k=1}^{g} {}^{\mathrm{t}}y_k Y^{-1} y_k\right) |J(w_1, \ldots, w_g)|,$$
$$\|\eta\|(z) = \det(Y)^{(g+5)/4} \exp\big(-\pi(g+1) {}^{\mathrm{t}}y Y^{-1} y\big) \cdot |\eta|(z),$$

were given by Faltings [4, p. 401], Guàrdia [6, Definition 2.1], respectively de Jong [2, Section 2]. Here we denote $Y = \mathrm{Im}(\tau)$, $y = \mathrm{Im}(z)$ and $y_k = \mathrm{Im}(w_k)$ for all k. Arakelov [1] has given a norm for the canonical section of the diagonal bundle $\mathcal{O}_{X^2}(\Delta)$. This norm is the Arakelov–Green function $G(\cdot,\cdot)$ defined by

$$\frac{\partial_p \overline{\partial}_p}{2\pi \mathrm{i}} \log G(p,q)^2 = \mu(p) - \delta_q(p) \qquad \text{and} \qquad \int_X \log G(p,q) \mu(p) = 0,$$

where $\mu = \frac{\mathrm{i}}{2g} \sum_{j,k=1}^{g} \big(Y^{-1}\big)_{jk} v_j v_k$ is the canonical $(1,1)$-form associated to X.

Faltings has given in [4, p. 402] an analogue of Fay's equation (2.2) for these norms to define his δ-invariant $\delta(X)$, which he used to prove an arithmetic Noether formula. Guàrdia [6, Corollary 3.6] has found an alternative description for $\delta(X)$, which is the following analogue of Proposition 2.1:

$$\|J\|(D_1, \ldots, D_g) = e^{-\delta(X)/8} \frac{\|\theta\|(D)^{g-1} \prod_{j<k} G(p_j, p_k)}{\prod_{j=1}^{g} G(p_j, q)^{g-1}}. \tag{3.1}$$

De Jong [2, Theorem 4.4] has given another formula, which can be expressed as the following analogue of Proposition 2.2:

$$\|\eta\|(D_l) = e^{-\delta(X)/4} \prod_{j=1}^{g-1} \frac{\|\theta\|(D_l + p_j - q)}{G(p_j, q)^g},$$

see also [9, equation (2.7)] for this expression. In the same way as in the proof of Theorem 1.1, we can combine both formulas to obtain an analogue of the formulas in the theorem.

Proposition 3.1. *With the notation as above, we have the following formulas:*

(*i*) $$\prod_{k=1}^{g} \|\eta\|(D_k) = \left(\frac{\|J\|(D_1, \ldots, D_g)}{\theta(D)^{g-1}}\right)^{2g} \prod_{j \neq k}^{g} \frac{\|\theta\|(D_j + p_k - q)^2}{\|\theta\|(D_j + p_k - p_j)},$$

(*ii*) $$\prod_{k=1}^{g} \|\eta\|((g-1)p_k) = \left(\frac{\|J\|(D_1, \ldots, D_g)}{\|\theta\|(D)^{g-1}}\right)^{2g} \prod_{j \neq k}^{g} \frac{\|\theta\|(gp_j - q)^2}{\|\theta\|(gp_j - p_k)},$$

(*iii*) $$\|\eta\|(D_g)^{g-1} = \prod_{k=1}^{g-1} \left(\|\eta\|((g-1)p_k) \left(\frac{\|\theta\|(D_g + p_j - q)}{\|\theta\|(gp_j - q)}\right)^{g-1}\right).$$

Of course, the proposition can also be obtained by taking the norms in the formulas of Theorem 1.1. But since the formulas for $\delta(X)$ have been known before, it shows how the proof of the theorem was motivated.

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