Symmetry, Integrability and Geometry: Methods and Applications SIGMA **20** (2024), 047, 70 pages

Unrestricted Quantum Moduli Algebras, II: Noetherianity and Simple Fraction Rings at Roots of 1

Stéphane BASEILHAC and Philippe ROCHE

IMAG, Univ Montpellier, CNRS, Montpellier, France
E-mail: *stephane.baseilhac@umontpellier.fr*, *philippe.roche@umontpellier.fr*

Received May 11, 2023, in final form May 07, 2024; Published online June 06, 2024
https://doi.org/10.3842/SIGMA.2024.047

Abstract. We prove that the quantum graph algebra and the quantum moduli algebra associated to a punctured sphere and complex semisimple Lie algebra $\mathfrak{g}$ are Noetherian rings and finitely generated rings over $\mathbb{C}(q)$. Moreover, we show that these two properties still hold on $\mathbb{C}\big[q,q^{-1}\big]$ for the integral version of the quantum graph algebra. We also study the specializations $\mathcal{L}_{0,n}^{\epsilon}$ of the quantum graph algebra at a root of unity ϵ of odd order, and show that $\mathcal{L}_{0,n}^{\epsilon}$ and its invariant algebra under the quantum group $U_\epsilon(\mathfrak{g})$ have classical fraction algebras which are central simple algebras of PI degrees that we compute.

Key words: quantum groups; invariant theory; character varieties; skein algebras; TQFT

2020 Mathematics Subject Classification: 16R30; 17B37; 20G42; 57M27; 57R56; 81R50

1 Introduction

This paper is the second part of our work, initiated in [18], on the quantum graph algebra $\mathcal{L}_{g,n}(\mathfrak{g})$ and the quantum moduli algebra $\mathcal{M}_{g,n}(\mathfrak{g})$, which are associated to a surface $\Sigma_{g,n+1}$ of genus g with $n+1$ punctures and a complex semisimple Lie algebra $\mathfrak{g}$. As in [18], we focus in this paper on punctured spheres ($g=0$, $n\geq 1$). From now on we fix $\mathfrak{g}$, and when no confusion may arise we omit it from the notations of the various algebras.

The algebras $\mathcal{L}_{g,n}$ and $\mathcal{M}_{g,n}$ are defined over the field $\mathbb{C}(q)$. They were introduced in the mid 90's by Alekseev–Grosse–Schomerus [2, 3] and Buffenoir–Roche [29, 30] by a method called *combinatorial quantization*. By this method, the pair formed by $\mathcal{L}_{g,n}$ and $\mathcal{M}_{g,n}$ appear naturally as a q-deformation of the Fock–Rosly [55] lattice model of the algebra of functions on the "classical" moduli space $\mathcal{M}_{g,n}^{cl}$ of flat $\mathfrak{g}$-connections on the surface $\Sigma_{g,n+1}$.

In [18], we showed that both $\mathcal{L}_{0,n}$ and $\mathcal{M}_{0,n}$ have integral forms $\mathcal{L}_{0,n}^{A}$ and $\mathcal{M}_{0,n}^{A}$ defined over the ring $A=\mathbb{C}\big[q,q^{-1}\big]$ (in fact we could have taken $\mathbb{Q}\big[q,q^{-1}\big]$ or $\mathbb{Z}\big[q,q^{-1}\big]$ as ground ring, see Section 1.1). One can thus consider the specializations of these algebras at $q=\epsilon\in\mathbb{C}^\times$, which we denote by $\mathcal{L}_{0,n}^{\epsilon}$ and $\mathcal{M}_{0,n}^{A,\epsilon}$ respectively. The algebra $\mathcal{L}_{0,n}^{A}$ is endowed with an action of the Lusztig integral form $U_A^{\mathrm{res}}=U_A^{\mathrm{res}}(\mathfrak{g})$ of the quantum group $U_q=U_q(\mathfrak{g})$, and $\mathcal{M}_{0,n}^{A}$ is the subalgebra of invariant elements under this action. Therefore,

$$\mathcal{M}_{0,n}^{A}:=\big(\mathcal{L}_{0,n}^{A}\big)^{U_A^{\mathrm{res}}},\qquad \mathcal{M}_{0,n}:=\mathcal{L}_{0,n}^{U_q}=\mathcal{M}_{0,n}^{A}\bigotimes_{A}\mathbb{C}(q).$$

The definition of $\mathcal{L}_{0,n}^{A}$ is based on the original combinatorial quantization method, together with twists of module-algebras and Lusztig's theory of canonical bases of quantum groups [83]. This allows us to address the structure and representation theory of $\mathcal{L}_{0,n}^{A}$ and $\mathcal{M}_{0,n}^{A}$ by means of quantum groups, following ideas of classical invariant theory. In particular, we obtained that $\mathcal{L}_{0,n}$ and $\mathcal{L}_{0,n}^{\epsilon}$ have no nontrivial zero divisors (and therefore do as well the subalgebras $\mathcal{M}_{0,n}$, $\mathcal{L}_{0,n}^{A}$, $\mathcal{M}_{0,n}^{A}$, and $\big(\mathcal{L}_{0,n}^{\epsilon}\big)^{U_\epsilon^{\mathrm{res}}}$, where $U_\epsilon^{\mathrm{res}}$ is the specialization of U_A^{res} at $q=\epsilon$).

Also, by extending the quantum coadjoint action of De Concini–Kac–Procesi [39, 40, 42], we described in the $\mathfrak{sl}_2$ case an action by derivations of the center $\mathcal{Z}(\mathcal{L}^{\epsilon}_{0,n})$ of $\mathcal{L}^{\epsilon}_{0,n}$ on $\mathcal{L}^{\epsilon}_{0,n}$, and we defined a subalgebra $\mathcal{Z}(\mathcal{L}^{\epsilon}_{0,n})^{\mathcal{G}} \subset \mathcal{Z}(\mathcal{L}^{\epsilon}_{0,n})$, which is a finite extension of the ring of regular functions on the character variety of the sphere with $(n+1)$ punctures (see [18, Corollary 7.20 and Theorem 8.8]). Moreover, from these results we derived an action by derivation of $\mathcal{Z}(\mathcal{L}^{\epsilon}_{0,n})^{\mathcal{G}}$ on $\mathcal{M}^{A,\epsilon}_{0,n}(\mathfrak{sl}_2)$.

Representations of a quotient (the semisimplification) of $\mathcal{M}^{A,\epsilon}_{g,n}$ were already constructed and classified in [4]; they involve only the irreducible representations of the finite-dimensional "small" quantum group $\mathfrak{u}_\epsilon(\mathfrak{g})$. Moreover, [4] deduced from these representations of $\mathcal{M}^{A,\epsilon}_{g,n}$ a family of representations of the mapping class groups of surfaces, that is equivalent to the one associated to the Witten–Reshetikhin–Turaev TQFT [95, 106]. Recently, representations of another, larger quotient of $\mathcal{M}^{A,\epsilon}_{g,n}$, and the corresponding representations of the mapping class groups of surfaces, were constructed in [52, 53]. These representations are equivalent to those previously obtained by Lyubashenko–Majid [85], and are associated to the TQFT defined in [44, 45]. In the $\mathfrak{sl}_2$ case, they involve the irreducible and also the principal indecomposable representations of the small quantum group $\mathfrak{u}_\epsilon(\mathfrak{sl}_2)$. The related link and 3-manifold invariants coincide with those of [21, 90].

In general, the representation theory of $\mathcal{M}^{A,\epsilon}_{g,n}$ is by now far from being understood. Because $\mathcal{M}^{A,\epsilon}_{g,n}$ deforms the classical moduli space $\mathcal{M}^{cl}_{g,n}$, it is natural to expect that its representation theory provides $(2+1)$-dimensional TQFTs for 3-manifolds endowed with general flat $\mathfrak{g}$-connections, extending the known TQFTs based on quantum groups (where purely topological ones correspond to the trivial connection). A family of such invariants, called *quantum hyperbolic invariants*, has already been defined for $\mathfrak{g} = \mathfrak{sl}_2$ by means of certain $6j$-symbols, *Deus ex machina*; they are closely connected to classical Chern–Simons theory, provide generalized volume conjectures, and contain quantum Teichmüller theory (see [9, 10, 11, 12, 13, 14, 15]). It is part of our present program, initiated in [8], to shed light on these invariants and to generalize them to arbitrary $\mathfrak{g}$ by developing the representation theory of $\mathcal{M}^{A,\epsilon}_{g,n}$.

The quantum moduli algebras have also been recognized as central objects from the viewpoints of factorization homology [22], multiplicative quiver varieties [58] and (stated) skein theory [16, 33, 36, 54]. In another direction, one may expect that the equivalence proved in [89] between combinatorial quantisation for the Drinfeld double $D(H)$ of a finite-dimensional semisimple Hopf algebra H, and Kitaev's lattice model in topological quantum computation, can be extended to the setup of quantum moduli algebras.

In the present paper, we study $\mathcal{L}_{0,n}$, its integral form $\mathcal{L}^{A}_{0,n}$, and the specialization $\mathcal{L}^{\epsilon}_{0,n}$ of $\mathcal{L}^{A}_{0,n}$ at $q = \epsilon$ a primitive root of unity of odd order. We study also the subalgebras of invariant elements $\mathcal{M}_{0,n} = \mathcal{L}^{U_q}_{0,n}$ and $(\mathcal{L}^{\epsilon}_{0,n})^{U_\epsilon}$. Here, U_ϵ is the specialization of U_A at $q = \epsilon$, where U_A is the De Concini–Kac integral form of U_q (see Section 1.1). Our results hold for every complex semisimple Lie algebra $\mathfrak{g}$. The main ones are proofs that $\mathcal{L}_{0,n}$, $\mathcal{L}^{A}_{0,n}$ and $\mathcal{M}_{0,n}$ are Noetherian and finitely generated rings (see Theorem 1.1), and that the classical fraction algebras of $\mathcal{L}^{\epsilon}_{0,n}$ and $(\mathcal{L}^{\epsilon}_{0,n})^{U_\epsilon}$ are central simple algebras of PI degrees l^{nN} and $l^{N(n-1)-m}$ respectively (see Theorem 1.3). Here, m and N are the rank and the number of positive roots of $\mathfrak{g}$.

In the sequel [16] to this paper, in collaboration with M. Faitg, we extend Theorem 1.1 to the algebras $\mathcal{L}_{g,n}$ and $\mathcal{M}_{g,n}$, associated to arbitrary finite type surfaces (arbitrary genus and number of punctures). Also, we show that $\mathcal{M}_{g,n}$ is isomorphic to the $\mathfrak{g}$-skein algebra of $\Sigma_{g,n+1}$, and $\mathcal{L}_{g,n}$ to the stated skein algebra of the compact surface $\overline{\Sigma}_{g,n+1}$ with one boundary component and one marked point on the boundary component. This was proved for $\mathfrak{g} = \mathfrak{sl}_2$ in [54]. In this specific case $\mathfrak{g} = \mathfrak{sl}_2$, the fact that the stated skein algebra of any finite type surface is Noetherian and finitely generated was proved in [80]. Still in the $\mathfrak{sl}_2$ case, for related results, e.g., on non-zero divisors and computation of PI degrees, see [23, 24, 57, 64, 73, 74, 75, 78]. For recent results on $\mathfrak{g} = \mathfrak{sl}_n$, see [79, 105].

By using the analysis developed in the present paper for $\mathcal{L}_{0,n}^A$, one can define the integral form $\mathcal{L}_{g,n}^A$ as well, and show that it is a Noetherian and finitely generated ring. We do not have a proof yet of these properties for the algebra $\mathcal{M}_{0,n}^A$, which seems to be much more difficult to handle. We note that there is a strict inclusion $\mathcal{M}_{0,n}^{A,\epsilon} \subset \big(\mathcal{L}_{0,n}^{\epsilon}\big)^{U_\epsilon}$. This is discussed after Theorem 1.2. In [17], we study further properties of $\big(\mathcal{L}_{g,n}^{\epsilon}\big)^{U_\epsilon}$, and we consider also the subalgebra $\mathcal{M}_{g,n}^{A,\epsilon}$.

1.1 Statement of results

Let us recall a few notations and facts from [18]. Let U_q be the simply-connected quantum group of $\mathfrak{g}$, defined over the field $\mathbb{C}(q)$. From U_q one can define a U_q-module algebra $\mathcal{L}_{0,n}$, called (quantum, daisy) *graph algebra*, where U_q acts by means of a right coadjoint action. The set of invariant elements of $\mathcal{L}_{0,n}$ for this action is an algebra; we denote it $\mathcal{M}_{0,n} := \mathcal{L}_{0,n}^{U_q}$ and call it *quantum moduli algebra*. As a $\mathbb{C}(q)$-module $\mathcal{L}_{0,n}$ is just $\mathcal{O}_q^{\otimes n}$, where $\mathcal{O}_q = \mathcal{O}_q(G)$ is the standard quantum function algebra of the connected and simply-connected Lie group G with Lie algebra $\mathfrak{g}$. The product of $\mathcal{L}_{0,n}$ is obtained by twisting both the product of each factor $\mathcal{O}_q$ and the product between them. It is equivariant with respect to a (right) coadjoint action of U_q, which defines the structure of U_q-module of $\mathcal{L}_{0,n}$.

The module algebra $\mathcal{L}_{0,n}$ has an integral form $\mathcal{L}_{0,n}^A$, which is defined over $A = \mathbb{C}\big[q, q^{-1}\big]$, and endowed with an (coadjoint) action of the Lusztig [82] integral form U_A^{res} of U_q. It is obtained by replacing $\mathcal{O}_q$ in the construction of $\mathcal{L}_{0,n}$ with the restricted dual $\mathcal{O}_A$ of the integral form U_A^{res}, or equivalently with the restricted dual of the integral form Γ of U_q defined by De Concini–Lyubashenko [41]. Since U_A^{res} contains the De Concini–Kac [39] integral form U_A, and U_A has the same set of invariant elements in $\mathcal{L}_{0,n}^A$, we systematically denote the latter

$$\mathcal{M}_{0,n}^A := \big(\mathcal{L}_{0,n}^A\big)^{U_A} \qquad \Big(= \big(\mathcal{L}_{0,n}^A\big)^{U_A^{\mathrm{res}}}\Big).$$

We call $\mathcal{M}_{0,n}^A$ the *integral* quantum moduli algebra.

A cornerstone of the theory of $\mathcal{M}_{0,n}$ is a map Φ_n originally due to Alekseev [1], building on works of Drinfeld [48] and Reshetikhin and Semenov-Tian-Shansky [94]. In [18], we showed that Φ_n eventually provides isomorphisms of module algebras and algebras respectively,

$$\Phi_n\colon\ \mathcal{L}_{0,n} \to \big(U_q^{\otimes n}\big)^{\mathrm{lf}}, \qquad \Phi_n\colon\ \mathcal{M}_{0,n} \to \big(U_q^{\otimes n}\big)^{U_q},$$

where $U_q^{\otimes n}$ is endowed with a right adjoint action of U_q, and $\big(U_q^{\otimes n}\big)^{\mathrm{lf}}$ is the subalgebra of locally finite elements with respect to this action. When $n = 1$ the algebra U_q^{lf} has been studied in great detail by Joseph–Letzter [61, 62, 63]; we will use simplified proofs of their results, obtained in [104].

All the material we need about the results discussed above is described in [18], and recalled in Sections 2.1 and 2.2.

Our first result, proved in Section 3, is the following.

Theorem 1.1. *$\mathcal{L}_{0,n}$, $\mathcal{M}_{0,n}$ and the integral form $\mathcal{L}_{0,n}^A$ are Noetherian rings, and finitely generated rings.*

It follows immediately from the theorem that the specializations $\mathcal{L}_{0,n}^{\epsilon}$, $\epsilon \in \mathbb{C}^\times$, are Noetherian and finitely generated rings as well. In [18] we proved that all these algebras (and therefore $\mathcal{M}_{0,n}^A$ and $\mathcal{M}_{0,n}^{A,\epsilon}$) have no nontrivial zero divisors.

Because the construction of the integral form $\mathcal{L}_{0,n}^A$ is based on the Kashiwara–Lusztig theory of canonical bases, we could have defined $\mathcal{L}_{0,n}^A$ over the ground ring $\mathbb{Z}\big[q, q^{-1}\big]$, and Theorem 1.1 for $\mathcal{L}_{0,n}^A$ holds true as well in this generality. Since we are mainly interested in the representation theory of the specializations $\mathcal{L}_{0,n}^{\epsilon}$ and $\mathcal{M}_{0,n}^{A,\epsilon}$, which will be addressed in [17], the choice of $A=\mathbb{C}\big[q,q^{-1}\big]$ is natural. Note however that the proof of Proposition 2.18 uses that $\mathbb{C}\big[q,q^{-1}\big]$ is a PID.

We describe the background material on canonical bases in Section 2.2.2; we have tried to make the exposition pedestrian and self-contained, so as to be more accessible to non experts.

After we finished this work, we discovered that [47] already proved that $\mathcal{L}_{0,1}(\mathfrak{gl}(n))$ and $\mathcal{L}_{0,n}(\mathfrak{gl}(2))$ are Noetherian and finitely generated rings. Our approach here is completely different. For $\mathcal{L}_{0,n}$, we adapt the proof given by Voigt–Yuncken [104] of a result of Joseph [61], which asserts that U_q^{lf} is a Noetherian ring (see Theorem 3.1). For $\mathcal{M}_{0,n}$, we deduce the result from the one for $\mathcal{L}_{0,n}$, by following a line of proof of the Hilbert–Nagata theorem in classical invariant theory (see Theorem 3.4).

At present, we do not have a proof that $\mathcal{M}_{0,n}^A$ is a Noetherian and finitely generated ring for arbitrary $\mathfrak{g}$ and $n\geq 1$, though it is natural to expect it is the case. Indeed, when $\mathfrak{g}=\mathfrak{sl}_2$, $\mathcal{M}_{0,n}^A(\mathfrak{sl}_2)$ is isomorphic to the skein algebra of a sphere with $n+1$ punctures (see [18, Theorem 8.6]), which is finitely generated and Noetherian by results of [32] and [93]. In our general situation, key arguments in the proof of Theorem 1.1 for $\mathcal{M}_{0,n}$ depend on the existence of a Reynolds operator on the U_q-module $\mathcal{L}_{0,n}$, and one can easily show there is no Reynolds operator on $\mathcal{L}_{0,n}^A$. This follows from the corresponding fact for the integral quantum coordinate ring $\mathcal{O}_A$ (see Remark 2.19).

From Section 4, we consider the specializations $\mathcal{L}_{0,n}^\epsilon$ of $\mathcal{L}_{0,n}^A$ at $q=\epsilon$, a primitive root of unity of odd order l (and coprime to 3 if $\mathfrak{g}$ has G_2 components). In [41], De Concini–Lyubashenko introduced a central subalgebra $\mathcal{Z}_0(\mathcal{O}_\epsilon)$ of $\mathcal{O}_\epsilon$ isomorphic to the coordinate ring $\mathcal{O}(G)$, and proved that the $\mathcal{Z}_0(\mathcal{O}_\epsilon)$-module $\mathcal{O}_\epsilon$ is projective of rank $l^{\dim\mathfrak{g}}$. As observed by Brown–Gordon–Stafford [28], Bass' cancellation theorem in K-theory and the fact that $K_0(\mathcal{O}(G))\cong\mathbb{Z}$, proved by Marlin [87], imply that this module is free. Alternatively, this follows also from the fact that $\mathcal{O}_\epsilon$ is a cleft extension of $\mathcal{O}(G)$ by the dual of the Frobenius–Lusztig kernel $\mathfrak{u}_\epsilon(\mathfrak{g})$, as proved by Andruskiewitsch–Garcia (see [6, Remark 2.18 (b)], and also [25, Section 3.2]; this argument was explained to us by K.A. Brown).

The Section 4 proves the analogous property for $\mathcal{L}_{0,n}^\epsilon$. Namely:

Theorem 1.2. *$\mathcal{Z}_0(\mathcal{O}_\epsilon)^{\otimes n}$ is a central subalgebra of $\mathcal{L}_{0,n}^\epsilon$, and $\mathcal{L}_{0,n}^\epsilon$ is a free $\mathcal{Z}_0(\mathcal{O}_\epsilon)^{\otimes n}$-module of rank $l^{n.\dim\mathfrak{g}}$, isomorphic to the $\mathcal{Z}_0(\mathcal{O}_\epsilon)^{\otimes n}$-module $\mathcal{O}_\epsilon^{\otimes n}$.*

In the sequel we systematically denote $\mathcal{Z}_0\big(\mathcal{L}_{0,n}^\epsilon\big):=\mathcal{Z}_0(\mathcal{O}_\epsilon)^{\otimes n}$. We prove the first and third claims of Theorem 1.2 in Proposition 4.1. The arguments use results of De Concini–Kac [39], De Concini–Procesi [40, 42], and De Concini–Lyubashenko [41], that we recall in Sections 2.3–2.5. Let us stress that the algebra structures of $\mathcal{L}_{0,n}^\epsilon$ and $\mathcal{O}_\epsilon^{\otimes n}$ are completely different.

Since $\mathcal{Z}_0(\mathcal{O}_\epsilon)\cong\mathcal{O}(G)$, we may deduce the second claim of Theorem 1.2 from the first and third claims together with the results of [41, 87], or [6], recalled above. Nevertheless, we give a self-contained proof that $\mathcal{L}_{0,1}^\epsilon$ is finite projective of rank $l^{\dim\mathfrak{g}}$ over $\mathcal{Z}_0\big(\mathcal{L}_{0,1}^\epsilon\big)$, by adapting the original arguments of De Concini–Lyubashenko [41, Theorem 7.2]. In particular, we study the coregular action of the braid group of $\mathfrak{g}$ on $\mathcal{L}_{0,1}^\epsilon$; by the way, in the appendix, we provide different proofs of some technical facts shown in [41]. Of course, it remains an exciting problem to describe the centralizing extension $\mathcal{O}(G)^{\otimes n}\subset\mathcal{L}_{0,n}^\epsilon$ (and similarly $\mathcal{O}(G)^{\otimes n}\subset\big(\mathcal{L}_{0,n}^\epsilon\big)^{U_\epsilon}$ below), aiming at generalizing the results of [6] and finding a direct, more structural proof of freeness in Theorem 1.2. Also, we note that bases of $\mathcal{L}_{0,n}^\epsilon$ over $\mathcal{Z}_0\big(\mathcal{L}_{0,n}^\epsilon\big)$ are complicated. The only case we know is for $\mathcal{O}_\epsilon(\mathfrak{sl}_2)$, described in [38], and it is far from being obvious (see (4.4)).

In Section 5, we turn to fraction rings. As mentioned above $\mathcal{L}_{0,n}^\epsilon$ has no nontrivial zero divisors. Therefore, its center $\mathcal{Z}\big(\mathcal{L}_{0,n}^\epsilon\big)$ is an integral domain. Denote by $Q\big(\mathcal{Z}\big(\mathcal{L}_{0,n}^\epsilon\big)\big)$ its fraction field. Denote by $\big(\mathcal{L}_{0,n}^\epsilon\big)^{U_\epsilon}$ the subring of $\mathcal{L}_{0,n}^\epsilon$ formed by the invariant elements of $\mathcal{L}_{0,n}^\epsilon$ with respect to the right coadjoint action of U_ϵ. The center $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)$ of $\mathcal{L}_{0,n}^\epsilon$ is contained in $\big(\mathcal{L}_{0,n}^\epsilon\big)^{U_\epsilon}$ (this follows from [18, Proposition 6.19]). Note also that we trivially have an inclusion

$\mathcal{M}_{0,n}^{A,\epsilon} \subset \big(\mathcal{L}_{0,n}^{\epsilon}\big)^{U_\epsilon}$, and these two algebras are distinct in general. For instance, when $n = 1$, we have $\big(\mathcal{L}_{0,1}^{\epsilon}\big)^{U_\epsilon} = \mathcal{Z}(\mathcal{L}_{0,1}^{\epsilon})$, which is a finite extension of $\mathcal{Z}_0(\mathcal{O}_\epsilon) \cong \mathcal{O}(G)$ (see Lemma 5.1). On another hand, $\mathcal{M}_{0,1}^{A,\epsilon}$ is the specialization at $q = \epsilon$ of $\mathcal{Z}\big(\mathcal{L}_{0,1}^{A}\big)$, a polynomial algebra in $\mathrm{rk}(\mathfrak{g})$ variables, which may be identified via Φ_1 with the center $\mathcal{Z}(U_A)$ of the integral form U_A.

Consider the rings

$$Q\big(\mathcal{L}_{0,n}^{\epsilon}\big) = Q\big(\mathcal{Z}\big(\mathcal{L}_{0,n}^{\epsilon}\big)\big) \bigotimes_{\mathcal{Z}(\mathcal{L}_{0,n}^{\epsilon})} \mathcal{L}_{0,n}^{\epsilon}, \qquad Q\big(\big(\mathcal{L}_{0,n}^{\epsilon}\big)^{U_\epsilon}\big) = Q\big(\mathcal{Z}\big(\mathcal{L}_{0,n}^{\epsilon}\big)\big) \bigotimes_{\mathcal{Z}(\mathcal{L}_{0,n}^{\epsilon})} \big(\mathcal{L}_{0,n}^{\epsilon}\big)^{U_\epsilon}.$$

In general, given a ring A with center $\mathcal{Z}(A)$ an integral domain we reserve the notation $Q(A)$ to the central localization of A, i.e., $Q(A) := Q(\mathcal{Z}(A)) \otimes_{\mathcal{Z}(A)} A$. Though the center $\mathcal{Z}\big(\big(\mathcal{L}_{0,n}^{\epsilon}\big)^{U_\epsilon}\big)$ of $\big(\mathcal{L}_{0,n}^{\epsilon}\big)^{U_\epsilon}$ is larger than $\mathcal{Z}\big(\mathcal{L}_{0,n}^{\epsilon}\big)$, the notation $Q\big(\big(\mathcal{L}_{0,n}^{\epsilon}\big)^{U_\epsilon}\big)$ is valid, for $\mathcal{Z}\big(\big(\mathcal{L}_{0,n}^{\epsilon}\big)^{U_\epsilon}\big)$ is an integral domain finite over $\mathcal{Z}\big(\mathcal{L}_{0,n}^{\epsilon}\big)$, and hence the central localization of $\big(\mathcal{L}_{0,n}^{\epsilon}\big)^{U_\epsilon}$ coincides with $Q\big(\big(\mathcal{L}_{0,n}^{\epsilon}\big)^{U_\epsilon}\big)$ as defined above. Throughout the paper, unless we mention it explicitly, we follow the conventions of McConnell–Robson [88] as regards the terminology of ring theory; in particular, for the notions of central simple algebras and PI degrees, see in [88, Sections 5.3 and 13.3.6–13.6.7].

Denote by m the rank of $\mathfrak{g}$, and by N the number of its positive roots. In Section 5, we prove the following.

Theorem 1.3.

(1) $Q\big(\mathcal{L}_{0,n}^{\epsilon}\big)$ *is a division algebra and a central simple algebra of PI degree* l^{nN}.

(2) $Q\big(\big(\mathcal{L}_{0,n}^{\epsilon}\big)^{U_\epsilon}\big)$, $n \geq 2$, *is a division algebra and a central simple algebra of PI degree* $l^{N(n-1)-m}$.

The second claim of (1) means that $Q\big(\mathcal{L}_{0,n}^{\epsilon}\big)$ is a complex subalgebra of a full matrix algebra $\mathrm{Mat}_d(\mathbb{F})$, where $d = l^{nN}$ and $\mathbb{F}$ is a finite extension of $Q(\mathcal{Z}\big(\mathcal{L}_{0,n}^{\epsilon}\big))$ such that

$$\mathbb{F} \bigotimes_{Q(\mathcal{Z}(\mathcal{L}_{0,n}^{\epsilon}))} Q\big(\mathcal{L}_{0,n}^{\epsilon}\big) = \mathrm{Mat}_d(\mathbb{F}).$$

That $Q\big(\mathcal{L}_{0,n}^{\epsilon}\big)$ is a division algebra and a central simple algebra follows from Theorem 1.2 and the fact that $\mathcal{L}_{0,n}^{\epsilon}$ has no nontrivial zero divisors (proved in [18]). The computation of $d = l^{nN}$ uses a lower bound coming from the representation theory of U_ϵ, and a lower bound for the degree of $Q\big(\mathcal{Z}\big(\mathcal{L}_{0,n}^{\epsilon}\big)\big)$ as a field extension of $Q\big(\mathcal{Z}_0\big(\mathcal{L}_{0,n}^{\epsilon}\big)\big)$, obtained by using specializations to $q = \epsilon$ of certain central elements in $\mathcal{L}_{0,n}$ (for q generic). In this computation a main role is played by results of De Concini–Kac [39].

We deduce (2) from (1), the double centralizer theorem for central simple algebras, a few results of [18, 41], and Theorem 1.2 again.

1.2 Basic notations

Given a ring R, we denote by $\mathcal{Z}(R)$ its center. When R is commutative and has no nontrivial zero divisors, $Q(R)$ denotes its fraction field.

Given a Hopf algebra H with product m and coproduct Δ, we denote by H^{cop} (resp. H_{op}) the Hopf algebra with the same algebra (resp. coalgebra) structure as H but the opposite coproduct $\Delta^{\mathrm{cop}} := \sigma \circ \Delta$ (resp. opposite product $m \circ \sigma$), where $\sigma(x \otimes y) = y \otimes x$, and the antipode S^{-1}. We use Sweedler's coproduct notation, $\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}$, $x \in H$, and we set $\Delta^{(1)} := \mathrm{id}$, $\Delta^{(2)} := \Delta$, and $\Delta^{(n)} := (\Delta \otimes \mathrm{id})\Delta^{(n-1)}$ for $n \geq 3$ (this is not the convention used in [18]).

The results of this paper hold true for any finite-dimensional complex semisimple Lie algebra $\mathfrak{g}$, but unless we state it differently, we will assume $\mathfrak{g}$ is simple. We will denote its rank by m, and its Cartan matrix by (a_{ij}). We fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and a basis of simple roots $\alpha_i \in \mathfrak{h}^*_{\mathbb{R}}$, and denote by $\mathfrak{b}_\pm$ the Borel subalgebras associated to it. We denote by N the number of positive roots of $\mathfrak{g}$, and by ρ half the sum of the positive roots.

We denote by $d_1, \dots, d_m$ the unique coprime positive integers such that the matrix $(d_i a_{ij})$ is symmetric, and $(\ , \ )$ the unique inner product on $\mathfrak{h}^*_{\mathbb{R}}$ such that $d_i a_{ij} = (\alpha_i, \alpha_j)$. For any root α, the coroot is $\check{\alpha} = 2\alpha/(\alpha,\alpha)$; in particular $\check{\alpha_i} = d_i^{-1}\alpha_i$. The root lattice Q is the $\mathbb{Z}$-lattice in $\mathfrak{h}^*_{\mathbb{R}}$ defined by $Q = \sum_{i=1}^m \mathbb{Z}\alpha_i$. The weight lattice P is the $\mathbb{Z}$-lattice formed by all $\lambda \in \mathfrak{h}^*_{\mathbb{R}}$ such that $(\lambda, \check{\alpha_i}) \in \mathbb{Z}$ for every $i = 1, \dots, m$. So $P = \sum_{i=1}^m \mathbb{Z}\varpi_i$, where ϖ_i is the fundamental weight dual to the simple coroot $\check{\alpha_i}$, which satisfies $(\varpi_i, \check{\alpha_j}) = \delta_{i,j}$. Note that $(\lambda, \alpha) \in \mathbb{Z}$ for every $\lambda \in P$, $\alpha \in Q$. We denote by D the cardinality of the quotient lattice P/Q. Then D is the smallest positive integer such that $D(\lambda, \mu) \in \mathbb{Z}$ for every $\lambda, \mu \in P$, that is, such that $DP \subset Q$.

We denote by

$$P_+ := \sum_{i=1}^m \mathbb{Z}_{\geq 0}\varpi_i$$

the cone of dominant integral weights, and we put

$$Q_+ := \sum_{i=1}^m \mathbb{Z}_{\geq 0}\alpha_i.$$

Though $Q \subset P$, it is not true that $Q_+ \subset P_+$, but we have $DP_+ \subset Q_+$. This last property is not trivial, and follows from the classical fact that the inverse of the Cartan matrix (a_{ij}) has coefficients in $D^{-1}\mathbb{N}$.

We will use the standard partial order relation $\leq$ on P, defined by: $\lambda, \mu \in P$ satisfy $\lambda \leq \mu$ if $\mu - \lambda \in Q_+$. In Section 3, we will also use the partial order relation $\preceq$ on P defined by: $\lambda \preceq \mu$ if $\mu - \lambda \in D^{-1}Q_+$.

We denote by $\mathcal{B}(\mathfrak{g})$ the braid group of $\mathfrak{g}$; we recall its standard defining relations in Appendix B.

We denote by G the connected and simply-connected algebraic group with Lie algebra $\mathfrak{g}$, and by T_G the maximal torus of G with Lie algebra $\mathfrak{h}$; $N(T_G)$ is the normalizer of T_G, $W = N(T_G)/T_G$ is the Weyl group, $B_\pm$ are the Borel subgroups of G with Lie algebra $\mathfrak{b}_\pm$, and $U_\pm \subset B_\pm$ are their unipotent subgroups.

We denote by $\mathcal{O}(G)$ the coordinate ring of G. It is a commutative Hopf algebra, which can be identified with the restricted dual of the universal enveloping algebra $U(\mathfrak{g})$ (see [76, 84]). Similarly we denote by $\mathcal{O}(B_\pm)$ the coordinate ring of $B_\pm$.

Let q be an indeterminate, let $q^{1/D}$ be such that $\left(q^{1/D}\right)^D = q$, set $A = \mathbb{C}\left[q, q^{-1}\right]$, $q_i = q^{d_i}$, $q_\beta = q^{(\beta,\beta)/2}$ for $\beta \in Q$, and given integers p, k with $0 \leq k \leq p$, we put

$$[p]_q = \frac{q^p - q^{-p}}{q - q^{-1}}, \qquad [0]_q! = 1, \qquad [p]_q! = [1]_q[2]_q \cdots [p]_q, \qquad \begin{bmatrix} p \\ k \end{bmatrix}_q = \frac{[p]_q!}{[p-k]_q![k]_q!},$$
$$(p)_q = \frac{q^p - 1}{q - 1}, \qquad (0)_q! = 1, \qquad (p)_q! = (1)_q(2)_q \cdots (p)_q, \qquad \begin{pmatrix} p \\ k \end{pmatrix}_q = \frac{(p)_q!}{(p-k)_q!(k)_q!}.$$

We denote by $\mathcal{A}_0 \subset \mathbb{C}(q)$ the ring of functions regular at $q = 0$; this ring is used only in Section 2.2.2.

We denote by ϵ a primitive l-th root of unity such that $\epsilon^{2d_i} \neq 1$ is also a primitive l-th root of unity for all $i \in \{1, \dots, m\}$. This means that l is odd, and coprime to 3 if $\mathfrak{g}$ is G_2. We put $\epsilon_i := \epsilon^{d_i}$.

In this paper, we use the definition of the unrestricted integral form $U_A(\mathfrak{g})$ given in [41, 42]; in [18] we used the one of [39, 40]. The two are (trivially) isomorphic, and have the same specialization at $q = \epsilon$. Also, we denote here by L_i the generators of $U_q(\mathfrak{g})$ we denoted by ℓ_i in [18].

In order to facilitate the comparison with the results of [41], we note that their generators denoted K_i, E_i and F_i, that we will denote by $\tilde{K}_i$, $\tilde{E}_i$ and $\tilde{F}_i$, can be written as K_i, $K_i^{-1}E_i$ and F_iK_i in our notations. They satisfy the same algebra relations.

2 Background results

2.1 On U_q, $\mathcal{O}_q$, $\mathcal{L}_{0,n}$, $\mathcal{M}_{0,n}$, and Φ_n

Except when stated differently, we refer to [18, Sections 2–4 and 6], and the references therein for details about the material of this section. We stress that the simply-connected quantum group, that we denote U_q below, was denoted $\tilde{U}_q$ in [18]. Also, the adjoint quantum group U_q^{ad} was denoted U_q.

The *simply-connected* quantum group $U_q = U_q(\mathfrak{g})$ is the Hopf algebra over $\mathbb{C}(q)$ with generators E_i, F_i, L_i, L_i^{-1}, $1 \leq i \leq m$, and defining relations

$$\begin{aligned}
&L_iL_j = L_jL_i, \qquad L_iL_i^{-1} = L_i^{-1}L_i = 1, \qquad L_iE_jL_i^{-1} = q_i^{\delta_{i,j}}E_j, \qquad L_iF_jL_i^{-1} = q_i^{-\delta_{i,j}}F_j,\\
&E_iF_j - F_jE_i = \delta_{i,j}\frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},\\
&\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_{q_i} E_i^{1-a_{ij}-r}E_jE_i^r = 0 \qquad \text{if} \quad i \neq j,\\
&\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_{q_i} F_i^{1-a_{ij}-r}F_jF_i^r = 0 \qquad \text{if} \quad i \neq j,
\end{aligned}$$

where for $\lambda = \sum_{i=1}^m m_i\varpi_i \in P$ we set $K_\lambda = \prod_{i=1}^m L_i^{m_i}$, and $K_i = K_{\alpha_i} = \prod_{j=1}^m L_j^{a_{ji}}$. The coproduct Δ, antipode S, and counit ε of U_q are given by

$$\begin{aligned}
&\Delta(L_i) = L_i \otimes L_i, \qquad \Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i, \qquad \Delta(F_i) = F_i \otimes 1 + K_i^{-1} \otimes F_i,\\
&S(E_i) = -E_iK_i^{-1}, \qquad S(F_i) = -K_iF_i, \qquad S(L_i) = L_i^{-1},\\
&\varepsilon(E_i) = \varepsilon(F_i) = 0, \qquad \varepsilon(L_i) = 1.
\end{aligned}$$

We fix a reduced expression $s_{i_1}\cdots s_{i_N}$ of the longest element w_0 of the Weyl group of $\mathfrak{g}$. It induces a total ordering of the positive roots,

$$\beta_1 = \alpha_{i_1}, \qquad \beta_2 = s_{i_1}(\alpha_{i_2}), \qquad \ldots, \qquad \beta_N = s_{i_1}\cdots s_{i_{N-1}}(\alpha_{i_N}).$$

The root vectors of U_q with respect to such an ordering are defined by

$$E_{\beta_k} = T_{i_1}\cdots T_{i_{k-1}}(E_{i_k}), \qquad F_{\beta_k} = T_{i_1}\cdots T_{i_{k-1}}(F_{i_k}), \tag{2.1}$$

where T_i is the Lusztig algebra automorphism of U_q associated to the simple root α_i [82, 83] (see also [35, Chapter 8]). The braid group $\mathcal{B}(\mathfrak{g})$ acts on U_q by means of the Lusztig automorphisms. In the appendix, we recall the relation between T_i and the generator $\hat{w}_i$ of the quantum Weyl group, which we will mostly use. Let us just recall here that the monomials $F_{\beta_1}^{r_1}\cdots F_{\beta_N}^{r_N}K_\lambda E_{\beta_N}^{t_N}\cdots E_{\beta_1}^{t_1}$ ($r_i, t_i \in \mathbb{N}$, $\lambda \in P$) form a basis of U_q, the *PBW basis*.

U_q is a *pivotal* Hopf algebra, with pivotal element $\ell := K_{2\rho} = \prod_{j=1}^m L_j^2$. So ℓ is group-like, and $S^2(x) = \ell x \ell^{-1}$ for every $x \in U_q$.

The *adjoint* quantum group $U_q^{\mathrm{ad}} = U_q^{\mathrm{ad}}(\mathfrak{g})$ is the Hopf subalgebra of U_q generated by the elements E_i, F_i $(i = 1, \dots, m)$ and K_α with $\alpha \in Q$; so $\ell \in U_q^{\mathrm{ad}}$. When $\mathfrak{g} = \mathfrak{sl}_2$, we simply write the above generators $E = E_1$, $F = F_1$, $L = L_1$, $K = K_1$.

We denote by $U_q(\mathfrak{n}_+)$, $U_q(\mathfrak{n}_-)$ and $U_q(\mathfrak{h})$ the subalgebras of U_q generated respectively by the E_i, the F_i, and the K_λ $(\lambda \in P)$, and by $U_q(\mathfrak{b}_+)$ and $U_q(\mathfrak{b}_-)$ the subalgebras generated by the E_i and the K_λ, and by the F_i and the K_λ, respectively. We do similarly with U_q^{ad}, where now $U_q^{\mathrm{ad}}(\mathfrak{h})$ is generated by the K_λ with $\lambda \in Q$.

The Hopf algebra U_q^{ad} is not braided in a strict sense, but it has braided categorical completions. Let us recall briefly what this means and implies. For details, we refer to [18, Sections 2 and 3] (see also [104, Section 3.10], where $\mathbb{U}_q$ below is formulated in terms of multiplier Hopf algebras).

A U_q^{ad}-module V is said *of type* 1 if it has finite dimension and the generators K_i are diagonalizable on V with eigenvalues in $q_i^{\mathbb{Z}}$. We denote by $\mathcal{C}$ the category of U_q^{ad}-modules of type 1, by Vect the category of finite-dimensional $\mathbb{C}(q)$-vector spaces, and by $F_{\mathcal{C}}\colon \mathcal{C} \to \mathrm{Vect}$ the forgetful functor. The category $\mathcal{C}$ is semisimple. The simple objects are highest weight U_q^{ad}-modules; we denote by V_μ the simple module with highest weight $\mu \in P_+$. In the case $\mathfrak{g} = \mathfrak{sl}_2$, we identify P_+ with $\mathbb{N}$, and denote by V_n the simple module of dimension $n+1$. Note that V_μ is canonically endowed with a structure of U_q-module of type 1, the generators L_i being diagonalizable with eigenvalues in $q_i^{\mathbb{Z}/D}$. The *categorical completion* $\mathbb{U}_q^{\mathrm{ad}}$ of U_q^{ad} is the set of natural transformations $F_{\mathcal{C}} \to F_{\mathcal{C}}$. An element of $\mathbb{U}_q^{\mathrm{ad}}$ is a collection $(a_V)_{V \in \mathrm{Ob}(\mathcal{C})}$, where $a_V \in \mathrm{End}_{\mathbb{C}(q)}(V)$ satisfies $F_{\mathcal{C}}(f) \circ a_V = a_W \circ F_{\mathcal{C}}(f)$ for any objects V, W of $\mathcal{C}$ and any arrow $f \in \mathrm{Hom}_{U_q^{\mathrm{ad}}}(V, W)$. It is not hard to see that $\mathbb{U}_q^{\mathrm{ad}}$ inherits from $\mathcal{C}$ a natural structure of (completed) Hopf algebra such that the map

$$\iota\colon\ U_q^{\mathrm{ad}} \longrightarrow \mathbb{U}_q^{\mathrm{ad}}, \qquad x \longmapsto (\pi_V(x))_{V \in \mathrm{Ob}(\mathcal{C})} \tag{2.2}$$

is a morphism of Hopf algebras, where $\pi_V\colon U_q^{\mathrm{ad}} \to \mathrm{End}(V)$ is the representation associated to a module V in $\mathcal{C}$. It is a theorem that this map is injective. From now on, let us extend the coefficient ring of the modules and morphisms in $\mathcal{C}$ to $\mathbb{C}(q^{1/D})$. Put $\mathbb{U}_q = \mathbb{U}_q^{\mathrm{ad}} \bigotimes_{\mathbb{C}(q)} \mathbb{C}(q^{1/D})$. The map ι above extends to an embedding of U_q in $\mathbb{U}_q$. The category $\mathcal{C}$, with coefficients extended to $\mathbb{C}(q^{1/D})$, is braided and ribbon; we postpone a discussion of that fact to Section 2.3, where it will be developed. As a consequence, we can regard $\mathbb{U}_q$ as a quasitriangular and ribbon Hopf algebra in a generalized sense (see [18]). The R-matrix of $\mathbb{U}_q$ is the family of morphisms

$$R = (R_{V,W})_{V,W \in \mathrm{Ob}(\mathcal{C})},$$

where $R_{V,W} \in \mathrm{End}(V \otimes W)$ is the endomorphism defined by the action of Drinfeld's universal R-matrix on $V \otimes W$. The ribbon element of $\mathbb{U}_q$ is defined similarly by Drinfeld's universal ribbon element. One defines the *categorical tensor product* $\mathbb{U}_q^{\hat{\otimes} 2}$ similarly as $\mathbb{U}_q$; in particular it contains all the infinite series of elements of $\mathbb{U}_q^{\otimes 2}$ having only a finite number of non-zero terms when evaluated on a given module $V \otimes W$ of $\mathcal{C}$. There is an expansion of R as an infinite series in $\mathbb{U}_q^{\hat{\otimes} 2}$. Adapting Sweedler's coproduct notation $\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}$, we find convenient to write this series as

$$R = \sum_{(R)} R_{(1)} \otimes R_{(2)}. \tag{2.3}$$

We put $R^+ := R$, $R^- := (\sigma \circ R)^{-1}$ where σ is the flip map $x \otimes y \mapsto y \otimes x$. We will not use any explicit formula of R, but the following factorization formula

$$R = \Theta \hat{R}, \tag{2.4}$$

where

$$\Theta = q^{\sum_{i,j=1}^m (B^{-1})_{ij} H_i \otimes H_j} \in \mathbb{U}_q^{\hat{\otimes} 2},$$

with $B \in M_m(\mathbb{Q})$ the matrix with entries $B_{ij} := d_j^{-1} a_{ij}$, and

$$\hat{R} = \sum_{(\hat{R})} \hat{R}_{(1)} \otimes \hat{R}_{(2)} \in \mathbb{U}_q(\mathfrak{n}_+) \hat{\otimes} \mathbb{U}_q(\mathfrak{n}_-)$$

(see [18, Section 3.2], and for details, e.g., [35, Theorem 8.3.9], or [104, Theorem 3.108]). If x, y are weight vectors of weights μ, ν respectively, then $\Theta(x \otimes y) = q^{(\mu,\nu)} x \otimes y$. Moreover, $\hat{R}$ has weight 0 for the adjoint action of $U_q(\mathfrak{h})$; that is, complementary components $\hat{R}_{(1)}$ and $\hat{R}_{(2)}$ have opposite weights.

Recall that we denote by G the connected and simply-connected algebraic group with Lie algebra $\mathfrak{g}$. The *quantum function Hopf algebra* $\mathcal{O}_q = \mathcal{O}_q(G)$ is defined as the restricted dual of U_q^{ad} with respect to the category $\mathcal{C}$, that is, the set of $\mathbb{C}(q)$-linear maps $f\colon U_q^{\mathrm{ad}} \to \mathbb{C}(q)$ such that $\mathrm{Ker}(f)$ contains a cofinite two sided ideal I (i.e., such that $I \oplus M = U_q$ for some finite-dimensional vector space M), and $\prod_{s=-r}^r (K_i - q_i^s) \in I$ for some $r \in \mathbb{N}$ and every i (see, e.g., [26, Chapter I.7]).

The space $\mathcal{O}_q$ is a Hopf algebra, with structure maps defined dually to those of U_q^{ad}. We denote by $\star$ its product. The algebras $\mathcal{O}_q(T_G)$, $\mathcal{O}_q(U_\pm)$, $\mathcal{O}_q(B_\pm)$ are defined similarly, by replacing U_q^{ad} with $U_q^{\mathrm{ad}}(\mathfrak{h})$, $U_q^{\mathrm{ad}}(\mathfrak{n}_\pm)$, $U_q^{\mathrm{ad}}(\mathfrak{b}_\pm)$, respectively. As a vector space, $\mathcal{O}_q$ is generated by the functionals $x \mapsto w(\pi_V(x)v)$, $x \in U_q^{\mathrm{ad}}$, for every object $V \in \mathrm{Ob}(\mathcal{C})$ and vectors $v \in V$, $w \in V^*$. Such functionals are called *matrix coefficients*. Because the morphism $\iota\colon U_q^{\mathrm{ad}} \to \mathbb{U}_q$ is injective (see (2.2)), the Hopf duality pairing $\langle \cdot,\cdot \rangle\colon \mathcal{O}_q \times U_q^{\mathrm{ad}} \to \mathbb{C}(q)$ is non degenerate. By extending the coefficient ring from $\mathbb{C}(q)$ to $\mathbb{C}(q^{1/D})$, we can uniquely extend it to a bilinear pairing

$$\langle \cdot,\cdot \rangle\colon\ \Big(\mathcal{O}_q \bigotimes_{\mathbb{C}(q)} \mathbb{C}\big(q^{1/D}\big)\Big) \times \mathbb{U}_q \to \mathbb{C}\big(q^{1/D}\big)$$

such that the following diagram is commutative:

$$\begin{tikzcd}
\mathcal{O}_q \otimes U_q^{\mathrm{ad}} \arrow[r, "\langle\cdot,\cdot\rangle"] \arrow[d, "\mathrm{id}\otimes\iota"'] & \mathbb{C}(q) \arrow[d] \\
\big(\mathcal{O}_q \bigotimes_{\mathbb{C}(q)} \mathbb{C}\big(q^{1/D}\big)\big) \otimes \mathbb{U}_q \arrow[r, "\langle\cdot,\cdot\rangle"] & \mathbb{C}\big(q^{1/D}\big).
\end{tikzcd}$$

This pairing is defined by $\langle {}_Y\phi_v^w, (a_X) \rangle = w(a_Y v)$ for every $(a_X) \in \mathbb{U}_q$ and ${}_Y\phi_v^w \in \mathcal{O}_q$. It is non degenerate.

The maps

$$\Phi^\pm\colon\ \mathcal{O}_q \longrightarrow U_q^{\mathrm{cop}}, \qquad \alpha \longmapsto (\alpha \otimes \mathrm{id})\big(R^\pm\big) = \sum_{(R^\pm)} \big\langle \alpha, R_{(1)}^\pm \big\rangle R_{(2)}^\pm \tag{2.5}$$

are well-defined morphisms of Hopf algebras. Here we stress that it is the simply-connected quantum group U_q^{cop} that is the range of $\Phi^\pm$. This will be explained with more details in Section 2.3.

Let us make two simple observations, for future reference. Firstly, because $\mathcal{O}_q$ is spanned by the matrix coefficients of the objects of $\mathcal{C}$, and $\mathcal{C}$ is semisimple with simple objects the U_q^{ad}-modules V_μ, $\mu \in P_+$, there is a decomposition of U_q-bimodule

$$\mathcal{O}_q = \bigoplus_{\mu \in P_+} C(\mu), \tag{2.6}$$

where $C(\mu) = V_\mu^* \otimes V_\mu$, the space of matrix coefficients of V_μ, is endowed with the left action on the factor V_μ and the right action on V_μ^*, and $\mathcal{O}_q$ has the left and right coregular actions $\lhd$ and $\rhd$, defined by

$$x \rhd \alpha := \sum_{(\alpha)} \alpha_{(1)} \langle \alpha_{(2)}, x \rangle, \qquad \alpha \lhd x := \sum_{(\alpha)} \langle \alpha_{(1)}, x \rangle \alpha_{(2)}$$

for all $x \in U_q$ and $\alpha \in \mathcal{O}_q$. Here we recall that each U_q^{ad}-module V_μ can be regarded as a U_q-module, so the above expressions make sense. The decomposition (2.6) is the *Peter–Weyl* decomposition of $\mathcal{O}_q$. It will be refined in Section 2.2.2.

Moreover, the algebra $\mathcal{O}_q$ is generated by the matrix coefficients of the simple U_q^{ad}-modules V_{ϖ_k} with highest weights the fundamental weights ϖ_k, $k = 1, \dots, m$; see, e.g., [26, Proposition I.7.8] for a proof. This relies on the standard fact that, for any $\mu, \nu \in P_+$ we have a direct sum decomposition of modules (where $m(\lambda) \in \mathbb{N}$)

$$V_\mu \otimes V_\nu = V_{\mu+\nu} \oplus \bigoplus_{\lambda < \mu+\nu} V_\lambda^{\oplus m(\lambda)}. \tag{2.7}$$

In particular, this decomposition implies that, up to scalar multiples, there is a unique non-zero morphism $V_{\mu+\nu} \to V_\mu \otimes V_\nu$, which is injective and splits. Dually, this means that, applying the product in $\mathcal{O}_q$ followed by the projection onto the subspace $C(\mu+\nu)$ we get a canonical projection map

$$C(\mu) \otimes C(\nu) \to C(\mu+\nu). \tag{2.8}$$

The *loop algebra* $\mathcal{L}_{0,1} = \mathcal{L}_{0,1}(\mathfrak{g})$ is defined by twisting the product $\star$ of $\mathcal{O}_q$, keeping the same underlying linear space. The new product is equivariant with respect to the right coadjoint action coad^r of U_q, defined by

$$\mathrm{coad}^r(x)(\alpha) = \sum_{(x)} S(x_{(2)}) \rhd \alpha \lhd x_{(1)}$$

for all $x \in U_q$ and $\alpha \in \mathcal{O}_q$. By equivariant we mean that $\mathcal{L}_{0,1}$ is a U_q-module algebra. Let us spell out its product and equivariance property. Using the fact that U_q can be regarded as a subspace of $\mathbb{U}_q$, the actions $\lhd$ and $\rhd$ extend naturally to actions of $\mathbb{U}_q$, and the product of $\mathcal{L}_{0,1}$ is expressed in terms of $\star$ by the formula (see [18, Proposition 4.1]):

$$\alpha\beta = \sum_{(R),(R)} (R_{(2')} S(R_{(2)}) \rhd \alpha) \star (R_{(1')} \rhd \beta \lhd R_{(1)}), \tag{2.9}$$

where $\sum_{(R)} R_{(1)} \otimes R_{(2)}$ and $\sum_{(R)} R_{(1')} \otimes R_{(2')}$ are expansions of two copies of $R \in \mathbb{U}_q^{\hat{\otimes} 2}$. Note that the sum in (2.9) has only a finite number of non-zero terms. By using that

$$R\Delta = \Delta^{\mathrm{cop}} R,$$

this product can equivalently be expressed as

$$\alpha\beta = \sum_{(R),(R)} (\beta \lhd R_{(1)} R_{(1')}) \star (S(R_{(2)}) \rhd \alpha \lhd R_{(2')}). \tag{2.10}$$

This product gives $\mathcal{L}_{0,1}$ (like $\mathcal{O}_q$) a structure of U_q-module algebra for the actions $\rhd$, $\lhd$, but also for coad^r (which is not the case of $\mathcal{O}_q$). Spelling this out for coad^r, this means

$$\mathrm{coad}^r(x)(\alpha\beta) = \sum_{(x)} \mathrm{coad}^r(x_{(1)})(\alpha) \mathrm{coad}^r(x_{(2)})(\beta).$$

The relations between $\mathcal{O}_q$, $\mathcal{L}_{0,1}$ and U_q are encoded by the map

$$\Phi_1\colon\ \mathcal{O}_q \longrightarrow \mathbb{U}_q, \qquad \alpha \longmapsto (\alpha\otimes \mathrm{id})(RR'), \tag{2.11}$$

where $R' = \sigma\circ R$, and as usual $\sigma\colon x\otimes y \mapsto y\otimes x$. Note that

$$\Phi_1 = m\circ\big(\Phi^+\otimes\big(S^{-1}\circ\Phi^-\big)\big)\circ\Delta. \tag{2.12}$$

We call Φ_1 the *RSD* map, for Drinfeld, Reshetikhin and Semenov-Tian-Shansky introduced it first (see [48, 86, 94]). It is a fundamental result of the theory (see [20, 34, 61]) that Φ_1 affords an isomorphism of U_q-modules $\Phi_1\colon \mathcal{O}_q \to U_q^{\mathrm{lf}}$. For full details on that result we refer to [104, Section 3.12]. Here, U_q^{lf} is the set of *locally finite* elements of U_q, endowed with the right adjoint action ad^r of U_q. It is defined by

$$U_q^{\mathrm{lf}} := \{x\in U_q \mid \mathrm{rk}_{\mathbb{C}(q)}(\mathrm{ad}^r(U_q)(x))<\infty\}$$

and

$$\mathrm{ad}^r(y)(x) = \sum_{(y)} S(y_{(1)})xy_{(2)}$$

for every $x,y\in U_q$. The action ad^r gives in fact U_q^{lf} a structure of right U_q-module algebra. It is also a right coideal, that is $\Delta\big(U_q^{\mathrm{lf}}\big)\subset U_q^{\mathrm{lf}}\otimes U_q$. Moreover, Φ_1 affords an isomorphism of U_q-module algebras $\Phi_1\colon \mathcal{L}_{0,1}\to U_q^{\mathrm{lf}}$. It is a fact that Φ_1 affords an isomorphism between the centers $\mathcal{Z}(\mathcal{L}_{0,1})$ of $\mathcal{L}_{0,1}$ and $\mathcal{Z}(U_q)$ of U_q [18, Proposition 6.24]. Since Φ_1 is an isomorphism of U_q-modules and $\mathcal{Z}(U_q) = U_q^{U_q}$, it follows that $\mathcal{Z}(\mathcal{L}_{0,1}) = \mathcal{L}_{0,1}^{U_q}$.

Let us recall a few fundamental results about U_q^{lf} that we will meet again later. Denote by $T\subset U_q$ the multiplicative Abelian group formed by the elements K_λ, $\lambda\in P$, and by $T_2\subset T$ the subgroup formed by the elements K_λ, $\lambda\in 2P$. Consider the subset $T_{2-}\subset T_2$ formed by the elements $K_{-\lambda}$, $\lambda\in 2P_+$. Clearly, $T_2 = T_{2-}^{-1}T_{2-}$ and $\mathrm{Card}(T/T_2) = 2^m$.

Theorem 2.1.

(1) $U_q^{\mathrm{lf}} = \bigoplus_{\lambda\in 2P_+}\mathrm{ad}^r(U_q)(K_{-\lambda})$.

(2) $U_q = T_{2-}^{-1}U_q^{\mathrm{lf}}[T/T_2]$, *so* U_q *is a free* $T_{2-}^{-1}U_q^{\mathrm{lf}}$*-module of rank* 2^m.

(3) *The ring* U_q^{lf} *is (left and right) Noetherian.*

These results were proved by Joseph–Letzter in [63, Theorem 4.10], [62, Theorem 6.4], and [61, Theorem 7.4.8], respectively (see also [61, Sections 7.1.6, 7.1.13 and 7.1.25]). For (1) and (3), we refer also to [104, Theorems 3.113 and 3.137], which provides simpler proofs. For instance, in the $\mathfrak{sl}_2$ case, we have

$$U_q(\mathfrak{sl}_2) = U_q(\mathfrak{sl}_2)^{\mathrm{lf}}[K]\oplus U_q(\mathfrak{sl}_2)^{\mathrm{lf}}[K].L.$$

The actual values of Φ_1 are complicated in general, however, there is a simple important one, that we describe now. Let $V_{-\lambda}$ be the type 1 simple U_q^{ad}-module of lowest weight $-\lambda\in -P_+$ (i.e., the highest weight U_q^{ad}-module $V_{-w_0(\lambda)}$ of highest weight $-w_0(\lambda)$, where w_0 is the longest element of the Weyl group; note that $-w_0$ permutes the simple roots). Let $v\in V_{-\lambda}$ be a lowest weight vector, and $v^*\in V_{-\lambda}^*$ be such that $v^*(v) = 1$ and v^* vanishes on a $U_q^{\mathrm{ad}}(\mathfrak{h})$-invariant complement of v. Define $\psi_{-\lambda}\in\mathcal{O}_q$ by $\langle\psi_{-\lambda},x\rangle = v^*(xv)$, $x\in U_q$. From the definition (2.11), it is quite easy to see that

$$\Phi_1(\psi_{-\lambda}) = K_{-2\lambda}. \tag{2.13}$$

In particular, $\Phi_1(\psi_{-\rho}) = \ell^{-1}$, where as usual ℓ is the pivotal element of U_q.

Remark 2.2. Since $\mathcal{L}_{0,1} = \mathcal{O}_q$ as a vector space, we still denote by $C(\mu)$, $\mu \in P^+$, the linear subspace generated by the matrix coefficients of V_μ, the U_q^{ad}-module of type 1 and highest weight μ. It can be proved (see [61, Section 7.1.22], or [104, p. 156], where different conventions are used) that Φ_1 yields an isomorphism of U_q-modules

$$\Phi_1\colon\ C(-w_0(\mu)) \to \mathrm{ad}^r(U_q)(K_{-2\mu}). \tag{2.14}$$

Therefore, the summands in (1) are finite-dimensional U_q-modules, and the action ad^r is completely reducible on U_q^{lf}. In fact, U_q^{lf} is the socle of ad^r on U_q.

Remark 2.3. Because $\ell = \prod_{j=1}^m L_j^2$ and $\Phi_1(\psi_{-\rho}) = \ell^{-1}$, a natural question is the factorization of $\psi_{-\rho}$ in $\mathcal{L}_{0,1}$ (see Corollary 2.23). This question is considered in [60], where $\mathcal{L}_{0,1}(\mathfrak{g})$ for $\mathfrak{g} = \mathfrak{gl}(r+1)$ is analysed and quantum minors are extensively studied. Let us review here some of their results in relation with $\psi_{-\rho}$.

First note that for $\mathfrak{g} = \mathfrak{sl}(r+1)$ the irreducible representation $V_{-\rho}$ of lowest weight $-\rho$ is isomorphic to the representation of highest weight V_ρ because $-w_0(\rho) = \rho$. By the Weyl formula, the dimension of this representation is

$$\prod_{\alpha>0} \frac{(2\rho,\alpha)}{(\rho,\alpha)} = 2^N.$$

In [71], a presentation of $U_q(\mathfrak{gl}(r+1))$ is given, which differs from our presentation of $U_q(\mathfrak{sl}(r+1))$ only by its subalgebra $U_q(\mathfrak{h})$, generated by $r+1$ elements $\mathbb{K}_1, \dots, \mathbb{K}_{r+1}$. The inclusion

$$U_q(\mathfrak{sl}(r+1)) \subset U_q(\mathfrak{gl}(r+1))$$

is such that $K_i = \mathbb{K}_i^2 \mathbb{K}_{i+1}^{-2}$, $i = 1, \dots, r$. The quantum minors, properly defined in [60], of the matrix of matrix elements of the natural representation of $U_q(\mathfrak{gl}(r+1))$ are denoted $\det_q(A_{\geq k})$ for $k = 1, \dots, r+1$. We have $\det_q(A_{\geq 1}) = 1$ in the case of $\mathfrak{sl}(r+1)$. Then [60] proves that $\det_q(A_{\geq k}) = (\mathbb{K}_k \cdots \mathbb{K}_{r+1})^2$, and there exists an element $\mathbb{K} \in U_q(\mathfrak{gl}(r+1))$ such that

$$\mathbb{K}^{-2\rho} = \det_q(A_{\geq 1})^{-r} \det_q(A_{\geq 2}) \cdots \det_q(A_{\geq r+1}).$$

This has to be interpreted as $K_{-2\rho} = \Phi_1(\det_q(A_{\geq 2}) \cdots \det_q(A_{\geq r+1}))$ in the case of $\mathfrak{sl}(r+1)$. As a result, this gives the equality

$$\psi_{-\rho} = \det_q(A_{\geq 2}) \cdots \det_q(A_{\geq r+1}).$$

The (*quantum*) *graph algebra* $\mathcal{L}_{0,n} = \mathcal{L}_{0,n}(\mathfrak{g})$ is the braided tensor product of n copies of $\mathcal{L}_{0,1}$ (considered as a U_q-module algebra). As a linear space and U_q-bimodule with actions $\lhd$ and $\rhd$, it coincides with $\mathcal{L}_{0,1}^{\otimes n}$, and thus with $\mathcal{O}_q^{\otimes n}$. It is also a right U_q-module algebra, with the following action of U_q (extending coad^r on $\mathcal{L}_{0,1}$):

$$\mathrm{coad}_n^r(y)\big(\alpha^{(1)} \otimes \cdots \otimes \alpha^{(n)}\big) = \sum_{(y)} \mathrm{coad}^r(y_{(1)})\big(\alpha^{(1)}\big) \otimes \cdots \otimes \mathrm{coad}^r(y_{(n)})\big(\alpha^{(n)}\big) \tag{2.15}$$

for all $y \in U_q$ and $\alpha^{(1)} \otimes \cdots \otimes \alpha^{(n)} \in \mathcal{L}_{0,n}$. The product of $\mathcal{L}_{0,n}$ can be expressed as follows. For every $1 \leq a \leq n$, define $\mathfrak{i}_a\colon \mathcal{L}_{0,1} \to \mathcal{L}_{0,n}$ by $\mathfrak{i}_a(x) = 1^{\otimes(a-1)} \otimes x \otimes 1^{\otimes(n-a)}$; $\mathfrak{i}_a$ is an embedding of U_q-module algebras. We will use the notations

$$\mathcal{L}_{0,n}^{(a)} := \mathrm{Im}(\mathfrak{i}_a), \qquad (\alpha)^{(a)} := \mathfrak{i}_a(\alpha). \tag{2.16}$$

Take $(\alpha)^{(a)}, (\alpha')^{(a)} \in \mathcal{L}_{0,n}^{(a)}$ and $(\beta)^{(b)}, (\beta')^{(b)} \in \mathcal{L}_{0,n}^{(b)}$ with $a < b$. Then the product of $\mathcal{L}_{0,n}$ is given by the following formula (see [18, Section 6]):

$$\begin{aligned}\big((\alpha)^{(a)} \otimes (\beta)^{(b)}\big)\big((\alpha')^{(a)} \otimes (\beta')^{(b)}\big) = \sum_{(R^1),\dots,(R^4)} &\Big(\alpha\big(S\big(R^3_{(1)}R^4_{(1)}\big) \rhd \alpha' \lhd R^1_{(1)}R^2_{(1)}\big)\Big)^{(a)} \\ &\otimes \Big(\big(S\big(R^1_{(2)}R^3_{(2)}\big) \rhd \beta \lhd R^2_{(2)}R^4_{(2)}\big)\beta'\Big)^{(b)}, \qquad (2.17)\end{aligned}$$

where $R^i = \sum_{(R^i)} R^i_{(1)} \otimes R^i_{(2)}$, $i \in \{1, 2, 3, 4\}$, are expansions of four copies of $R \in \mathbb{U}_q^{\hat{\otimes}2}$, and on the right-hand side the product is componentwise that of $\mathcal{L}_{0,1}$. Later we will use the fact that the product of $\mathcal{L}_{0,n}$ is obtained from the standard (componentwise) product of $\mathcal{L}_{0,1}^{\otimes n}$ by a process that may be inverted. Indeed, (2.17) can be rewritten as

$$\big((\alpha)^{(a)} \otimes (\beta)^{(b)}\big)\big((\alpha')^{(a)} \otimes (\beta')^{(b)}\big) = \sum_{(F)} (\alpha)^{(a)}\big((\alpha')^{(a)} \cdot F_{(2)}\big) \otimes \big((\beta)^{(b)} \cdot F_{(1)}\big)(\beta')^{(b)}, \quad (2.18)$$

where $F = \sum_{(F)} F_{(1)} \otimes F_{(2)} := (\Delta \otimes \Delta)(R')$, and the symbol "$\cdot$" stands for the right action of $\mathbb{U}_q^{\hat{\otimes}2}$ on $\mathcal{L}_{0,1}$ that may be read from (2.17). The tensor F is known as a twist. Then, by replacing F with its inverse $\bar{F} = (\Delta \otimes \Delta)\big(R'^{-1}\big)$, one can express the product of $\mathcal{L}_{0,1}^{\otimes n}$ in terms of the product of $\mathcal{L}_{0,n}$ by

$$(\alpha)^{(a)}(\alpha')^{(a)} \otimes (\beta)^{(b)}(\beta')^{(b)} = \sum_{(\bar{F})} \big((\alpha)^{(a)} \otimes \big((\beta)^{(b)} \cdot \bar{F}_{(1)}\big)\big)\big(\big((\alpha')^{(a)} \cdot \bar{F}_{(2)}\big) \otimes (\beta')^{(b)}\big). \quad (2.19)$$

We call *quantum moduli algebra* and denote by $\mathcal{M}_{0,n} = \mathcal{L}_{0,n}^{U_q}$ the subalgebra of $\mathcal{L}_{0,n}$ formed by the U_q-invariant elements.

The map Φ_1 can be extended to $\mathcal{L}_{0,n}$ as follows. Consider the following action of U_q on the tensor product algebra $U_q^{\otimes n}$, which extends ad^r on U_q:

$$\mathrm{ad}^r_n(y)(x) = \sum_{(y)} \Delta^{(n)}(S(y_{(1)}))x\Delta^{(n)}(y_{(2)})$$

for all $y \in U_q$, $x \in U_q^{\otimes n}$. This action gives $U_q^{\otimes n}$ a structure of right U_q-module algebra. In [1], Alekseev introduced a morphism of U_q-module algebras $\Phi_n \colon \mathcal{L}_{0,n} \to U_q^{\otimes n}$ which extends Φ_1. In [18, Proposition 6.7], we showed that Φ_n affords isomorphisms

$$\Phi_n \colon\ \mathcal{L}_{0,n} \to \big(U_q^{\otimes n}\big)^{\mathrm{lf}}, \qquad \Phi_n \colon\ \mathcal{M}_{0,n} \to \big(U_q^{\otimes n}\big)^{U_q}, \quad (2.20)$$

where $\big(U_q^{\otimes n}\big)^{\mathrm{lf}}$ is the set of ad^r_n-locally finite elements of $U_q^{\otimes n}$. We call Φ_n the *Alekseev map*; we do not recall here the definition of Φ_n, for we will not use it. It is a key argument of the proof of (2.20) that the set of locally finite elements of $U_q^{\otimes n}$ for $(\mathrm{ad}^r)^{\otimes n} \circ \Delta^{(n)}$ coincides with $\big(U_q^{\mathrm{lf}}\big)^{\otimes n}$; this follows from the main result of [72]. Using that the map

$$\psi_n = \Phi_n \circ \big(\Phi_1^{-1}\big)^{\otimes n} \colon\ \big(U_q^{\mathrm{lf}}\big)^{\otimes n} \to \big(U_q^{\otimes n}\big)^{\mathrm{lf}} \quad (2.21)$$

intertwines the actions $(\mathrm{ad}^r)^{\otimes n} \circ \Delta^{(n-1)}$ and ad^r_n, we deduced that $\mathrm{Im}(\Phi_n) = \big(U_q^{\otimes n}\big)^{\mathrm{lf}}$.

Remark 2.4. We have $\big(U_q^{\mathrm{lf}}\big)^{\otimes n} \neq \big(U_q^{\otimes n}\big)^{\mathrm{lf}}$ and in fact there is not even an inclusion. Indeed, let $\Omega = \big(q - q^{-1}\big)^2 FE + qK + q^{-1}K^{-1}$ be the Casimir element of $U_q(\mathfrak{sl}_2)$. We trivially have $\Delta(\Omega) \in \big(U_q^{\otimes 2}\big)^{\mathrm{lf}}$ but

$$\Delta(\Omega) = \big(q - q^{-1}\big)^2\big(K^{-1}E \otimes FK + F \otimes E\big) + \Omega \otimes K + K^{-1} \otimes \Omega - \big(q + q^{-1}\big)K^{-1} \otimes K$$

and therefore $\Delta(\Omega) \notin \big(U_q^{\mathrm{lf}}\big)^{\otimes 2}$, since $K \notin U_q^{\mathrm{lf}}$ (see, e.g., Theorem 2.1 (2)). This reflects the fact that U_q^{lf} is only a right coideal of U_q (and not a subcoalgebra).

As in Remark 2.2, denote by $C(\mu)$, $\mu \in P^+$, the linear subspace of $\mathcal{L}_{0,1}$ generated by the matrix coefficients of V_μ. For every tuple $[\mu] = (\mu_1, \dots, \mu_n) \in P_+^n$ put

$$C([\mu]) = C(\mu_1) \otimes \cdots \otimes C(\mu_n). \tag{2.22}$$

Then $\mathcal{L}_{0,n} = \bigoplus_{[\mu]\in P_+^n} C([\mu])$. Each space $C([\mu])$ is a finite-dimensional U_q-module under the action coad_n^r, whence it is completely reducible. Therefore, $\mathcal{L}_{0,n} = \mathcal{M}_{0,n} \oplus I$ as U_q-modules, where I is the sum of nontrivial isotypical components of $\mathcal{L}_{0,n}$. The $\mathbb{C}(q)$-linear projection map

$$\mathcal{R}\colon\ \mathcal{L}_{0,n} \to \mathcal{M}_{0,n}, \qquad \mathrm{Ker}(\mathcal{R}) = I \tag{2.23}$$

is called the *Reynolds operator*. For all $\alpha \in \mathcal{M}_{0,n}$, $\beta \in \mathcal{L}_{0,n}$ it satisfies $\mathcal{R}(\alpha\beta) = \alpha\mathcal{R}(\beta)$. This property will be crucial in the sequel, so let us recall a (classical) proof of it. We can write $\beta = \mathcal{R}(\beta) + \gamma$ with $\gamma \in I$, and then we have to show $\alpha\gamma \in I$. We can reduce to the case where γ is contained in a simple summand V of I. Multiplication by the invariant element α yields a surjective map $V \to \alpha V$, which is a morphism of U_q-modules. Since V is simple, it is either the 0 map, or an isomorphism. In either cases it follows $\alpha V \subset I$ (in fact the first case cannot happen, for $\mathcal{L}_{0,n}$ has no nontrivial zero divisors, as explained after (2.25)).

We can formulate the Reynolds operator in the following way. Recall that $\mathcal{O}_q$ has a unique left (or right, or 2-sided) Haar integral, that is a linear map $h\colon \mathcal{O}_q \to \mathbb{C}(q)$ such that

$$h(1) = 1 \qquad \text{and} \qquad (\mathrm{id} \otimes h)\Delta(\alpha) = h(\alpha)1, \qquad \forall \alpha \in \mathcal{O}_q.$$

(See, e.g., [35, Proposition 13.3.6].) It vanishes on all matrix coefficients except the one of the trivial representation, to which it gives the value 1. Denote by $\Delta_{\mathcal{L}}\colon \mathcal{L}_{0,n} \to \mathcal{L}_{0,n} \otimes \mathcal{O}_q$ the right coaction dual to the action coad_n^r of U_q on $\mathcal{L}_{0,n}$. Then, in analogy with the formula of the averaging operator $\mathcal{C}^\infty(G) \to \mathcal{C}^\infty(G)^G$, $f \to [f] = \int_G f\big(g^{-1} \cdot g\big)\mathrm{d}\mu(g)$, for a locally compact group G with Haar measure $\mathrm{d}\mu(g)$, it is straightforward that

$$\mathcal{R} = (\mathrm{id} \otimes h)\Delta_{\mathcal{L}}. \tag{2.24}$$

Note that the complete reducibility of $\mathcal{L}_{0,n}$ discussed after (2.22) follows also from Theorem 2.1 (1), since by (2.21) we have an isomorphism of U_q-modules

$$\mathcal{L}_{0,n} \xrightarrow{\Phi_n} \big(U_q(\mathfrak{g})^{\otimes n}\big)^{\mathrm{lf}} \xrightarrow{\psi_n^{-1}} U_q^{\mathrm{lf}}(\mathfrak{g})^{\otimes n},$$

where lf means respectively locally finite for the action ad_n^r of $U_q(\mathfrak{g})$ on $U_q(\mathfrak{g})^{\otimes n}$, and locally finite for the action ad^r of $U_q(\mathfrak{g})$ on $U_q(\mathfrak{g})$. An explicit basis of $\mathcal{M}_{0,n}$ is described in [18, Proposition 6.22].

Finally, let us point out here two important consequences of (2.20). First, Φ_n yields isomorphisms between centers, $\mathcal{Z}(\mathcal{L}_{0,n}) \cong \mathcal{Z}(U_q)^{\otimes n}$ and $\mathcal{Z}\big(\mathcal{L}_{0,n}^{U_q}\big) \cong \mathcal{Z}\big(\big(U_q^{\otimes n}\big)^{U_q}\big)$, where one can show that [18, Lemma 6.29]

$$\mathcal{Z}\big(\big(U_q^{\otimes n}\big)^{U_q}\big) \cong \Delta^{(n)}(\mathcal{Z}(U_q)) \bigotimes_{\mathbb{C}(q)} \mathcal{Z}(U_q)^{\otimes n}. \tag{2.25}$$

Second, $\mathcal{L}_{0,n}$ (and therefore $\mathcal{M}_{0,n}$) has no nontrivial zero divisors because of the isomorphisms $\Phi_n\colon \mathcal{L}_{0,n} \to \big(U_q^{\otimes n}\big)^{\mathrm{lf}} \subset U_q^{\otimes n}$ and $U_q^{\otimes n} \cong U_q\big(\mathfrak{g}^{\oplus n}\big)$, and the fact that $U_q\big(\mathfrak{g}^{\oplus n}\big)$ has no nontrivial zero divisors (proved, e.g., in [39]).

2.2 Integral forms and specializations

Let $A = \mathbb{C}\big[q, q^{-1}\big]$. We call integral form of a (Hopf) $\mathbb{C}(q)$-algebra H a (Hopf) A-subalgebra ${}_AH$ such that the canonical map ${}_AH \bigotimes_A \mathbb{C}(q) \to H$ is an isomorphism. Note that the standard notion of integral form of $\mathbb{C}(q)$-algebra uses $\mathbb{Z}\big[q, q^{-1}\big]$ instead of $\mathbb{C}\big[q, q^{-1}\big]$; our choice is made for simplicity ($\mathbb{C}\big[q, q^{-1}\big]$ is a principal ideal domain, whereas $\mathbb{Z}\big[q, q^{-1}\big]$ is not).

2.2.1 Definitions

The *unrestricted* integral form of U_q is the A-subalgebra $U_A = U_A(\mathfrak{g})$ introduced by De Concini–Kac–Procesi in [42, Section 12] (and in a differently normalized form in [39, 40]). It is the smallest A-subalgebra of U_q which contains the elements $(i = 1, \dots, m)$

$$\bar{E}_i = \big(q_i - q_i^{-1}\big)E_i, \qquad \bar{F}_i = \big(q_i - q_i^{-1}\big)F_i, \qquad L_i, \qquad L_i^{-1} \tag{2.26}$$

and is stable under the action of $\mathcal{B}(\mathfrak{g})$ given by the Lusztig automorphisms (see (2.1)). Recall the root vectors E_{β_k}, F_{β_k} defined in (2.1). Let us put $q_\beta := q^{(\beta,\beta)/2}$. The algebra U_A is a free A-module with basis the monomials $\bar{E}_{\beta_1}^{p_1} \cdots \bar{E}_{\beta_N}^{p_N} K_\lambda \bar{F}_{\beta_N}^{n_N} \cdots \bar{F}_{\beta_1}^{n_1}$, where $\lambda \in P$ and we set

$$\bar{E}_{\beta_k} = \big(q_{\beta_k} - q_{\beta_k}^{-1}\big)E_{\beta_k}, \qquad \bar{F}_{\beta_k} = \big(q_{\beta_k} - q_{\beta_k}^{-1}\big)F_{\beta_k}.$$

We denote $U_A^{\mathrm{lf}} := U_A \cap U_q^{\mathrm{lf}}$. The unrestricted integral form of U_q^{ad} is defined similarly, as the smallest A-subalgebra $U_A^{\mathrm{ad}} \subset U_A$ which contains the elements $\bar{E}_i$, $\bar{F}_i$ and $K_i^{\pm 1}$, for $i = 1, \dots, m$, and is stable under the Lusztig action of $\mathcal{B}(\mathfrak{g})$.

For β a positive root, we define the divided powers

$$E_\beta^{(k)} = \frac{E_\beta^k}{[k]_{q_\beta}!}, \qquad F_\beta^{(k)} = \frac{F_\beta^k}{[k]_{q_\beta}!}, \qquad k \in \mathbb{N}.$$

The Lusztig *restricted* integral form of U_q^{ad} [82, 83] (see also [35, Chapter 9.3]) is the A-subalgebra U_A^{res} generated by the elements $(i = 1, \dots, m,\ k \in \mathbb{N}^*)$

$$E_i^{(k)} = \frac{E_i^k}{[k]_{q_i}!}, \qquad F_i^{(k)} = \frac{F_i^k}{[k]_{q_i}!}, \qquad K_i, \qquad K_i^{-1}.$$

The algebra U_A^{res} is a free A-module with Poincaré–Birkhoff–Witt (PBW) basis

$$E_{\beta_1}^{(p_1)} \cdots E_{\beta_N}^{(p_N)} \prod_{i=1}^m K_i^{\sigma_i} [K_i; t_i]_{q_i} F_{\beta_N}^{(n_N)} \cdots F_{\beta_1}^{(n_1)},$$

where $\sigma_i \in \{0, 1\}$, $n_i, p_i, t_i \in \mathbb{N}$, and we set $[K_i; 0]_{q_i} := 1$ and

$$[K_i; t]_{q_i} = \prod_{s=1}^t \frac{K_i q_i^{-s+1} - K_i^{-1} q_i^{s-1}}{q_i^s - q_i^{-s}}.$$

The integral forms $U_A(\mathfrak{h})$, $U_A(\mathfrak{b}_\pm)$ and $U_A^{\mathrm{res}}(\mathfrak{h})$, $U_A^{\mathrm{res}}(\mathfrak{b}_\pm)$ associated to the subalgebras $\mathfrak{h}$, $\mathfrak{b}_\pm \subset \mathfrak{g}$ are the subalgebras of U_A and U_A^{res}, respectively, defined in the obvious way. For instance, the "Cartan" subalgebra $U_A^{\mathrm{res}}(\mathfrak{h}) = U_q(\mathfrak{h}) \cap U_A^{\mathrm{res}}$ is generated as a A-module by the elements $\prod_{i=1}^m K_i^{\sigma_i}[K_i; t_i]_{q_i}$.

Denote by $\mathcal{C}_A$ the category of U_A^{res}-modules of *type* 1, i.e., free A-modules of finite rank which have a basis where the elements K_i act diagonally with eigenvalues of the form q_i^k, $k \in \mathbb{Z}$ (in general, finiteness of the rank imposes eigenvalues of the form $\pm q_i^k$, $k \in \mathbb{Z}$). The category $\mathcal{C}_A$ is a rigid and tensor category. It is not semisimple, and this makes the study of $\mathcal{C}_A$ a complicated task; for this, see [18], and Section 2.2.2 below. Every type 1 finite-dimensional simple U_q-module V_μ, $\mu \in P_+$, has a U_A^{res}-invariant full A-sublattice, that we denote by ${}_AV_\mu$. These U_A^{res}-modules form the simple objects of $\mathcal{C}_A$. Moreover, $\mathcal{C}_A \otimes \mathbb{C}\big[q^{1/D}, q^{-1/D}\big]$ is a ribbon category (see Section 2.3).

The *integral* quantum function Hopf algebra $\mathcal{O}_A = \mathcal{O}_A(G)$ is the (type 1) restricted dual of U_A^{res}, that is, the A-span of the matrix coefficients $x \mapsto v^i(\pi_V(x)v_i)$, $x \in U_A^{\mathrm{res}}$, for every module V in $\mathcal{C}_A$, where (v_i) is an A-basis of V and $\big(v^i\big)$ the dual A-basis of the dual module V^* (compare with the definition of $\mathcal{O}_q$). We can also regard $\mathcal{O}_A$ as the set of A-linear maps $f\colon U_A^{\mathrm{res}} \to A$

such that $\mathrm{Ker}(f)$ contains a cofinite two sided ideal I, and $\prod_{s=-r}^{r}(K_i - q_i^s) \in I$ for some $r \in \mathbb{N}$ and every i. Because of the inclusions of $U_A^{\mathrm{res}}(\mathfrak{h})$, $U_A^{\mathrm{res}}(\mathfrak{n}_\pm)$, $U_A^{\mathrm{res}}(\mathfrak{b}_\pm)$ in U_A^{res}, there are Hopf epimorphisms from $\mathcal{O}_A$ to the A-duals of these subalgebras, that we denote by $\mathcal{O}_A(T_G)$, $\mathcal{O}_A(U_\pm)$ and $\mathcal{O}_A(B_\pm)$, respectively.

The algebra $\mathcal{O}_A$ has been introduced by Lusztig in [82, 83]. It is an integral form of $\mathcal{O}_q$, so $\mathcal{O}_q = \mathcal{O}_A \bigotimes_A \mathbb{C}(q)$.

$\mathcal{O}_A$ is also the restricted dual of the integral form $\Gamma = \Gamma(\mathfrak{g})$ of U_q^{ad} introduced by De Concini–Lyubashenko in [41, Sections 2 and 3]; Γ is the A-subalgebra of U_q^{ad} generated by the elements $(i = 1, \dots, m)$

$$E_i^{(k)} = \frac{E_i^k}{[k]_{q_i}!}, \qquad F_i^{(k)} = \frac{F_i^k}{[k]_{q_i}!}, \qquad (K_i;t)_{q_i} = \prod_{s=1}^{t} \frac{K_i q_i^{-s+1} - 1}{q_i^s - 1}, \qquad K_i^{-1},$$

where $k \in \mathbb{N}$, $t \in \mathbb{N}$ (setting $(K_i;0)_{q_i} = 1$ by convention). Note that the definition of Γ is less symmetric than that of U_A^{res}. However, Γ contains the elements K_i, and the commutation relations between the generators $E_i^{(k)}$, $F_i^{(k)}$ imply that the symmetrized elements $[K_i;t]_{q_i}$ belong to Γ. In fact, let us denote $\Gamma(\mathfrak{h}) = U_q(\mathfrak{h}) \cap \Gamma$ and $\Gamma(\mathfrak{b}_\pm) = U_q(\mathfrak{b}_\pm) \cap \Gamma$. It is proved in [41, Theorem 3.1] that $\Gamma(\mathfrak{h})$ contains $U_A^{\mathrm{res}}(\mathfrak{h})$ and that the elements $\prod_{i=1}^m K_i^{-\sigma(t_i)}(K_i;t_i)_{q_i}$, $t_i \in \mathbb{N}$, where $\sigma(t)$ is the integer part of $t/2$, is an A-basis of $\Gamma(\mathfrak{h})$. A PBW basis of Γ is formed by the monomials

$$E_{\beta_1}^{(p_1)} \cdots E_{\beta_N}^{(p_N)} \prod_{i=1}^{m} K_i^{-\sigma(t_i)}(K_i;t_i)_{q_i} F_{\beta_N}^{(n_N)} \cdots F_{\beta_1}^{(n_1)}.$$

The inclusion $U_A^{\mathrm{res}} \subset \Gamma$ is strict, for the elements $(K_i;t)_{q_i}$, $t \neq 0$, do not belong to U_A^{res}. However, the restriction functor $\mathcal{C}_\Gamma \to \mathcal{C}_A$ is obviously an equivalence, where $\mathcal{C}_\Gamma$ is the category of Γ-modules of *type* 1, i.e., free A-modules of finite rank which have a basis where the elements K_i act diagonally with eigenvalues of the form q_i^k, $k \in \mathbb{Z}$. Therefore, we can identify the two categories, and $\mathcal{O}_A$ with the (type 1) restricted dual of Γ. We will thus consider the U_A^{res}-modules ${}_AV_\mu$, $\mu \in P_+$, equally as Γ-modules. We will sometimes use Γ instead of U_A^{res} in order to make direct the connection with results of De Concini–Lyubashenko about the integral pairings $\pi_A^\pm$ considered in Section 2.3.

The *integral* form $\mathcal{L}_{0,1}^A$ of $\mathcal{L}_{0,1}$ is defined as the U_A^{res}-module $\mathcal{O}_A$ endowed with the product of $\mathcal{L}_{0,1}$. The *integral* form $\mathcal{L}_{0,n}^A$ of $\mathcal{L}_{0,n}$ is the braided tensor product of n copies of $\mathcal{L}_{0,1}^A$; in particular, $\mathcal{L}_{0,n}^A = \mathcal{O}_A^{\otimes n}$ as U_A^{res}-modules. That the products of $\mathcal{L}_{0,1}$ and $\mathcal{L}_{0,n}$ are well defined over A was shown in [18, Proposition 6.9].

The *integral* quantum moduli algebra is

$$\mathcal{M}_{0,n}^A := \left(\mathcal{L}_{0,n}^A\right)^{U_A^{\mathrm{res}}} = \left(\mathcal{L}_{0,n}^A\right)^{U_A}.$$

Finally, given $q = \epsilon \in \mathbb{C}^\times$ we define the *specializations* U_ϵ, Γ_ϵ, $\mathcal{O}_\epsilon$, $\mathcal{L}_{0,n}^\epsilon$ and $\mathcal{M}_{0,n}^{A,\epsilon}$ as the $\mathbb{C}$-algebras obtained by tensoring U_A, Γ, $\mathcal{O}_A$, $\mathcal{L}_{0,n}^A$ and $\mathcal{M}_{0,n}^A$ respectively with $\mathbb{C}_\epsilon$, the A-module $\mathbb{C}$ where q acts by multiplication by ϵ. Each one can also be defined as the quotient by the ideal generated by $q - \epsilon$. We find convenient to use the notations

$$\left(U_A^{\otimes n}\right)_\epsilon^{U_A} := \left(U_A^{\otimes n}\right)^{U_A} \bigotimes_A \mathbb{C}_\epsilon, \qquad \left(U^{\otimes n}\right)_\epsilon^{\mathrm{lf}} := \left(U_A^{\otimes n}\right)^{\mathrm{lf}} \bigotimes_A \mathbb{C}_\epsilon. \tag{2.27}$$

Let us stress here that when ϵ is a root of unity, taking the locally finite part and taking the specialization at ϵ are non commuting operations. Indeed, as shown by Theorem 2.27 below, U_ϵ is finite over $\mathcal{Z}_0(U_\epsilon)$ and therefore all its elements are locally finite for ad^r; on another hand $U_\epsilon^{\mathrm{lf}} = U_A^{\mathrm{lf}} \bigotimes_A \mathbb{C}_\epsilon$ does not contain the elements L_i.

Similarly, taking invariants and taking the specialization at ϵ are non commuting operations when ϵ is a root of unity: indeed, it is easily checked that in this case $\big(U_A^{\otimes n}\big)^{U_A}_\epsilon$ and $\big(U_\epsilon^{\otimes n}\big)^{U_\epsilon}$, or $\mathcal{M}_{0,n}^{A,\epsilon} = \mathcal{M}_{0,n}^A \bigotimes_A \mathbb{C}_\epsilon$ and $\big(\mathcal{L}_{0,n}^\epsilon\big)^{U_\epsilon}$, are distinct spaces. When ϵ is a root of unity, we will not consider the algebras $\mathcal{M}_{0,n}^{A,\epsilon}$ in this paper.

Arguments similar to those mentioned at the end of Section 2.1 imply that the algebras $\mathcal{L}_{0,n}^A$, $\mathcal{M}_{0,n}^A$ and $\mathcal{L}_{0,n}^{\epsilon'}$, $\mathcal{M}_{0,n}^{A,\epsilon'}$, $\epsilon' \in \mathbb{C}^\times$, have no nontrivial zero divisors (see [18, Propositions 6.11 and 6.30]).

2.2.2 Canonical bases and modified quantum groups

Because the category $\mathcal{C}_A$ is not semisimple, it is not clear from the above definition of $\mathcal{O}_A$ whether or not it is a finitely generated algebra, if $\mathcal{M}_{0,n}^A$ is a direct summand of the A-module $\mathcal{L}_{0,n}^A$, or if the projection map (2.8) may be refined to a morphism between underlying A-modules.

Such properties, using the appropriate formalism developed by Kashiwara–Lusztig, indeed hold true, and will play a key role later. We state them precisely in Proposition 2.10, Theorem 2.15 and Proposition 2.12. These results are consequences of the existence of an A-basis of $\mathcal{O}_A$ with favourable properties, which implies in particular that $\mathcal{O}_A$ is a free A-module. In order to introduce this A-basis it is necessary to consider a variant of $U_q^{\rm ad}$ introduced by Lusztig [83], called *modified quantum group*, and use the Kashiwara–Lusztig theory of canonical bases [65, 66, 67, 83]. We are going to recall the background material step by step.

The Lusztig *modified quantum group* is the $\mathbb{C}(q)$-algebra $\dot{\mathbf{U}}$ obtained by replacing $U_q^{\rm ad}(\mathfrak{h})$ with the direct sum of infinitely many one-dimensional algebras, generated by orthogonal idempotents 1_λ indexed by the elements λ of the weight lattice P [83, Chapter 23]. Namely, as a vector space $\dot{\mathbf{U}} = \bigoplus_{\lambda',\lambda''\in P} {}_{\lambda'}\dot{\mathbf{U}}_{\lambda''}$, where

$$ {}_{\lambda'}\dot{\mathbf{U}}_{\lambda''} = U_q^{\rm ad}\Big/\bigg(\sum_{\alpha\in Q}\big(K_\alpha - q^{(\alpha,\lambda')}\big)U_q^{\rm ad} + \sum_{\alpha\in Q}U_q^{\rm ad}\big(K_\alpha - q^{(\alpha,\lambda'')}\big)\bigg). $$

Denote by $\pi_{\lambda',\lambda''}\colon U_q^{\rm ad} \to {}_{\lambda'}\dot{\mathbf{U}}_{\lambda''}$ the canonical projection. The product of $\dot{\mathbf{U}}$ is given by $\pi_{\lambda_1',\lambda_1''}(s)\pi_{\lambda_2',\lambda_2''}(t) = \pi_{\lambda_1',\lambda_2''}(st)$ if $\lambda_1'' = \lambda_2'$ and zero otherwise. Set $1_\lambda := \pi_{\lambda,\lambda}(1)$. The algebra $\dot{\mathbf{U}}$ has not unit, but the family $(1_\lambda)_{\lambda\in P}$ can be regarded as a substitute of it. Denote by Δ the collection of maps

$$ \Delta_{\lambda_1',\lambda_2',\lambda_1'',\lambda_2''}\colon {}_{\lambda_1'+\lambda_2'}\dot{\mathbf{U}}_{\lambda_1''+\lambda_2''} \to {}_{\lambda_1'}\dot{\mathbf{U}}_{\lambda_1''} \otimes {}_{\lambda_2'}\dot{\mathbf{U}}_{\lambda_2''} $$

such that

$$ \Delta_{\lambda_1',\lambda_2',\lambda_1'',\lambda_2''}\pi_{\lambda_1'+\lambda_2',\lambda_1''+\lambda_2''} = (\pi_{\lambda_1',\lambda_1''}\otimes\pi_{\lambda_2',\lambda_2''})\Delta_{U_q^{\rm ad}}, \tag{2.28} $$

where $\Delta_{U_q^{\rm ad}}$ is the coproduct of $U_q^{\rm ad}$. We can regard Δ as a (categorically completed) coproduct $\Delta\colon \dot{\mathbf{U}} \to \dot{\mathbf{U}}^{\hat{\otimes}2}$. There is a natural structure of $U_q^{\rm ad}$-bimodule on $\dot{\mathbf{U}}$, defined by

$$ t'\pi_{\lambda',\lambda''}(s)t'' = \pi_{\lambda'+\nu',\lambda''-\nu''}(t'st'') \tag{2.29} $$

for all $s \in U_q^{\rm ad}$ and all elements $t', t'' \in U_q^{\rm ad}$ of respective weights ν', ν''. This structure affords a triangular decomposition of $\dot{\mathbf{U}}$: given bases $\{b^\pm\}$ of $U_q^{\rm ad}(\mathfrak{n}_\pm)$, the set of elements $b^+1_\lambda b^-$ (or $b^-1_\lambda b^+$, or $b^+b^-1_\lambda$), where $\lambda \in P$, is a basis of $\dot{\mathbf{U}}$.

Given any $U_q^{\rm ad}$-module X of type 1, and any weight subspace $X^\lambda \subset X$ of weight $\lambda \in P$, one can define the action of an element $u1_\lambda \in \dot{\mathbf{U}}$, $u \in U_q^{\rm ad}$, on X as the projection onto X^λ followed by the action of u. This way, one can identify the category $\mathcal{C}$ with the one of finite-dimensional *unital* $\dot{\mathbf{U}}$-modules, where unital means that all elements 1_λ act as 0 but a finite number of them, and $\sum_{\lambda\in P} 1_\lambda$ acts as the identity. It is proved in [83, Section 29.5.1], that

$$\mathcal{O}_q = \left\{ f\colon \dot{\mathbf{U}} \to \mathbb{C}(q) \,\middle|\, \begin{array}{c} f \text{ is } \mathbb{C}(q)\text{-linear and vanishes on some} \\ \text{two-sided ideal of finite codimension of } \dot{\mathbf{U}} \end{array} \right\}.$$

There is an analogous realization of $\mathcal{O}_A$, of the form (see [83, Sections 23.2 and 29.5.2], and [84])

$$\mathcal{O}_A = \left\{ f\colon \dot{\mathbf{U}}_A \to A \,\middle|\, \begin{array}{c} f \text{ is } A\text{-linear and vanishes on some} \\ \text{two-sided ideal of finite corank of } \dot{\mathbf{U}}_A \end{array} \right\},$$

where $\dot{\mathbf{U}}_A$ is the A-subalgebra of $\dot{\mathbf{U}}$ generated by the elements $E_i^{(k)}1_\lambda$ and $F_i^{(k)}1_\lambda$, for all $i \in \{1,\dots,m\}$, $k\in\mathbb{N}$ and $\lambda\in P$. It is a U_A^{res}-subbimodule of $\dot{\mathbf{U}}$, and the coproduct restricts to a map $\Delta\colon \dot{\mathbf{U}}_A \to \dot{\mathbf{U}}_A{}^{\hat{\otimes}2}$. The above identification of the category $\mathcal{C}$ with the one of finite-dimensional unital $\dot{\mathbf{U}}$-modules yields an identification of the category $\mathcal{C}_A$ of U_A^{res}-modules of type 1 with the category of $\dot{\mathbf{U}}_A$-modules of finite rank.

The key advantage of this realization of $\mathcal{O}_A$ is that $\dot{\mathbf{U}}_A$ can be equipped with a canonical A-basis $\dot{\mathbf{B}}$. The construction of $\dot{\mathbf{B}}$ is described in [83, Chapter 25]. It relies on the Kashiwara–Lusztig *canonical basis* of $U_A^{\mathrm{res}}(\mathfrak{n}_-)$. This last basis, denoted by $\mathbf{B}^-$, is defined in [83, Chapter 14], and [65] (a review can be found in [35, Chapter 14]). It enjoys the following nice properties. Denote by ${}^-\colon \mathbb{C}(q)\to\mathbb{C}(q)$ the field involution such that $\overline{q} = q^{-1}$, and by ${}^-\colon U_q^{\mathrm{ad}} \to U_q^{\mathrm{ad}}$ the homomorphism of $\mathbb{C}$-algebras such that

$$\bar{E}_i = E_i, \qquad \bar{F}_i = F_i, \qquad \bar{K}_\lambda = K_{-\lambda}, \qquad \overline{fx} = \bar{f}\bar{x}$$

for all $f\in\mathbb{C}(q)$, $x\in U_q^{\mathrm{ad}}$ ($\bar{E}_i$ and $\bar{F}_i$ above, which will not appear elsewhere, should not be confused with the normalized elements in (2.26)). The map ${}^-$ yields a $\mathbb{C}$-algebra homomorphism ${}^-\colon \dot{\mathbf{U}}\to\dot{\mathbf{U}}$. Then, we have

(1) the elements of $\mathbf{B}^-$ are weight vectors under the adjoint action of $U_q^{\mathrm{ad}}(\mathfrak{h})$;
(2) for every $b\in\mathbf{B}^-$, $\bar{b} = b$;
(3) for every $b,b'\in\mathbf{B}^-$, $bb' = \sum_{b''\in\mathbf{B}^-} N_{b''}^{bb'} b''$ where $N_{b''}^{bb'}\in\mathbb{Z}\big[q,q^{-1}\big]$;
(4) for every $b,b'\in\mathbf{B}^-$, $\Delta(b) = \sum_{b',b''\in\mathbf{B}^-} C_{b'b''}^{b} b'\otimes b''$ where $C_{b'b''}^{b}\in\mathbb{Z}\big[q,q^{-1}\big]$;
(5) for every $\mu\in P^+$, denoting by v_μ the highest weight vector of the U_A^{res}-module ${}_AV_\mu$, the elements bv_μ which are non-zero, where $b\in\mathbf{B}^-$, form an A-basis of ${}_AV_\mu$.

When $\mathfrak{g}$ is simply laced, the coefficients $N_{b''}^{bb'}$ and $C_{b'b''}^{b}$ belong to $\mathbb{N}\big[q,q^{-1}\big]$ [83, Theorem 14.3.13]. In the case of $\mathfrak{g}=\mathfrak{sl}_2$, the elements of $\mathbf{B}^-$ are just the divided powers $F^{(k)}$, $k\in\mathbb{N}$. Formulas in terms of PBW basis elements are known also for $\mathfrak{g}=\mathfrak{sl}_3$ and $\mathfrak{sl}_4$, and an algorithm exists in the general case (see [43] and the references therein).

Correspondingly to $\mathbf{B}^-$, the set $\mathbf{B}^+ = \omega(\mathbf{B}^-)$ is a basis of $U_A^{\mathrm{res}}(\mathfrak{n}_+)$, where $\omega\colon U_q^{\mathrm{ad}}\to U_q^{\mathrm{ad}}$ is the ($\mathbb{C}(q)$-linear) *Cartan automorphism*, defined by

$$\omega(E_i) = F_i, \qquad \omega(F_i) = E_i, \qquad \omega(K_i) = K_i^{-1}$$

for $i=1,\dots,m$. The triangular decomposition of $\dot{\mathbf{U}}$ implies that the elements $b^+1_\lambda b'^-$, where $b^+\in\mathbf{B}^+$, $b'^-\in\mathbf{B}^-$ and $\lambda\in P$, form a basis of $\dot{\mathbf{U}}$. They form in fact an A-basis of $\dot{\mathbf{U}}_A$, and its elements are fixed by the involution ${}^-\colon \dot{\mathbf{U}}\to\dot{\mathbf{U}}$.

Lusztig has constructed another A-basis of $\dot{\mathbf{U}}_A$, denoted $\dot{\mathbf{B}}$, and called the *canonical basis* of $\dot{\mathbf{U}}_A$. It satisfies numerous properties that we now review. Its elements are denoted by $b\diamondsuit_\lambda b'$, where $b, b' \in \mathbf{B}^-$ and $\lambda \in P$, and are related to the elements $b^+ b'^- 1_\lambda$, where $b^+ := \omega(b)$ and $b'^- := b'$, by a specific trigonal change of basis with coefficients in A. Although we always have $b^+ 1_\lambda$, $b'^- 1_\lambda \in \dot{\mathbf{B}}$, to our knowledge explicit formulas of the elements of $\dot{\mathbf{B}}$ as linear combinations of elements $b^+ 1_\lambda b'^-$ or $b'^- 1_\lambda b^+$ are known only for $\mathfrak{g} = \mathfrak{sl}_2$ or $\mathfrak{sl}_3$ (see [83, Section 25.3] and [37]). In the former case, identifying P with $\mathbb{Z}$ and Q with $2\mathbb{Z}$ the canonical basis $\dot{\mathbf{B}}$ is formed by the elements

$$E^{(k)} 1_{-n} F^{(l)} \quad \text{and} \quad F^{(l)} 1_n E^{(k)}, \qquad k, l, n \in \mathbb{N}, \qquad n \geq k + l,$$

where $E^{(k)} 1_{-n} F^{(l)} = F^{(l)} 1_n E^{(k)}$ for $n = k + l$.

We are going to review Lusztig's construction of $\dot{\mathbf{B}}$, its canonical partition $\dot{\mathbf{B}} = \bigcup_{\lambda \in P_+} \dot{\mathbf{B}}[\lambda]$, the dual basis $\dot{\mathbf{B}}^*$, and Kashiwara's approach to $\dot{\mathbf{B}}^*$ [66, 67]. The latter is stated in Theorem 2.6 below. At first we need to recall the notions of based module and balanced triple; for details on these notions we refer to [83, Chapter 27] and [66] (see also [68], [104, Sections 3.15 and 3.16], or [35, Chapter 14] for overviews).

Denote by $\mathcal{A}_0 \subset \mathbb{C}(q)$ the ring of rational functions regular at $q = 0$. By applying the involution $^-$, put $\mathcal{A}_\infty = \overline{\mathcal{A}}_0$. Since $\mathcal{A}_0$ is the localization of $\mathbb{C}[q]$ at $q = 0$, we may regard $\mathcal{A}_\infty$ as the localization of $\mathbb{C}\left[q^{-1}\right]$ at $q = \infty$.

Let us recall briefly the definition of crystal basis (see [65]). Denote by $U_q^{\mathrm{ad}}(\mathfrak{g})_i$ the subalgebra of $U_q^{\mathrm{ad}}(\mathfrak{g})$ generated by E_i, F_i and $K_i^{\pm 1}$; thus $U_q^{\mathrm{ad}}(\mathfrak{g})_i$ is isomorphic to $U_{q_i}(\mathfrak{sl}_2)$. Let M be a U_q^{ad}-module of type 1. Denote M^ζ the subspace of M of weight $\zeta \in P$. For every $i = 1, \ldots, m$, we can regard M as a $U_q^{\mathrm{ad}}(\mathfrak{g})_i$-module, so $M \cong \bigoplus_j V_{\lambda_j}$ for some simple $U_q^{\mathrm{ad}}(\mathfrak{g})_i$-modules V_{λ_j}. These being generated by primitive weight vectors, the PBW basis of $U_q^{\mathrm{ad}}(\mathfrak{g})_i$ yields

$$M = \bigoplus_{\zeta \in P} \bigoplus_{0 \leq n \leq (\check{\alpha}_i, \zeta)} F_i^{(n)} \big(\mathrm{Ker}(E_i) \cap M^\zeta\big).$$

The *Kashiwara operators* $\tilde{e}_i$, $\tilde{f}_i$ are the endomorphisms of M defined by, for every $v \in \mathrm{Ker}(E_i) \cap M^\zeta$ and $0 \leq n \leq (\check{\alpha}_i, \zeta)$,

$$\tilde{f}_i\big(F_i^{(n)} v\big) = F_i^{(n+1)} v, \qquad \tilde{e}_i\big(F_i^{(n)} v\big) = F_i^{(n-1)} v.$$

A *crystal basis of M at $q = 0$* consists of a pair $(\mathcal{L}, \mathcal{B})$, where

- $\mathcal{L}$ is a free $\mathcal{A}_0$-sublattice of M such that the canonical map $\mathcal{L} \bigotimes_{\mathcal{A}_0} \mathbb{C}(q) \to M$ is an isomorphism;
- $\mathcal{B}$ is a basis of the $\mathbb{C}$-vector space $\mathcal{L}/q\mathcal{L}$;
- $\mathcal{L} = \bigoplus_{\zeta \in P} \mathcal{L}^\zeta$ and $\mathcal{B} = \coprod_{\zeta \in P} \left(\mathcal{B} \cap \mathcal{L}^\zeta / q\mathcal{L}^\zeta\right)$, where $\mathcal{L}^\zeta = \mathcal{L} \cap M^\zeta$;
- for every $i = 1, \ldots, m$ the Kashiwara operators $\tilde{e}_i$, $\tilde{f}_i$ preserve $\mathcal{L}$, and the induced maps on $\mathcal{L}/q\mathcal{L}$ send $\mathcal{B}$ into $\mathcal{B} \cup \{0\}$, and satisfy $b' = \tilde{f}_i(b)$ if and only if $b = \tilde{e}_i(b')$ for every $b, b' \in B$.

Crystal bases at $q = \infty$ are defined similarly, by replacing $\mathcal{A}_0$ with $\mathcal{A}_\infty$ and q with q^{-1}.

A *based module* consists of a pair (M, B) where M is a U_q^{ad}-module of type 1 endowed with a $\mathbb{C}(q)$-basis B such that the following conditions hold:

(i) For every weight $\zeta \in P$, the set $B \cap M^\zeta$ is a basis of the weight subspace $M^\zeta \subset M$.

(ii) The A-module ${}_A M$ generated by B is stable under U_A^{res}.

We will denote by $\mathcal{L}_M$ the $\mathcal{A}_0$-submodule of M generated by B, and by $\bar{\mathcal{L}}_M$ the $\mathcal{A}_\infty$-submodule of M generated by B.

(iii) The $\mathbb{C}$-linear involution $^- : M \to M$ defined by $\overline{fb} = \overline{f} b$ for all $f \in \mathbb{C}(q)$ and $b \in B$ is compatible with the action of U_q^{ad} in the sense that $\overline{xm} = \bar{x}\bar{m}$ for all $x \in U_q^{\mathrm{ad}}$, $m \in M$.

(iv) The $\mathcal{A}_\infty$-submodule $\bar{\mathcal{L}}_M$ of M together with the image of B in $\bar{\mathcal{L}}_M/q^{-1}\bar{\mathcal{L}}_M$ forms a crystal basis of M at $q=\infty$.

If (M,B) is a based module, we will denote by $\overline{\mathcal{B}}$ the image of B in $\bar{\mathcal{L}}_M/q^{-1}\bar{\mathcal{L}}_M$. From the notion of balanced triple that we recall now, denoting by $\mathcal{B}$ the image of B in $\mathcal{L}_M/q\mathcal{L}_M$, we see that $(\mathcal{L}_M,\mathcal{B})$ is a crystal basis at $q=0$.

Indeed, consider more generally a $\mathbb{C}(q)$-vector space V, finite-dimensional or not, a sub-A-module ${}_AV$, a sub-$\mathcal{A}_0$-module ${}_{\mathcal{A}_0}V$ and a sub-$\mathcal{A}_\infty$-module ${}_{\mathcal{A}_\infty}V$ satisfying the conditions (all isomorphisms being the canonical maps)

$$V\cong\mathbb{C}(q)\bigotimes_{A}{}_AV,\qquad V\cong\mathbb{C}(q)\bigotimes_{\mathcal{A}_0}{}_{\mathcal{A}_0}V,\qquad V\cong\mathbb{C}(q)\bigotimes_{\mathcal{A}_\infty}{}_{\mathcal{A}_\infty}V.$$

Consider the $\mathbb{C}$-vector space $E:={}_AV\cap{}_{\mathcal{A}_0}V\cap{}_{\mathcal{A}_\infty}V$. Then $({}_AV,{}_{\mathcal{A}_0}V,{}_{\mathcal{A}_\infty}V)$ is a *balanced triple* [65, 66] if the canonical maps

$$A\bigotimes_{\mathbb{C}}E\to{}_AV,\qquad \mathcal{A}_0\bigotimes_{\mathbb{C}}E\to{}_{\mathcal{A}_0}V,\qquad \mathcal{A}_\infty\bigotimes_{\mathbb{C}}E\to{}_{\mathcal{A}_\infty}V \tag{2.30}$$

are isomorphisms. Equivalently, $({}_AV,{}_{\mathcal{A}_0}V,{}_{\mathcal{A}_\infty}V)$ is balanced if and only if the canonical map $E\to{}_{\mathcal{A}_0}V/q{}_{\mathcal{A}_0}V$ is an isomorphism, if and only if the canonical map $E\to{}_{\mathcal{A}_\infty}V/q^{-1}{}_{\mathcal{A}_\infty}V$ is an isomorphism [66, Lemma 2.1.1].

Given a based module (M,B), the elements of B are weight vectors and $\bar{b}=b$ for every $b\in B$. Also, if an element $m\in{}_AM$ satisfies $\overline{m}=m$ and $m\in B+q^{-1}\bar{\mathcal{L}}_M$, then $m\in B$ (see [83, Section 27.1.5] for details on this fact). It follows that the canonical quotient map

$${}_AM\cap\mathcal{L}_M\cap\bar{\mathcal{L}}_M\to\bar{\mathcal{L}}_M/q^{-1}\bar{\mathcal{L}}_M \tag{2.31}$$

is an isomorphism of $\mathbb{C}$-vector spaces. This provides another way of viewing based modules: by (2.31), $\left({}_AM,\mathcal{L}_M,\bar{\mathcal{L}}_M\right)$ is a balanced triple, and by (2.30) the A-lattice ${}_AM$ is completely determined by the crystal base $\left(\bar{\mathcal{L}}_M,\overline{\mathcal{B}}\right)$. We will say that $\left(\bar{\mathcal{L}}_M,\overline{\mathcal{B}}\right)$ (or the corresponding crystal base $(\mathcal{L}_M,\mathcal{B})$ at $q=0$) is *melted into* the based module (M,B).

We will indifferently apply the notion of based module to finite-dimensional unital $\dot{\mathbf{U}}$-modules, since they are equivalent to U_q^{ad}-modules of type 1.

Every module V_μ, $\mu\in P^+$, supports a structure of based module (see [83, Section 14.4.10] and [65]); the corresponding basis, called *canonical basis* and that we will denote by $\underline{\mathbf{B}}_\mu$, is formed by the elements $bv_\mu\in{}_AV_\mu$ which are non-zero, where $b\in\mathbf{B}^-$ and v_μ is the canonical highest weight vector of V_μ, corresponding to the coset of $1\in U_q^{\mathrm{ad}}(\mathfrak{n}_-)$ in the Verma module construction of V_μ. Note that the involution $^-\colon V_\mu\to V_\mu$ defined by (iii) above is indeed an automorphism, for the space V_μ with action of U_q^{ad} defined by $x\cdot v:=\bar{x}v$, for all $x\in U_q^{\mathrm{ad}}$, $v\in V_\mu$, has highest weight μ, and is thus isomorphic to V_μ via the map $^-$. The crystal base $\left(\mathcal{L}_\mu^{\mathrm{low}},\mathcal{B}_\mu^{\mathrm{low}}\right)$ at $q=0$ is formed by the $\mathcal{A}_0$-sublattice $\mathcal{L}_\mu^{\mathrm{low}}$ of V_μ generated by $\underline{\mathbf{B}}_\mu$ (which is eventually the same as the $\mathcal{A}_0$-sublattice generated by the vectors of the form $\tilde{f}_{i_1}\circ\cdots\circ\tilde{f}_{i_k}(v_\mu)$, where $i_1,\dots,i_k\in\{1,\dots,m\}$), and $\mathcal{B}_\mu^{\mathrm{low}}$ is the set of non-zero images of these vectors in $\mathcal{L}_\mu^{\mathrm{low}}/q\mathcal{L}_\mu^{\mathrm{low}}$.

There is the following uniqueness result [65, Theorem 3].

Theorem 2.5. *Let M be a U_q^{ad}-module of type 1, and $(\mathcal{L},\mathcal{B})$ a crystal base at $q=0$ of M. Then there exists a $\mathbb{C}(q)$-isomorphism $M\to\bigoplus_j V_{\lambda_j}$ by which $(\mathcal{L},\mathcal{B})$ is $\mathcal{A}_0$-isomorphic to $\bigoplus_j\left(\mathcal{L}_{\lambda_j}^{\mathrm{low}},\mathcal{B}_{\lambda_j}^{\mathrm{low}}\right)$.*

The based modules form a category. Given based modules (M,B) and (M',B'), a morphism of U_q^{ad}-modules $f\colon M\to M'$ is a morphism of based modules if

(a) $f(b)\in B'\cup\{0\}$ for any $b\in B$;

(b) $B\cap\mathrm{Ker}(f)$ is a basis of $\mathrm{Ker}(f)$.

The direct sum of based modules (M,B) and (M',B') is a based module $(M\oplus M', B\cup B')$; and a submodule M' of a based module (M,B) spanned over $\mathbb{C}(q)$ by a subset B' of B forms a based module (M',B'). The quotient module M/M' together with the image of $B\setminus B'$ is then a based module.

The tensor product of based modules (M,B), (M',B') is also defined. Namely, consider the $\mathbb{C}$-linear map $\Psi\colon M\otimes M' \to M\otimes M'$ defined by

$$\Psi(m\otimes m') = \hat{R}^{-1}(\bar{m}\otimes \bar{m}'),$$

where $\hat{R}=\Theta^{-1}R$, see (2.4) (note that, as we use the coproduct opposite to [83] our quasi-R-matrix is $\hat{R}^{-1}$). It can be checked that Ψ is an involution compatible with the action of $\dot{\mathbf{U}}$ in the sense of (iii) above in the definition of based module. Moreover, denote by $\mathcal{L}_{M,M'}$ the $\mathbb{C}[q^{-1}]$-submodule of $M\otimes M'$ spanned by the basis elements $b\otimes b'$, where $b\in B$, $b'\in B'$. It is shown in [83, Section 27.3], that for every pair $(b,b')\in B\times B'$ there is a unique element $b\diamondsuit b'\in \mathcal{L}_{M,M'}$ such that

(a) $\Psi(b\diamondsuit b') = b\diamondsuit b'$,

(b) $b\diamondsuit b' - b\otimes b' \in q^{-1}\mathcal{L}_{M,M'}$.

Moreover, $B_\diamondsuit = \{b\diamondsuit b', b\in B, b'\in B'\}$ is a basis of $M\otimes M'$, a $\mathbb{C}[q^{-1}]$-basis of $\mathcal{L}_{M,M'}$, a $\mathbb{C}\big[q,q^{-1}\big]$-basis of the $\mathbb{C}\big[q,q^{-1}\big]$-module ${}_A\mathcal{L}_{M,M'}$ of $M\otimes M'$ generated by the elements $b\otimes b'$, where $b\in B$, $b'\in B'$, and $(M\otimes M', B_\diamondsuit)$ is a based module.

This construction of $B_\diamondsuit$ is associative. Since $(V_\mu, \underline{\mathbf{B}}_\mu)$ is for every $\mu\in P_+$ a based module, it follows that any tensor product M of a finite number of the simple modules V_μ is naturally a based module. The corresponding basis is called *the canonical basis* of M. These canonical basis have been computed explicitly in [56] in the case $\mathfrak{g}=\mathfrak{sl}_2$.

Consider now the $U_q^{\rm ad}$-module ${}^\omega V_\mu$ with underlying space V_μ, $\mu\in P_+$, and action defined by $x._\omega v := \omega(x)v$, for every $x\in U_q^{\rm ad}$ and $v\in V_\mu$ (as usual $\omega\colon U_q^{\rm ad}\to U_q^{\rm ad}$ is the Cartan automorphism). Note that there are isomorphisms ${}^\omega V_\mu \cong V_{-w_0(\mu)} \cong V_\mu^*$ (endowed with the standard left action of $U_q^{\rm ad}$). Let us denote by ${}^\omega v_\mu$ the vector v_μ regarded in ${}^\omega V_\mu$ (i.e., its canonical lowest weight vector), and by ${}^\omega\underline{\mathbf{B}}_\mu := \{b._\omega{}^\omega v_\mu,\ b\in \mathbf{B}^+\}\setminus\{0\}$ its canonical basis; note that ${}^\omega\underline{\mathbf{B}}_\mu = \{\omega(b)v_\mu,\ b\in\omega(\mathbf{B}^-)\}\setminus\{0\} = \{bv_\mu,\ b\in\mathbf{B}^-\}\setminus\{0\} = \underline{\mathbf{B}}_\mu$. Then ${}^\omega V_{\mu'}\otimes V_{\mu''}$ has the canonical basis $\underline{\mathbf{B}}_{\mu',\mu''} := \{\underline{b}'\diamondsuit \underline{b}'',\ \underline{b}'\in{}^\omega\underline{\mathbf{B}}_{\mu'},\ \underline{b}''\in\underline{\mathbf{B}}_{\mu''}\}$. Since $\underline{b}'\diamondsuit\underline{b}''$ is canonically determined by the elements $b', b''\in\mathbf{B}^-$ such that $\underline{b}'=\omega(b')._\omega{}^\omega v_{\mu'}$, $\underline{b}''=b''v_{\mu''}$, following Lusztig we denote it by $(b'\diamondsuit b'')_{\mu',\mu''}$.

Denote by $v_{w_0(\mu)}$ the canonical lowest weight vector of V_μ, and by ${}^\omega v_{w_0(\mu)}$ the vector $v_{w_0(\mu)}$ regarded in ${}^\omega V_\mu$. It is a crucial observation that ${}^\omega v_{w_0(\mu')}\otimes v_{w_0(\mu'')}$ is a cyclic vector of ${}^\omega V_{\mu'}\otimes V_{\mu''}$ (see, e.g., [83, Proposition 23.3.6]; note that ${}^\omega v_{w_0(\mu')}\otimes v_{w_0(\mu'')}$ plays the role of $\xi_{-\mu'}\otimes\eta_{\mu''} := {}^\omega v_{\mu'}\otimes v_{\mu''}$ in [83], because we use opposite coproducts on $U_q^{\rm ad}$ but the factors ${}^\omega V_{\mu'}$ and $V_{\mu''}$ are ordered in the same way).

We can now state the definition of the canonical basis $\dot{\mathbf{B}}$ of $\dot{\mathbf{U}}$: each element u of $\dot{\mathbf{B}}$ belongs to $\dot{\mathbf{U}}_A 1_\zeta$ for some (unique) $\zeta\in P$, and it is then uniquely determined by the property that, for every μ', $\mu''\in P^+$ such that $w_0(\mu''-\mu')=\zeta$, we have

$$u({}^\omega v_{w_0(\mu')}\otimes v_{w_0(\mu'')}) = (b'\diamondsuit b'')_{\mu',\mu''} \tag{2.32}$$

for some $(b'\diamondsuit b'')_{\mu',\mu''}\in\underline{\mathbf{B}}_{\mu',\mu''}$ [83, Section 25.2]. We write $u=b'\diamondsuit_\zeta b''$, and as in [84] we denote by $\dot{\mathbf{B}}_{\mu',\mu''}$ the finite subset of $\dot{\mathbf{B}}$ which is in bijection with $\underline{\mathbf{B}}_{\mu',\mu''}$ under the map $u\mapsto u({}^\omega v_{w_0(\mu')}\otimes v_{w_0(\mu'')})$. So

$$\dot{\mathbf{B}} = \bigcup_{\mu',\mu''\in P_+} \dot{\mathbf{B}}_{\mu',\mu''}. \tag{2.33}$$

Note in particular that $\dot{\mathbf{B}}$ is formed by weight vectors for the left and right action of $U_q^{\mathrm{ad}}(\mathfrak{h})$ (defined as usual by (2.29)).

In a sense, one can view $\dot{\mathbf{U}}$ as the projective limit of an inverse system formed by the $\big(U_q^{\mathrm{ad}} \otimes U_q^{\mathrm{ad}}\big)$-modules ${}^{\omega}V_{\mu'} \otimes V_{\mu''}$, where $\mu', \mu'' \in P^+$; then $\dot{\mathbf{B}}$ is the basis resulting from the corresponding inverse system of basis $\{\dot{\mathbf{B}}_{\mu',\mu''}\}_{\mu',\mu''}$.

Lusztig has produced a partition of $\dot{\mathbf{B}}$ as follows. First, consider the situation of a based module (M, B). For every $\lambda \in P_+$ denote by $M[\lambda]$ the sum of the simple submodules of M isomorphic to V_λ (i.e., its isotypical component). Set

$$M[\geq \lambda] = \bigoplus_{\lambda' \geq \lambda} M[\lambda']. \tag{2.34}$$

Then, for every base element $b \in B$ there is a unique $\lambda \in P_+$ such that $b \in M[\geq \lambda]$ and λ is maximal with this property [83, Section 27.2]. Denote by $B[\lambda]$ the set of all $b \in B$ that give rise to $\lambda \in P_+$ in this way. Clearly, the sets $B[\lambda]$, $\lambda \in P_+$, form a partition of B.

Now, given $b \in \dot{\mathbf{B}}$, let $\zeta \in P$ be the unique weight such that $b \in \dot{\mathbf{U}}_A 1_\zeta$, and let $\mu', \mu'' \in P^+$ be such that $w_0(\mu'' - \mu') = \zeta$, and $(\check{\alpha}_i, \mu')$ is large enough for all $i = 1, \dots, m$ so that $u({}^{\omega}v_{w_0(\mu')} \otimes v_{w_0(\mu'')})$ is non-zero. This element belongs to the canonical basis $\underline{\mathbf{B}}_{\mu',\mu''}$ of ${}^{\omega}V_{\mu'} \otimes V_{\mu''}$, and therefore to one of the subsets $\underline{\mathbf{B}}_{\mu',\mu''}[\lambda]$, for a unique $\lambda \in P_+$. It is a result that λ does not depend on the choice of μ', μ'' (see [83, Section 29.1.1]). Hence there is a well-defined map $\dot{\mathbf{B}} \to P_+$, $b \mapsto \lambda$. Denoting by $\dot{\mathbf{B}}[\lambda]$ the fiber of this map, we thus obtain a partition

$$\dot{\mathbf{B}} = \coprod_{\lambda \in P_+} \dot{\mathbf{B}}[\lambda]. \tag{2.35}$$

The sets $\dot{\mathbf{B}}[\lambda]$ are called 2-*sided cells*. They are finite sets and have the following remarkable properties. For every $\lambda \in P_+$ denote by $\dot{\mathbf{U}}[\geq \lambda]$ and $\dot{\mathbf{U}}[> \lambda]$ the subspaces of $\dot{\mathbf{U}}$ spanned by $\coprod_{\lambda' \geq \lambda} \dot{\mathbf{B}}[\lambda']$ and $\coprod_{\lambda' > \lambda} \dot{\mathbf{B}}[\lambda']$ respectively. Then $\dot{\mathbf{U}}[\geq \lambda]$ (respectively $\dot{\mathbf{U}}[> \lambda]$) consists of the elements $u \in \dot{\mathbf{U}}$ such that if u acts on V_μ by a non-zero linear map, then $\mu \geq \lambda$ (respectively $\mu > \lambda$) [83, Lemmas 29.1.3 and 29.1.4]. Both $\dot{\mathbf{U}}[\geq \lambda]$ and $\dot{\mathbf{U}}[> \lambda]$ are two-sided ideals of $\dot{\mathbf{U}}$. Moreover, the algebra homomorphism $\pi_\lambda \colon \dot{\mathbf{U}}[\geq \lambda] \to \mathrm{End}(V_\lambda)$ given by the $\dot{\mathbf{U}}$-module structure on V_λ descends to an algebra and U_q^{ad}-bimodule isomorphism (keeping the same notation) [83, Proposition 29.2.2]

$$\bar{\pi}_\lambda \colon\ \dot{\mathbf{U}}[\geq \lambda]/\dot{\mathbf{U}}[> \lambda] \to \mathrm{End}(V_\lambda). \tag{2.36}$$

For instance, when $\mathfrak{g} = \mathfrak{sl}_2$ the 2-sided cell $\dot{\mathbf{B}}[n]$ associated to the simple $U_q^{\mathrm{ad}}(\mathfrak{sl}_2)$-module of type 1 and dimension $n+1$ is the set of cardinality $(n+1)^2$ given by [83, Section 29.4.3]

$$\dot{\mathbf{B}}[n] = \big\{E^{(k)} 1_{-n} F^{(l)},\ n \geq k + l\big\} \cup \big\{F^{(l)} 1_n E^{(k)},\ n \geq k + l\big\}, \tag{2.37}$$

with the identification $E^{(k)} 1_{-n} F^{(l)} = F^{(l)} 1_n E^{(k)}$ when $n = k + l$. As we are mainly interested in $\mathcal{O}_A$, we are going to describe the dual partition of $\dot{\mathbf{B}}^*$, see Theorem 2.6. The duality with (2.35) is discussed after that theorem.

First, we follow the approach of Kashiwara [66, 67]. For every $\lambda \in P_+$, denote by V_λ^r the dual space of V_λ endowed with its natural structure of right U_q^{ad}-module, defined by $(fx)(v) = f(xv)$ for every $f \in V_\lambda^r$, $x \in U_q^{\mathrm{ad}}$, $v \in V_\lambda$. Clearly, V_λ^r is a simple module of highest weight λ. Let $\varphi \colon U_q^{\mathrm{ad}} \to U_q^{\mathrm{ad}}$ be the anti-automorphism of $\mathbb{C}(q)$-algebra given by $\varphi(E_i) = F_i$, $\varphi(F_i) = E_i$, $\varphi(K_\lambda) = K_\lambda$. By using φ, any right U_q^{ad}-module can be considered as a left U_q^{ad}-module. In particular, by the Verma module construction of V_λ it follows

$$V_\lambda^r \cong U_q^{\mathrm{ad}} \bigg/ \bigg(\sum_{\mu \in P_+} \big(K_\mu - q^{(\lambda,\mu)}\big) U_q^{\mathrm{ad}} + \sum_{i=1}^m E_i^{1+(\check{\alpha}_i,\lambda)} U_q^{\mathrm{ad}} \bigg),$$

and φ affords an isomorphism of the right module V_λ^r with the *left* module V_λ. We will denote by f_λ the unique highest weight vector of V_λ^r satisfying $\langle f_\lambda, v_\lambda\rangle = 1$.

The space $V_\lambda^r \otimes V_\lambda$ can be identified with $\mathrm{End}(V_\lambda)^*$, and thus acquires by duality a natural structure of U_q^{ad}-bimodule $\big($or equivalently left $U_q^{\mathrm{ad}} \otimes \big(U_q^{\mathrm{ad}}\big)^{\mathrm{op}}$-module$\big)$; the left and right actions are given by

$$x(f \otimes v)y = fy \otimes xv \tag{2.38}$$

for every $x, y \in U_q^{\mathrm{ad}}$, $f \in V_\lambda^r$, $v \in V_\lambda$. The space $V_\lambda^r \otimes V_\lambda$ also acquires by duality a natural "upper" crystal structure over $U_q^{\mathrm{ad}} \otimes \big(U_q^{\mathrm{ad}}\big)^{\mathrm{op}}$, as we explain now. Denote by $\langle\ ,\ \rangle_\lambda \colon V_\lambda \times V_\lambda \to \mathbb{C}(q)$ the unique symmetric bilinear form such that

$$\langle v_\lambda, v_\lambda\rangle_\lambda = 1 \qquad \text{and} \qquad \langle \varphi(x)u, v\rangle_\lambda = \langle u, xv\rangle_\lambda \tag{2.39}$$

for every $u, v \in V_\lambda$ and $x \in U_q^{\mathrm{ad}}$. Recall the crystal base $\big(\mathcal{L}_\mu^{\mathrm{low}}, \mathcal{B}_\mu^{\mathrm{low}}\big)$ at $q = 0$ introduced before Theorem 2.5. In Kashiwara's terminology [65, 66], the pair $\big(\mathcal{L}_\lambda^{\mathrm{low}}, \mathcal{B}_\lambda^{\mathrm{low}}\big)$ is the *lower crystal base* of V_λ at $q = 0$. Applying the involution $^{-} \colon V_\lambda \to V_\lambda$, one obtains the lower crystal base $\big(\overline{\mathcal{L}_\lambda^{\mathrm{low}}}, \overline{\mathcal{B}_\lambda^{\mathrm{low}}}\big)$ at $q = \infty$. Because the canonical bases are determined by the crystal bases (see the discussion about (2.31)), we call $(V_\lambda, \underline{\mathbf{B}}_\lambda)$ the *lower* based module of V_λ, and $\underline{\mathbf{B}}_\lambda$ the *lower canonical basis* of V_λ.

Put

$$\begin{aligned} &{}_AV_\lambda^{\mathrm{up}} := \{v \in V_\lambda, \langle v, {}_AV_\lambda\rangle_\lambda \subset A\}, \qquad \mathcal{L}_\lambda^{\mathrm{up}} := \big\{v \in V_\lambda, \langle v, \mathcal{L}_\lambda^{\mathrm{low}}\rangle_\lambda \subset \mathcal{A}_0\big\}, \\ &\overline{\mathcal{L}_\lambda^{\mathrm{up}}} := \big\{v \in V_\lambda, \langle v, \overline{\mathcal{L}_\lambda^{\mathrm{low}}}\rangle_\lambda \subset \mathcal{A}_\infty\big\}. \end{aligned} \tag{2.40}$$

Then $\big({}_AV_\lambda^{\mathrm{up}}, \mathcal{L}_\lambda^{\mathrm{up}}, \overline{\mathcal{L}_\lambda^{\mathrm{up}}}\big)$ is a balanced triple [66, Lemma 4.2.1]. Denote by $\mathcal{B}_\lambda^{\mathrm{up}}$ the basis of $\mathcal{L}_\lambda^{\mathrm{up}}/q\mathcal{L}_\lambda^{\mathrm{up}}$ dual to $\mathcal{B}_\lambda^{\mathrm{low}}$ by the induced pairing $\langle\ ,\ \rangle_\lambda \colon \mathcal{L}_\lambda^{\mathrm{up}}/q\mathcal{L}_\lambda^{\mathrm{up}} \times \mathcal{L}_\lambda^{\mathrm{low}}/q\mathcal{L}_\lambda^{\mathrm{low}} \to \mathbb{C}$. The pair $\big(\mathcal{L}_\lambda^{\mathrm{up}}, \mathcal{B}_\lambda^{\mathrm{up}}\big)$ is the *upper crystal base* of V_λ at $q = 0$. The weight spaces of the $\mathcal{A}_0$-modules $\mathcal{L}_\lambda^{\mathrm{low}}$ and $\mathcal{L}_\lambda^{\mathrm{up}}$ are related by

$$\big(\mathcal{L}_\lambda^{\mathrm{up}}\big)^\mu = q^{\frac{(\lambda,\lambda)}{2} - \frac{(\mu,\mu)}{2}} \big(\mathcal{L}_\lambda^{\mathrm{low}}\big)^\mu, \qquad \mu \in P. \tag{2.41}$$

Correspondingly, denoting $\big(\mathcal{B}_\lambda^{\mathrm{up}}\big)^\mu := \mathcal{B}_\lambda^{\mathrm{up}} \cap \big(\mathcal{L}_\lambda^{\mathrm{up}}\big)^\mu$ and $\big(\mathcal{B}_\lambda^{\mathrm{low}}\big)^\mu := \mathcal{B}_\lambda^{\mathrm{low}} \cap \big(\mathcal{L}_\lambda^{\mathrm{low}}\big)^\mu$, we have (see [65] and [66, equation (4.2.9)])

$$\big(\mathcal{B}_\lambda^{\mathrm{up}}\big)^\mu = q^{\frac{(\lambda,\lambda)}{2} - \frac{(\mu,\mu)}{2}} \big(\mathcal{B}_\lambda^{\mathrm{low}}\big)^\mu.$$

The A-module ${}_AV_\lambda^{\mathrm{up}}$ is characterized by the following two properties [66, equations (4.2.10)–(4.2.12)]:

$$\big({}_AV_\lambda^{\mathrm{up}}\big)^\lambda = \mathbb{C}\big[q, q^{-1}\big]v_\lambda, \qquad \big({}_AV_\lambda^{\mathrm{up}}\big)^\mu = \big\{v \in V_\lambda \mid U_A^{\mathrm{res}}\big(\mathfrak{n}^+\big)^{\lambda-\mu} v \in \mathbb{C}\big[q, q^{-1}\big]v_\lambda\big\},$$

where $U_A^{\mathrm{res}}\big(\mathfrak{n}^+\big)^\gamma = \big\{u \in U_A^{\mathrm{res}}\big(\mathfrak{n}^+\big) \mid \forall \nu \in P,\, K_\nu u K_\nu^{-1} = q^{(\nu,\gamma)}u\big\}$. Denote by $\underline{\mathbf{B}}_\lambda^{\mathrm{up}}$ the inverse image of $\mathcal{B}_\lambda^{\mathrm{up}}$ by the isomorphism ${}_AV_\lambda^{\mathrm{up}} \cap \mathcal{L}_\lambda^{\mathrm{up}} \cap \overline{\mathcal{L}_\lambda^{\mathrm{up}}} \to \mathcal{L}_\lambda^{\mathrm{up}}/q\mathcal{L}_\lambda^{\mathrm{up}}$. By (2.30), the set $\underline{\mathbf{B}}_\lambda^{\mathrm{up}}$ is a basis of ${}_AV_\lambda^{\mathrm{up}}$; we call it the *upper canonical basis* of V_λ. In the appendix, we describe in details the $\mathfrak{sl}_2$ case.

Similarly, the right module V_λ^r with its canonical basis $\underline{\mathbf{B}}_\lambda^r = \{f_\lambda b,\, b \in \mathbf{B}^+\} \setminus \{0\}$ has the lower crystal base $\big(\mathcal{L}_\lambda^{r\,\mathrm{low}}, \mathcal{B}_\lambda^{r\,\mathrm{low}}\big)$, and it supports a balanced triple $\big({}_AV_\lambda^{r\,\mathrm{up}}, \mathcal{L}_\lambda^{r\,\mathrm{up}}, \overline{\mathcal{L}_\lambda^{r\,\mathrm{up}}}\big)$ defined again by duality. We denote by $(\mathcal{L}_\lambda^{r\,\mathrm{up}}, \mathcal{B}_\lambda^{r\,\mathrm{up}})$ and $\underline{\mathbf{B}}_\lambda^{r\,\mathrm{up}}$ the corresponding crystal base and upper canonical basis of V_λ^r, respectively.

It follows that $\big({}_AV_\lambda^{r\,\mathrm{up}} \bigotimes_A {}_AV_\lambda^{\mathrm{up}}, \mathcal{L}_\lambda^{r\,\mathrm{up}} \bigotimes_{\mathcal{A}_0} \mathcal{L}_\lambda^{\mathrm{up}}, \overline{\mathcal{L}_\lambda^{r\,\mathrm{up}}} \bigotimes_{\mathcal{A}_\infty} \overline{\mathcal{L}_\lambda^{\mathrm{up}}}\big)$ is a balanced triple; equivalently $V_\lambda^r \otimes V_\lambda$ with the bimodule structure (2.38) and the basis $\underline{\mathbf{B}}_\lambda^{r\,\mathrm{up}} \otimes \underline{\mathbf{B}}_\lambda^{\mathrm{up}}$ is a based $\big(U_q^{\mathrm{ad}} \otimes (U_q^{\mathrm{ad}})^{\mathrm{op}}\big)$-module.

Denote again by $\langle\cdot,\cdot\rangle\colon \mathcal{O}_q\times\dot{\mathbf{U}}\to\mathbb{C}(q)$ the pairing of U_q^{ad}-bimodules induced by the canonical pairing $\langle\ ,\ \rangle\colon \mathcal{O}_q\times U_q^{\mathrm{ad}}\to\mathbb{C}(q)$, and let $\Phi_\lambda\colon V_\lambda^r\otimes V_\lambda\to\mathcal{O}_q$, $\lambda\in P_+$, be the "matrix coefficient" map, i.e.,

$$\langle\Phi_\lambda(f\otimes v),x\rangle=\langle f,xv\rangle_\lambda \tag{2.42}$$

for every $f\in V_\lambda^r$, $x\in U_q^{\mathrm{ad}}$, $v\in V_\lambda$. The map $\Phi:=\bigoplus_{\lambda\in P_+}\Phi_\lambda$ is an isomorphism of U_q^{ad}-bimodules, so let us use it to identify $\mathcal{O}_q$ with $\bigoplus_{\lambda\in P_+}V_\lambda^r\otimes V_\lambda$ (which is the content of the Peter–Weyl decomposition (2.6)). Define

$$\mathcal{L}(\mathcal{O}_q)=\bigoplus_{\lambda\in P_+}\Big(\mathcal{L}_\lambda^{r\,\mathrm{up}}\bigotimes_{\mathcal{A}_0}\mathcal{L}_\lambda^{\mathrm{up}}\Big),\qquad \mathcal{B}(\mathcal{O}_q):=\coprod_{\lambda\in P_+}\mathcal{B}_\lambda^{r\,\mathrm{up}}\otimes\mathcal{B}_\lambda^{\mathrm{up}},$$
$$\overline{\mathcal{L}}(\mathcal{O}_q)=\bigoplus_{\lambda\in P_+}\Big(\overline{\mathcal{L}_\lambda^{r\,\mathrm{up}}}\bigotimes_{\mathcal{A}_\infty}\overline{\mathcal{L}_\lambda^{\mathrm{up}}}\Big),\qquad \overline{\mathcal{B}}(\mathcal{O}_q):=\coprod_{\lambda\in P_+}\overline{\mathcal{B}_\lambda^{r\,\mathrm{up}}}\otimes\overline{\mathcal{B}_\lambda^{\mathrm{up}}}.$$

Theorem 2.6.

(i) The triple $\big(\mathcal{O}_A,\mathcal{L}(\mathcal{O}_q),\overline{\mathcal{L}}(\mathcal{O}_q)\big)$ is balanced. Therefore, denoting by G the inverse of the canonical map $\mathcal{O}_A\cap\mathcal{L}(\mathcal{O}_q)\cap\overline{\mathcal{L}}(\mathcal{O}_q)\to\mathcal{L}(\mathcal{O}_q)/q\mathcal{L}(\mathcal{O}_q)$, we have

$$\mathcal{O}_A=\bigoplus_{b\in\mathcal{B}(\mathcal{O}_q)}AG(b).$$

(ii) The basis $G(\mathcal{B}(\mathcal{O}_q)):=\{G(b),\,b\in\mathcal{B}(\mathcal{O}_q)\}$ coincides with the dual canonical basis $\dot{\mathbf{B}}^*$, i.e., the elements $a^*\in\mathcal{O}_A$, for every $a\in\dot{\mathbf{B}}$, defined by $a^*(a')=\delta_{a,a'}$ for every $a'\in\dot{\mathbf{B}}$. Therefore,

$$\mathcal{O}_A=\bigoplus_{b\in\dot{\mathbf{B}}}Ab^*.$$

The statement (i) is [66, Theorem 1], and (ii) is [67, Theorem 10.1 and Proposition 10.2.2] and [83, Section 29.5]. The basis $G(\mathcal{B}(\mathcal{O}_q))=\dot{\mathbf{B}}^*$ is called the *global basis*, or *canonical basis*, of $\mathcal{O}_q$. The proof of Theorem 2.6 (ii) in [67] (see also [68]) exhibits an isomorphism of crystals over $U_q^{\mathrm{ad}}\otimes\big(U_q^{\mathrm{ad}}\big)^{\mathrm{op}}$,

$$\psi\colon\ \mathcal{B}(\mathcal{O}_q)\to\mathcal{B}\big(\dot{\mathbf{U}}\big), \tag{2.43}$$

where $\big(\mathcal{L}\big(\dot{\mathbf{U}}\big),\mathcal{B}\big(\dot{\mathbf{U}}\big)\big)$ is the crystal base of $\dot{\mathbf{U}}$ associated to the canonical basis $\dot{\mathbf{B}}$. The isomorphism ψ satisfies $\langle G(b),G(b')\rangle=\delta_{\psi(b),b'}$ for every $b\in\mathcal{B}(\mathcal{O}_q)$, $b'\in\mathcal{B}\big(\dot{\mathbf{U}}\big)$. The unit 1 of $\mathcal{O}_A$ is $(1_0)^*$; the constant structures of $\mathcal{O}_A$ are studied in [83, 84].

The canonical basis of $\mathcal{O}_A$ when $\mathfrak{g}=\mathfrak{sl}_2$. Denote by a, b, c, d the matrix coefficients in the canonical basis $(v_+,v_-:=Fv_+)$ of V_1, the simple $U_q^{\mathrm{ad}}(\mathfrak{sl}_2)$-module of type 1 and dimension two, read from the top left to the bottom right. In that case of V_1 the upper canonical basis $\underline{\mathbf{B}}_1^{r\,\mathrm{up}}$ and $\underline{\mathbf{B}}_1^{\mathrm{up}}$ coincide with the lower ones (this is not true in general, see Example 2.17). The basis $\dot{\mathbf{B}}^*(\mathfrak{sl}_2)$ is formed by the monomials $c^sa^pb^r$ where $p,r,s\in\mathbb{N}$, and $c^sd^pb^r$ where $p,r,s\in\mathbb{N}$ and $p>0$; this is stated in [66, Proposition 9.1.1] (in [41, Proposition 1.3], similar monomials are shown to form an A-basis of $\mathcal{O}_A(\mathrm{SL}_2)$, but without reference to the canonical basis; see the comments before (4.3) below). More precisely, recall the 2-sided cells (2.37). We verified by a tedious though straightforward computation that we have the duality pairing

$$\big\langle c^sd^pb^r,E^{(i)}1_{-k}F^{(j)}\big\rangle=\delta_{p+r+s,k}\delta_{r,i}\delta_{s,j},\qquad \big\langle c^sd^pb^r,F^{(j)}1_kE^{(i)}\big\rangle=0,$$
$$\big\langle c^sa^pb^r,E^{(i)}1_{-k}F^{(j)}\big\rangle=0,\qquad \big\langle c^sa^pb^r,F^{(j)}1_kE^{(i)}\big\rangle=\delta_{p+r+s,k}\delta_{r,i}\delta_{s,j}.$$

Therefore,

$$
\begin{aligned}
\dot{\mathbf{B}}[n]^* := &\{c^s a^p b^r,\, p,r,s \in \mathbb{N},\, p+r+s=n\} \\
&\cup \{c^s d^p b^r,\, p,r,s \in \mathbb{N},\, p>0,\, p+r+s=n\}.
\end{aligned}
$$

A description of $\dot{\mathbf{B}}^*$ in the case of $\mathfrak{g} = \mathfrak{sl}_n$ can be found in [49]. Moreover, denote by V_n the simple $U_q^{\mathrm{ad}}(\mathfrak{sl}_2)$-module of type 1 and dimension $n+1$, by (v_k) the canonical basis of V_n, by $\left(v^k\right)$ the dual basis, and by $\pi_n\colon \dot{\mathbf{U}}(\mathfrak{sl}_2) \to \mathrm{End}(V_n)$ the representation morphism. By using the above pairing, it is readily checked that for every $0 \le l, m \le n$, we have

$$
\begin{aligned}
&v^l(\pi_n(\cdot)\ v_m) \\
&\quad= \sum_{\substack{0\le i,j,k\\ i+j\le k\le n\\ j-i=l-m}} \delta_{-k,n-2(m+j)} \begin{bmatrix} m+j \\ j \end{bmatrix}_q \begin{bmatrix} n-m+i-j \\ i \end{bmatrix}_q \left(E^{(i)} 1_{-k} F^{(j)}\right)^* \\
&\qquad+ \sum_{\substack{0\le i,j,k\\ i+j<k\le n\\ j-i=l-m}} \delta_{k,n-2(m-i)} \begin{bmatrix} m-i+j \\ j \end{bmatrix}_q \begin{bmatrix} n-m+i \\ i \end{bmatrix}_q \left(F^{(j)} 1_{+k} E^{(i)}\right)^*.
\end{aligned}
\tag{2.44}
$$

In particular, we see in this case of $\mathfrak{g} = \mathfrak{sl}_2$ that in general the matrix coefficients of simple U_A^{res}-modules of type 1 are not elements of the dual canonical basis $\dot{\mathbf{B}}^*$. Moreover, these matrix coefficients do not form a basis of $\mathcal{O}_A$. For instance, it follows from (2.44) that the matrix of matrix coefficients of V_2 has the following form:

$$
\begin{pmatrix}
a^2 & [2]_q ab & b^2 \\
ca & [2]_q bc + 1 & db \\
c^2 & [2]_q cd & d^2
\end{pmatrix}.
\tag{2.45}
$$

The matrix coefficient $v_0^* \otimes v_0$ being equal to $[2]_q bc + 1$, this shows bc cannot be expressed as a linear combination over A of matrix coefficients of simple modules.

The refined Peter–Weyl theorem. Let us discuss the U_A^{res}-bimodule structure of $\mathcal{O}_A$, and its relation with the partition (2.35). For every $\lambda \in P_+$, put

$$
{}_A\dot{C}(\lambda) := \bigoplus_{b \in \dot{\mathbf{B}}[\lambda]} A b^*
\tag{2.46}
$$

and

$$
\mathcal{O}_A(\le \lambda) := \bigoplus_{\lambda' \le \lambda} {}_A\dot{C}(\lambda'), \qquad \mathcal{O}_A(< \lambda) := \bigoplus_{\lambda' < \lambda} {}_A\dot{C}(\lambda').
$$

In particular, in the $\mathfrak{sl}_2$ case the A-module ${}_A\dot{C}(n\varpi_1)$ has basis $\dot{\mathbf{B}}[n]^*$ given above, of cardinality $(n+1)^2$.

Recall that $\dot{\mathbf{U}}[\ge \lambda]$ and $\dot{\mathbf{U}}[> \lambda]$ are two-sided ideals of $\dot{\mathbf{U}}$, and the algebra (whence U_q^{ad}-bimodule) isomorphism $\bar{\pi}_\lambda\colon \dot{\mathbf{U}}[\ge \lambda]/\dot{\mathbf{U}}[> \lambda] \to \mathrm{End}(V_\lambda)$ (see (2.36)). In [83, Section 29.3], Lusztig groups this isomorphism and its properties under the general term of *refined Peter–Weyl theorem*. We are going to reinterpret it in terms of $\mathcal{O}_A$. First observe that

Lemma 2.7. *The A-modules $\mathcal{O}_A(\le \lambda)$ and $\mathcal{O}_A(< \lambda)$ are U_A^{res}-bimodules, and the surjective map*

$$
d_\lambda\colon\ \mathcal{O}_A(\le \lambda) \longrightarrow \mathrm{Hom}\big(\dot{\mathbf{U}}_A[\ge \lambda]/\dot{\mathbf{U}}_A[> \lambda], A\big), \qquad \alpha \longmapsto \langle \alpha, \cdot\ \rangle
\tag{2.47}
$$

descends to an isomorphism of U_A^{res}-bimodules $\bar{d}_\lambda$ on $\mathcal{O}_A(\le \lambda)/\mathcal{O}_A(< \lambda)$.

Proof. For every $\alpha \in \mathcal{O}_A(\leq \lambda)$, $x, y \in U_A^{\rm res}$, and $b \in \dot{\mathbf{B}}[\mu]$ with $\mu \nleq \lambda$, we have $xby \in \dot{\mathbf{U}}_A[\geq \mu]$. Since $\dot{\mathbf{U}}_A[\geq \mu] = \bigoplus_{\eta \geq \mu} A\dot{\mathbf{B}}[\eta]$ and $\eta \geq \mu$ implies $\eta \nleq \lambda$, it follows that $\langle xby, \alpha\rangle = 0$, i.e., $(x \triangleright \alpha \triangleleft y)(b) = 0$. This shows $x \triangleright \alpha \triangleleft y \in \mathcal{O}_A(\leq \lambda)$. The same proof applies as well to $\mathcal{O}_A(< \lambda)$, whence the first claim. Since $\dot{\mathbf{U}}[\geq \lambda]$ and $\dot{\mathbf{U}}[> \lambda]$ are two-sided ideals of $\dot{\mathbf{U}}$, $\dot{\mathbf{B}}$ is a basis of $\dot{\mathbf{U}}_A$, and the A-modules $\dot{\mathbf{U}}_A[\geq \lambda]$ and $\dot{\mathbf{U}}_A[> \lambda]$ are spanned by $\coprod_{\lambda' \geq \lambda} \dot{\mathbf{B}}[\lambda']$ and $\coprod_{\lambda' > \lambda} \dot{\mathbf{B}}[\lambda']$, both are two-sided ideals of $\dot{\mathbf{U}}_A$, and $\dot{\mathbf{U}}_A[\geq \lambda]/\dot{\mathbf{U}}_A[> \lambda]$ inherits the quotient $U_A^{\rm res}$-bimodule structure. Clearly, the map d_λ is well defined, it is a morphism of $U_A^{\rm res}$-bimodules, and its kernel contains $\mathcal{O}_A(< \lambda)$. Bijectivity of $\bar{d}_\lambda$ comes by comparing the cardinality of canonical bases: $\mathcal{O}_A(\leq \lambda)/\mathcal{O}_A(< \lambda)$ has the basis formed by the cosets of the elements of the basis $(\dot{\mathbf{B}}[\lambda])^*$ of ${}_A\dot{C}(\lambda')$, and $\dot{\mathbf{U}}_A[\geq \lambda]/\dot{\mathbf{U}}_A[> \lambda]$ the basis formed by the cosets of the elements of $\dot{\mathbf{B}}[\lambda]$, all cosets being non-zero and pairwise distinct. ∎

Since $\dot{\mathbf{U}}_A$ preserves the canonical basis $\underline{\mathbf{B}}_\lambda$ of ${}_AV_\lambda$, $\bar{\pi}_\lambda$ descends to an isomorphism of $U_A^{\rm res}$-bimodules $\bar{\pi}_\lambda \colon \dot{\mathbf{U}}_A[\geq \lambda]/\dot{\mathbf{U}}_A[> \lambda] \to \mathrm{End}({}_AV_\lambda)$. We thus get exact sequences of $U_A^{\rm res}$-bimodules

$$0 \longrightarrow \dot{\mathbf{U}}_A[> \lambda] \longrightarrow \dot{\mathbf{U}}_A[\geq \lambda] \xrightarrow{\bar{\pi}_\lambda} \mathrm{End}({}_AV_\lambda) \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{O}_A(< \lambda) \longrightarrow \mathcal{O}_A(\leq \lambda) \xrightarrow{(\bar{\pi}_\lambda^{-1})^* \circ d_\lambda} (\mathrm{End}({}_AV_\lambda))^* \longrightarrow 0. \tag{2.48}$$

They split as sequences of A-modules but not as sequences of bimodules. In fact,

$$\begin{aligned}(\mathrm{End}({}_AV_\lambda))^* &:= \mathrm{Hom}(\mathrm{End}({}_AV_\lambda), A)\\ &\cong \mathrm{Hom}\Big({}_A^\omega V_\lambda \bigotimes_A {}_AV_\lambda, A\Big) = {}_AV_\lambda^{\rm up} \bigotimes_A ({}_A^\omega V_\lambda)^{\rm up},\end{aligned} \tag{2.49}$$

with the " up " structure defined in (2.40), and corresponding basis $\underline{\mathbf{B}}_\lambda^{\rm up} \otimes ({}^\omega\underline{\mathbf{B}}_\lambda)^{\rm up}$. Moreover, the exact sequence (2.48) shows that this A-module of matrix coefficients, regarded as an A-submodule of $\mathcal{O}_A$ by means of the coefficient map $\Phi := \bigoplus_{\lambda \in P_+} \Phi_\lambda$ (see (2.42)), is contained in $\mathcal{O}_A(\leq \lambda)$. This for all $\lambda' \leq \lambda$ yields $\bigoplus_{\lambda' \leq \lambda} (\mathrm{End}({}_AV_{\lambda'}))^* \subset \mathcal{O}_A(\leq \lambda)$. Now, using the isomorphism $\bar{\pi}_\lambda$, we get

$$\mathrm{rank}_A(\mathcal{O}_A(\leq \lambda)) = \sum_{\lambda' \leq \lambda} \mathrm{Card}\big(\dot{\mathbf{B}}[\lambda']\big) = \sum_{\lambda' \leq \lambda} \mathrm{rank}({}_AV_{\lambda'})^2$$

and therefore

$$\dim_{\mathbb{C}(q)}\Big(\mathcal{O}_A(\leq \lambda) \bigotimes_A \mathbb{C}(q)\Big) = \sum_{\lambda' \leq \lambda} \dim(V_{\lambda'})^2 = \sum_{\lambda' \leq \lambda} \dim((C(\lambda')), \tag{2.50}$$

where as usual $C(\lambda')$ denotes the space of matrix coefficients of $V_{\lambda'}$ (see (2.22)). It follows

$$\mathcal{O}_A(\leq \lambda) \bigotimes_A \mathbb{C}(q) = \bigoplus_{\lambda' \leq \lambda} C(\lambda'), \qquad \mathcal{O}_A(< \lambda) \bigotimes_A \mathbb{C}(q) = \bigoplus_{\lambda' < \lambda} C(\lambda'). \tag{2.51}$$

However, in general ${}_A\dot{C}(\lambda) \bigotimes_A \mathbb{C}(q)$ is not equal to $C(\lambda)$, ${}_A\dot{C}(\lambda)$ is not an A-sublattice of $C(\lambda)$, and ${}_A\dot{C}(\lambda)$ is not a $U_A^{\rm res}$-bimodule (it is because of this discrepancy that we have introduced the dot notation " $\dot{}$ "). For instance, we can see the first two facts in the case of $\mathfrak{g} = \mathfrak{sl}_2$, by inverting the system of identities (2.44) for all $0 \leq l, m \leq n$ (or more simply by considering the identity $v_0^* \otimes v_0 = [2]_q bc + 1$ from (2.45)). For the third fact, we have $1_2E \in \dot{\mathbf{B}}[2]$ (see (2.37)), so $((1_2E)^* \triangleleft E)(1_0) = \langle \Delta((1_2E)^*), E \otimes 1_0\rangle = \langle (1_2E)^*, E1_0\rangle = \langle (1_2E)^*, 1_2E\rangle = 1$ since $E1_0 = 1_2E$. Therefore, $(1_2E)^* \triangleleft E \notin {}_A\dot{C}(2)$.

From the formulas (2.44) and Appendix A, we can observe the isomorphism (2.49) in the case of $\mathfrak{g} = \mathfrak{sl}_2$. More simply, by projecting the matrix (2.45) onto $(\mathrm{End}(_AV_2))^*$ the entries are unchanged except the $(1,1)$ entry, which becomes $[2]_q bc$. All factors $[2]_q$ in the middle column disappear if one uses matrix coefficients in the upper canonical basis of V_2, which is $v_0^{\mathrm{up}} := v_0$, $v_1^{\mathrm{up}} := [2]_q^{-1}v_1$, $v_2^{\mathrm{up}} := v_2$ in the notations of (2.44), since we have $v^l(\pi_2(\cdot)\ v_m) = [\delta_{m,1}+1]_q\langle v_l^{\mathrm{up}}, \cdot v_m^{\mathrm{up}}\rangle$ for $l,m \in \{0,1,2\}$, where $\langle\ ,\ \rangle$ is the pairing (2.39). Thus, in this particular example of $(\mathrm{End}(_AV_2))^*$ we see explicitly the identification of the basis $(\bar\pi_2^*)^{-1}\circ d_2(\dot{\mathbf{B}}[2]^*)$ and $\underline{\mathbf{B}}_2^{\mathrm{up}} \otimes ({}^\omega\underline{\mathbf{B}}_2)^{\mathrm{up}}$.

Summing up this discussion, the Lusztig refined Peter–Weyl theorem of [83, Section 29.3], implies the following.

Theorem 2.8. *As an A-module we have a direct sum decomposition*

$$\mathcal{O}_A = \bigoplus_{\lambda\in P_+} {}_A\dot{C}(\lambda), \tag{2.52}$$

as U_A^{res}-bimodules we have a (directed by inclusion, and non direct) sum

$$\mathcal{O}_A = \sum_{\lambda\in P_+} \mathcal{O}_A(\leq\lambda), \tag{2.53}$$

and the composition factors of $\mathcal{O}_A$ are the bimodules

$$(\mathrm{End}(_AV_\lambda))^* \cong ({}_A^\omega V_\lambda \otimes {}_AV_\lambda)^* \tag{2.54}$$

for every $\lambda \in P_+$, each of multiplicity 1.

Remark 2.9. The above filtration and its composition factors appear in disguised manner as *good filtration* in [5] and [91] (see also [103]).

Because $\dot{\mathbf{B}}$ is formed by weight vectors for the left and right action of $U_q^{\mathrm{ad}}(\mathfrak{h})$ (see (2.33)), the same is true of $\dot{\mathbf{B}}^*$ and (2.52) can thus be refined into a weight space decomposition

$$\mathcal{O}_A = \bigoplus_{\mu,\nu\in P}\bigoplus_{\lambda\in P_+} \left({}_A\dot{C}(\lambda)\right)_{\mu,\nu}. \tag{2.55}$$

Now recall the property (2.33). Consider in particular the finite subsets $\dot{\mathbf{B}}_{0,\varpi_i}$ and $\dot{\mathbf{B}}_{\varpi_i,0}$ associated to the fundamental weights ϖ_i, $i=1,\dots,m$. The map $u \mapsto u({}^\omega v_0 \otimes v_{w_0(\varpi_i)})$, $u \in \dot{\mathbf{U}}$, allows one to identify $\dot{\mathbf{B}}_{0,\varpi_i}$ with the canonical basis $\underline{\mathbf{B}}_{\varpi_i}$ of ${}^\omega V_0 \otimes V_{\varpi_i} \cong V_{\varpi_i}$, and therefore with a uniquely determined finite subset $\mathbf{B}_{\varpi_i}$ of the canonical basis $\mathbf{B}^-$ of $U_q^{\mathrm{ad}}(\mathfrak{n}_-)$; similarly, one can identify $\dot{\mathbf{B}}_{\varpi_i,0}$ with a uniquely determined finite subset ${}^\omega\mathbf{B}_{\varpi_i}$ of the canonical basis $\mathbf{B}^+$ of $U_q^{\mathrm{ad}}(\mathfrak{n}_+)$. The elements of $\dot{\mathbf{B}}_{0,\varpi_i}$ and $\dot{\mathbf{B}}_{\varpi_i,0}$ are respectively of the form $b^-1_{\varpi_i}$ and $b^+1_{-\varpi_i}$, where $b^- \in \mathbf{B}_{\varpi_i}$ and $b^+ \in {}^\omega\mathbf{B}_{\varpi_i}$, and we have (see [84, Proposition 3.3 and Section 3.4]):

Proposition 2.10. *The algebra $\mathcal{O}_A$ is finitely generated. A system of generators is provided by the elements $a^* \in \dot{\mathbf{B}}^*$, where $a \in \bigcup_{i=1}^m(\dot{\mathbf{B}}_{0,\varpi_i} \cup \dot{\mathbf{B}}_{\varpi_i,0})$.*

Note that the above system of generators of $\mathcal{O}_A$ has $2\sum_{i=1}^m \dim(V_{\varpi_i})$ elements. In fact, recall that $\varphi\colon U_q^{\mathrm{ad}} \to U_q^{\mathrm{ad}}$ is the anti-automorphism given by $\varphi(E_i) = F_i$, $\varphi(F_i) = E_i$, $\varphi(K_\lambda) = K_\lambda$. Denote by $v_{-\varpi_i}$ and $f_{-\varpi_i}$ the canonical lowest-weight vectors of the highest weight modules $V_{-w_0(\varpi_i)}$ and $V^r_{-w_0(\varpi_i)}$, respectively, and put the superscript “ up ” for the upper canonical basis vectors.

Lemma 2.11. *For every $b^- \in \mathbf{B}_{\varpi_i}$ and $b^+ \in {}^\omega\mathbf{B}_{\varpi_i}$, we have*

$$(b^-1_{\varpi_i})^* = \Phi_{\varpi_i}((f_{\varpi_i}\varphi(b^-))^{\mathrm{up}} \otimes v_{\varpi_i}), \tag{2.56}$$

$$\big(b^+1_{-\varpi_i}\big)^* = \Phi_{-w_0(\varpi_i)}\big(\big(f_{-\varpi_i}\varphi\big(b^+\big)\big)^{\mathrm{up}} \otimes v_{-\varpi_i}\big). \tag{2.57}$$

In other words, $(b^-1_{\varpi_i})^*$ and $\big(b^+1_{-\varpi_i}\big)^*$ are the matrix coefficients lying on the first and last columns of the matrix representations in the upper canonical bases of the spaces V_{ϖ_i}, $i = 1, \dots, m$.

Proof. This can be checked by using the isomorphism (2.43). The key observation is that

$$\langle \Phi_\lambda(f_\lambda \otimes v_\lambda), 1_\mu\rangle = \langle f_\lambda, 1_\mu v_\lambda\rangle_\lambda = \delta_{\lambda,\mu}$$

for every $\lambda \in P_+$, $\mu \in P$, and therefore $\Phi_\lambda(f_\lambda \otimes v_\lambda) = 1_\lambda^*$. Then the computation proceeds by using the equivariance of Φ under the action of $U_q^{\mathrm{ad}} \otimes (U_q^{\mathrm{ad}})^{\mathrm{op}}$, the fact that $\langle\cdot,\cdot\rangle$ dualizes the bimodules structures on $\mathcal{O}_q$ and $\dot{\mathbf{U}}$, and the description of the associated Kashiwara operators on $\mathcal{B}(\mathcal{O}_q)$ and $\mathcal{B}(\dot{\mathbf{U}})$. Here is an alternative argument. By the very definition of the sets $\dot{\mathbf{B}}[\lambda]$ we have $b^-1_{\varpi_i} \in \dot{\mathbf{B}}[\varpi_i]$, $b^+1_{-\varpi_i} \in \dot{\mathbf{B}}[-\omega_0(\varpi_i)]$. We wish to check if their duals $(b^-1_{\varpi_i})^*$, $(b^+1_{-\varpi_i})^*$ coincide with the elements of $\mathcal{O}_A$ on the right sides of (2.56) and (2.57). As already noticed after (2.48), by the isomorphism $\mathcal{O}_A(\leq\lambda)/\mathcal{O}_A(<\lambda) \cong \mathrm{End}({}_AV_\lambda)^*$ every matrix coefficient of ${}_AV_\lambda$ belongs to $\mathcal{O}_A(\leq\lambda)$. Now, the A-modules $\mathcal{O}_A(\leq\varpi_i)$ and $\mathcal{O}_A(\leq -w_0(\varpi_i))$ are generated by $\dot{\mathbf{B}}[\varpi_i]^*$ and $\dot{\mathbf{B}}[-w_0(\varpi_i)]^*$, respectively. Because $((\bar{\pi}_\lambda^*)^{-1} \circ d_\lambda)(\dot{\mathbf{B}}[\lambda]^*)$ coincides with $\underline{\mathbf{B}}_\lambda^{\mathrm{up}} \otimes ({}^\omega\underline{\mathbf{B}}_\lambda)^{\mathrm{up}}$, the conclusion follows. ■

Note that the same argument implies that, for every $\lambda \in P_+$, any matrix coefficient of V_λ in the upper canonical basis and vanishing on the elements of $\dot{\mathbf{B}}[\lambda']$ for $\lambda' < \lambda$ must belong to $\dot{\mathbf{B}}[\lambda]^*$. For instance, in the $\mathfrak{sl}_2$ case, $\mathcal{O}_A(\leq 2)$ has canonical basis $\dot{\mathbf{B}}[0]^* \coprod \dot{\mathbf{B}}[2]^*$, so the matrix coefficients of V_2 vanishing on 1_0 belong to $\dot{\mathbf{B}}[2]^*$. This can be observed in (2.45), using the comments in the paragraph before (2.52).

Though the A-module ${}_AV_\mu \bigotimes_A {}_AV_\nu$ has no decomposition like (2.7), we can refine the map $C(\mu)\otimes C(\nu) \to C(\mu+\nu)$ in (2.8) to an A-linear map defined on ${}_A\dot{C}(\mu) \bigotimes_A {}_A\dot{C}(\nu)$. Indeed, there is a unique injective morphism of U_A^{res}-modules $\mathfrak{T}_{\mu,\nu}\colon {}_AV_{\mu+\nu} \to {}_AV_\mu \bigotimes_A {}_AV_\nu$, which is given by $\mathfrak{T}_{\mu,\nu}(v_{\mu+\nu}) = v_\mu \otimes v_\nu$ [83, Proposition 25.1.2 (a)–(b)]. It defines a morphism of based modules

$$(V_{\mu+\nu}, \underline{\mathbf{B}}_{\mu+\nu}) \to (V_\mu \otimes V_\nu, \underline{\mathbf{B}}_\mu \diamondsuit \underline{\mathbf{B}}_\nu),$$

where $\underline{\mathbf{B}}_\mu \diamondsuit \underline{\mathbf{B}}_\nu := \{b \diamondsuit b', b \in \underline{\mathbf{B}}_\mu, b' \in \underline{\mathbf{B}}_\nu\}$ [83, Proposition 27.1.7]. Hence, $\mathfrak{T}_{\mu,\nu}$ is a split A-linear map, i.e., there exists a A-linear map $\mathfrak{S}_{\mu,\nu}\colon {}_AV_\mu \bigotimes_A {}_AV_\nu \to {}_AV_{\mu+\nu}$ such that $\mathfrak{S}_{\mu,\nu} \circ \mathfrak{T}_{\mu,\nu} = \mathrm{id}$. Note that $\mathfrak{S}_{\mu,\nu}$ is not a U_A^{res}-morphism. Similarly, the unique morphism of U_A^{res}-modules ${}^\omega\mathfrak{T}_{\mu,\nu}\colon {}_A^\omega V_{\mu+\nu} \to {}_A^\omega V_\mu \bigotimes_A {}_A^\omega V_\nu$ is a split injection. Define $\rho_{\mu',\mu''}\colon \dot{\mathbf{U}}_A \to {}_A^\omega V_{\mu'} \bigotimes_A {}_AV_{\mu''}$ by

$$\rho_{\mu',\mu''}(u) = u\bigg({}^\omega v_{w_0(\mu')} \bigotimes_A v_{w_0(\mu'')}\bigg),$$

and $\rho_{\mu',\mu'',\nu',\nu''}\colon \dot{\mathbf{U}}_A{}^{\hat{\otimes}2} \to {}_A^\omega V_{\mu'} \bigotimes_A {}_AV_{\mu''} \bigotimes_A {}_A^\omega V_{\nu'} \bigotimes_A {}_AV_{\nu''}$ by

$$\rho_{\mu',\mu'',\nu',\nu''}(u) = u\bigg({}^\omega v_{w_0(\mu')} \bigotimes_A v_{w_0(\mu'')} \bigotimes_A {}^\omega v_{w_0(\nu')} \bigotimes_A v_{w_0(\nu'')}\bigg).$$

Define $\tau_{\mu',\mu'',\nu',\nu''}\colon {}_A^\omega V_{\mu'+\nu'} \bigotimes_A {}_AV_{\mu''+\nu''} \to {}_A^\omega V_{\mu'} \bigotimes_A {}_AV_{\mu''} \bigotimes_A {}_A^\omega V_{\nu'} \bigotimes_A {}_AV_{\nu''}$ by

$$\tau_{\mu',\mu'',\nu',\nu''} = \big(1 \otimes \hat{R}^{-1} \otimes 1\big)\big({}^\omega\mathfrak{T}_{\mu',\nu'} \otimes \mathfrak{T}_{\mu'',\nu''}\big).$$

It is an injective morphism of U_A^{res}-modules. In [84, Section 1.13], Lusztig proved that $\tau_{\mu',\mu'',\nu',\nu''}$ is a split A-linear map ([84] uses $\hat{R}$ instead of $\hat{R}^{-1}$, since our coproducts on U_q^{ad} are opposite), and that it satisfies

$$\tau_{\mu',\mu'',\nu',\nu''}\rho_{\mu'+\mu'',\nu'+\nu''} = \rho_{\mu',\mu'',\nu',\nu''}\Delta, \tag{2.58}$$

where Δ is the coproduct of $\dot{\mathbf{U}}_A$, see (2.28).

Now take $\mu := \mu' = \mu''$, $\nu := \nu' = \nu'' \in P_+$, and put $\tau_{\mu,\nu} := \tau_{\mu,\mu,\nu,\nu}$. It follows from the classical decomposition (2.7) over $\mathbb{C}(q)$, and (2.8) and (2.51), that the product of $\mathcal{O}_A$ yields a map $m\colon \mathcal{O}_A(\leq \mu) \bigotimes_A \mathcal{O}_A(\leq \nu) \to \mathcal{O}_A(\leq \mu+\nu)$.

Denote the projection map $p_{\mu+\nu}\colon \mathcal{O}_A(\leq \mu+\nu) \to {}_A\dot{C}(\mu+\nu)$, define ${}_A\dot{\tau}_{\mu,\nu} := p_{\mu+\nu}\circ m$, and put

$$\pi'_\lambda\colon\ \mathcal{O}_A(\leq\lambda) \longrightarrow \mathcal{O}_A(\leq\lambda)/\mathcal{O}_A(<\lambda) \xrightarrow{(\bar{\pi}_\lambda^*)^{-1}\circ\bar{d}_\lambda} (\mathrm{End}({}_AV_\lambda))^*\ ,$$

where the first map is the quotient map. Consider the diagram

$$\begin{tikzcd}
{}_A\dot{C}(\mu)\otimes{}_A\dot{C}(\nu) \arrow[r, "{}_A\dot{\tau}_{\mu,\nu}"] \arrow[d, "\pi'_\mu\otimes\pi'_\nu"] & {}_A\dot{C}(\mu+\nu) \arrow[d, "\pi'_{\mu+\nu}"] \\
(\mathrm{End}({}_AV_\mu))^*\otimes(\mathrm{End}({}_AV_\nu))^* \arrow[r, "\tau^t_{\mu,\nu}"] & (\mathrm{End}({}_AV_{\mu+\nu}))^*,
\end{tikzcd}$$

where $\tau^t_{\mu,\nu}$ is the transpose of Lusztig's map $\tau_{\mu,\nu}$.

Proposition 2.12. *The map ${}_A\dot{\tau}_{\mu,\nu}\colon {}_A\dot{C}(\mu)\bigotimes_A{}_A\dot{C}(\nu) \to {}_A\dot{C}(\mu+\nu)$ is split as an A-linear map and the above diagram is commutative.*

Proof. The commutativity of the diagram comes from equation (2.58). The epimorphism π'_λ is injective on ${}_A\dot{C}(\lambda)$, and maps the canonical basis elements to the elements of the upper canonical basis $\underline{\mathbf{B}}_\lambda^{\mathrm{up}}\otimes({}^\omega\underline{\mathbf{B}}_\lambda)^{\mathrm{up}}$. By Lusztig's results recalled above, the epimorphism $\tau^t_{\mu,\nu}$ splits as an A-linear map. Therefore, the same is true of ${}_A\dot{\tau}_{\mu,\nu}$. ■

We stress that ${}_A\dot{\tau}_{\mu,\nu}$ plays for $\mathcal{O}_A$ the same role as the map (2.8) for $\mathcal{O}_q$.

Finally, we consider for any $n\geq 1$ the invariant elements of $\mathcal{O}_A^{\otimes n}$ endowed with the action coad_n^r of U_A^{res}, see (2.15) $\big($recall that $\mathcal{L}_{0,n} = \mathcal{O}_q^{\otimes n}$ as U_q^{ad}-module$\big)$.

First note that, by definition, $\mathcal{O}_A(G^n)$ is the restricted dual of the Hopf algebra $U_A^{\mathrm{res}}\big(\mathfrak{g}^{\oplus n}\big)$, associated to its category of type 1 modules. By ordering the summands of $\mathfrak{g}^{\oplus n}$ we get an isomorphism $U_A^{\mathrm{res}}\big(\mathfrak{g}^{\oplus n}\big) \cong U_A^{\mathrm{res}}(\mathfrak{g})^{\otimes n}$, and any type 1 simple $U_A^{\mathrm{res}}(\mathfrak{g})^{\otimes n}$-module is isomorphic to $V_{[\lambda]} := \bigotimes_{i=1}^n V_{\lambda_i}$ endowed with the componentwise action, for some $[\lambda] := (\lambda_1,\dots,\lambda_n)\in P_+^n$ (this is a classical fact; see, e.g., [51, Theorem 3.10.2]). Therefore, we have an isomorphism $\mathcal{O}_A(G^n)\cong\mathcal{O}_A^{\otimes n}$. With the same notation $[\lambda] := (\lambda_1,\dots,\lambda_n)\in P_+^n$, let us put

$$\begin{aligned}
{}_A\dot{C}([\lambda]) &:= \bigotimes_{i=1}^n {}_A\dot{C}(\lambda_i) = \bigoplus_{b\in\bigotimes_{i=1}^n \dot{\mathbf{B}}[\lambda_i]^*} Ab,\\
\mathcal{O}_A(\leq[\lambda]) &:= \bigotimes_{i=1}^n \mathcal{O}_A(\leq\lambda_i) = \bigoplus_{[\lambda']\in P_+^n,\lambda'_i\leq\lambda_i} {}_A\dot{C}([\lambda']).
\end{aligned}$$

We thus obtain a decomposition into based $(U_A^{\mathrm{res}}\otimes(U_A^{\mathrm{res}})^{\mathrm{op}})^{\otimes n}$-modules

$$\mathcal{O}_A^{\otimes n} = \sum_{[\lambda]\in P_+^n} \mathcal{O}_A(\leq[\lambda]).$$

Now $\mathrm{coad}^r_n = (\mathrm{coad}^r)^{\otimes n} \circ \Delta^{(n-1)}$ gives structures of U_A^{res}-modules to $\mathcal{O}_A^{\otimes n}$ and $\mathcal{O}_A(\leq [\lambda])$. In order to make it a based module, we give it the "$\Diamond$" product of the canonical bases of the factors $\mathcal{O}_A(\leq \lambda_i)$, i.e.,

$$\dot{\mathbf{B}}[[\lambda]]^* := \Diamond_{i=1}^n \Bigg(\coprod_{\lambda_i' \leq \lambda_i} \dot{\mathbf{B}}[\lambda_i']^* \Bigg).$$

We thus obtain a decomposition into based U_A^{res}-modules

$$\mathcal{O}_A^{\otimes n} = \sum_{[\lambda] \in P_+^n} (\mathcal{O}_A(\leq [\lambda]), \dot{\mathbf{B}}[[\lambda]]^*), \tag{2.59}$$

with composition factors $\bigotimes_{i=1}^n (\mathrm{End}({}_AV_{\lambda_i}))^*$. By the properties of "$\Diamond$" products of bases of based modules, the underlying A-module is

$$\mathcal{O}_A^{\otimes n} = \bigoplus_{[\lambda] \in P_+^n} {}_A\dot{C}([\lambda]). \tag{2.60}$$

Finally, we state the last property of based modules we need. Let (M, B) be a based module. Recall the notations introduced around (2.34). It is proved in [83, Proposition 27.1.8] that for every $\lambda \in P_+$ the submodule $M[\geq \lambda]$ is a sub-based module of M, and that it has the basis

$$B \cap M[\geq \lambda] = \bigcup_{\lambda' \geq \lambda} B[\lambda']. \tag{2.61}$$

Consider $M[\neq 0] := \bigoplus_{\lambda \neq 0} M[\lambda]$, the largest proper submodule of M that contains no non-zero invariant element. Recall that the space of *coinvariants of* M is

$$M_{U_q^{\mathrm{ad}}} = M/M[\neq 0] = M/\mathbb{C}(q)\big\{um - \varepsilon(u)m,\, m \in M,\, u \in U_q^{\mathrm{ad}}\big\}$$

that is, the largest quotient of M with trivial action, where $\varepsilon\colon U_q^{\mathrm{ad}} \to \mathbb{C}(q)$ is the counit. It follows from (2.61) that $M[\neq 0]$ is a sub-based module of M, with the basis $\bigcup_{\lambda \neq 0} B[\lambda]$, and we have (this is, [83, Proposition 27.2.6]):

Proposition 2.13. *The quotient map $\pi\colon M \to M_{U_q^{\mathrm{ad}}}$ is a morphism of based modules, where $M_{U_q^{\mathrm{ad}}}$ is endowed with the basis $B_{U_q^{\mathrm{ad}}} := \pi(B[0])$.*

Keeping the same notations, let ${}_AM \subset M$ be the A-module generated by B, and let ${}_AM^* \subset M^*$ be the A-module generated by B^*. They are U_A^{res}-modules. Denote by $({}_AM^*)^{U_A^{\mathrm{res}}}$ the submodule of U_A^{res}-invariant elements of ${}_AM^*$, regarded as a right module in the natural way.

Lemma 2.14. *We have a direct sum decomposition of A-modules*

$$ {}_AM^* = ({}_AM^*)^{U_A^{\mathrm{res}}} \bigoplus_A {}_AN, \tag{2.62}$$

where ${}_AN \subset {}_AM^*$ is the A-submodule generated by $\bigcup_{\lambda \neq 0} B[\lambda]^*$.

Proof. By Proposition 2.13, the transpose map $\pi^t\colon (M_{U_q^{\mathrm{ad}}})^* \to M^*$ is a monomorphism mapping the dual basis $B^*_{U_q^{\mathrm{ad}}}$ to the subset $B[0]^*$ of B^*. The image of π^t is $(M^*)^{U_q^{\mathrm{ad}}}$. If we set ${}_AM_{U_A^{\mathrm{res}}} = \pi({}_AM)$, then $\pi^t(({}_AM_{U_A^{\mathrm{res}}})^*) = ({}_AM^*)^{U_A^{\mathrm{res}}}$ is generated by $B[0]^*$, which concludes the proof. ∎

Note that, since $B[0]$ is in general not invariant under the action of $U_A^{\rm res}$, ${}_AN$ need not be stable under this action.

We are now ready to draw consequences of this discussion and the previous results. As usual denote by $\big(\mathcal{O}_A^{\otimes n}\big)^{U_A^{\rm res}}$ the subspace of invariant elements of $\mathcal{O}_A^{\otimes n}$ for the action coad_n^r. In the case $n=1$, it is just the center $\mathcal{Z}(\mathcal{O}_A)$.

Theorem 2.15. *$\big(\mathcal{O}_A^{\otimes n}\big)^{U_A^{\rm res}}$ is a direct summand of the A-module $\mathcal{O}_A^{\otimes n}$ for any $n\geq 1$.*

Proof. By equation (2.59), it is enough to show that for every $[\lambda]\in P_+^n$ the invariant elements of $\mathcal{O}_A(\leq[\lambda])$ form a direct summand, and these summands are compatible with non-empty intersections $\mathcal{O}_A(\leq[\lambda])\cap\mathcal{O}_A(\leq[\lambda'])$. Using that $\mathcal{O}_A(G^n)\cong\mathcal{O}_A^{\otimes n}$ and viewing P_+^n as the weight lattice of G^n, it is enough to prove these claims for $n=1$. Given $\lambda\in P_+$ put

$$P_\lambda=\{\lambda'\in P_+,\,\lambda'\not\leq\lambda\},$$

and denote by $\dot{\mathbf{U}}_A[P_\lambda]$ the A-submodule of $\dot{\mathbf{U}}_A$ generated by $\coprod_{\lambda'\in P_\lambda}\dot{\mathbf{B}}[\lambda']$. Also, let us put $\dot{\mathbf{U}}[P_\lambda]=\dot{\mathbf{U}}_A[P_\lambda]\bigotimes_A\mathbb{C}(q)$. The complement $P_+\setminus P_\lambda$ is finite, and if $\lambda'\in P_\lambda$ and $\lambda''\geq\lambda'$, then $\lambda''\in P_\lambda$. By the results of [83, Section 29.2], $\dot{\mathbf{U}}[P_\lambda]$ is a two-sided ideal, and the quotient algebra $\dot{\mathbf{U}}/\dot{\mathbf{U}}[P_\lambda]$ is finite-dimensional with unit the coset of $\sum_{\lambda'\leq\lambda}1_{\lambda'}$, and it is semisimple, isomorphic to $\bigoplus_{\lambda'\leq\lambda}\mathrm{End}(V_{\lambda'})$ (whereas $\dot{\mathbf{U}}_A/\dot{\mathbf{U}}_A[P_\lambda]$ has indecomposable modules, see Example 2.17). It inherits from $\dot{\mathbf{U}}$ a canonical basis, formed by the non-zero cosets of elements of $\dot{\mathbf{B}}$, and with this basis $\dot{\mathbf{U}}/\dot{\mathbf{U}}[P_\lambda]$ is a based module for the right adjoint action ad^r. Similarly as for (2.47), we have a morphism of $U_A^{\rm res}$-modules

$$\tilde{d}_\lambda\colon\ \mathcal{O}_A(\leq\lambda)\longrightarrow\mathrm{Hom}\big(\dot{\mathbf{U}}_A/\dot{\mathbf{U}}_A[P_\lambda],A\big),\qquad\alpha\longmapsto\langle\alpha,\cdot\,\rangle,$$

which is an isomorphism by (2.50) and the computation $\dim\big(\dot{\mathbf{U}}/\dot{\mathbf{U}}[P_\lambda]\big)=\sum_{\lambda'\leq\lambda}\dim(V_{\lambda'})^2$ in [83, Section 29.2]. Applying Proposition 2.13 and (2.62) to the based module $M=\dot{\mathbf{U}}/\dot{\mathbf{U}}[P_\lambda]$, we obtain that the invariant elements of $\mathcal{O}_A(\leq\lambda)$ form a direct summand. Finally, for any $\lambda,\lambda'\in P_+$ we have $\mathcal{O}_A(\leq\lambda)\cap\mathcal{O}_A(\leq\lambda')\cong\mathrm{Hom}\big(\dot{\mathbf{U}}_A/\big(\dot{\mathbf{U}}_A[P_\lambda]+\dot{\mathbf{U}}_A[P_{\lambda'}]\big),A\big)$. Applying Proposition 2.13 and (2.62) to the based module $M:=\dot{\mathbf{U}}/(\dot{\mathbf{U}}[P_\lambda]+\dot{\mathbf{U}}[P_{\lambda'}])$, we obtain that the invariant elements $({}_AM^*)^{U_A^{\rm res}}$ of $\mathcal{O}_A(\leq\lambda)\cap\mathcal{O}_A(\leq\lambda')$ form a direct A-summand. Since the latter is a based $U_A^{\rm res}$-submodule of $\mathcal{O}_A(\leq\lambda)$ and $\mathcal{O}_A(\leq\lambda')$, this summand is also a direct A-summand of $\mathcal{O}_A(\leq\lambda)^{U_A^{\rm res}}$ and $\mathcal{O}_A(\leq\lambda')^{U_A^{\rm res}}$. This shows the A-modules $\mathcal{O}_A(\leq\lambda)^{U_A^{\rm res}}$ for all $\lambda\in P_+$ match to form the A-summand $(\mathcal{O}_A)^{U_A^{\rm res}}$ of $\mathcal{O}_A$, and thus concludes the proof. ■

Remark 2.16. Let (M,B), (M',B') be based modules, with tensor product $(M\otimes M',B_\Diamond)$, and $B_\Diamond[0]\subset B_\Diamond$ the subset in bijection with the canonical basis of the space of coinvariants $(M\otimes M')_{U_q^{\rm ad}}$ (see Proposition 2.13). This subset is described in [83, Proposition 27.3.8] in terms of B and B'. Since $\dot{\mathbf{U}}/\dot{\mathbf{U}}[P_\lambda]$ is semisimple with known summands, and the construction of the "$\Diamond$" product of canonical bases is associative, one can recursively compute the subset of the canonical basis of $\bigotimes_{i=1}^n\dot{\mathbf{U}}/\dot{\mathbf{U}}[P_{\lambda_i}]$ (endowed with the action dual to coad_n^r) which is in bijection with the canonical basis of the space of coinvariants. Therefore, a complete (though highly nontrivial) characterization of the basis of $\big(\mathcal{O}_A^{\otimes n}\big)^{U_A^{\rm res}}$ can be obtained. Examples can be found in [83, Section 27.3.10]. In the case $\mathfrak{g}=\mathfrak{sl}_2$, the canonical basis of the dual space $\mathrm{End}\big(V_1^{\otimes n}\big)^*$ has been identified in [56] with the canonical basis of the Temperley–Lieb algebra $TL_n(q)$.

Example 2.17. The simplest case is already instructive. Namely, consider V_1 and V_2, the simple $U_q^{\rm ad}(\mathfrak{sl}_2)$-modules of type 1 and dimension two and three.

On V_1, we have the lower canonical basis vectors v_+ and v_-, such that $Kv_+=qv_+$, $Ev_+=0$, $v_-=Fv_+$. The canonical lower and upper bases of V_1 are both $\{v_+,v_-\}$. Using the relation (2.32), we see that the elements of $\dot{\mathbf{B}}_{0,1}$ and $\dot{\mathbf{B}}_{1,0}$ are 1_1, $F1_1$ and 1_{-1}, $E1_{-1}$, respectively;

the dual linear forms generate $\mathcal{O}_A(\mathrm{SL}_2)$, they are the matrix coefficients a, c, d and b respectively. By (2.37), we have $\dot{\mathbf{B}}[1] = \dot{\mathbf{B}}_{0,1} \coprod \dot{\mathbf{B}}_{1,0}$.

Next consider V_2. On V_2, we have the canonical highest weight vector v_0 of weight 2, and lower canonical basis $\underline{\mathbf{B}}_2 = \{v_0, v_1, v_2\}$, where $v_1 = Fv_0$ and $v_2 = F^{(2)}v_0$. We have $\underline{\mathbf{B}}_2^{\mathrm{up}} = \{v_0, [2]_q^{-1}v_1, v_2\}$ (see Appendix A). We can identify the ambient space of the right module V_2^r with that of V_2; its highest weight vector is then v_0, and its canonical lower and upper bases are $\underline{\mathbf{B}}_2^r = \{v_0, v_1, v_2\}$ and $\underline{\mathbf{B}}_2^{r\,\mathrm{up}} = \{v_0, [2]_q^{-1}v_1, v_2\}$.

Consider now the module ${}^{\omega}V_1 \otimes V_1$. We have

$$\hat{R} = \sum_{n=0}^{\infty} \frac{(q-q^{-1})^n}{[n]_q!} q^{n(n-1)/2} E^n \otimes F^n,$$

so the matrix of the involution $\Psi = \hat{R}^{-1} \circ^-$ in the basis $v_+ \otimes v_+, v_+ \otimes v_-, v_- \otimes v_+, v_- \otimes v_-$ is

$$\left(\hat{R}^{-1} \circ^-\right)_{{}^{\omega}V_1,V_1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ q^{-1}-q & 0 & 0 & 1 \end{pmatrix}.$$

Therefore, the canonical basis $\underline{\mathbf{B}}_{1,1}$ is formed by the vectors $v_+ \diamondsuit v_+ = v_+ \otimes v_+ + q^{-1} v_- \otimes v_-$ and $v_+ \diamondsuit v_- = v_+ \otimes v_-$, $v_- \diamondsuit v_+ = v_- \otimes v_+$, $v_- \diamondsuit v_- = v_- \otimes v_-$. Consider the partition $\underline{\mathbf{B}}_{1,1} = \underline{\mathbf{B}}_{1,1}[2] \cup \underline{\mathbf{B}}_{1,1}[0]$. We have $\underline{\mathbf{B}}_{1,1}[2] = \{v_- \diamondsuit v_+, v_+ \diamondsuit v_+, , v_+ \diamondsuit v_-\}$, which is a basis of the three-dimensional submodule W_2 of $V_1 \otimes V_1$. Since $\underline{\mathbf{B}}_{1,1}$ is an A-basis of ${}^{\omega}_{A}V_1 \bigotimes_A {}_AV_1$, it follows that the epimorphism $\tau_{1,1}^t \colon {}_A\dot{C}(1) \bigotimes_A {}_A\dot{C}(1) \to {}_A\dot{C}(2)$ splits (see Proposition 2.12). The vector $v_- \diamondsuit v_-$ is cyclic, so $\underline{\mathbf{B}}_{1,1}[0] = \{v_- \diamondsuit v_-\}$. By the definitions, we have $v_+ \diamondsuit v_+ = (1 \diamondsuit_0 1)_{1,1}$, $v_+ \diamondsuit v_- = (1 \diamondsuit_0 F)_{1,1}$, $v_- \diamondsuit v_+ = (F \diamondsuit_0 1)_{1,1}$, $v_- \diamondsuit v_- = (F \diamondsuit_0 F)_{1,1}$, so the corresponding elements of $\dot{\mathbf{B}}_{1,1} \subset \dot{\mathbf{B}}$ are respectively 1_0, $1_{-2}F$, 1_2E, and $F1_2E = E1_{-2}F$.

The invariant submodule W_0 of ${}^{\omega}V_1 \otimes V_1$ is generated by $v' = v_- \otimes v_- - q^{-1} v_+ \otimes v_+$. The U_A^{res}-modules ${}^{\omega}_{A}V_1 \bigotimes_A {}_AV_1$ and $W_2 \oplus W_0$ are not equal, though they are by extending scalars to $\mathbb{C}(q)$. Indeed, we have

$$v_+ \otimes v_+ = [2]_q^{-1}(qv_+ \diamondsuit v_+ - v') \notin W_2 \oplus W_0.$$

The module of coinvariants is $({}^{\omega}V_1 \otimes V_1)_{U_q^{\mathrm{ad}}} = \mathbb{C}(q)\{\pi(v_- \otimes v_-)\}$, where as usual $\pi \colon {}^{\omega}V_1 \otimes V_1 \to ({}^{\omega}V_1 \otimes V_1)_{U_q^{\mathrm{ad}}}$ is the quotient map. The transpose map $\pi^t \colon (({}^{\omega}V_1 \otimes V_1)_{U_q^{\mathrm{ad}}})^* \to ({}^{\omega}V_1 \otimes V_1)^*$ sends $(v_- \diamondsuit v_-)^*$ to the unique U_q^{ad}-invariant linear map

$$\mathrm{ev}_1 \colon {}^{\omega}V_1 \otimes V_1 \to \mathbb{C}(q)$$

such that $\mathrm{ev}_1(v_- \otimes v_-) = 1$.

Note that, since elements of $\dot{\mathbf{U}}_A[\lambda > 2]$ act trivially on modules with all isotypical components of highest weight ≤ 2, ${}^{\omega}_{A}V_1 \bigotimes_A {}_AV_1$ is an indecomposable module over $\dot{\mathbf{U}}_A/\dot{\mathbf{U}}_A[\lambda > 2]$ (that is, $\dot{\mathbf{U}}_A/\dot{\mathbf{U}}_A[P_2]$ in the notations of Theorem 2.15).

2.2.3 Some consequences on $\mathcal{L}_{0,n}^A$ and $\mathcal{M}_{0,n}^A$

Recall from Section 2.2.1 the definition of the integral forms $\mathcal{L}_{0,n}^A$ and $\mathcal{M}_{0,n}^A$.

Proposition 2.18. *$\mathcal{L}_{0,n}^A$ and $\mathcal{M}_{0,n}^A$ are free A-modules, and $\mathcal{M}_{0,n}^A$ is a direct summand of the A-module $\mathcal{L}_{0,n}^A$. Moreover, $\mathcal{L}_{0,n}^A$ is a finitely generated ring.*

Proof. Since $\mathcal{L}_{0,n}^A = \mathcal{O}_A^{\otimes n}$ as $U_A^{\rm res}$-modules, by (2.60) it has the basis $\bigcup_{[\lambda]\in P_+^n} \dot{\mathbf{B}}[[\lambda]]^*$. Therefore, $\mathcal{L}_{0,n}^A$ is a free A-module. Since A is a principal ideal domain, it follows that $\mathcal{M}_{0,n}^A$ is a free A-submodule [77, Appendix 2.2]. By Theorem 2.15, there is a direct sum decomposition as A-module

$$\mathcal{L}_{0,n}^A = \mathcal{M}_{0,n}^A \oplus {}_AN, \tag{2.63}$$

and the proof identifies a basis of $\mathcal{M}_{0,n}^A$ as a subset of $\bigcup_{[\lambda]\in P_+^n} \dot{\mathbf{B}}[[\lambda]]^*$.

Next, consider the question of finite generation. By the formula (2.17), it is enough to verify this for $\mathcal{L}_{0,1}^A$, but $\mathcal{L}_{0,1}^A = \mathcal{O}_A$ as an A-module, and $\mathcal{O}_A$ is finitely generated by the matrix coefficients of the fundamental $U_A^{\rm res}$-modules ${}_AV_{\varpi_k}$, $k\in\{1,\dots,m\}$ (see (2.56) and (2.57)). Any monomials in these generators can be written as a A-linear combination of monomials in the same generators but with the product of $\mathcal{L}_{0,1}^A$, instead of the product $\star$. This follows from the integrality properties of the R-matrix, and the formula inverse to (2.9) (see in [18, Section 3.3 and the formulas (4.6)–(4.8)]). ■

Remark 2.19.

(a) As noted in (2.62), the A-module ${}_AN$ in the decomposition (2.63) is in general not a $U_A^{\rm res}$-module. Therefore, the A-linear projection map $\mathcal{R}_A\colon \mathcal{L}_{0,n}^A \to \mathcal{M}_{0,n}^A$ such that $\mathrm{Ker}(\mathcal{R}_A) = {}_AN$ is not a Reynolds operator, for it does not satisfy the identity $\mathcal{R}_A(\alpha\beta) = \alpha\mathcal{R}_A(\beta)$ for all $\alpha\in\mathcal{M}_{0,n}^A$, $\beta\in\mathcal{L}_{0,n}^A$.

(b) Recall (2.24). In coherence with (a) above, there is no normalized Haar measure on $\mathcal{O}_A$ taking values in A. Indeed, by extending scalars over $\mathbb{C}(q)$ it should otherwise coincide with the Haar measure $h\colon \mathcal{O}_q\to\mathbb{C}(q)$, but in the notations of Example 2.17 (see also the comments after (2.44)), since $h(v_0^*\otimes v_0) = 0$ we have $h(bc) = -1/\big(q+q^{-1}\big)$, whence h cannot be defined on $\mathcal{O}_A$.

(c) The Haar measure yields a well-defined $\mathcal{A}_0$-linear map $h\colon \mathcal{L}(\mathcal{O}_q)\to\mathcal{A}_0$ (and analogously $\mathcal{A}_0$-linear and $\mathcal{A}_\infty$-linear maps $h\colon \mathcal{L}_\Diamond\big(\mathcal{O}_q^{\otimes n}\big)\to\mathcal{A}_0$ and $\bar{h}\colon \bar{\mathcal{L}}_\Diamond\big(\mathcal{O}_q^{\otimes n}\big)\to\mathcal{A}_\infty$ for any $n\geq 1$, where $\big(\mathcal{L}_\Diamond\big(\mathcal{O}_q^{\otimes n}\big), \mathcal{B}[[\lambda]]^*\big)$ is the crystal basis at $q=0$ underlying the based $U_q^{\rm ad}$-module (2.59)). Indeed, by (2.41) the lattice $\mathcal{L}_\lambda^{r\,\rm up}\bigotimes_{\mathcal{A}_0}\mathcal{L}_\lambda^{\rm up}$ is generated by the matrix coefficients in the canonical bases of V_λ^r and V_λ. Since the normalisation by powers of q is vacuous on the trivial module $V_0^*\otimes V_0$, and h vanishes on $V_\lambda^*\otimes V_\lambda$ for $\lambda\in P_+\setminus\{0\}$, the claim follows.

2.3 Perfect pairings

We will need to restrict the morphisms Φ^+, Φ^- in (2.5) on the integral forms $\mathcal{O}_A(B_+)$, $\mathcal{O}_A(B_-)$. We collect their properties in Theorem 2.20 and the discussion thereafter. In order to state it, we recall first a few facts about R-matrices and related pairings.

Recall that $\mathcal{C}_A$ is the category of $U_A^{\rm res}$-modules of type 1. In [82, 83], Lusztig proved that $\mathcal{C}_A\bigotimes_A\mathbb{C}[q^{\pm 1/D}]$ is braided and ribbon, with braiding given by the collection of endomorphisms

$$R = (R_{V,W})_{V,W\in \mathrm{Ob}(\mathcal{C}_A)}.$$

Actually, $R_{V,W}$ is represented by a matrix with coefficients in $q^{\mathbb{Z}/D}\mathbb{C}\big[q^{\pm1}\big]$ on the tensor product of the lower canonical bases of V and W (see [83, Corollary 24.1.5]).

This can be rephrased as follows in Hopf algebra terms. Denote by $\mathbb{U}_\Gamma$ the categorical completion of Γ, i.e., the Hopf algebra of natural transformations $F_{\mathcal{C}_A}\to F_{\mathcal{C}_A}$, where $F_{\mathcal{C}_A}\colon\mathcal{C}_A\to A$-$Mod_f$ is the forgetful functor towards the category A-Mod_f of finite rank A-modules. Then

$\mathbb{U}_\Gamma \bigotimes_A \mathbb{C}[q^{\pm 1/D}]$ is quasi-triangular and ribbon with R-matrix

$$R \in \mathbb{U}_\Gamma^{\hat{\otimes} 2} \bigotimes_A \mathbb{C}\big[q^{\pm 1/D}\big].$$

As in (2.3), we can write

$$R^\pm = \sum_{(R)} R^\pm_{(1)} \otimes R^\pm_{(2)}.$$

There are pairings of Hopf algebras naturally related to the R-matrix R, considered as an element of $\mathbb{U}_q^{\hat{\otimes} 2}$. What follows is standard (see, e.g., [69, 70, 81]), for details we refer to [104, Proposition 3.73, Lemma 3.75, Theorem 3.92, Propositions 3.106 and 3.107]:

- There is a unique pairing of Hopf algebras $\rho\colon U_q(\mathfrak{b}_-)^{\mathrm{cop}} \otimes U_q(\mathfrak{b}_+) \to \mathbb{C}\big(q^{1/D}\big)$ such that, for every $\alpha, \lambda \in P$ and $l, k \in U_q(\mathfrak{h})$,

$$\begin{aligned} &\rho(K_\lambda, K_\alpha) = q^{(\lambda,\alpha)}, \qquad \rho(F_i, E_j) = \delta_{i,j}\big(q_i - q_i^{-1}\big)^{-1},\\ &\rho(l, E_j) = \rho(F_i, k) = 0. \end{aligned} \tag{2.64}$$

- The *Drinfeld pairing* $\tau\colon U_q(\mathfrak{b}_+)^{\mathrm{cop}} \otimes U_q(\mathfrak{b}_-) \to \mathbb{C}\big(q^{1/D}\big)$ is the bilinear map defined by $\tau(X, Y) = \rho(S(Y), X)$; it satisfies

$$\begin{aligned} &\tau(K_\lambda, K_\alpha) = q^{-(\lambda,\alpha)}, \qquad \tau(E_j, F_i) = -\delta_{i,j}\big(q_i - q_i^{-1}\big)^{-1},\\ &\tau(l, F_i) = \tau(E_j, k) = 0. \end{aligned} \tag{2.65}$$

- ρ and τ are perfect pairings; this means that they yield *isomorphisms* of Hopf algebras $i_\pm\colon U_q(\mathfrak{b}_\pm) \to \mathcal{O}_q(B_\mp)_{\mathrm{op}}$ (with coefficients *a priori* extended to $\mathbb{C}\big(q^{1/D}\big)$, but see below) defined by, for every $X \in U_q(\mathfrak{b}_+)$, $Y \in U_q(\mathfrak{b}_-)$,

  $$\langle i_+(X), Y\rangle = \tau(S(X), Y), \qquad \langle i_-(Y), X\rangle = \tau(X, Y).$$

  Since $\mathcal{O}_q(B_\mp)_{\mathrm{op}}$ is equipped with the *inverse* of the antipode of $\mathcal{O}_q(B_\mp)$, which is induced by the antipode $S_{\mathcal{O}_q}$ of $\mathcal{O}_q$, it follows that $i_\pm \circ S = S_{\mathcal{O}_q}^{-1} \circ i_\pm$.

- Denote by $p_\pm\colon \mathcal{O}_q(G) \to \mathcal{O}_q(B_\pm)$ the canonical projection map, i.e., the Hopf algebra homomorphism dual to the inclusion map $U_q(\mathfrak{b}_\pm) \hookrightarrow U_q(\mathfrak{g})$. For every $\alpha, \beta \in \mathcal{O}_q(G)$, we have

$$\langle \alpha \otimes \beta, R\rangle = \tau\big(i_+^{-1}(p_-(\beta)), i_-^{-1}(p_+(\alpha))\big). \tag{2.66}$$

Note that it is the use of weights $\alpha, \lambda \in P$ that forces the pairings ρ, τ to be defined over $\mathbb{C}\big(q^{1/D}\big)$, instead of $\mathbb{C}(q)$. Then, let us consider the restrictions π_q^+ of ρ, and π_q^- of τ defined by the formulas (2.64) and (2.65), where now $\alpha \in Q$ and $k \in U_q^{\mathrm{ad}}(\mathfrak{h})$. They take values in $\mathbb{C}(q)$, and define pairings

$$\pi_q^+\colon\ U_q(\mathfrak{b}_-)^{\mathrm{cop}} \otimes U_q^{\mathrm{ad}}(\mathfrak{b}_+) \to \mathbb{C}(q), \qquad \pi_q^-\colon\ U_q(\mathfrak{b}_+)^{\mathrm{cop}} \otimes U_q^{\mathrm{ad}}(\mathfrak{b}_-) \to \mathbb{C}(q).$$

By the same arguments as for ρ and τ (e.g., in [104, Proposition 3.92]), it follows that $\pi_q^\pm$ are perfect pairings. Note also that $\pi_q^- = \kappa \circ \pi_q^+ \circ (\kappa \otimes \kappa)$, where $\kappa\colon U_q \to U_q$ is the $\mathbb{C}$-linear automorphism extending ${}^-\colon U_q^{\mathrm{ad}} \to U_q^{\mathrm{ad}}$ in Section 2.2.2, so defined by

$$\kappa(E_i) = F_i, \qquad \kappa(F_i) = E_i, \qquad \kappa(K_\lambda) = K_{-\lambda}, \qquad \kappa(q) = q^{-1}. \tag{2.67}$$

In [41], De Concini–Lyubashenko described integral forms of $\pi_q^{\pm}$ as follows. Denote by $m^*\colon \mathcal{O}_A \to \mathcal{O}_A(B_+)\otimes\mathcal{O}_A(B_-)$ the map dual to the multiplication map $\Gamma(\mathfrak{b}_+)\otimes\Gamma(\mathfrak{b}_-)\to\Gamma$, so $m^*=(p_+\otimes p_-)\circ\Delta_{\mathcal{O}_A}$. Let $U_A(G^*)$ be the smallest A-subalgebra of $U_A(\mathfrak{b}_-)^{\mathrm{cop}}\otimes U_A(\mathfrak{b}_+)^{\mathrm{cop}}$ which contains the elements

$$1\otimes K_i^{-1}\bar{E}_i,\qquad \bar{F}_iK_i\otimes 1,\qquad L_i^{\pm1}\otimes L_i^{\mp1},\qquad i=1,\dots,m,$$

and is stable under the diagonal action of $\mathcal{B}(\mathfrak{g})$. The reason for the notation $U_A(G^*)$ will be explained at the beginning of Section 2.5. Note that $U_A(G^*)$ is free over A, a Hopf subalgebra, and that a basis is given by the elements

$$\bar{F}_{\beta_1}^{n_1}\cdots\bar{F}_{\beta_N}^{n_N}K_{n_1\beta_1+\cdots+n_N\beta_N}K_\lambda\otimes K_{-\lambda}K_{-p_1\beta_1-\cdots-p_N\beta_N}\bar{E}_{\beta_1}^{p_1}\cdots\bar{E}_{\beta_N}^{p_N},\tag{2.68}$$

where $\lambda\in P$ and $n_1,\dots,n_N,p_1,\dots,p_N\in\mathbb{N}$.

Now, let v be a lowest weight vector of the lowest weight Γ-module ${}_AV_{-\lambda}$, $\lambda\in P_+$. As after Theorem 2.1, denote by $v^*\in{}_AV_{-\lambda}^*$ the dual vector, and by $\psi_{-\lambda}\in\mathcal{O}_A$ the matrix coefficient defined by $\langle\psi_{-\lambda},x\rangle=v^*(xv)$ for every $x\in\Gamma$. Consider the maps $j_q^{\pm}\colon\mathcal{O}_q(B_\pm)\to U_q(\mathfrak{b}_\mp)^{\mathrm{cop}}$ defined by

$$\langle\alpha_+,X\rangle=\pi_q^+\bigl(j_q^+(\alpha_+),X\bigr),\qquad \langle\alpha_-,Y\rangle=\pi_q^-(j_q^-(\alpha_-),Y),$$

where $\alpha_\pm\in\mathcal{O}_q(B_\pm)$, $X\in U_q^{\mathrm{ad}}(\mathfrak{b}_+)$, and $Y\in U_q^{\mathrm{ad}}(\mathfrak{b}_-)$.

The following theorem summarizes results proved in [41, Sections 3 and 4]. Denote by $\mathcal{O}_A[\psi_{-\rho}^{-1}]$ the localization of $\mathcal{O}_A$ by the element $\psi_{-\rho}$; this localization is well defined, for the set $\{\psi_{-\rho}^n\}_{n\in\mathbb{N}}$ is a left and right multiplicative Ore subset of $\mathcal{O}_A$ (see Corollary 2.23 below for an analogous statement for $\mathcal{L}_{0,1}^A$). For the sake of clarity, let us spell out the correspondence of notations between statements: π_q^+, π_q^-, $U_q(\mathfrak{b}_\mp)^{\mathrm{cop}}$, $U_A(\mathfrak{b}_\mp)^{\mathrm{cop}}$, $\mathcal{O}_A(B_\pm)$, $U_A(G^*)$ and Φ are denoted in [41] respectively by π'', $\bar{\pi}''$, $U_q(\mathfrak{b}_\mp)_{\mathrm{op}}$, $R_q[B_\pm]''$, $R_q[B_\pm]$, A'' and μ'' (the definition of $j_A^{\pm}$ is implicit in [41, Section 4.2]).

Theorem 2.20.

(1) $\pi_q^{\pm}$ *restricts to a perfect Hopf pairing between the unrestricted and restricted integral forms,* $\pi_A^{\pm}\colon U_A(\mathfrak{b}_\mp)^{\mathrm{cop}}\otimes\Gamma(\mathfrak{b}_\pm)\to A$.

(2) $j_q^{\pm}$ *yields an isomorphism of Hopf algebras* $j_A^{\pm}\colon\mathcal{O}_A(B_\pm)\to U_A(\mathfrak{b}_\mp)^{\mathrm{cop}}$*, satisfying* $\langle\alpha_\pm,x_\pm\rangle=\pi_A^{\pm}\bigl(j_A^{\pm}(\alpha_\pm),x_\pm\bigr)$ *for every* $\alpha_\pm\in\mathcal{O}_A(B_\pm)$, $x_\pm\in\Gamma(\mathfrak{b}_\pm)$.

(3) *The map* $\Phi:=\bigl(j_A^+\otimes j_A^-\bigr)\circ m^*\colon\mathcal{O}_A\to U_A(G^*)\subset U_A(\mathfrak{b}_-)^{\mathrm{cop}}\otimes U_A(\mathfrak{b}_+)^{\mathrm{cop}}$ *is an embedding of Hopf algebras, and it extends to an isomorphism* $\Phi\colon\mathcal{O}_A\bigl[\psi_{-\rho}^{-1}\bigr]\to U_A(G^*)$.

For our purposes, it is necessary to reformulate this result. Consider the morphisms of Hopf algebras $\Phi^{\pm}\colon\mathcal{O}_A(B_\pm)\to U_A(\mathfrak{b}_\mp)^{\mathrm{cop}}$, $\alpha\mapsto(\alpha\otimes\mathrm{id})(R^{\pm})$.

Lemma 2.21. *We have* $\Phi^{\pm}=j_A^{\pm}$.

Proof. By definitions, for every $X\in U_q(\mathfrak{b}_+)^{\mathrm{cop}}$, $Y\in U_q^{\mathrm{ad}}(\mathfrak{b}_-)$, we have $\langle i_+(S^{-1}(X)),Y\rangle=\pi_q^-(X,Y)$, and similarly for every $X\in U_q^{\mathrm{ad}}(\mathfrak{b}_+)$, $Y\in U_q(\mathfrak{b}_-)^{\mathrm{cop}}$, we have $\langle i_-(S^{-1}(Y)),X\rangle=\pi_q^+(Y,X)$. By keeping these notations for X and Y, we deduce $j_q^-(i_+(S^{-1}(X)))=X$ and $j_q^+(i_-(S^{-1}(Y)))=Y$, i.e., $j_q^{\pm}=S\circ i_\mp^{-1}$. Because $S_{\mathcal{O}_q}^{-1}\circ i_\pm=i_\pm\circ S$, it follows that

$$j_q^{\pm}\circ S_{\mathcal{O}_q}=S^{-1}\circ j_q^{\pm}.\tag{2.69}$$

Also, for every $\alpha_-\in\mathcal{O}_q(B_-)$, we have

$$\begin{aligned}\bigl\langle\alpha_-,\Phi^+(i_-(Y))\bigr\rangle&=\langle i_-(Y)\otimes\alpha_-,R\rangle=\tau\bigl(i_+^{-1}(\alpha_-),Y\bigr)\\&=\pi_q^-(j_q^-(S_{\mathcal{O}_q}(\alpha_-)),Y)=\langle\alpha_-,S(Y)\rangle,\end{aligned}$$

where the first equality is by definition of Φ^+ (see (2.5)), the second is (2.66), the third follows from (2.69), and the last from the definition of j_q^-. Similarly, for every $\alpha_+ \in \mathcal{O}_q(B_+)$, we have

$$\begin{aligned}\langle \alpha_+, \Phi^-(i_+(X))\rangle &= \langle i_+(X)\otimes\alpha_+, R^-\rangle = \big\langle \alpha_+\otimes S_{\mathcal{O}_q}^{-1}\circ i_+(X), R\big\rangle = \langle \alpha_+\otimes i_+(S(X)), R\rangle\\ &= \tau\big(S(X), i_-^{-1}(\alpha_+)\big) = \pi_q^+\big(S\big(i_-^{-1}(\alpha_+)\big), S(X)\big)\\ &= \pi_q^+\big(j_q^+(\alpha_+), S(X)\big) = \langle \alpha_+, S(X)\rangle.\end{aligned}$$

These computations imply $\Phi^{\pm} = S\circ i_{\mp}^{-1} = j_q^{\pm}$, and the result follows by taking integral forms. ■

2.4 Integral form and specialization of Φ_n

Recall the isomorphism of U_q-module algebras $\Phi_1\colon \mathcal{L}_{0,1}\to U_q^{\mathrm{lf}}$, and that $U_A^{\mathrm{lf}} = U_A\cap U_q^{\mathrm{lf}}$. We have:

Corollary 2.22. *The map Φ_1 affords an embedding of U_A^{res}-module algebras $\Phi_1\colon \mathcal{L}_{0,1}^A\to U_A^{\mathrm{lf}}$.*

Proof. The only thing to be proved is that $\Phi_1(\mathcal{O}_A)\subset U_A^{\mathrm{lf}}$, since $\mathcal{L}_{0,1}^A = \mathcal{O}_A$ as A-module. But Lemma 2.21 and (2.12) imply $\Phi_1 = m\circ\big(\mathrm{id}\otimes S^{-1}\big)\circ\Phi$, and Φ maps $\mathcal{O}_A$ into $U_A(\mathfrak{b}_-)^{\mathrm{cop}}\otimes U_A(\mathfrak{b}_+)^{\mathrm{cop}}$ by Theorem 2.20. The conclusion follows. ■

Let us denote

$$d = \psi_{-\rho}\in\mathcal{L}_{0,1}^A.$$

(The linear forms $\psi_{-\lambda}$ have been introduced before Theorem 2.20.) When $\mathfrak{g} = \mathfrak{sl}_2$ the element d is one of the "standard" generators of $\mathcal{L}_{0,1}(\mathfrak{sl}_2)$ (see (4.5) below). In this case we have shown in [18, Lemma 5.7] that $\mathcal{L}_{0,1}^A$ has a well-defined localization $\mathcal{L}_{0,1}^A\big[d^{-1}\big]$, and that $\Phi_1\colon \mathcal{L}_{0,1}^A\big[d^{-1}\big]\to U_A^{\mathrm{ad}} = T_{2-}^{-1}U_A^{\mathrm{lf}}$ is an isomorphism of algebras. A generalization of these facts to any $\mathfrak{g}$ is provided by the following statement. As usual $\ell = K_{2\rho}$, the pivotal element.

Corollary 2.23.

(1) *The set $\{d^n\}_{n\in\mathbb{N}}$ is a left and right multiplicative Ore set in $\mathcal{L}_{0,1}^A$. We can therefore define the localization $\mathcal{L}_{0,1}^A\big[d^{-1}\big]$.*

(2) *Φ_1 extends to an embedding of U_A^{res}-module algebras $\Phi_1\colon \mathcal{L}_{0,1}^A\big[d^{-1}\big]\to U_A^{\mathrm{lf}}[\ell]$, and $U_A^{\mathrm{lf}}[\ell] = T_{2-}^{-1}U_A^{\mathrm{lf}}$.*

Proof. (1) Because $\mathcal{L}_{0,1}^A$ has no nontrivial zero divisors, d is a regular element. We have to show that for all $x\in\mathcal{L}_{0,1}^A$ there exists elements $y, y'\in\mathcal{L}_{0,1}^A$ and $d', d''\in\{d^n\}_{n\in\mathbb{N}}$ such that $xd' = dy$ and $d''x = y'd$. In fact, $d' = d'' = d$ in the present situation. Indeed by (2.13), we have $\Phi_1(x)\Phi_1(d) = \Phi_1(x)K_{-2\rho} = K_{-2\rho}\mathrm{ad}^r(K_{2\rho})(\Phi_1(x))$, and $\mathrm{ad}^r(K_{2\rho})(\Phi_1(x)) = \Phi_1(\mathrm{coad}^r(K_{2\rho})(x))$. Therefore, the left Ore condition is satisfied with $y = \mathrm{coad}^r(K_{2\rho})(x)$. Similarly, one finds y'.

(2) The first claim follows immediately from Corollary 2.22 and $\Phi_1(d) = \ell^{-1}$, which is a regular element of U_A. For the second claim, since $K_{-2\rho} = \prod_{j=1}^m L_j^{-2}$, localizing in d we obtain

$$L_j^2 = \prod_{k\neq j} L_k^{-2}\Phi_1\big(d^{-1}\big) = \Phi_1\left(\prod_{k\neq j}\psi_{-\varpi_k}d^{-1}\right)\in\Phi_1\big(\mathcal{L}_{0,1}^A\big[d^{-1}\big]\big).$$

Therefore, $T_{2-}^{-1}\subset\Phi_1\big(\mathcal{L}_{0,1}^A\big[d^{-1}\big]\big)$, which implies the assertion (2). ■

We expect that the inclusion $\Phi_1(\mathcal{O}_A)\subset U_A^{\mathrm{lf}}$ is an equality, but have no proof yet. However, recall Joseph–Letzter's Theorem 2.1 (1) and (2).

Proposition 2.24. *We have*

$$U_A = T_{2-}^{-1}U_A^{\mathrm{lf}}[T/T_2] = \Phi_1\big(\mathcal{L}_{0,1}^A\big[d^{-1}\big]\big)[T/T_2],$$

and therefore $\Phi_1\colon \mathcal{L}_{0,1}^A\big[d^{-1}\big] \to T_{2-}^{-1}U_A^{\mathrm{lf}}$ *is an isomorphism. Moreover,*

$$\Phi_1(\mathcal{O}_A) = \bigoplus_{\lambda\in 2P_+} \mathrm{ad}^r\big(U_A^{\mathrm{res}}\big)(K_{-\lambda}).$$

Proof. The inclusions $T \subset U_A$, $U_A^{\mathrm{lf}} \subset U_A$ and $\Phi_1\big(\mathcal{L}_{0,1}^A\big[d^{-1}\big]\big) \subset T_{2-}^{-1}U_A^{\mathrm{lf}}$ imply

$$\Phi_1\big(\mathcal{L}_{0,1}^A\big[d^{-1}\big]\big)[T/T_2] \subset T_{2-}^{-1}U_A^{\mathrm{lf}}[T/T_2] \subset U_A.$$

For the inverse inclusion, it is enough to show that any PBW basis vector of U_A lies in $\Phi_1\big(\mathcal{L}_{0,1}^A\big[d^{-1}\big]\big)[T/T_2]$. This will follow at once if this is true of all root vectors $\bar{E}_{\beta_k}$, $\bar{F}_{\beta_k}$. Let us show this explicitly for the simple root vectors $\bar{E}_i$ and $\bar{F}_i$. For every positive root α, define elements $\psi_{-\lambda}^{\alpha}, \psi_{-\lambda}^{-\alpha} \in \mathcal{O}_A$ by the formulas

$$\langle\psi_{-\lambda}^{\alpha}, x\rangle = v^*(xE_\alpha v), \qquad \langle\psi_{-\lambda}^{-\alpha}, x\rangle = v^*(F_\alpha x v),$$

where $x \in \Gamma$. It is shown in [41, Lemma 4.5] that

$$\Phi(\psi_{-\lambda}) = K_{-\lambda}\otimes K_\lambda, \qquad \Phi\big(\psi_{-\varpi_j}^{\alpha_i}\big) = -\delta_{i,j}q_iL_i^{-1}\otimes L_iK_i^{-1}\bar{E}_i,$$
$$\Phi\big(\psi_{-\varpi_j}^{-\alpha_i}\big) = \delta_{i,j}q_i^{-1}\bar{F}_iK_iL_i^{-1}\otimes L_i.$$

$\big($Note that the generators denoted by E_i and F_i in [41] are respectively $K_i^{-1}E_i$ and F_iK_i in our notations, which explains the factors q_i, q_i^{-1} in the formulas below; also κ in (2.67) maps $\bar{E}_i$, $\bar{F}_i$ to $-\bar{F}_i$, $-\bar{E}_i$, whence the sign for the expression of $\Phi\big(\psi_{-\varpi_j}^{\alpha_i}\big)$.$\big)$ Since $\Phi_1 = m\circ\big(\mathrm{id}\otimes S^{-1}\big)\circ\Phi$, we have

$$\Phi_1(\psi_{-\lambda}) = K_{-2\lambda}, \qquad \Phi_1\big(\psi_{-\varpi_j}^{\alpha_i}\big) = \delta_{i,j}L_i^{-2}\bar{E}_i, \qquad \Phi_1\big(\psi_{-\varpi_j}^{-\alpha_i}\big) = \delta_{i,j}q_i^{-1}\bar{F}_iK_iL_i^{-2}. \tag{2.70}$$

Therefore,

$$\bar{E}_i, \bar{F}_i, L_i^{\pm1} \in T_{2-}^{-1}\Phi_1\big(\mathcal{L}_{0,1}^A\big)[T/T_2] = \Phi_1\big(\mathcal{L}_{0,1}^A\big[d^{-1}\big]\big)[T/T_2].$$

These elements do not generate U_A; it is necessary to consider general root vectors. By the stability of $U_A(G^*)$ under $\mathcal{B}(\mathfrak{g})$ and the isomorphism $\mathcal{O}_A\big[\psi_{-\rho}^{-1}\big] \to U_A(G^*)$ of Theorem 2.20 (3), for every positive root β_k, we have $1\otimes K_{\beta_k}^{-1}\bar{E}_{\beta_k}$, $\bar{F}_{\beta_k}K_{\beta_k}\otimes 1 \in \Phi\big(\mathcal{O}_A\big[\psi_{-\rho}^{-1}\big]\big) = \Phi\big(\mathcal{L}_{0,1}^A\big[d^{-1}\big]\big)$. Therefore, $\bar{F}_{\beta_k}K_{\beta_k}, S^{-1}\big(\bar{E}_{\beta_k}\big)K_{\beta_k} \in \Phi_1\big(\mathcal{L}_{0,1}^A\big[d^{-1}\big]\big)$, and $\bar{F}_{\beta_k}, S^{-1}\big(\bar{E}_{\beta_k}\big) \in \Phi_1\big(\mathcal{L}_{0,1}^A\big[d^{-1}\big]\big)[T/T_2]$. The sets $S^{-1}\big(\bar{E}_{\beta_k}\big)U_A(\mathfrak{h})$ generate the subalgebra $U_A(\mathfrak{b}_+)$ of U_A (in fact, let us quote that a formula of $S^{-1}(\bar{E}_{\beta_k})$ is given in [107]). From the triangular decomposition $U_A = U_A(\mathfrak{n}_-)U_A(\mathfrak{h})U_A(\mathfrak{n}_+)$, the inclusion $U_A \subset \Phi_1\big(\mathcal{L}_{0,1}^A\big[d^{-1}\big]\big)[T/T_2]$ follows, whence the equality too. In particular, U_A is a free $\Phi_1\big(\mathcal{L}_{0,1}^A\big[d^{-1}\big]\big)$-module with a basis formed by representatives of the cosets in T/T_2. By the uniqueness of this free decomposition, we find $\Phi_1\big(\mathcal{L}_{0,1}^A\big[d^{-1}\big]\big) = T_{2-}^{-1}U_A^{\mathrm{lf}}$. Therefore, Φ_1 in Corollary 2.23 (2) is surjective.

For the third claim, recall the isomorphism $\Phi_1\colon C(-w_0(\mu)) \to \mathrm{ad}^r(U_q)(K_{-2\mu})$ (see (2.14)), and that $\psi_{-\mu}$ is the matrix coefficient dual to the vector ${}^\omega v_{-\mu}\otimes v_{-\mu} \in \mathrm{End}_A(V_{-w_0(\mu)})$. This vector is cyclic by (2.32), so by equivariance $\Phi_1\colon {}_AC(-w_0(\mu)) \to \mathrm{ad}^r\big(U_A^{\mathrm{res}}\big)(K_{-2\mu})$ is an isomorphism of U_A^{res}-modules. The second claim follows from this and (2.60) for $n=1$. ■

Recall from (2.20) the isomorphisms of U_q-module algebras $\Phi_n\colon \mathcal{L}_{0,n} \to \big(U_q^{\otimes n}\big)^{\mathrm{lf}}$ and of algebras $\Phi_n\colon \mathcal{M}_{0,n} \to (U_q^{\otimes n})^{U_q}$, and from (2.27) the notations for specializations. Corollary 2.22 can be extended to Φ_n as follows:

Corollary 2.25. *The map Φ_n affords embeddings of module algebras $\Phi_n\colon \mathcal{L}_{0,n}^A \to \big(U_A^{\otimes n}\big)^{\rm lf}$ and $\Phi_n\colon \mathcal{L}_{0,n}^{\epsilon'} \to \big(U^{\otimes n}\big)^{\rm lf}_{\epsilon'}$, $q=\epsilon'\in\mathbb{C}^\times$.*

Proof. For the first claim, the only thing to prove is the inclusion $\Phi_n\big(\mathcal{L}_{0,n}^A\big)\subset U_A^{\otimes n}$. It follows from Corollary 2.22 and the expression of Φ_n in terms of Φ_1 and R-matrices (in particular, the fact that they preserve integrality, see [18, Lemma 6.10]). For the specialization at $q=\epsilon'\in\mathbb{C}^\times$, we have to justify that Φ_n is injective. One uses the fact, to be developed in Theorem 2.29 below, that $\Phi\colon \mathcal{O}_\epsilon \to U_\epsilon(G^*)$ is an embedding. The algebra $U_\epsilon(G^*)$ has the basis elements (2.68), and the map $m\circ\big(\mathrm{id}\otimes S^{-1}\big)$ sends this basis to a free family of U_ϵ. Therefore, $\Phi_1\colon \mathcal{L}_{0,1}^\epsilon \to U_\epsilon$ is injective. Since Φ_n differs from $\Phi_1^{\otimes n}$ by a linear isomorphism (induced by the conjugation action of R-matrices on the components ${}_A\dot{C}([\lambda])$ of $\mathcal{L}_{0,n}^A$ in (2.60), see [18, equation (6.10)]), $\Phi_n\colon \mathcal{L}_{0,n}^\epsilon \to U_\epsilon^{\otimes n}$ is an embedding as well. ∎

Remark 2.26.

(1) It is a natural problem to determine the image of Φ_n. One may expect that it would be $\big(T_{2-}^{-1}U_A^{\rm lf}\big)^{\otimes n}$, because this is true for $n=1$, as well as for any n in the $\mathfrak{sl}_2$ case, as shown in [18]. Unfortunately, this is not so. This comes from the fact, e.g., for $n=2$, that the matrix elements of $R_{02}R_{01}R'_{01}R_{02}^{-1}$ do not belong to $\big(T_{2-}^{-1}U_A^{\rm lf}\big)^{\otimes 2}$ as can be shown by an explicit computation in the $\mathfrak{sl}(3)$ case.

(2) In the case of $\mathfrak{g}=\mathfrak{sl}_2$, we defined in [18] an algebra ${}_{\rm loc}\mathcal{L}_{0,n}^A$ generalizing $\mathcal{L}_{0,1}^A\big[d^{-1}\big]$ above, containing $\mathcal{L}_{0,n}^A$ as a subalgebra, and such that Φ_n extends to ${}_{\rm loc}\mathcal{L}_{0,n}^A$ and yields an isomorphism $\Phi_n\colon {}_{\rm loc}\mathcal{L}_{0,n}^A \to U_A^{\rm ad}(\mathfrak{sl}_2)^{\otimes n}$. The definition of ${}_{\rm loc}\mathcal{L}_{0,n}^A$ involves elements $\xi^{(i)}\in\mathcal{L}_{0,n}^A$ $(i=1,\dots,n)$ such that $\Phi_n\big(\xi^{(i)}\big)=\big(K^{-1}\big)^{(i)}\cdots\big(K^{-1}\big)^{(n)}$. It may be of interest to study a similar extension of Φ_n for general $\mathfrak{g}$.

2.5 Structure theorems for U_ϵ and $\mathcal{O}_\epsilon$

As usual, we denote by ϵ a primitive l-th root of unity, where l is odd, and coprime to 3 if $\mathfrak{g}$ has G_2-components.

Recall the subgroups T_G, $U_\pm$ and $B_\pm$ of G. Let $G^0=B_+B_-$ (the *big cell* of G), and define the subgroup

$$G^*=\big\{\big(u_+t,u_-t^{-1}\big),\ t\in T_G,\ u_\pm\in U_\pm\big\}\subset B_+^{\rm op}\times B_-^{\rm op},$$

where $B_\pm^{\rm op}$ is the group $B_\pm$ with opposite multiplication. The group G^* can be naturally identified with the Poisson–Lie dual of G with its standard structure.

Recall also that there is an injective homomorphism $\gamma_q^{-1}\circ h_q\colon \mathcal{Z}(U_q)\to U_q(\mathfrak{h})$, defined by means of the quantum Harish-Chandra homomorphism (see, e.g., [35, Section 9.1.C], or [104, Section 3.13]). The image of $\gamma_q^{-1}\circ h_q$ is the set $U_q(\mathfrak{h})^{\tilde W}$ of invariant elements under $\tilde W$, the subgroup of $W\ltimes P_2^*$ generated by the conjugates $\sigma W\sigma$ of W by elements $\sigma\in P_2^*$. Here, P_2^* is the group of homomorphisms $P\to\mathbb{Z}/2\mathbb{Z}$, and the semidirect product $W\ltimes P_2^*$ acts on $U_q(\mathfrak{h})$ by the standard action of the Weyl group W, and by the action of P_2^* given by $\sigma\cdot K_\lambda:=\sigma(\lambda)K_\lambda$.

Consider the inverse map $h_q^{-1}\circ\gamma_q\colon U_q(\mathfrak{h})^{\tilde W}\to\mathcal{Z}(U_q)$. The elements of the domain and target, when expanded in the PBW basis, have coefficients in $\mathbb{C}(q)$. It was shown in [42, Section 21.1] that if an element of $U_q(\mathfrak{h})^{\tilde W}$ has no coefficient with a pole at $q=\epsilon$, then its image by $h_q^{-1}\circ\gamma_q$ has no coefficient with a pole at $q=\epsilon$. We therefore have a well-defined injection

$$U_\epsilon(\mathfrak{h})^{\tilde W}\to\mathcal{Z}(U_\epsilon).$$

We denote its image by $\mathcal{Z}_1(U_\epsilon)$. For instance, when $U_\epsilon=U_\epsilon(\mathfrak{sl}_2)$, $\mathcal{Z}_1(U_\epsilon)$ is the polynomial algebra generated by the Casimir element $\Omega=\big(\epsilon-\epsilon^{-1}\big)^2FE+\epsilon K+\epsilon^{-1}K^{-1}$.

Denote by $\mathcal{Z}_0(U_\epsilon) \subset U_\epsilon$ the smallest subalgebra containing the elements E_i^l, F_i^l, K_α^l, for $i \in \{1,\dots,m\}$, $\alpha \in P$, and stable under $\mathcal{B}(\mathfrak{g})$; it is also the subalgebra generated by $E_{\beta_k}^l$, $F_{\beta_k}^l$, $L_i^{\pm l}$, for $k \in \{1,\dots,N\}$ and $i \in \{1,\dots,m\}$ [42, Section 18]. We will denote by $\mathcal{Z}_0(U_\epsilon(\mathfrak{n}_-))$, $\mathcal{Z}_0(U_\epsilon(\mathfrak{h}))$ and $\mathcal{Z}_0(U_\epsilon(\mathfrak{n}_+))$ the subalgebras of $\mathcal{Z}_0(U_\epsilon)$ generated by the elements $F_{\beta_k}^l$, K_λ^l ($\lambda \in P$), and $E_{\beta_k}^l$, respectively. In [39, Sections 1.8, 3.3 and 3.8] and [42, Theorem 14.1 and Sections 20–21], the following results are proved:

Theorem 2.27.

(1) *U_ϵ has no nontrivial zero divisors, $\mathcal{Z}_0(U_\epsilon)$ is a central Hopf subalgebra of U_ϵ, and U_ϵ is a free $\mathcal{Z}_0(U_\epsilon)$-module of rank $l^{\dim\mathfrak{g}}$. Moreover, the classical fraction algebra $Q(U_\epsilon) = Q(\mathcal{Z}(U_\epsilon)) \bigotimes_{\mathcal{Z}(U_\epsilon)} U_\epsilon$ is a central simple algebra of PI degree l^N, and U_ϵ is a maximal order of $Q(U_\epsilon)$.*

(2) Maxspec$(\mathcal{Z}_0(U_\epsilon))$ *is a group isomorphic to G^* above, and the multiplication map yields an isomorphism $\mathcal{Z}_0(U_\epsilon) \bigotimes_{\mathcal{Z}_0(U_\epsilon)\cap\mathcal{Z}_1(U_\epsilon)} \mathcal{Z}_1(U_\epsilon) \to \mathcal{Z}(U_\epsilon)$.*

By this theorem, the dimension of $Q(U_\epsilon)$ over its center $Q(\mathcal{Z}(U_\epsilon))$ is l^{2N}, and its dimension over $Q(\mathcal{Z}_0(U_\epsilon))$ is $l^{\dim\mathfrak{g}} = l^{m+2N}$. Therefore, the field $Q(\mathcal{Z}(U_\epsilon))$ is an extension of $Q(\mathcal{Z}_0(U_\epsilon))$ of degree l^m.

Note that, because $\mathcal{Z}_0(U_\epsilon)$ is an affine and commutative algebra, the maximal spectrum Maxspec$(\mathcal{Z}_0(U_\epsilon))$, viewed as the set of characters of $\mathcal{Z}_0(U_\epsilon)$, acquires by duality a structure of affine algebraic group. Thus, the first claim of (2) in the theorem means precisely that this group can be identified with G^*. See, for instance, [18, Section 7.2.1] for an explicit description in the $\mathfrak{sl}_2$ case.

In addition, Maxspec$(\mathcal{Z}_0(U_\epsilon))$ and G^* have natural Poisson structures which correspond one to the other under the isomorphism of (2), and we have the following identifications (see [42, Section 21.2]). The dual isomorphism $\mathcal{O}(G^*)\to\mathcal{Z}_0(U_\epsilon)$ identifies $\mathcal{O}(T_G)$ with $\mathcal{Z}_0(U_\epsilon)\cap U_\epsilon(\mathfrak{h}) = \mathbb{C}[lP]$, where as usual $U_\epsilon(\mathfrak{h}) = U_A(\mathfrak{h}) \bigotimes_A \mathbb{C}_\epsilon$. Therefore, we can identify $\mathbb{C}[P]$ with $\mathcal{O}(\tilde{T}_G)$, the coordinate ring of the l^m-fold covering space $\tilde{T}_G \to T_G$. The quantum Harish-Chandra isomorphism identifies $\mathcal{Z}_1(U_\epsilon)$ with $\mathbb{C}[2P]^W \cong \mathcal{O}(\tilde{T}_G/(2))^W$, where we denote by (2) the subgroup of 2-torsion elements in $\tilde{T}_G$. Consider the map

$$\sigma\colon\ B_+ \times B_- \longrightarrow G^0, \qquad (b_+, b_-) \longmapsto b_+ b_-^{-1}.$$

The restriction of σ to G^* is an unramified covering map of degree 2^m. Composing $\sigma\colon G^* \to G^0$ with the quotient map under conjugation, $G^0 \hookrightarrow G \to G//G$, we get dually an embedding of $\mathcal{O}(G//G) = \mathcal{O}(G)^G$ in $\mathcal{O}(G^*)$. Collecting these observations, we see that the isomorphism of Theorem 2.27 (2) affords identifications

$$\mathcal{Z}_0(U_\epsilon) \cap \mathcal{Z}_1(U_\epsilon) \cong \mathcal{O}(G)^G$$

as a subalgebra of $\mathcal{Z}_0(U_\epsilon) \cong \mathcal{O}(G^*)$, and

$$\mathcal{Z}_0(U_\epsilon) \cap \mathcal{Z}_1(U_\epsilon) = \mathbb{C}[2lP]^W \cong \mathcal{O}\big(\tilde{T}_G/(2l)\big)^W \cong \mathcal{O}(T_G/(2))^W$$

as a subalgebra of $\mathcal{Z}_1(U_\epsilon) \cong \mathcal{O}\big(\tilde{T}_G/(2)\big)^W$.

We will use the following obvious though crucial fact. Note that U_A^{ad} is naturally a subalgebra of U_A^{res}, and therefore acts on $U_\epsilon^{\mathrm{res}}$-modules. Denote by $\mathcal{Z}_0\big(U_A^{\mathrm{ad}}\big) \subset U_A^{\mathrm{ad}}$ the subalgebra generated by the elements $\bar{E}_{\beta_k}^l$, $\bar{F}_{\beta_k}^l$, $K_i^{\pm l}$, for $k \in \{1,\dots,N\}$ and $i \in \{1,\dots,m\}$.

Lemma 2.28. *For every U_A^{res}-module V of type 1, the action of $\mathcal{Z}_0\big(U_A^{\mathrm{ad}}\big)$ on the specialization $V_\epsilon := V \bigotimes_A \mathbb{C}_\epsilon$ is trivial.*

Proof. This comes from $E_i^l = [l]_{q_i}! E_i^{(l)}$, $F_i^l = [l]_{q_i}! F_i^{(l)}$ and the fact that K_i acts on V by powers of q_i. Specializing to $q = \epsilon$ ends the proof. ∎

A result similar to Theorem 2.27 holds true for $\mathcal{O}_\epsilon$. Namely, take the specializations at $q = \epsilon$ in Theorem 2.20. Denote by $\mathcal{Z}_0(U_\epsilon(G^*))$ the subalgebra of $U_\epsilon(G^*)$ generated by the elements ($k \in \{1, \dots, N\}, i \in \{1, \dots, m\}$)

$$1 \otimes K_{-l\beta_k} E_{\beta_k}^l, \qquad F_{\beta_k}^l K_{l\beta_k} \otimes 1, \qquad L_i^{\pm l} \otimes L_i^{\mp l}.$$

It is a central Hopf subalgebra. Recall that the coordinate ring $\mathcal{O}(G)$ can be identified as a Hopf algebra with $U(\mathfrak{g})^\circ$, where as usual $U(\mathfrak{g})^\circ$ denotes the restricted dual of the enveloping algebra $U(\mathfrak{g})$ over $\mathbb{C}$. In [41, Section 6], De Concini–Lyubashenko introduced an epimorphism of Hopf algebras $\eta\colon \Gamma_\epsilon \to U(\mathfrak{g})$ (essentially a version of Lusztig's "Frobenius" epimorphism in [82]), defined by

$$\eta\big(E_i^{(p)}\big) = \begin{cases} \dfrac{e_i^{p/l}}{(p/l)!} & \text{if } l \text{ divides } p, \\ 0 & \text{otherwise}, \end{cases} \qquad \eta\big(F_i^{(p)}\big) = \begin{cases} \dfrac{f_i^{p/l}}{(p/l)!} & \text{if } l \text{ divides } p, \\ 0 & \text{otherwise}, \end{cases}$$
$$\eta(K_i) = 1, \qquad \eta((K_i; p)_{q_i}) = \begin{cases} \dfrac{h_i(h_i - 1) \cdots (h_i - (p/l) + 1)}{(p/l)!} & \text{if } l \text{ divides } p, \\ 0 & \text{otherwise}, \end{cases} \tag{2.71}$$

where $p \in \mathbb{N}$, and e_i, f_i and h_i, $i \in \{1, \dots, m\}$, denote the standard generators of $U(\mathfrak{g})$. The kernel of η is generated by the elements E_i, F_i, $K_i - 1$, and $(K_i; p)_{q_i}$ where l does not divide p. Put

$$\mathcal{Z}_0(\mathcal{O}_\epsilon) := \eta^*(\mathcal{O}(G)), \tag{2.72}$$

where $\eta^*\colon U(\mathfrak{g})^\circ \to \Gamma_\epsilon^\circ$ is the monomorphism dual to η. Let us define special matrix coefficients, analogous to those introduced in Theorem 2.20. Denote by v_{ϖ_i} and $v_{w_0(\varpi_i)}$ a highest weight vector and a lowest weight vector of the Γ-module ${}_AV_{\varpi_i}$. Denote also by $v^*_{w_0(\varpi_i)}$ and $v^*_{\varpi_i}$ a highest and lowest weight vector of the dual module Γ-module ${}_AV^*_{\varpi_i} \cong {}_AV_{-w_0(\varpi_i)}$. Define the matrix coefficients $b_{\varpi_i}, c_{\varpi_i} \in \mathcal{O}_A$ by

$$b_{\varpi_i}(x) = v^*_{\varpi_i}(x v_{w_0(\varpi_i)}), \qquad c_{\varpi_i}(x) = v^*_{w_0(\varpi_i)}(x v_{\varpi_i})$$

for all $x \in \Gamma$. We consider them as elements of $\mathcal{O}_\epsilon$. Denote by $\mathcal{Z}_1(\mathcal{O}_\epsilon)$ the subalgebra of $\mathcal{O}_\epsilon$ generated by the elements $b_{\varpi_i}^k c_{\varpi_i}^{l-k}$ for $1 \leq i \leq m$ and $0 \leq k \leq l$.

Theorem 2.29.

(1) *$\mathcal{Z}_0(\mathcal{O}_\epsilon)$ is a central Hopf subalgebra of $\mathcal{O}_\epsilon \subset \Gamma_\epsilon^\circ$, and $Q(\mathcal{Z}(\mathcal{O}_\epsilon))$ is an extension of $Q(\mathcal{Z}_0(\mathcal{O}_\epsilon))$ of degree l^m.*

(2) *$\psi_{-l\rho} \in \mathcal{Z}_0(\mathcal{O}_\epsilon)$, and $\mathcal{Z}_0(\mathcal{O}_\epsilon)$ is generated by matrix coefficients of irreducible Γ-modules of highest weight $l\lambda$, $\lambda \in P_+$. Moreover, the multiplication map yields an isomorphism*

$$\mathcal{Z}_0(\mathcal{O}_\epsilon) \bigotimes_{\mathcal{Z}_0(\mathcal{O}_\epsilon) \cap \mathcal{Z}_1(\mathcal{O}_\epsilon)} \mathcal{Z}_1(\mathcal{O}_\epsilon) \to \mathcal{Z}(\mathcal{O}_\epsilon),$$

and the map Φ in Theorem 2.20 affords an algebra embedding $\mathcal{Z}_0(\mathcal{O}_\epsilon) \to \mathcal{Z}_0(U_\epsilon(G^*))$ and algebra isomorphisms $\mathcal{Z}_0(\mathcal{O}_\epsilon)\big[\psi_{-l\rho}^{-1}\big] \to \mathcal{Z}_0(U_\epsilon(G^*))$ and $\mathcal{O}_\epsilon\big[\psi_{-l\rho}^{-1}\big] \to U_\epsilon(G^*)$.

(3) *$\mathcal{O}_\epsilon$ has no nontrivial zero divisors, and it is a free $\mathcal{Z}_0(\mathcal{O}_\epsilon)$-module of rank $l^{\dim \mathfrak{g}}$. Moreover, the classical fraction algebra $Q(\mathcal{O}_\epsilon) = Q(\mathcal{Z}(\mathcal{O}_\epsilon)) \bigotimes_{\mathcal{Z}(\mathcal{O}_\epsilon)} \mathcal{O}_\epsilon$ is a central simple algebra of PI degree l^N, and $\mathcal{O}_\epsilon$ is a maximal order of $Q(\mathcal{O}_\epsilon)$.*

For the proof, see [41]: Proposition 6.4 for the first claim of (1) (where $\mathcal{Z}_0(\mathcal{O}_\epsilon)$ and $\mathcal{Z}_0(U_\epsilon(G^*))$ are denoted F_0 and A_0 respectively), the appendix of Enriquez and [50] for the second claim of (1) and (2), Propositions 6.4 and 6.5 for the other claims of (2), Theorem 7.2 (where $\mathcal{O}_\epsilon$ is shown to be projective over $\mathcal{Z}_0(\mathcal{O}_\epsilon)$) and [28] (which provides the additional K-theoretic arguments to deduce that $\mathcal{O}_\epsilon$ is free), or [6, Remark 2.18 (b)], for the second claim of (3), and Corollary 7.3 and Theorem 7.4 for the third claim. The fact that $\mathcal{O}_\epsilon$ has no nontrivial zero divisors follows from the embedding $\mathcal{O}_\epsilon \to U_\epsilon(G^*)$ via Φ.

As above for U_ϵ, it follows directly from (3) that $Q(\mathcal{Z}(\mathcal{O}_\epsilon))$ has degree l^m over $Q(\mathcal{Z}_0(\mathcal{O}_\epsilon))$. For a more complete description of $\mathcal{Z}(\mathcal{O}_\epsilon)$ we refer to [50] and Enriquez' appendix in [41], as well as [27].

We do not know a basis of $\mathcal{O}_\epsilon$ over $\mathcal{Z}_0(\mathcal{O}_\epsilon)$ for general G, but see [38] for the case of SL_2. We will recall the known results in this case of SL_2 before Lemma 4.5.

Finally, there is a natural action of the braid group $\mathcal{B}(\mathfrak{g})$ on $\mathcal{O}_\epsilon$, that we will use. Namely, let $n_i \in N(T_G)$ be a representative of the reflection $s_i \in W = N(T_G)/T_G$ associated to the simple root α_i. In [98, 102], Soibelman–Vaksman introduced functionals $t_i \colon \mathcal{O}_q \to \mathbb{C}(q)$ which quantize the elements n_i. They correspond dually to generators of the quantum Weyl group of $\mathfrak{g}$; in the appendix, we recall their main properties, in particular, they map $\mathcal{O}_A$ to A (see also [35, Section 8.2], and [41, 69, 70, 81, 102]). Denote by $\lhd$ the natural right action of functionals on $\mathcal{O}_A$, namely (using Sweedler's notation)

$$\alpha \lhd h = \sum_{(\alpha)} h(\alpha_{(1)})\alpha_{(2)}$$

for every $\alpha \in \mathcal{O}_A$ and $h \in \mathcal{O}_A \to A$. Let us identify $\mathcal{Z}_0(\mathcal{O}_\epsilon)$ with $\mathcal{O}(G)$ by means of (2.72). We have [41, Proposition 7.1]:

Proposition 2.30. *The maps $\lhd t_i$ on $\mathcal{O}_\epsilon$ preserve $\mathcal{Z}_0(\mathcal{O}_\epsilon)$, and satisfy $(f \lhd t_i)(a) = f(n_i a)$ and $(f \star \alpha) \lhd t_i = (f \lhd t_i)(\alpha \lhd t_i)$ for every $f \in \mathcal{Z}_0(\mathcal{O}_\epsilon)$, $a \in G$, $\alpha \in \mathcal{O}_\epsilon$.*

We provide an alternative, non computational, proof of this result in Appendix C.

3 Noetherianity and finiteness

In this section, we prove Theorem 1.1. Recall that by Noetherian we mean right and left Noetherian. We begin with

Theorem 3.1. *The algebras $\mathcal{L}_{0,n}$, $\mathcal{L}_{0,n}^A$ and $\mathcal{L}_{0,n}^{\epsilon'}$, $\epsilon' \in \mathbb{C}^\times$, are Noetherian.*

By Proposition 2.18, each of the algebras in this theorem is finitely generated.

Theorem 3.1 for $\mathcal{L}_{0,1}$ and any $\mathfrak{g}$ follows immediately from Joseph–Letzter's Theorem 2.1, claim (3), by identifying $\mathcal{L}_{0,1}$ with U_q^{lf} via Φ_1. The method of proof uses filtration arguments. An alternative proof in the case of $\mathfrak{sl}(n)$, which works also for $\mathcal{L}_{0,1}^A$, was obtained by Domokos–Lenagan in [47], by exhibiting special sequences of generators of $\mathcal{L}_{0,1}^A$ satisfying *polynormal* relations, as we define now.

Definition 3.2 (see [104, Proposition 3.133])**.** Let R be a Noetherian Abelian ring, and B a finitely generated R-algebra with product $\circ$. We call polynormal a set of relations between generators $u_1, \dots, u_M$ of B, of the form

$$u_i \circ u_j - q_{ij} u_j \circ u_i = \sum_{s=1}^{j-1} \sum_{t=1}^{M} \big(\alpha_{ij}^{st} u_s \circ u_t + \beta_{ij}^{st} u_t \circ u_s \big) \tag{3.1}$$

for all $1 \leq j < i \leq M$, where $\alpha_{ij}^{st}, \beta_{ij}^{st} \in R$, and the elements $q_{ij} \in R$ are invertible.

Note that this definition is more restrictive than the more standard one, e.g., in [26, Definition II.4.1]. If such a set of relations exists in B, then B can be endowed with an algebra filtration such that the associated graded algebra is a quotient of a skew-polynomial algebra [26, Proposition I.8.17]. By classical results, we have (see, e.g., [88, Theorems 1.2.9, 1.6.9 and Examples 1.6.11], or [104, Lemmas 3.130–3.131]):

Theorem 3.3. *If the algebra filtration is well founded, then B is a Noetherian ring.*

In [47], Theorem 3.1 is also proved for any $n \geq 1$ in the case of $\mathfrak{g} = \mathfrak{sl}_2$ by considering $\mathcal{L}_{0,n}^A(\mathfrak{sl}_2)$ as an iterated overring of $\mathcal{L}_{0,1}(\mathfrak{sl}_2)$.

The proof of Theorem 3.1 that we develop for any $\mathfrak{g}$ and $n \geq 1$ is also based on polynormal relations. In our proof, the generating set of $\mathcal{L}_{0,n}$ that we will consider is evident, as they are matrix coefficients in the modules V_{ϖ_k}, $k \in \{1, \dots, m\}$; the task is then to exhibit a set of polynormal relations between them, that hold in a certain graded algebra associated to $\mathcal{L}_{0,n}$. Indeed, as explained above this will imply that the graded algebra is Noetherian, and that $\mathcal{L}_{0,n}$ is Noetherian as well. In the case of $\mathcal{L}_{0,n}^A$, the proof is formally similar, but it needs the use of canonical bases discussed in Section 2.2.2.

Proof of Theorem 3.1. First, we develop the proof for $\mathcal{L}_{0,n}$, and then for $\mathcal{L}_{0,n}^A$; the result for

$$\mathcal{L}_{0,n}^{\epsilon'} = \mathcal{L}_{0,n}^A / (q - \epsilon') \mathcal{L}_{0,n}^A$$

follows immediately by lifting ideals by the quotient map $\mathcal{L}_{0,n}^A \to \mathcal{L}_{0,n}^{\epsilon'}$.

We adapt the proof of Theorem 2.1 (3) given in [104, Theorem 3.137]. Let us begin by recalling these arguments. In doing this, let us stress that [104] takes on $\mathcal{O}_q$ and $\mathcal{L}_{0,1}$ the product opposite to ours, and below in (3.7) and (3.8) we respect their convention.

As usual, let $C(\mu)$ be the vector space generated by the matrix coefficients of V_μ, the simple U_q^{ad}-module of highest weight $\mu \in P_+$. Denote by $C(\mu)_\lambda \subset C(\mu)$ the subspace of weight λ for the left coregular action of $U_q(\mathfrak{h})$; so $\alpha \in C(\mu)_\lambda$ if $K_\nu \rhd \alpha = q^{(\nu,\lambda)} \alpha$, $\nu \in P$. Consider the semigroup

$$\Lambda = \{(\mu, \lambda) \in P_+ \times P,\, \lambda \text{ is a weight of } V_\mu\}.$$

Recall that the partial order $\preceq$ on P is defined by $\mu \preceq \mu'$ if and only if $\mu' - \mu \in D^{-1} Q_+$. Define $\preceq$ on Λ by: $(\mu, \lambda) \preceq (\mu', \lambda')$ if and only if $\mu' - \mu \in D^{-1} Q_+$ and $\lambda' - \lambda \in D^{-1} Q_+$. If $(\mu, \lambda) \preceq (\mu', \lambda')$ and $(\mu, \lambda) \neq (\mu', \lambda')$, we write $(\mu, \lambda) \prec (\mu', \lambda')$. Since $\mathcal{L}_{0,1}$ and $\mathcal{O}_q$ are isomorphic vector spaces, we have $\mathcal{L}_{0,1} = \bigoplus_{\mu \in P_+} C(\mu) = \bigoplus_{(\mu,\lambda) \in \Lambda} C(\mu)_\lambda$. Consider the family of subspaces

$$\mathcal{F}_2^{\mu,\lambda} := \bigoplus_{(\mu',\lambda') \preceq (\mu,\lambda)} C(\mu')_{\lambda'}, \qquad \mathcal{F}_2^{\prec \mu,\lambda} := \bigoplus_{(\mu',\lambda') \prec (\mu,\lambda)} C(\mu')_{\lambda'}, \qquad (\mu, \lambda) \in \Lambda.$$

We have

$$\mathcal{L}_{0,1} = \bigcup_{(\mu,\lambda) \in \Lambda} \mathcal{F}_2^{\mu,\lambda}. \tag{3.2}$$

Indeed, clearly

$$\mathcal{L}_{0,1} = \sum_{(\mu,\lambda)\in\Lambda} \mathcal{F}_2^{\mu,\lambda},$$

so (3.2) follows from the following fact: for every $(\mu,\lambda),(\mu',\lambda')\in\Lambda$, the element $(\mu'',\lambda''):=(\mu+\mu',\lambda+\lambda')$ is such that

$$\mathcal{F}_2^{\mu,\lambda}+\mathcal{F}_2^{\mu',\lambda'}\subset\mathcal{F}_2^{\mu'',\lambda''}.$$

Note that in general, since $Q_+\not\subseteq P_+$ (but $P_+\subset D^{-1}Q_+$), it is not true that there exists an element (μ'',λ'') satisfying such an inclusion if one replaces $\preceq$ with the standard "product" partial order $\leq$ on Λ, defined by $(\mu,\lambda)\leq(\mu',\lambda')$ if and only if $\mu'-\mu\in Q_+$ and $\lambda'-\lambda\in Q_+$. Note also that $\preceq$ is finer than $\leq$, in the sense that if $\mu\leq\mu'$, then $\mu\preceq\mu'$. Again, this would not be true if we had replaced $D^{-1}Q_+$ by P_+ in the definition of $\preceq$.

The family $\mathcal{F}_2:=\left\{\mathcal{F}_2^{\mu,\lambda}\right\}_{(\mu,\lambda)\in\Lambda}$ is a filtration of the vector space $\mathcal{L}_{0,1}$, which is clearly well founded (i.e., every subset of Λ contains a minimal element, or equivalently any decreasing infinite sequence of elements in Λ is eventually constant).

Consider the associated graded vector space $\mathrm{Gr}_{\mathcal{F}_2}(\mathcal{L}_{0,1}):=\bigoplus_{(\mu,\lambda)}\mathcal{F}_2^{\mu,\lambda}/\mathcal{F}_2^{\prec\mu,\lambda}$. By identifying an element $x\in C(\mu)_\lambda$ with its coset $\bar{x}\in\mathcal{F}_2^{\mu,\lambda}/\mathcal{F}_2^{\prec\mu,\lambda}$, we get an equality of vector spaces $\mathrm{Gr}_{\mathcal{F}_2}(\mathcal{L}_{0,1})=\bigoplus_{(\mu,\lambda)\in\Lambda}C(\mu)_\lambda$. Now, one has the following facts:

(i) Taking the product in $\mathcal{L}_{0,1}$, we have

$$\alpha\beta\in\mathcal{F}_2^{\mu_1+\mu_2,\lambda_1+\lambda_2}\qquad\text{for}\quad\alpha\in C(\mu_1)_{\lambda_1},\quad\beta\in C(\mu_2)_{\lambda_2}.\tag{3.3}$$

This follows from (2.7) and the fact that, for every $\nu\in P_+$ and every summand of the formula (2.9), denoting by $-r\in-Q_+$ the weight of the R-matrix component $R_{(2)}$ we have

$$\begin{aligned}K_\nu\rhd&\big((R_{(2')}S(R_{(2)})\rhd\alpha)\star(R_{(1')}\rhd\beta\lhd R_{(1)})\big)\\&=q^{(\nu,\lambda_1+\lambda_2-r)}(R_{(2')}S(R_{(2)})\rhd\alpha)\star(R_{(1')}\rhd\beta\lhd R_{(1)}).\end{aligned}$$

(Details of a similar computation are given below (3.12).) It follows from (3.3) that $\mathcal{F}_2$ is an algebra filtration of $\mathcal{L}_{0,1}$, and then $\mathrm{Gr}_{\mathcal{F}_2}(\mathcal{L}_{0,1})$ is a graded algebra.

(ii) Denote by $\alpha\circ\beta$ the product in $\mathrm{Gr}_{\mathcal{F}_2}(\mathcal{L}_{0,1})$ of $\alpha,\beta\in\mathcal{L}_{0,1}$. The space $C(\mu_1+\mu_2)$ has multiplicity one in $C(\mu_1)\otimes C(\mu_2)$ (again by (2.7)), therefore if $\alpha\in C(\mu_1)_{\lambda_1}$ and $\beta\in C(\mu_2)_{\lambda_2}$, then $\alpha\circ\beta$ is the projection of $\alpha\beta$ onto $C(\mu_1+\mu_2)_{\lambda_1+\lambda_2}$. Denote by $\bar{\star}$ the product $\star$ of $\mathcal{O}_q$ followed by the projection onto the component $C(\mu+\nu)$. Then, we have

$$C(\mu)\circ C(\nu)=C(\mu)\,\bar{\star}\,C(\nu)=C(\mu+\nu).\tag{3.4}$$

This follows from the formula (2.9), and the fact that it is given by an invertible twist of the product $\star$.

(iii) For every $\mu\in P_+$, fix a basis of weight vectors $e_1^\mu,\ldots,e_{d(\mu)}^\mu$ of V_μ. Denote by $e_\mu^1,\ldots,e_\mu^{d(\mu)}\in V_\mu^*$ the dual basis, and by $w(e_i^\mu)$ the weight of e_i^μ. Consider the matrix coefficients ${}_\mu\phi_j^i(x):=e_\mu^i\big(\pi_V(x)(e_j^\mu)\big)$, $x\in U_q$. By using the formula (2.9) and the explicit form of the R-matrix, one can check that

$$\begin{aligned}{}_\mu\phi_j^i\circ{}_\nu\phi_l^k&=\sum_{j',l'}{}'\,c_{j',l'}^{ikjl}\;{}_\mu\phi_{j'}^i\;\bar{\star}\;{}_\nu\phi_{l'}^k\\&=q^{(w(e_j^\mu),w(e_l^\nu)-w(e_k^\nu))}{}_\mu\phi_j^i\;\bar{\star}\;{}_\nu\phi_l^k+\sum_{\substack{j',l'\\j'\neq j,l'\neq l}}{}'\,d_{j',l'}^{ikjl}\;{}_\mu\phi_{j'}^i\circ{}_\nu\phi_{l'}^k,\end{aligned}\tag{3.5}$$

where $\sum'_{j',l'}$ is the sum over indices with weights satisfying

$$w(e^\mu_j) + w(e^\nu_l) = w(e^\mu_{j'}) + w(e^\nu_{l'}), \qquad w(e^\mu_{j'}) \leq w(e^\mu_j) \qquad \text{and} \qquad w(e^\nu_{l'}) \geq w(e^\nu_l),$$

and the coefficient $c^{ikjl}_{j,l}$, equal to $q^{(w(e^\mu_j),w(e^\nu_l)-w(e^\nu_k))}$, is computed from the term Θ in the R-matrix factorization (2.4). In general, all the coefficients $c^{ikjl}_{j',l'}$ and $d^{ikjl}_{j',l'}$ belong to $\mathbb{C}(q)$ (see [18, Proposition 4.1]); in particular $q^{(w(e^\mu_j),w(e^\nu_l)-w(e^\nu_k))} \in q^{\mathbb{Z}}$ since $w(e^\nu_l) - w(e^\nu_k) \in Q$. The second equality follows by repeated use of the first and (3.4). Similarly, by using (2.10) one gets

$$\begin{aligned}
{}_\nu\phi^k_l \circ {}_\mu\phi^i_j &= \sum_{i',k'}{}' e^{kilj}_{i',k'}\ {}_\mu\phi^{i'}_j \mathbin{\bar{\star}} {}_\nu\phi^{k'}_l \\
&= q^{(w(e^\mu_i),w(e^\nu_k)-w(e^\nu_l))}{}_\mu\phi^i_j \mathbin{\bar{\star}} {}_\nu\phi^k_l + \sum_{\substack{i',k' \\ i'\neq i,\, k'\neq k}}{}' e^{kilj}_{i',k'}\ {}_\mu\phi^{i'}_j \mathbin{\bar{\star}} {}_\nu\phi^{k'}_l \\
&= q^{(w(e^\mu_i),w(e^\nu_k)-w(e^\nu_l))}{}_\mu\phi^i_j \mathbin{\bar{\star}} {}_\nu\phi^k_l + \sum_{\substack{i',k',j',l' \\ i'\neq i,\, k'\neq k}}{}' f^{kilj}_{i',k'}\ {}_\mu\phi^{i'}_{j'} \circ {}_\nu\phi^{k'}_{l'},
\end{aligned}$$

where $e^{kilj}_{i',k'}$, $f^{kilj}_{i',k'} \in \mathbb{C}(q)$, and $\sum'_{i',k'}$ is the sum over indices with weights satisfying

$$\begin{aligned}
&w(e^\mu_i) + w(e^\nu_k) = w(e^\mu_{i'}) + w(e^\nu_{k'}), \qquad w(e^\mu_{i'}) \leq w(e^\mu_i), \\
&w(e^\nu_{k'}) \geq w(e^\nu_k), \qquad e^{kilj}_{i,k} = q^{(w(e^\mu_i),w(e^\nu_k)-w(e^\nu_l))}.
\end{aligned}$$

The third equality comes from the second and (3.5); the sum is over indices with weights satisfying

$$\begin{aligned}
&w(e^\mu_i) + w(e^\nu_k) = w(e^\mu_{i'}) + w(e^\nu_{k'}), \\
&w(e^\mu_{i'}) < w(e^\mu_i), \qquad w(e^\nu_{k'}) > w(e^\nu_k), \qquad w(e^\mu_{j'}) \leq w(e^\mu_j), \qquad w(e^\nu_{l'}) \geq w(e^\nu_l).
\end{aligned}$$

By eliminating the leading term ${}_\mu\phi^i_j \mathbin{\bar{\star}} {}_\nu\phi^k_l$, one deduces

$$ {}_\nu\phi^k_l \circ {}_\mu\phi^i_j - q_{ijkl}\ {}_\mu\phi^i_j \circ {}_\nu\phi^k_l = \sum_{\substack{i',k',j',l' \\ i'\neq i,\, k'\neq k}}{}' f^{kilj}_{i',k'}\ {}_\mu\phi^{i'}_{j'} \circ {}_\nu\phi^{k'}_{l'} - \sum_{\substack{j',l' \\ j'\neq j,\, l'\neq l}}{}' q_{ijkl} d^{ikjl}_{j',l'}\ {}_\mu\phi^i_{j'} \circ {}_\nu\phi^k_{l'}, \tag{3.6}$$

where $q_{ijkl} = q^{(w(e^\mu_j)+w(e^\mu_i),w(e^\nu_k)-w(e^\nu_l))}$.

(iv) We can always reorder the weight vectors $e^\mu_1, \dots, e^\mu_{d(\mu)}$ so that $w(e^\mu_i) > w(e^\mu_j)$ implies $i < j$; then (3.6) reads

$$\begin{aligned}
{}_\nu\phi^k_l \circ {}_\mu\phi^i_j - q_{ijkl}\ {}_\mu\phi^i_j \circ {}_\nu\phi^k_l = &\sum_{r=i}^{d(\mu)} \sum_{s=1}^{k} \sum_{u=1}^{l-1} \sum_{v=j+1}^{d(\mu)} \delta^{ijkl}_{rsuv}\ {}_\mu\phi^r_v \circ {}_\nu\phi^s_u \\
&- \sum_{r=i+1}^{d(\mu)} \sum_{s=1}^{k-1} q_{ijkl}\gamma^{ijkl}_{rs}\ {}_\mu\phi^r_j \circ {}_\nu\phi^s_l,
\end{aligned} \tag{3.7}$$

where $\gamma^{ijkl}_{rs}, \delta^{ijkl}_{rsuv} \in \mathbb{C}(q)$ are such that $\gamma^{ijkl}_{rs} = 0$ unless $w(e^\mu_r) < w(e^\mu_i)$ and $w(e^\nu_s) > w(e^\nu_k)$, and $\delta^{ijkl}_{rsuv} = 0$ unless $w(e^\nu_u) > w(e^\nu_l)$, $w(e^\mu_v) < w(e^\mu_j)$, $w(e^\mu_r) \leq w(e^\mu_i)$ and $w(e^\nu_s) \geq w(e^\nu_k)$. Now, from (3.7) one can extract a defining set of polynormal relations for $\mathrm{Gr}_{\mathcal{F}_2}(\mathcal{L}_{0,1})$, as in (3.1). Indeed, like $\mathcal{L}_{0,1}$ the algebra $\mathrm{Gr}_{\mathcal{F}_2}(\mathcal{L}_{0,1})$ is generated by the matrix coefficients ${}_{\varpi_k}\phi^j_i$ of the fundamental representations V_{ϖ_k}. One can list these matrix coefficients, say M in number, in an ordered sequence $u_1, \dots, u_M$ such that the following condition holds: if $w(e^{\varpi_s}_k) < w(e^{\varpi_r}_i)$, or $w(e_k^{\varpi_s}) = w(e_i^{\varpi_r})$ and $w(e_l^{\varpi_s}) < w(e_j^{\varpi_r})$, then $u_a := {}_{\varpi_r}\phi_j^i$ and $u_b := {}_{\varpi_s}\phi_l^k$ satisfy $b < a$. Then denoting ${}_\mu\phi_j^i$, ${}_\nu\phi_l^k$ in (3.7) by u_j, u_i, respectively, and assuming $u_j < u_i$, one finds that all terms $u_s := {}_\mu\phi_v^r$, ${}_\mu\phi_j^r$ in the sums are $< u_j$. Therefore, for all $1 \le j < i \le M$ it takes the form

$$u_i \circ u_j - q_{ij} u_j \circ u_i = \sum_{s=1}^{j-1} \sum_{t=1}^{M} \alpha_{ij}^{st} u_s \circ u_t \tag{3.8}$$

for some $q_{ij} \in q^{\mathbb{Z}}$ and $\alpha_{ij}^{st} \in \mathbb{C}(q)$. As explained after (3.1), it follows that $\mathrm{Gr}_{\mathcal{F}_2}(\mathcal{L}_{0,1})$ is a Noetherian ring, and since the filtration $\mathcal{F}_2$ is well founded, it implies that $\mathcal{L}_{0,1}$ is Noetherian too.

We are going to extend all these facts to $\mathcal{L}_{0,n}$, $n > 1$. First, we need to refine the filtration $\mathcal{F}_2$ on $\mathcal{L}_{0,1}$. Consider the action of $U_q(\mathfrak{h})$ on $C(\mu)_\lambda$ given by

$$K_\nu.\alpha := \mathrm{coad}\big(K_\nu^{-1}\big)(\alpha), \qquad \nu \in P, \quad \alpha \in C(\mu)_\lambda. \tag{3.9}$$

Denote by $C(\mu)_{\lambda,\gamma} \subset C(\mu)_\lambda$ the subspace of weight γ for this action; so $\alpha \in C(\mu)_{\lambda,\gamma}$ if $K_\nu.\alpha = q^{(\nu,\gamma)}\alpha$. Consider the semigroup

$$\Lambda_P = \{(\mu,\lambda,\gamma) \in P_+ \times P^2,\ \lambda \text{ is a weight of } V_\mu \text{ for } \triangleright,\ \gamma \text{ is a weight of } V_\mu \text{ for } .\}$$

with the partial order $(\mu,\lambda,\gamma) \preceq (\mu',\lambda',\gamma')$ if and only if $\mu'-\mu, \lambda'-\lambda, \gamma'-\gamma \in D^{-1}Q_+$. Define

$$[\Lambda_P] = \big\{([\mu],[\lambda],[\gamma]) \in P_+^n \times P^n \times P^n \\ \mid (\mu_i,\lambda_i,\gamma_i) \in \Lambda_P,\ [\mu] = (\mu_i)_{i=1}^n, [\lambda] = (\lambda_i)_{i=1}^n, [\gamma] = (\gamma_i)_{i=1}^n\big\}.$$

Let us put the following lexicographic order on $[\Lambda_P]$, starting from the tail: $([\mu'],[\lambda'],[\gamma']) \preceq ([\mu],[\lambda],[\gamma])$ if $(\mu_n',\lambda_n',\gamma_n') \prec (\mu_n,\lambda_n,\gamma_n)$, or $(\mu_n,\lambda_n,\gamma_n) = (\mu_n',\lambda_n',\gamma_n')$ and $(\mu_{n-1}',\lambda_{n-1}',\gamma_{n-1}') \prec (\mu_{n-1},\lambda_{n-1},\gamma_{n-1}), \dots,$ or $(\mu_k,\lambda_k,\gamma_k) = (\mu_k',\lambda_k',\gamma_k')$ for all $1 < k \le n$ and $(\mu_1',\lambda_1',\gamma_1') \preceq (\mu_1,\lambda_1,\gamma_1)$. (As usual, we write $([\mu'],[\lambda'],[\gamma']) \prec ([\mu],[\lambda],[\gamma])$ for $([\mu'],[\lambda'],[\gamma']) \preceq ([\mu],[\lambda],[\gamma])$ and $([\mu'],[\lambda'],[\gamma']) \ne ([\mu],[\lambda],[\gamma])$.)

Now recall that $\mathcal{L}_{0,n} = \mathcal{L}_{0,1}^{\otimes n} = \mathcal{O}_q^{\otimes n}$ as vector spaces. For every $([\mu],[\lambda],[\gamma]) \in [\Lambda_P]$, consider the subspace $C([\mu])_{[\lambda],[\gamma]} \subset \mathcal{L}_{0,n}$ defined by

$$C([\mu]) = C(\mu_1) \otimes \cdots \otimes C(\mu_n), \qquad C([\mu])_{[\lambda],[\gamma]} = C(\mu_1)_{\lambda_1,\gamma_1} \otimes \cdots \otimes C(\mu_n)_{\lambda_n,\gamma_n}.$$

Then $\mathcal{L}_{0,n} = \bigoplus_{[\mu]\in P_+^n} C([\mu])$ and $C([\mu]) = \bigoplus_{([\lambda],[\gamma])} C([\mu])_{[\lambda],[\gamma]}$. For every $([\mu],[\lambda],[\gamma]) \in [\Lambda_P]$ define

$$\mathcal{F}_3^{[\mu],[\lambda],[\gamma]} = \bigoplus_{([\mu'],[\lambda'],[\gamma'])\preceq([\mu],[\lambda],[\gamma])} C([\mu'])_{[\lambda'],[\gamma']}, \\ \mathcal{F}_3^{\prec[\mu],[\lambda],[\gamma]} = \bigoplus_{([\mu'],[\lambda'],[\gamma'])\prec([\mu],[\lambda],[\gamma])} C([\mu'])_{[\lambda'],[\gamma']}. \tag{3.10}$$

Clearly, $\mathcal{L}_{0,n}$ is the union of the subspaces $\mathcal{F}_3^{[\mu],[\lambda],[\gamma]}$ over all $([\mu],[\lambda],[\gamma]) \in [\Lambda_P]$, so these form a vector space filtration of $\mathcal{L}_{0,n}$. Let us denote it $\mathcal{F}_3$, and define

$$\mathrm{Gr}_{\mathcal{F}_3}(\mathcal{L}_{0,n})_{[\mu],[\lambda],[\gamma]} = \mathcal{F}_3^{[\mu],[\lambda],[\gamma]} / \mathcal{F}_3^{\prec[\mu],[\lambda],[\gamma]}.$$

This space is canonically identified with $C([\mu])_{[\lambda],[\gamma]}$, so the graded vector space associated to $\mathcal{F}_3$ is

$$\mathrm{Gr}_{\mathcal{F}_3}(\mathcal{L}_{0,n}) = \bigoplus_{([\mu],[\lambda],[\gamma])\in[\Lambda_P]} \mathrm{Gr}_{\mathcal{F}_3}(\mathcal{L}_{0,n})_{[\mu],[\lambda],[\gamma]} = \bigoplus_{([\mu],[\lambda],[\gamma])\in[\Lambda_P]} C([\mu])_{[\lambda],[\gamma]}. \tag{3.11}$$

We claim that $\mathcal{F}_3$ is an algebra filtration with respect to the product of $\mathcal{L}_{0,n}$, and therefore $\mathrm{Gr}_{\mathcal{F}_3}(\mathcal{L}_{0,n})$ is a graded algebra.

For notational simplicity, let us prove it for $n = 2$, the general case being strictly similar. Recall the R-matrix factorization (2.4). Take tuples $([\mu],[\lambda],[\gamma]) = ((\mu_1,\mu_2),(\lambda_1,\lambda_2),(\gamma_1,\gamma_2))$ and $([\mu'],[\lambda'],[\gamma']) = ((\mu'_1,\mu'_2),(\lambda'_1,\lambda'_2),(\gamma'_1,\gamma'_2))$ in $[\Lambda_P]$, and elements $\alpha\otimes\beta \in C([\mu])_{[\lambda],[\gamma]}$ and $\alpha'\otimes\beta' \in C([\mu'])_{[\lambda'],[\gamma']}$. Recall from (2.17) that the product of $\mathcal{L}_{0,2}$ is given by the formula

$$(\alpha\otimes\beta)(\alpha'\otimes\beta') \\ = \sum_{(R^1),\dots,(R^4)} \alpha\big(S(R^3_{(1)}R^4_{(1)}) \rhd \alpha' \lhd R^1_{(1)}R^2_{(1)}\big) \otimes \big(S(R^1_{(2)}R^3_{(2)}) \rhd \beta \lhd R^2_{(2)}R^4_{(2)}\big)\beta'. \tag{3.12}$$

For every $\nu \in P$ and any of the components $R^1_{(2)},\dots,R^4_{(2)}$, denoting by $-r_j \in -Q_+$ the weight of $R^j_{(2)}$, we have

$$\begin{aligned} &K_\nu \rhd \big(S\big(R^1_{(2)}R^3_{(2)}\big) \rhd \beta \lhd R^2_{(2)}R^4_{(2)}\big) \\ &\quad = \sum_{(\beta)} \beta_{(1)}\big(R^2_{(2)}R^4_{(2)}\big)\big(K_\nu S\big(R^1_{(2)}R^3_{(2)}\big) \rhd \beta_{(2)}\big) \\ &\quad = q^{-(\nu,r_1+r_3)} \sum_{(\beta)} \beta_{(1)}\big(R^2_{(2)}R^4_{(2)}\big)\big(S\big(R^1_{(2)}R^3_{(2)}\big)K_\nu \rhd \beta_{(2)}\big) \\ &\quad = q^{(\nu,\lambda_2-r_1-r_3)} \sum_{(\beta)} \beta_{(1)}\big(R^2_{(2)}R^4_{(2)}\big)\big(S\big(R^1_{(2)}R^3_{(2)}\big) \rhd \beta_{(2)}\big) \\ &\quad = q^{(\nu,\lambda_2-r_1-r_3)}\big(S\big(R^1_{(2)}R^3_{(2)}\big) \rhd \beta \lhd R^2_{(2)}R^4_{(2)}\big). \end{aligned}$$

By similar computations for the action $\mathrm{coad}(K_\nu^{-1})$, and for all terms in the right-hand side of (3.12), and using (3.3) componentwisely, we find that

$$\alpha\big(S\big(R^3_{(1)}R^4_{(1)}\big) \rhd \alpha' \lhd R^1_{(1)}R^2_{(1)}\big) \otimes \big(S\big(R^1_{(2)}R^3_{(2)}\big) \rhd \beta \lhd R^2_{(2)}R^4_{(2)}\big)\beta' \in \mathcal{F}_3^{[\mu]+[\mu'],[\lambda''],[\gamma'']},$$

where

$$\begin{aligned} &\lambda'' = (\lambda_1+\lambda'_1+r_3+r_4, \lambda_2+\lambda'_2-r_1-r_3), \\ &\gamma'' = (\gamma_1+\gamma'_1+r_1+r_2+r_3+r_4, \gamma_2+\gamma'_2-r_1-r_2-r_3-r_4). \end{aligned}$$

Since $r_1+r_2+r_3+r_4 = 0$ implies $r_1=r_2=r_3=r_4=0$, by the order we have put on $[\Lambda_P]$, we deduce

$$(\alpha\otimes\beta)(\alpha'\otimes\beta') \in \mathcal{F}_3^{[\mu]+[\mu'],[\lambda]+[\lambda'],[\gamma]+[\gamma']}.$$

Note that the filtration $\mathcal{F}_3$, taking the action (3.9) into account, is crucial for this argument to work. Similar arguments work for any $n \geq 2$. This proves that $\mathrm{Gr}_{\mathcal{F}_3}(\mathcal{L}_{0,n})$ is a graded algebra. We denote its product by $\circ_n$.

Next, we show that (3.4) implies the analogous property for the product $\circ_n$. For simplicity of notations let us again assume that $n = 2$. Recall that the product $\circ_2$ is defined on homogeneous elements $\overline{\alpha\otimes\beta} \in \mathrm{Gr}_{\mathcal{F}_3}(\mathcal{L}_{0,n})_{[\mu],[\lambda]}$ and $\overline{\alpha'\otimes\beta'} \in \mathrm{Gr}_{\mathcal{F}_3}(\mathcal{L}_{0,n})_{[\mu'],[\lambda']}$ by

$$\overline{\alpha\otimes\beta} \circ_n \overline{\alpha'\otimes\beta'} = (\alpha\otimes\beta)(\alpha'\otimes\beta') + \mathcal{F}_3^{\prec[\mu+\mu'],[\lambda+\lambda']}.$$

Clearly, (3.4) gives $(C(\mu_1)\circ C(\mu'_1)) \otimes (C(\mu_2)\circ C(\mu'_2)) = C([\mu+\mu'])$, and (3.12) gives

$$C([\mu]) \circ_n C([\mu']) \subset (C(\mu_1)\circ C(\mu'_1)) \otimes (C(\mu_2)\circ C(\mu'_2)).$$

The converse inclusion holds true as well, as one can see by expressing, reciprocally, the (componentwise) product of $\mathcal{L}_{0,1}^{\otimes n}$ in terms of the product of $\mathcal{L}_{0,n}$ via the formula (2.19). In conclusion,

$$C([\mu]) \circ_n C([\mu']) = C([\mu+\mu']).$$

We are left to show that (3.7) generalizes to $\mathcal{L}_{0,n}$. First, note that for every $1 \le a \le n$ the embedding $\mathfrak{i}_a\colon \mathcal{L}_{0,1} \to \mathcal{L}_{0,n}$ in (2.16) is a morphism of the filtered algebras $(\mathcal{L}_{0,1}, \mathcal{F}_2)$ and $(\mathcal{L}_{0,n}, \mathcal{F}_3)$, in the sense that

$$\mathfrak{i}_a\big(\mathcal{F}_2^{\mu,\lambda}\big) \subset \sum_{\gamma\in P} \mathcal{F}_3^{[\mu_a],[\lambda_a],[\gamma_a]},$$

where by definition $[\mu_a] = (0,\dots,0,\mu,0,\dots,0)$ with μ on the a-th entry, and similarly $[\lambda_a] = (0,\dots,0,\lambda,0,\dots,0)$ and $[\gamma_a] = (0,\dots,0,\gamma,0,\dots,0)$. Therefore, the relation (3.7) yields in $\mathrm{Gr}_{\mathcal{F}_3}(\mathcal{L}_{0,n})$ similar relations between elements of the form (matrix coefficient)$\otimes 1$, or $1\otimes$(matrix coefficient).

We now consider the case of "mixed" products. We give the details when $n=2$, the general case being similar. Let us write the twist F in (2.18) as

$$F = \sum_{(F)} F_{(1)} \otimes F_{(2)} = \sum_{(F)} F_{(1)1} \otimes F_{(1)2} \otimes F_{(2)1} \otimes F_{(2)2},$$

that is, we set $F_{(1)1} := R^2_{(2)}R^4_{(2)}$, $F_{(1)2} := R^1_{(2)}R^3_{(2)}$, $F_{(2)1} := R^1_{(1)}R^2_{(1)}$, $F_{(2)2} := R^3_{(1)}R^4_{(1)}$. Put $d(\mu) := \dim(V_\mu)$, $\mu \in P_+$, and

$$\Delta^{(2)}\big({}_{\mu_2}\phi^{k_2}_{l_2}\big) = \sum_{p,s=1}^{d(\mu_2)} {}_{\mu_2}\phi^{k_2}_{p} \otimes {}_{\mu_2}\phi^{p}_{s} \otimes {}_{\mu_2}\phi^{s}_{l_2}, \qquad \Delta^{(2)}\big({}_{\mu_1'}\phi^{k_1'}_{l_1'}\big) = \sum_{p',s'=1}^{d(\mu_1')} {}_{\mu_1'}\phi^{k_1'}_{p'} \otimes {}_{\mu_1'}\phi^{p'}_{s'} \otimes {}_{\mu_1'}\phi^{s'}_{l_1'}.$$

From (3.12), one obtains

$$\begin{aligned}\big(1 \otimes {}_{\mu_2}\phi^{k_2}_{l_2}\big)\big({}_{\mu_1'}\phi^{k_1'}_{l_1'} \otimes 1\big) = \sum_{(F)} \sum_{p,s=1}^{d(\mu_2)} \sum_{p',s'=1}^{d(\mu_1')} &\big({}_{\mu_1'}\phi^{p'}_{s'}\big({}_{\mu_1'}\phi^{k_1'}_{p'}(F_{(2)1})\,{}_{\mu_1'}\phi^{s'}_{l_1'}(S(F_{(2)2}))\big)\big)\\ &\otimes \big({}_{\mu_2}\phi^{p}_{s}\big({}_{\mu_2}\phi^{k_2}_{p}(F_{(1)1})\,{}_{\mu_2}\phi^{s}_{l_2}(S(F_{(1)2}))\big)\big).\end{aligned} \tag{3.13}$$

It is immediate that

$${}_{\mu_1'}\phi^{p'}_{s'} \otimes {}_{\mu_2}\phi^{p}_{s} \in C(\mu_1')_{w(e^{\mu_1'}_{s'}),\,w(e^{\mu_1'}_{s'})-w(e^{\mu_1'}_{p'})} \otimes C(\mu_2)_{w(e^{\mu_2}_{s}),\,w(e^{\mu_2}_{s})-w(e^{\mu_2}_{p})}.$$

As in (iv) above, for every $\mu \in P_+$ we order the weight vectors $e^\mu_1,\dots,e^\mu_m$ so that $w\big(e^\mu_i\big) > w\big(e^\mu_j\big)$ implies $i<j$. With such an ordering the factorization $R = \Theta\hat{R}$ (see (2.4)) implies

$${}_{\mu_2}\phi^{k_2}_{p}(F_{(1)1})\,{}_{\mu_2}\phi^{s}_{l_2}(S(F_{(1)2})) = 0 \qquad \text{unless } k_2 \ge p \text{ and } s \ge l_2,$$

and

$${}_{\mu_1'}\phi^{k_1'}_{p'}(F_{(2)1})\,{}_{\mu_1'}\phi^{s'}_{l_1'}(S(F_{(2)2})) = 0 \qquad \text{unless } k_1' \le p' \text{ and } s' \le l_1'.$$

Since $s > l_2$, we have $w\big(e^{\mu_2}_s\big) \le w\big(e^{\mu_2}_{l_2}\big)$, and if $w\big(e^{\mu_2}_s\big) < w\big(e^{\mu_2}_{l_2}\big)$, then ${}_{\mu_2}\phi^p_s \in \mathcal{F}_2^{<\mu_2,w(e^{\mu_2}_{l_2})}$. In this last situation, the summands ${}_{\mu_1'}\phi^{p'}_{s'} \otimes {}_{\mu_2}\phi^p_s$ in the sum above vanish in $\mathrm{Gr}_{\mathcal{F}_3}(\mathcal{L}_{0,2})$. In order to find all the non-zero summands, we have to consider also the weights with respect to the action (3.9).

Since $k_2 \geq p$ implies $w\big(e^{\mu_2}_{k_2}\big) \leq w\big(e^{\mu_2}_{p}\big)$, we have $w\big(e^{\mu_2}_{s}\big) - w\big(e^{\mu_2}_{p}\big) \leq w\big(e^{\mu_2}_{l_2}\big) - w\big(e^{\mu_2}_{k_2}\big)$. Therefore, the summands which are non-zero in $\mathrm{Gr}_{\mathcal{F}_3}(\mathcal{L}_{0,2})$ have both weights $w\big(e^{\mu_2}_{s}\big) = w\big(e^{\mu_2}_{l_2}\big)$ and $w\big(e^{\mu_2}_{p}\big) = w\big(e^{\mu_2}_{k_2}\big)$. Doing similarly with the weights of ${}_{\mu'_1}\phi^{p'}_{s'}$, we find that also $w\big(e^{\mu'_1}_{s'}\big) = w\big(e^{\mu'_1}_{l'_1}\big)$ and $w\big(e^{\mu'_1}_{p'}\big) = w\big(e^{\mu'_1}_{k'_1}\big)$. When all these conditions on weights are satisfied, the corresponding components $F_{(1)1}$, $F_{(1)2}$, $F_{(2)1}$, $F_{(2)2}$ have zero weight. Therefore, the sum reduces to

$$\sum_{(F)} {}_{\mu_2}\phi^{k_2}_{k_2}(F_{(1)1})\,{}_{\mu_2}\phi^{l_2}_{l_2}(S(F_{(1)2}))\,{}_{\mu'_1}\phi^{k'_1}_{k'_1}(F_{(2)1})\,{}_{\mu'_1}\phi^{l'_1}_{l'_1}(S(F_{(2)2}))$$
$$= \big\langle {}_{\mu_2}\phi^{k_2}_{k_2} \otimes {}_{\mu_2}\phi^{l_2}_{l_2} \otimes {}_{\mu'_1}\phi^{k'_1}_{k'_1} \otimes {}_{\mu'_1}\phi^{l'_1}_{l'_1}, \Theta_{13}\Theta_{14}^{-1}\Theta_{24}\Theta_{23}^{-1}\big\rangle = q^{(w(e^{\mu_2}_{k_2}) - w(e^{\mu_2}_{l_2}), w(e^{\mu'_1}_{k'_1}) - w(e^{\mu'_1}_{l'_1}))}.$$

Denoting by $q'_{k_2 l_2 k'_1 l'_1}$ this scalar, it follows

$$\big(1 \otimes {}_{\mu_2}\phi^{k_2}_{l_2}\big) \circ_2 \big({}_{\mu'_1}\phi^{k'_1}_{l'_1} \otimes 1\big) = q'_{k_2 l_2 k'_1 l'_1}\ {}_{\mu'_1}\phi^{k'_1}_{l'_1} \otimes {}_{\mu_2}\phi^{k_2}_{l_2} = q'_{k_2 l_2 k'_1 l'_1}\big({}_{\mu'_1}\phi^{k'_1}_{l'_1} \otimes 1\big) \circ_2 \big(1 \otimes {}_{\mu_2}\phi^{k_2}_{l_2}\big).$$

This is the relation analogous to (3.7) for mixed products in $\mathrm{Gr}_{\mathcal{F}_3}(\mathcal{L}_{0,2})$.

Recall that in (3.8) we denoted by $u_1, \dots, u_M$ the ordered list of matrix coefficients ${}_{\varpi_k}\phi^j_i$. Let us order in a lexicographic way the elements $u_i \otimes u_j$, i.e., as a sequence $u^{(2)}_1, \dots, u^{(2)}_{M^2}$ such that the following condition holds: if ${}_{\varpi_{l'}}\phi^{t'}_{s'} < {}_{\varpi_{k'}}\phi^{j'}_{i'}$, or ${}_{\varpi_{l'}}\phi^{t'}_{s'} = {}_{\varpi_{k'}}\phi^{j'}_{i'}$ and ${}_{\varpi_l}\phi^t_s < {}_{\varpi_k}\phi^j_i$, then $u^{(2)}_a := {}_{\varpi_k}\phi^j_i \otimes {}_{\varpi_{k'}}\phi^{j'}_{i'}$ and $u^{(2)}_b := {}_{\varpi_l}\phi^t_s \otimes {}_{\varpi_{l'}}\phi^{t'}_{s'}$ satisfy $u^{(2)}_b < u^{(2)}_a$. Then, for this ordering the polynormal relations (3.8) hold true for all $1 \leq u^{(2)}_j < u^{(2)}_i \leq M^2$. As described after (3.1), it follows that $\mathrm{Gr}_{\mathcal{F}_3}(\mathcal{L}_{0,n})$ is Noetherian. The filtration $\mathcal{F}_3$ being well founded, it implies that $\mathcal{L}_{0,n}$ is Noetherian too.

Finally, we consider the A-algebra $\mathcal{L}^A_{0,n}$, and prove it is Noetherian. We proceed in exactly the same way as for $\mathcal{L}_{0,n}$, changing the generators and replacing key arguments of the steps (i)–(iv) by the corresponding results over A. Let us describe these modifications step by step.

First, consider the case $n = 1$. Recall the A-lattices ${}_A\dot{C}(\lambda)$ (see (2.46)), and the decomposition (2.55) of $\mathcal{O}_A$ into weight subspaces. In particular, have a decomposition into weight subspaces for the left coregular action,

$${}_A\dot{C}(\lambda) = \bigoplus_{\lambda' \in P} {}_A\dot{C}(\lambda)_{\lambda'}.$$

Define

$${}_A\mathcal{F}^{\mu,\lambda}_2 := \bigoplus_{(\mu',\lambda') \preceq (\mu,\lambda)} {}_A\dot{C}(\mu')_{\lambda'}.$$

Recall that every A-module of matrix coefficients $(\mathrm{End}({}_AV_\mu))^*$, $\mu \in P_+$, is contained in $\mathcal{O}_A(\leq \mu)$, and by inverting over $\mathbb{C}(q)$ the corresponding linear triangular system between basis elements, and using that the order relation $\preceq$ is finer than $\leq$, we obtain an inclusion

$$\bigoplus_{\mu' \preceq \mu} {}_A\dot{C}(\mu') \subset \bigoplus_{\mu' \preceq \mu} C(\mu')$$

(see (2.48)–(2.51)). It follows that ${}_A\mathcal{F}^{\mu,\lambda}_2 = \mathcal{F}^{\mu,\lambda}_2 \cap \mathcal{O}_A$, and therefore, like $\mathcal{F}_2$ the family ${}_A\mathcal{F}_2 := \big\{{}_A\mathcal{F}^{\mu,\lambda}_2\big\}_{(\mu,\lambda)\in\Lambda}$ is a well-founded filtration of $\mathcal{O}_A$. Put ${}_A\mathcal{F}^{\prec\mu,\lambda}_2 = \mathcal{F}^{\prec\mu,\lambda}_2 \cap \mathcal{O}_A$, and consider the graded A-module $\mathrm{Gr}_{{}_A\mathcal{F}_2}\big(\mathcal{L}^A_{0,1}\big)$ associated to ${}_A\mathcal{F}_2$. By (2.52)–(2.54) and the fact that $\mathcal{O}_A = \mathcal{L}^A_{0,1}$ as an A-module, we have the A-module decomposition

$$\mathrm{Gr}_{{}_A\mathcal{F}_2}\big(\mathcal{L}^A_{0,1}\big) = \bigoplus_{(\mu,\lambda)\in\Lambda} {}_AC(\mu)_\lambda,$$

where ${}_AC(\mu)_\lambda$ is the submodule of weight λ (for the left coregular action) of

$${}_AC(\mu) := (\mathrm{End}({}_AV_\mu))^*.$$

Then, we can proceed as before. By step (i), we deduce that ${}_A\mathcal{F}_2$ is an algebra filtration of $\mathcal{L}_{0,1}^A$. By Proposition 2.12, the A-module ${}_A\dot{C}(\mu_1+\mu_2)$ has multiplicity one in ${}_A\dot{C}(\mu_1)\otimes {}_A\dot{C}(\mu_2)$. In fact, by step (ii), ${}_AC(\mu_1+\mu_2)$ has multiplicity one in ${}_AC(\mu_1)\bigotimes_A {}_AC(\mu_2)$, so exactly in the same way as (3.4), we obtain in $\mathrm{Gr}_{{}_A\mathcal{F}_2}\big(\mathcal{L}_{0,1}^A\big)$ the equality

$${}_AC(\mu)\circ {}_AC(\nu) = {}_AC(\mu+\nu).$$

In step (iii), we fixed a basis of each space $C(\mu)$, consisting of a set of matrix coefficients $\big\{{}_\mu\phi_j^i\big\}$ with respect to dual basis of weight vectors of the modules V_μ and V_μ^*. In step (iv), the basis elements of V_μ and V_μ^* were ordered by means of the weights, and we used the fact that the matrix coefficients in the spaces $C(\varpi_1),\dots,C(\varpi_m)$ form a generating set of the algebra $\mathrm{Gr}_{\mathcal{F}_2}(\mathcal{L}_{0,1})$. The only property of the matrix coefficients used in the computations was that they are weight vectors for the left coregular action (and later, in the case $n>1$, for the action (3.9)).

We can proceed exactly in the same manner by working with the A-modules of matrix coefficients ${}_AC(\mu)$. If one wishes to work at the lever of $\mathcal{O}_A$, recall that any set of generators of $\mathcal{O}_A$ generates $\mathcal{L}_{0,1}^A$ as well (see the proof of Proposition 2.18). Then, one can replace the basis $\big\{{}_\mu\phi_j^i\big\}$ of each space $C(\mu)$ with the canonical basis $\dot{\mathbf{B}}[\mu]^*$ of ${}_A\dot{C}(\mu)$, and take the generating set of $\mathcal{O}_A$ formed by the elements in $\dot{\mathbf{B}}[\varpi_i]^*$, $i=1,\dots,m$ (see Proposition 2.10 and the comments thereafter). By the integrality properties satisfied by the R-matrix and the twists, all the computations in the proof of steps (iii) and (iv) can be done using such basis elements, and eventually take place over A (see [18, Propositions 4.10 and 6.9]). Therefore, we obtain a relation like (3.8) with coefficients $\alpha_{ij}^{st}\in A$. Since A is a Noetherian ring, again this proves $\mathrm{Gr}_{{}_A\mathcal{F}_2}\big(\mathcal{L}_{0,1}^A\big)$, whence $\mathcal{L}_{0,1}^A$, are Noetherian rings.

This being done, the adaptation of the proof when $n>1$ is immediate. The filtration $\mathcal{F}_3$ has to be replaced with ${}_A\mathcal{F}_3 := \big\{{}_A\mathcal{F}_3^{[\mu],[\lambda],[\gamma]}\big\}_{([\mu],[\lambda],[\gamma])}$, where ${}_A\mathcal{F}_3^{[\mu],[\lambda],[\gamma]}$ is the A-module defined by

$${}_A\mathcal{F}_3^{[\mu],[\lambda],[\gamma]} = \bigoplus_{([\mu'],[\lambda'],[\gamma'])\preceq([\mu],[\lambda],[\gamma])} {}_A\dot{C}([\mu'])_{[\lambda'],[\gamma']},$$

where

$${}_A\dot{C}([\mu])_{[\lambda],[\gamma]} = {}_A\dot{C}(\mu_1)_{\lambda_1,\gamma_1}\bigotimes_A\cdots\bigotimes_A {}_A\dot{C}(\mu_n)_{\lambda_n,\gamma_n},$$

and ${}_A\dot{C}(\mu)_{\lambda,\gamma}$ is the subspace of ${}_A\dot{C}(\mu)_\lambda$ of weight γ for the action (3.9). Then the proof proceeds in exactly the same way, replacing in (3.13) and all subsequent computations the matrix coefficients by the generators of $\mathcal{O}_A$ provided by Proposition 2.10. This concludes the proof. ■

Theorem 3.4. *The algebra $\mathcal{M}_{0,n}=\mathcal{L}_{0,n}^{U_q}$ is Noetherian and generated over $\mathbb{C}(q)$ by a finite number of elements.*

Our method of proof follows closely that of the Hilbert–Nagata theorem (see [46]). Let us recall one version of this theorem. Let K be an arbitrary field, $\mathfrak{A}$ a commutative algebra over K finitely generated by elements $a_1,\dots,a_n$, and G a group of algebra automorphisms of $\mathfrak{A}$.

Theorem 3.5. *If the action of G on $\mathfrak{A}$ is completely reducible on finite-dimensional representations, then the ring $\mathfrak{A}^G$ of invariants of $\mathfrak{A}$ with respect to G is Noetherian and a finitely generated algebra over K.*

We recall here the main steps of the proof that we will adapt in order to prove Theorem 3.4:

(a) From the complete reducibility of the action of G on $\mathfrak{A}$, one can define a linear map

$$R\colon\ \mathfrak{A} \to \mathfrak{A}^G$$

namely the projection onto the space of invariant elements along the sum of nontrivial isotypical components of $\mathfrak{A}$. This linear map is the Reynolds operator; we already discussed it in (2.23) in the case of U_q acting on $\mathcal{L}_{0,n}$. By the same arguments we developed there, it satisfies $R(hf) = hR(f)$ for every $f \in \mathfrak{A}$, $h \in \mathfrak{A}^G$.

(b) Let I be an ideal of $\mathfrak{A}^G$. Then $I = R(\mathfrak{A}I) = \mathfrak{A}I \cap \mathfrak{A}^G$. Because $\mathfrak{A}I$ is an ideal of $\mathfrak{A}$, and $\mathfrak{A}$ is Noetherian, there exist elements $b_1, \dots, b_s$, that can be chosen in $I \subset \mathfrak{A}^G$, such that $\mathfrak{A}I = \mathfrak{A}b_1 + \cdots + \mathfrak{A}b_s$. Since $I = R(\mathfrak{A}I) = R(\mathfrak{A}b_1 + \cdots + \mathfrak{A}b_s) = \mathfrak{A}^G b_1 + \cdots + \mathfrak{A}^G b_s$, I is finitely generated over $\mathfrak{A}^G$. Therefore, $\mathfrak{A}^G$ is Noetherian.

(c) Let $\mathfrak{B}$ be an algebra graded over $\mathbb{N}$ (for simplicity of notations): $\mathfrak{B} = \bigoplus_{n=0}^{+\infty} \mathfrak{B}_n$, with $\mathfrak{B}_m.\mathfrak{B}_n \subset \mathfrak{B}_{m+n}$. The augmentation ideal of $\mathfrak{B}$ is $\mathfrak{B}^+ := \bigoplus_{n=1}^{+\infty} \mathfrak{B}_n$. If $\mathfrak{B}^+$ is a Noetherian ideal of $\mathfrak{B}$, then $\mathfrak{B}$ is a finitely generated algebra over $\mathfrak{B}_0$. This is [99, Lemma 2.4.5] (in that statement $\mathfrak{B}$ is commutative, but this hypothesis is not necessary for the proof).

(d) Assume that $\mathfrak{A}^G$ is graded over $\mathbb{N}$ (for simplicity of notations): $\mathfrak{A}^G = \bigoplus_{n=0}^{+\infty} \mathfrak{A}^G_n$ with $\mathfrak{A}^G_0 = K$. Then $\mathfrak{A}^{G+} = \bigoplus_{n=1}^{+\infty} \mathfrak{A}^G_n$ is an ideal of $\mathfrak{A}^G$, which is Noetherian by (b) above. Applying (c), we deduce that $\mathfrak{A}^G$ is a finitely generated algebra over K.

Proof of Theorem 3.4. Consider the filtration $\mathcal{F}$ of $\mathcal{L}_{0,n}$ by the subspaces

$$\mathcal{F}^{[\mu]} = \bigoplus_{[\mu'] \preceq [\mu]} C([\mu']), \qquad \mu \in P_+^n,$$

where P_+^n is given the lexicographic partial order induced from $[\Lambda]$. It is easily seen that $\mathcal{F}$ is an algebra filtration: the coregular actions $\triangleright$, $\triangleleft$ fix globally each component $C(\mu)$ of $\mathcal{L}_{0,1}$, so the claim follows from (2.9), (2.17) and the fact that $C(\mu) \star C(\nu) \subset C(\mu + \nu)$ for all $\mu, \nu \in P_+$. Denote by $\mathrm{Gr}_{\mathcal{F}}(\mathcal{L}_{0,n})$ the corresponding graded algebra. As a vector space, we have

$$\mathrm{Gr}_{\mathcal{F}}(\mathcal{L}_{0,n}) = \bigoplus_{[\mu] \in P_+^n} C([\mu]). \tag{3.14}$$

Because each space $C([\mu])$ is stabilized by the coadjoint action of U_q, (3.14) has a key advantage on the refined decomposition (3.11). Indeed, since $\mathcal{L}_{0,n}$ is a U_q-module algebra, the action of U_q is well defined on $\mathrm{Gr}_{\mathcal{F}}(\mathcal{L}_{0,n})$ and gives it a structure of U_q-module algebra. As vector spaces, we have

$$\mathrm{Gr}_{\mathcal{F}}(\mathcal{L}_{0,n})^{U_q} = \bigoplus_{[\mu] \in P_+^n} C([\mu])^{U_q}.$$

Now we can adapt the different steps (a)–(d) recalled above:

(a′) The action of U_q on $\mathrm{Gr}_{\mathcal{F}}(\mathcal{L}_{0,n})$ is completely reducible. This follows from (3.14) and the fact that the spaces $C(\mu)$ are finite-dimensional and thus completely reducible U_q-modules. We can therefore define the Reynolds operator as in (a),

$$R\colon\ \mathrm{Gr}_{\mathcal{F}}(\mathcal{L}_{0,n}) \to \mathrm{Gr}_{\mathcal{F}}(\mathcal{L}_{0,n})^{U_q}.$$

(b′) $\mathrm{Gr}_{\mathcal{F}}(\mathcal{L}_{0,n})$ is Noetherian, because (3.14) shows it is filtered by $\mathcal{F}_3$, and the associated graded algebra $\mathrm{Gr}_{\mathcal{F}_3}(\mathrm{Gr}_{\mathcal{F}}(\mathcal{L}_{0,n})) = \mathrm{Gr}_{\mathcal{F}_3}(\mathcal{L}_{0,n})$ is Noetherian by Theorem 3.1. As in (b), we deduce that $\mathrm{Gr}_{\mathcal{F}}(\mathcal{L}_{0,n})^{U_q}$ is Noetherian. But $\mathrm{Gr}_{\mathcal{F}}(\mathcal{L}_{0,n})^{U_q} = \mathrm{Gr}_{\mathcal{F}}\big(\mathcal{L}_{0,n}^{U_q}\big)$, which implies that $\mathcal{L}_{0,n}^{U_q}$ is Noetherian.

(c′) (and (d′)) Then we can apply the steps (c)–(d). As a result $\mathrm{Gr}_{\mathcal{F}}(\mathcal{L}_{0,n})^{U_q}$ is finitely generated, say by k non-zero elements $\bar{x}_1,\dots,\bar{x}_k$, which we may assume homogeneous.

(e′) We can now deduce that $\mathcal{L}_{0,n}^{U_q}$ is generated by elements x_i with leading terms the $\bar{x}_i$'s. Indeed, let $x\in\mathcal{L}_{0,n}^{U_q}$, and $[\mu]\in P_+^n$ such that $x\in\mathcal{F}^{[\mu]}\setminus\mathcal{F}^{\prec[\mu]}$, where $\mathcal{F}^{\prec[\mu]}:=\bigoplus_{[\mu']\prec[\mu]}C([\mu'])$. In $\mathrm{Gr}_{\mathcal{F}}(\mathcal{L}_{0,n})^{U_q}_{[\mu]}=\mathcal{F}^{[\mu]}/\mathcal{F}^{\prec[\mu]}$, we have

$$\bar{x}=\sum_{(i_1,\dots,i_k)\in I}\lambda_{(i_1,\dots,i_k)}\bar{x}_1^{i_1}\cdots\bar{x}_k^{i_k}$$

for some finite set $I\subset\mathbb{N}^k$, scalars $\lambda_{(i_1,\dots,i_k)}\in\mathbb{C}(q)$, and monomials $\bar{x}_1^{i_1}\cdots\bar{x}_k^{i_k}$ of degree $[\mu]$. By definition of the product in $\mathrm{Gr}_{\mathcal{F}}(\mathcal{L}_{0,n})^{U_q}$,

$$\bar{x}_1^{i_1}\cdots\bar{x}_k^{i_k}=x_1^{i_1}\cdots x_k^{i_k}+\mathcal{F}^{\prec[\mu]},$$

so $x_1^{i_1}\cdots x_k^{i_k}\in\mathcal{F}^{[\mu]}\setminus\mathcal{F}^{\prec[\mu]}$, whence $\bar{x}_1^{i_1}\cdots\bar{x}_k^{i_k}=\overline{x_1^{i_1}\cdots x_k^{i_k}}$ and

$$x-\sum_{(i_1,\dots,i_k)\in I}\lambda_{(i_1,\dots,i_k)}x_1^{i_1}\cdots x_k^{i_k}\in\mathcal{F}^{\prec[\mu]}.$$

The conclusion follows by decreasing induction on $[\mu]$, since at last we terminate at $\mathcal{F}^{[0]}\cong\mathbb{C}(q)$.

By combining the steps (a′) to (e′), we get that $\mathcal{M}_{0,n}$ is a Noetherian and finitely generated ring. ■

Remark 3.6.

(1) Because $\mathcal{L}_{0,1}^{U_q}$ is the center of $\mathcal{L}_{0,1}$, (e′) proves it is finitely generated. Of course this follows also from the isomorphism $\mathcal{L}_{0,1}\cong U_q^{\mathrm{lf}}$ and the fact that the center of U_q^{lf} is the center of U_q (by Theorem 2.1), plus the well-known description of the latter.

(2) In the $\mathfrak{sl}_2$ case the filtration $\mathcal{F}$ on $\mathcal{L}_{0,n}^{U_q}$ should be related via the Wilson loop isomorphism (defined in [18, Section 8.2]) to the filtration of skein algebras of spheres with $n+1$ punctures used in [93].

4 Proof of Theorem 1.2

As usual we let ϵ be a primitive l-th root of unity with l odd and $l>d_i$ for all $i\in\{1,\dots,m\}$. We now consider the specialization $\mathcal{L}_{0,n}^{\epsilon}$ of $\mathcal{L}_{0,n}$ at $q=\epsilon$, defined in Section 2.2.1. Recall the isomorphism of algebras $\eta^*\colon\mathcal{O}(G)\to\mathcal{Z}_0(\mathcal{O}_\epsilon)$ (see (2.71)), and that $\mathcal{L}_{0,n}^{\epsilon}=\mathcal{O}_\epsilon^{\otimes n}$ as a vector space. Consider the linear subspace of $\mathcal{L}_{0,n}^{\epsilon}$ defined by $\mathcal{Z}_0\big(\mathcal{L}_{0,n}^{\epsilon}\big):=\mathcal{Z}_0(\mathcal{O}_\epsilon)^{\otimes n}$. This space is naturally a subalgebra of $\mathcal{O}_\epsilon^{\otimes n}$ (endowed with the componentwise product $\star$). In fact, we also have the following.

Proposition 4.1.

(1) *$\mathcal{Z}_0\big(\mathcal{L}_{0,n}^{\epsilon}\big)$ is a central subalgebra of the algebra $\mathcal{L}_{0,n}^{\epsilon}$, and the $\mathcal{Z}_0\big(\mathcal{L}_{0,n}^{\epsilon}\big)$-modules $\mathcal{L}_{0,n}^{\epsilon}$ and $\mathcal{O}_\epsilon^{\otimes n}$, with actions defined by the respective products of these algebras, do coincide.*

(2) *$\mathcal{L}_{0,n}^{\epsilon}$ is a free $\mathcal{Z}_0\big(\mathcal{L}_{0,n}^{\epsilon}\big)$-module of rank $l^{n.\dim\mathfrak{g}}$.*

(3) *$\big(\eta^{*-1}\big)^{\otimes n}\colon\mathcal{Z}_0\big(\mathcal{L}_{0,n}^{\epsilon}\big)\to\mathcal{O}(G)^{\otimes n}$ is an isomorphism of algebras, and $\mathcal{Z}_0\big(\mathcal{L}_{0,n}^{\epsilon}\big)$ is a Noetherian ring.*

(4) *The $\mathcal{Z}_0\big(\mathcal{L}_{0,n}^{\epsilon}\big)$-module $\mathcal{L}_{0,n}^{\epsilon}$ is finite and Noetherian. Therefore, $\mathcal{L}_{0,n}^{\epsilon}$ is a Noetherian ring.*

Note that the proof we give in (4) of the fact that $\mathcal{L}_{0,n}^{\epsilon}$ is Noetherian is independent from the proof of Theorem 3.1.

Proof. (1) Let us show that $\mathcal{Z}_0(\mathcal{L}_{0,n}^{\epsilon})$ is a central subalgebra of $\mathcal{L}_{0,n}^{\epsilon}$. In the case $n=1$, the formula (2.9) implies that $\alpha\beta = \alpha\star\beta$ for all $\alpha\in\mathcal{Z}_0(\mathcal{O}_\epsilon)$ and $\beta\in\mathcal{L}_{0,1}^{\epsilon}$. Indeed, by (2.9) we have

$$\begin{aligned}\alpha\beta &= \sum_{(R),(R)} (R_{(2')}S(R_{(2)})\rhd\alpha)\star(R_{(1')}\rhd\beta\lhd R_{(1)})\\ &= \sum_{(R),(R),(\alpha),(\beta)} \alpha_{(1)}\star(\beta_{(1)}(R_{(1)}\alpha_{(3)}(S(R_{(2)}))\beta_{(3)}(R_{(1')}\alpha_{(2)}(R_{(2')}))\beta_{(2)}),\end{aligned}$$

where all components $\alpha_{(1)},\alpha_{(2)},\alpha_{(3)}\in\mathcal{Z}_0(\mathcal{O}_\epsilon)$, since $\mathcal{Z}_0(\mathcal{O}_\epsilon)$ is a Hopf subalgebra of $\mathcal{O}_\epsilon$. But

$$\sum_{(R)} R_{(1)}\alpha_{(3)}(S(R_{(2)})) = S^{-1}(\Phi^-(S_{\mathcal{O}_\epsilon}(\alpha_{(3)})))\in\mathcal{Z}_0(U_\epsilon),$$

since $\Phi^-(S_{\mathcal{O}_\epsilon}(\alpha_{(3)}))\in\mathcal{Z}_0(U_\epsilon)$ by Theorem 2.29 (2). Similarly, $\sum_{(R)} R_{(1')}\alpha_{(2)}(R_{(2')})\in\mathcal{Z}_0(U_\epsilon)$. In general, these elements belong to $\mathcal{Z}_0(U_\epsilon)$ and not $\mathcal{Z}_0(U_\epsilon^{\mathrm{ad}})$ because of the "diagonal" factor Θ of the R-matrix in (2.4). By Lemma 2.28, $\mathcal{Z}_0(U_A^{\mathrm{ad}})$ acts by the trivial character ε (the counit) on specializations of Γ-modules. The action of $\mathcal{Z}_0(U_A)$ is the counit ε multiplied with some powers of $\epsilon^{1/D}$. However, [18, Propositions 4.1 and 4.10] show that such powers of $\epsilon^{1/D}$ eventually disappear in the sum above; this is because the sum can be rewritten in terms of copies of the quasi R-matrix $\hat{R}$ in (2.4) and the pivotal element ℓ, instead of copies of R. Therefore,

$$\alpha\beta = \sum_{(\alpha),(\beta)} \alpha_{(1)}\star(\varepsilon(\beta_{(1)})\varepsilon(\alpha_{(3)})\varepsilon(\beta_{(3)})\varepsilon(\alpha_{(2)})\beta_{(2)}) = \alpha\star\beta. \tag{4.1}$$

This shows $\mathcal{L}_{0,1}^{\epsilon}$ and $\mathcal{O}_\epsilon$ coincide as modules over $\mathcal{Z}_0(\mathcal{L}_{0,1}^{\epsilon}) = \mathcal{Z}_0(\mathcal{O}_\epsilon)$. Next, we show that the subalgebras $\mathcal{Z}_0(\mathcal{O}_\epsilon)^{(a)}$ are central in $\mathcal{L}_{0,n}^{\epsilon}$ for all $a=1,\dots,n$. This fact will conclude the proof that $\mathcal{L}_{0,n}^{\epsilon}$ and $\mathcal{O}_\epsilon^{\otimes n}$ coincide as $\mathcal{Z}_0(\mathcal{L}_{0,n}^{\epsilon})$-modules, because the subalgebras $\mathcal{Z}_0(\mathcal{O}_\epsilon)^{(a)}$ generate the space $\mathcal{Z}_0(\mathcal{L}_{0,n}^{\epsilon})$ in $(\mathcal{L}_{0,1}^{\epsilon})^{\otimes n}$, and hence in $\mathcal{L}_{0,n}^{\epsilon}$ (this follows from the comment before (2.18)).

In order to show that $\mathcal{Z}_0(\mathcal{O}_\epsilon)^{(a)}$ is central in $\mathcal{L}_{0,n}^{\epsilon}$ for all $a=1,\dots,n$, it is enough to show $\mathcal{Z}_0(\mathcal{O}_\epsilon)^{(a)}$ commutes with the elements of $\mathcal{L}_{0,n}^{\epsilon}$ supported by the tensor factors $(\mathcal{L}_{0,1}^{\epsilon})^{(b)}$ with $b\neq a$. Since $(\alpha)^{(a)}\otimes(\beta)^{(b)} = ((\alpha)^{(a)}\otimes 1)(1\otimes(\beta)^{(b)})$ by the definition, we have to show that $(1\otimes(\beta)^{(b)})((\alpha)^{(a)}\otimes 1) = (\alpha)^{(a)}\otimes(\beta)^{(b)}$ whenever $\alpha\in\mathcal{Z}_0(\mathcal{O}_\epsilon)$. We have (denoting $\sum_{(\alpha),(\alpha),(\alpha),(\alpha)}$ by $\sum_{(\alpha)^4}$, $\Delta(\alpha_{(1)}) = \sum_{(\alpha)}\alpha_{(1)(1)}\otimes\alpha_{(1)(2)}$ etc.):

$$\begin{aligned}(1\otimes(\beta)^{(b)})((\alpha)^{(a)}\otimes 1) &= \sum_{(R^i)} \big(S(R^3_{(1)}R^4_{(1)})\rhd\alpha\lhd R^1_{(1)}R^2_{(1)}\big)^{(a)}\\ &\quad\otimes\big(S(R^1_{(2)}R^3_{(2)})\rhd\beta\lhd R^2_{(2)}R^4_{(2)}\big)^{(b)}\\ &= \sum_{(R^i),(\alpha)^4,(\beta)^2} (\alpha_{(2)})^{(a)}\otimes(\beta_{(2)})^{(b)}\\ &\quad\times\beta_{(1)}\big(\alpha_{(1)(2)}\big(R^2_{(1)}\big)R^2_{(2)}\alpha_{(3)(1)}\big(S\big(R^4_{(1)}\big)\big)R^4_{(2)}\big)\\ &\quad\times\beta_{(3)}\big(\alpha_{(3)(2)}\big(R^3_{(1)}\big)R^3_{(2)}\alpha_{(1)(1)}\big(R^1_{(1)}\big)S\big(R^1_{(2)}\big)\big).\end{aligned}$$

By Theorem 2.29 (2), it follows that

$$\alpha_{(1)(2)}\big(R^2_{(1)}\big)R^2_{(2)} = \Phi^+(\alpha_{(1)(2)})\in\mathcal{Z}_0(U_\epsilon),$$

and similarly

$$\alpha_{(3)(1)}\big(S\big(R^4_{(1)}\big)\big)R^4_{(2)},\alpha_{(3)(2)}\big(R^3_{(1)}\big)R^3_{(2)},\alpha_{(1)(1)}\big(R^1_{(1)}\big)S\big(R^1_{(2)}\big)\in\mathcal{Z}_0(U_\epsilon).$$

Denote by z any such element; $\mathcal{Z}_0\big(U_\epsilon^{\mathrm{ad}}\big)$ acts by the trivial character (the counit ε) on specializations of Γ-modules. By using [18, Proposition 6.2], arguing as above (4.1), we obtain that the expression of z in terms of the corresponding $\alpha_{(i)(j)}$ involves $\varepsilon(z) = \varepsilon(\alpha_{(i)(j)})$ only $\big($no root $\epsilon^{1/D}\big)$. It follows

$$\begin{aligned}&\beta_{(1)}\big(\alpha_{(1)(2)}\big(R^2_{(1)}\big)R^2_{(2)}\alpha_{(3)(1)}\big(S\big(R^4_{(1)}\big)\big)R^4_{(2)}\big)\\&\qquad = \varepsilon(\alpha_{(1)(2)}\alpha_{(3)(1)})\beta_{(1)}(1) = \varepsilon(\alpha_{(1)(2)})\varepsilon(\alpha_{(3)(1)})\varepsilon(\beta_{(1)}),\\&\beta_{(3)}\big(\alpha_{(3)(2)}\big(R^3_{(1)}\big)R^3_{(2)}\alpha_{(1)(1)}\big(R^1_{(1)}\big)S\big(R^1_{(2)}\big)\big) = \varepsilon(\alpha_{(3)(2)})\varepsilon(\alpha_{(1)(1)})\varepsilon(\beta_{(3)}).\end{aligned}$$

Therefore, $\big(1\otimes(\beta)^{(b)}\big)\big((\alpha)^{(a)}\otimes 1\big) = (\alpha)^{(a)}\otimes(\beta)^{(b)}$. It follows that $\mathcal{L}^\epsilon_{0,n} = \mathcal{O}_\epsilon^{\otimes n}$ as modules over $\mathcal{Z}_0\big(\mathcal{L}^\epsilon_{0,n}\big)$; for instance when $n=2$, given $\alpha',\beta'\in\mathcal{Z}_0\big(\mathcal{L}^\epsilon_{0,1}\big)$ we have $(\alpha'\otimes\beta')(\alpha\otimes\beta) = (\alpha'\otimes 1)(1\otimes\beta')(\alpha\otimes 1)(1\otimes\beta)$ immediately by (2.17), and $(1\otimes\beta')(\alpha\otimes 1) = \alpha\otimes\beta' = (\alpha\otimes 1)(1\otimes\beta')$ as above. Then $(\alpha'\otimes\beta')(\alpha\otimes\beta) = \alpha'\alpha\otimes\beta'\beta$. In particular, $\mathcal{Z}_0\big(\mathcal{L}^\epsilon_{0,n}\big)$ is a central subalgebra of $\mathcal{L}^\epsilon_{0,n}$.

(2) Since $\mathcal{L}^\epsilon_{0,n}$ and $\mathcal{O}_\epsilon^{\otimes n}$ coincide as modules over $\mathcal{Z}_0\big(\mathcal{L}^\epsilon_{0,n}\big) = \mathcal{Z}_0(\mathcal{O}_\epsilon^{\otimes n})$, the claim follows from Theorem 2.29, that is, from [41, Theorem 7.2], which shows that $\mathcal{O}_\epsilon$ is a finitely generated projective module of rank $l^{\dim\mathfrak{g}}$ over $\mathcal{Z}_0(\mathcal{O}_\epsilon)$, and from the arguments of [28] (using that $K_0(\mathcal{O}(G)) = \mathbb{Z}$ by [87]), which imply that this module is free. Alternatively, it follows from the fact that $\mathcal{O}_\epsilon$ is a cleft extension of $\mathcal{O}(G)$ (see [6, Remark 2.18 (b)], and [25, Section 3.2]).

(3) The linear isomorphism $\big(\eta^{*-1}\big)^{\otimes n}\colon \mathcal{Z}_0\big(\mathcal{L}^\epsilon_{0,n}\big)\to\mathcal{O}(G)^{\otimes n}$ is an isomorphism of algebras because $\mathcal{Z}_0\big(\mathcal{L}^\epsilon_{0,n}\big)$ is central in $\mathcal{L}^\epsilon_{0,n}$. It implies that $\mathcal{Z}_0\big(\mathcal{L}^\epsilon_{0,n}\big)$ is a Noetherian ring, since $\mathcal{O}(G)^{\otimes n} = \mathcal{O}(G^n)$ and G^n is an affine algebraic variety.

(4) The fact that $\mathcal{L}^\epsilon_{0,n}$ is a finitely generated $\mathcal{Z}_0\big(\mathcal{L}^\epsilon_{0,n}\big)$-module follows from (2); an alternative proof of this fact will be provided at the end of the proof of Theorem 4.9. Since $\mathcal{L}^\epsilon_{0,n}$ is finite over $\mathcal{Z}_0\big(\mathcal{L}^\epsilon_{0,n}\big)$ and $\mathcal{Z}_0\big(\mathcal{L}^\epsilon_{0,n}\big)$ is Noetherian, $\mathcal{L}^\epsilon_{0,n}$ is a Noetherian $\mathcal{Z}_0\big(\mathcal{L}^\epsilon_{0,n}\big)$-module (e.g., by [7, Proposition 6.5]). It follows that $\mathcal{L}^\epsilon_{0,n}$ is a Noetherian ring (e.g., by [88, Chapter 1, Section 1.3]). ■

Recall that we denote $U_\epsilon^{\mathrm{lf}} = U_A^{\mathrm{lf}}\bigotimes_A\mathbb{C}_\epsilon$ (see (2.27)), and $\mathcal{Z}_0(U_\epsilon)\subset U_\epsilon$ is the central polynomial subalgebra generated by $E^l_{\beta_k}$, $F^l_{\beta_k}$, $L_i^{\pm l}$, for $k\in\{1,\dots,N\}$ and $i\in\{1,\dots,m\}$. Since $\Phi_1\colon\mathcal{L}^\epsilon_{0,1}\to U_\epsilon^{\mathrm{lf}}$ is an embedding of algebras (see Corollary 2.25), it identifies $\mathcal{Z}_0\big(\mathcal{L}^\epsilon_{0,1}\big)$ with a central subalgebra of U_ϵ^{lf}. Put $\mathcal{Z}_0\big(U_\epsilon^{\mathrm{lf}}\big) := \Phi_1\big(\mathcal{Z}_0\big(\mathcal{L}^\epsilon_{0,1}\big)\big)$. Recall Theorem 2.1, Proposition 2.24, and let $T^{(l)}$, $T_{2-}^{(l)}$ and $T_2^{(l)}$ be the subsets of T, T_{2-} and T_2 formed by the elements $K_{\lambda l}$ with $\lambda\in P$, $\lambda\in -2P_+$ and $\lambda\in 2P$, respectively.

Proposition 4.2. *We have $U_\epsilon = T_{2-}^{-1}U_\epsilon^{\mathrm{lf}}[T/T_2] = \Phi_1\big(\mathcal{L}^\epsilon_{0,1}\big[d^{-1}\big]\big)[T/T_2]$, and therefore the map $\Phi_1\colon\mathcal{L}^\epsilon_{0,1}\big[d^{-1}\big]\to T_{2-}^{-1}U_\epsilon^{\mathrm{lf}}$ is an isomorphism.*

Moreover, $\mathcal{Z}\big(U_\epsilon^{\mathrm{lf}}\big) = U_\epsilon^{\mathrm{lf}}\cap\mathcal{Z}(U_\epsilon)$, and

$$\mathcal{Z}_0(U_\epsilon) = T_{2-}^{(l)-1}\mathcal{Z}_0\big(U_\epsilon^{\mathrm{lf}}\big)\big[T^{(l)}/T_2^{(l)}\big],\qquad \mathcal{Z}(U_\epsilon) = T_{2-}^{(l)-1}\mathcal{Z}\big(U_\epsilon^{\mathrm{lf}}\big)\big[T^{(l)}/T_2^{(l)}\big].$$

Proof. The first claim follows immediately from Proposition 2.24 by specialization at $q=\epsilon$. For the second claim, the inclusion $U_\epsilon^{\mathrm{lf}}\cap\mathcal{Z}(U_\epsilon)\subset\mathcal{Z}\big(U_\epsilon^{\mathrm{lf}}\big)$ is clear, and for the converse inclusion we only have to show that the elements of $\mathcal{Z}\big(U_\epsilon^{\mathrm{lf}}\big)$ commute with T. They commute with $T_2\subset U_\epsilon^{\mathrm{lf}}$, so the conjugation action by elements of T on $\mathcal{Z}\big(U_\epsilon^{\mathrm{lf}}\big)$ has order at most 2. Let $x\in\mathcal{Z}\big(U_\epsilon^{\mathrm{lf}}\big)$ with decomposition $x=\sum_i c_ix_i$ with all $c_i\in\mathbb{C}$ and x_i PBW basis vectors, and let $\lambda\in P$. We have $K_\lambda xK_{-\lambda} = \sum_i c_iq(x_i)x_i$, where $q(x_i)\in\epsilon^{\mathbb{Z}}$ satisfies $q(x_i)^2=1$ for all i. Because ϵ has odd order the only possibility is $q(x_i)=1$, whence $K_\lambda xK_{-\lambda}=x$. The conclusion follows.

The inclusion $\mathcal{Z}_0\big(U_\epsilon^{\mathrm{lf}}\big)\subset\mathcal{Z}_0(U_\epsilon)$ follows from the definition $\mathcal{Z}_0\big(\mathcal{L}^\epsilon_{0,1}\big) = \mathcal{Z}_0(\mathcal{O}_\epsilon)$, the formula $\Phi_1 = m\circ\big(\mathrm{id}\otimes S^{-1}\big)\circ\Phi$, and the fact that Φ affords an embedding $\mathcal{Z}_0(\mathcal{O}_\epsilon)\to\mathcal{Z}_0(U_\epsilon(G^*))$

(see Theorem 2.29 (2)). Since $T^{(l)} \subset \mathcal{Z}_0(U_\epsilon)$, we obtain

$$T_{2-}^{(l)-1}\mathcal{Z}_0\big(U_\epsilon^{\mathrm{lf}}\big)\big[T^{(l)}/T_2^{(l)}\big] \subset \mathcal{Z}_0(U_\epsilon).$$

The proof of the converse inclusion is similar to that in Proposition 2.24. The isomorphism $\mathcal{Z}_0(\mathcal{O}_\epsilon)\big[\psi_{-l\rho}^{-1}\big] \to \mathcal{Z}_0(U_\epsilon(G^*))$ of Theorem 2.29 (2) implies

$$F_{\beta_k}^l K_{\beta_k}^l \otimes 1, 1 \otimes K_{\beta_k}^{-l} E_{\beta_k}^l \in \Phi\big(\mathcal{Z}_0(\mathcal{O}_\epsilon)\big[\psi_{-l\rho}^{-1}\big]\big)$$

for every positive root β_k. Since $\psi_{-l\rho} = \Phi_1^{-1}(K_{-2l\rho}) = \psi_{-\rho}^l$ (the l-th power of $\psi_{-\rho}$ in $\mathcal{L}_{0,1}^\epsilon$), and

$$\Phi_1\big(\mathcal{Z}_0\big(\mathcal{L}_{0,1}^\epsilon\big)\big[\psi_{-\rho}^{-l}\big]\big) = T_{2-}^{(l)-1}\mathcal{Z}_0\big(U_\epsilon^{\mathrm{lf}}\big),$$

it follows that

$$F_{\beta_k}^l K_{\beta_k}^l, S^{-1}\big(E_{\beta_k}^l\big)K_{\beta_k}^l \in T_{2-}^{(l)-1}\mathcal{Z}_0\big(U_\epsilon^{\mathrm{lf}}\big).$$

Hence $F_{\beta_k}^l, S^{-1}\big(E_{\beta_k}^l\big) \in T_{2-}^{(l)-1}\mathcal{Z}_0\big(U_\epsilon^{\mathrm{lf}}\big)\big[T^{(l)}/T_2^{(l)}\big]$. The sets $S^{-1}\big(E_{\beta_k}^l\big)\mathcal{Z}_0(U_\epsilon(\mathfrak{h}))$, $k = 1, \dots, N$, generate the subalgebra $\mathcal{Z}_0(U_\epsilon(\mathfrak{b}_+))$ of $\mathcal{Z}_0(U_\epsilon)$, so from the triangular decomposition $\mathcal{Z}_0(U_\epsilon) = \mathcal{Z}_0(U_\epsilon(\mathfrak{n}_-))\mathcal{Z}_0(U_\epsilon(\mathfrak{h}))\mathcal{Z}_0(U_\epsilon(\mathfrak{n}_+))$ this proves the inclusion $\mathcal{Z}_0(U_\epsilon) \subset T_{2-}^{(l)-1}\mathcal{Z}_0\big(U_\epsilon^{\mathrm{lf}}\big)\big[T^{(l)}/T_2^{(l)}\big]$. From the isomorphism

$$\mathcal{Z}_0(U_\epsilon) \bigotimes_{\mathcal{Z}_0(U_\epsilon)\cap\mathcal{Z}_1(U_\epsilon)} \mathcal{Z}_1(U_\epsilon) \to \mathcal{Z}(U_\epsilon)$$

(see Theorem 2.27), and the fact that $\mathcal{Z}(U_q) \subset U_q^{\mathrm{lf}}$ $\big($whence $\mathcal{Z}_1(U_\epsilon) \subset \mathcal{Z}\big(U_\epsilon^{\mathrm{lf}}\big)\big)$, the equality $\mathcal{Z}(U_\epsilon) = T_{2-}^{(l)-1}\mathcal{Z}\big(U_\epsilon^{\mathrm{lf}}\big)\big[T^{(l)}/T_2^{(l)}\big]$ follows at once. ■

Remark 4.3. Let us explain how this can be used to give an interpretation of the isomorphism $\mathcal{Z}_0\big(\mathcal{L}_{0,1}^\epsilon\big) \cong \mathcal{O}(G)$. Recall the notations introduced around Theorem 2.27. Since $G^* = U_+T_GU_-$, we have $\mathcal{O}(G^*) = \mathcal{O}(U_+)\mathcal{O}(T_G)\mathcal{O}(U_-)$, and the map σ yields an identification

$$\mathcal{O}\big(G^0\big) = \mathcal{O}(U_+)\mathcal{O}(T_G/(2))\mathcal{O}(U_-). \tag{4.2}$$

We can identify $\mathcal{O}\big(G^0\big)$ with the subalgebra $(\sigma_{|G^*})^*\big(\mathcal{O}\big(G^0\big)\big) \subset \mathcal{O}(G^*)$. Consider the exterior power $V = \wedge^N\mathfrak{g}$ endowed with the action $\wedge^N\mathrm{Ad}$ of G. Put on $\mathfrak{g}$ a basis consisting of one element e_α per root space $\mathfrak{g}_\alpha$, along with a basis of $\mathfrak{h}$. Let $v \in V$ be the exterior power of the e_α's for α negative, and v^* a dual vector such that $v^*(v) = 1$ and v^* vanishes on a T_G-invariant complement of v. It is classical that $G \setminus G^0$ has defining equation $\delta(g) = 0$, where δ is the matrix coefficient $\delta(g) = v^*(\pi_V(g)v)$ (see, e.g., [59, p. 174]). Hence $\mathcal{O}\big(G^0\big) = \mathcal{O}(G)\big[\delta^{-1}\big]$. On G^0 we have $\delta(u_+tu_-) = \chi_{-2\rho}(t)$, where $\chi_{-2\rho}$ is the character of T_G associated to the root -2ρ. Now we can make the connection with U_ϵ. The isomorphism $\mathcal{Z}_0(U_\epsilon) \cong \mathcal{O}(G^*)$ of Theorem 2.27 (2) identifies $\mathcal{Z}_0(U_\epsilon(\mathfrak{h})) = \mathbb{C}\big[T^{(l)}\big]$ with $\mathcal{O}(T_G)$ by mapping $K_{\lambda l}$ to the character of T_G associated to λ. Therefore, it maps $\mathbb{C}\big[T_2^{(l)}\big]$ to $\mathcal{O}(T_G/(2))$, and $T_{2-}^{(l)-1}\mathcal{Z}_0\big(U_\epsilon^{\mathrm{lf}}\big)$ to $\mathcal{O}\big(G^0\big)$ by (4.2) and Proposition 4.2. Since $\mathcal{O}\big(G^0\big) = \mathcal{O}(G)\big[\delta^{-1}\big]$ and $T_{2-}^{(l)-1}\mathcal{Z}_0\big(U_\epsilon^{\mathrm{lf}}\big) = \mathcal{Z}_0\big(U_\epsilon^{\mathrm{lf}}\big)\big[\ell^l\big]$, it follows that $\mathcal{Z}_0\big(U_\epsilon^{\mathrm{lf}}\big)$ and $\mathcal{O}(G)$ coincide after localization by ℓ^l and δ respectively. By using the Bruhat decomposition of G as in (4.6) in the proof of Theorem 4.9 below, one can deduce $\mathcal{Z}_0\big(U_\epsilon^{\mathrm{lf}}\big) \cong \mathcal{O}(G)$, whence $\mathcal{Z}_0\big(\mathcal{L}_{0,1}^\epsilon\big) \cong \mathcal{O}(G)$ by injectivity of Φ_1.

Let us make the following observation. Since $\mathcal{L}_{0,n}^\epsilon = \mathcal{L}_{0,n}^A \bigotimes_A \mathbb{C}_\epsilon$, with $\mathcal{L}_{0,n}^A = \mathcal{O}_A^{\otimes n}$ as an A-module, and a generating system of $\mathcal{O}_A^{\otimes n}$ is also a generating system of $\mathcal{L}_{0,n}^A$, it follows from Proposition 2.10 and the identities (2.56)–(2.57) that $\mathcal{L}_{0,n}^\epsilon$ is generated by elements of the form $\alpha_1 \otimes \cdots \otimes \alpha_n$, where $\alpha_1, \dots, \alpha_n$ belong to the set C_{gen} of matrix coefficients lying on the first and last columns of the matrix representations of U_A^{res} in the canonical bases of the modules ${}_AV_{\varpi_i}$, $i = 1, \dots, m$. Denote by $\alpha^{\star k}$, $k \in \mathbb{N}$, the k-th power of an element $\alpha \in \mathcal{O}_A$.

Lemma 4.4. *For all* $\alpha \in C_{\rm gen}$, $\alpha^{\star l} \in \mathcal{Z}_0\big(\mathcal{L}_{0,1}^{\epsilon}\big)$.

Proof. Recall that the Frobenius epimorphism $\eta\colon U_A^{\rm res}\bigotimes_A \mathbb{C}_\epsilon \to U(\mathfrak{g})$ in (2.71) has kernel the ideal I generated by the elements E_i, F_i, K_i-1, and $(K_i;p)_{q_i}$ where l does not divide p, $i=1,\dots,m$. It follows that an element of $\mathcal{O}_\epsilon$ belongs to $\mathcal{Z}_0(\mathcal{O}_\epsilon)=\eta^*(\mathcal{O}(G))$ if and only if it vanishes on I. But this is immediate to check for the elements of the form $\alpha^{\star l}$ with $\alpha \in C_{\rm gen}$, using that K_i is grouplike and the pure summands of $\Delta(E_i)$ and $\Delta(F_i)$ have one component equal to 1 or $K_i^{\pm 1}$ and the other component equal to E_i or F_i. For instance,

$$\psi_{\varpi_i}^{\star l}(K_i-1)=\psi_{\varpi_i}(K_i)^l-1=\epsilon^{l(\alpha_i,\varpi_i)}-1=0.$$

Similarly, for every $\alpha\in C_{\rm gen}$, we find

$$\alpha^{\star l}(E_i)=\alpha^{\otimes l}\big(\Delta^{(l)}(E_i)\big)=0 \qquad \text{and} \qquad \alpha^{\star l}(F_i)=\alpha^{\star l}(K_i-1)=0. \qquad \blacksquare$$

We need below explicit descriptions of the centers of $\mathcal{O}_\epsilon(\mathrm{SL}_2)$ and $\mathcal{L}_{0,1}^{\epsilon}(\mathfrak{sl}_2)$ and their $\mathcal{Z}_0$-subalgebras. Denote by a, b, c, d the standard generators of $\mathcal{O}_q(\mathrm{SL}_2)$, i.e., the matrix coefficients in the basis of weight vectors v_0, $v_1=F.v_0$ of the 2-dimensional irreducible representation V_1 of $U_q(\mathfrak{sl}_2)$. As above, denote by $x^{\star k}$, $k\in\mathbb{N}$, the k-th power of an element $x\in\mathcal{O}_A(\mathrm{SL}_2)$. The algebra $\mathcal{O}_A(\mathrm{SL}_2)$ is generated by a, b, c, d; the monomials $a^{\star i}\star b^{\star j}\star d^{\star r}$ and $a^{\star i}\star c^{\star k}\star d^{\star r}$, $i,j,k,r\in\mathbb{N}, k>0$, form an A-basis of $\mathcal{O}_A(\mathrm{SL}_2)$. The algebra $\mathcal{Z}_0(\mathcal{O}_\epsilon(\mathrm{SL}_2))$ is generated by $a^{\star l}$, $b^{\star l}$, $c^{\star l}$, $d^{\star l}$; the monomials $a^{\star il}\star b^{\star jl}\star d^{\star rl}$ and $a^{\star il}\star c^{\star kl}\star d^{\star rl}$ form a basis of $\mathcal{Z}_0(\mathcal{O}_\epsilon(\mathrm{SL}_2))$, and $\mathcal{Z}(\mathcal{O}_\epsilon(\mathrm{SL}_2))$ is generated by $\mathcal{Z}_0(\mathcal{O}_\epsilon(\mathrm{SL}_2))$ and the elements $b^{\star(l-k)}\star c^{\star k}$, $k=0,\dots,l$ (see [41, Proposition 1.4 and the appendix]). We have the relation

$$a^{\star l}\star d^{\star l}-b^{\star l}\star c^{\star l}=1 \tag{4.3}$$

and the Frobenius isomorphism of Parshall–Wang (see [92, Chapter 7]) coincides with the map

$$\mathrm{Fr}_{\rm PW}\colon\ \mathcal{O}(\mathrm{SL}_2)\to\mathcal{Z}_0(\mathcal{O}_\epsilon(\mathrm{SL}_2))$$

induced by η^*; it sends the standard generators $\underline{a}$, $\underline{b}$, $\underline{c}$, $\underline{d}$ of $\mathcal{O}(\mathrm{SL}_2)=\mathcal{O}_1(\mathrm{SL}_2)$ respectively to $a^{\star l}$, $b^{\star l}$, $c^{\star l}$, $d^{\star l}$. Finally, we have seen that $\mathcal{O}_\epsilon(\mathrm{SL}_2)$ is a free $\mathcal{Z}_0(\mathcal{O}(\mathrm{SL}_2))$-module of rank l^3 (see Theorem 2.29 (3)). In [38], it is shown that a basis of this module is formed by the monomials $a^m b^n c^{s'}$ and $b^n c^{s''} d^r$, with the integers m, n, r, s', s'' in the range

$$1\le m\le l-1,\qquad 0\le n,r\le l-1,\qquad m\le s'\le l-1,\qquad 0\le s''\le l-r-1. \tag{4.4}$$

Now consider $\mathcal{L}_{0,1}^{A}(\mathfrak{sl}_2)$. Recall that $\mathcal{L}_{0,1}^{A}=\mathcal{O}_A$ as U_A-modules. The algebra $\mathcal{L}_{0,1}^{A}(\mathfrak{sl}_2)$ is also generated by a, b, c, d; a set of defining relations is (see [18, Section 5]):

$$\begin{aligned} &ad=da,\qquad db=q^2bd,\qquad cd=q^2dc,\qquad ab-ba=-\big(1-q^{-2}\big)bd,\\ &cb-bc=\big(1-q^{-2}\big)\big(da-d^2\big),\qquad ac-ca=\big(1-q^{-2}\big)dc,\qquad ad-q^2bc=1. \end{aligned} \tag{4.5}$$

The element $\omega:=qa+q^{-1}d$ is central. Let T_k, $k\in\mathbb{N}$, be such that $T_k(x)/2$ is the k-th Chebyshev polynomial of the first type in the variable $x/2$. We have (see [18, Proposition 7.2], for the generalization to $\mathcal{L}_{0,n}^{\epsilon}(\mathfrak{sl}_2)$):

Lemma 4.5. *Let $\mathcal{I}$ be the ideal of $\mathbb{C}\big[\omega,b^l,c^l,d^l\big]$ generated by $\big(T_l(\omega)-d^l\big)d^l-b^lc^l-1$, we have*

$$\mathcal{Z}(\mathcal{L}_{0,1}^{\epsilon}(\mathfrak{sl}_2))=\mathbb{C}\big[\omega,b^l,c^l,d^l\big]/\mathcal{I}\qquad\text{and}\qquad\mathcal{Z}_0(\mathcal{L}_{0,1}^{\epsilon}(\mathfrak{sl}_2))=\mathbb{C}\big[T_l(\omega),b^l,c^l,d^l\big]/\mathcal{I}.$$

Here b^l, c^l, d^l are the l-th powers of b, c, d computed using the product of $\mathcal{L}_{0,1}^{A}(\mathfrak{sl}_2)$, not the product $\star$ of $\mathcal{Z}_0(\mathcal{O}_\epsilon(\mathrm{SL}_2))$. The above generator of $\mathcal{I}$ can be interpreted as a determinant, and ω as a quantum trace on V_1. The following has also been observed in [75].

Lemma 4.6. *Viewed as elements of $\mathcal{O}_A(\mathrm{SL}_2)$, $T_l(\omega) - d^l = a^{\star l}$ and $x^l = x^{\star l}$, $x \in \{b, c, d\}$.*

Proof. Let α and ϖ be the simple root and fundamental weight of $\mathfrak{sl}_2$. In the notations of (2.70), we have $b = \psi_{-\varpi}^{-\alpha}$, $c = \psi_{-\varpi}^{\alpha}$, $d = \psi_{-\varpi}$; the formulas of Φ give

$$\Phi_1\big(b^{\star l}\big) = \big(q - q^{-1}\big)^l F^l, \qquad \Phi_1\big(c^{\star l}\big) = \big(q - q^{-1}\big)^l E^l K^{-l}, \qquad \Phi_1\big(d^{\star l}\big) = K^{-l}.$$

These coincide respectively with $\Phi_1\big(b^l\big)$, $\Phi_1(c^l)$, $\Phi_1\big(d^l\big)$ (see [18, equation (5.3)]). By passing to the localization $\mathcal{O}_A(\mathrm{SL}_2)\big[d^{-1}\big]$, and using Parshall–Wang's relation (4.3), one deduces easily

$$\Phi_1\big(a^{\star l}\big) = K^l + \big(q - q^{-1}\big)^{2l} F^l E^l = T_l(\Omega) - K^{-l},$$

where $\Omega = \big(\epsilon - \epsilon^{-1}\big)^2 FE + \epsilon K + \epsilon^{-1} K^{-1}$ is the Casimir element, and $T_l(x)/2$ is the l-th Chebyshev polynomial of the first type in the variable $x/2$. We have $\Phi_1(\omega) = \Omega$, so $\Phi_1\big(a^{\star l}\big) = T_l(\omega) - d^l$. The conclusion follows from the injectivity of Φ_1. ∎

This lemma proves that we have a commutative diagram

$$\begin{tikzcd}
\mathcal{O}(\mathrm{SL}_2) \arrow[r, "\mathrm{Fr}_{\mathrm{PW}}"] \arrow[dr, "\mathrm{Fr}"'] & \mathcal{Z}_0(\mathcal{O}_\epsilon(\mathrm{SL}_2)) \arrow[r, hook] \arrow[d] & \mathcal{O}_\epsilon(\mathrm{SL}_2) \arrow[d] \\
& \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon(\mathfrak{sl}_2)) \arrow[r, hook] & \mathcal{L}_{0,1}^\epsilon(\mathfrak{sl}_2),
\end{tikzcd}$$

where $\mathrm{Fr}_{\mathrm{PW}}$ is Parshall–Wang's Frobenius isomorphism recalled above, Fr is the Frobenius isomorphism introduced in [18, Definition 7.1], and the vertical arrows are the identifications as vector spaces (the middle one proved by Proposition 4.1).

Remark 4.7. By Lemma 4.5, the monomials $T_l(\omega)^i b^{jl} d^{rl}$ and $T_l(\omega)^i c^{kl} d^{rl}$, for $i, j, k, r \in \mathbb{N}$ and $k > 0$, form an A-basis of $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon(\mathfrak{sl}_2))$. It is straightforward (though cumbersome) to express these basis elements in terms of the basis elements $a^{\star il} \star b^{\star jl} \star d^{\star rl}$ and $a^{\star il} \star c^{\star kl} \star d^{\star rl}$ of $\mathcal{Z}_0(\mathcal{O}_\epsilon(\mathrm{SL}_2))$, and conversely; this can be done by using Lemma 4.6, the formula (2.9) or the inverse one (expressing $\star$ in terms of the product of $\mathcal{L}_{0,1}$, see [18, equation (4.6)]), and the formula of the coproduct $\Delta\colon \mathcal{L}_{0,1}^\epsilon(\mathfrak{sl}_2)) \to \mathcal{L}_{0,2}^\epsilon(\mathfrak{sl}_2))$ restricted to $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon(\mathfrak{sl}_2))$ (given in [18, Lemma 7.5]).

Since $\mathcal{L}_{0,1}^A = \mathcal{O}_A$ as an A-module, the functionals t_i in Proposition 2.30 can be seen as maps $t_i\colon \mathcal{L}_{0,1}^A \to A$. Though the algebra structures of $\mathcal{O}_\epsilon$ and $\mathcal{L}_{0,1}^\epsilon$ are very different, $\mathcal{L}_{0,1}^\epsilon$ satisfies a result analogous to Proposition 2.30:

Proposition 4.8. *The maps $\lhd t_i$ preserve $\mathcal{Z}_0\big(\mathcal{L}_{0,1}^\epsilon\big)$, and they satisfy $(f \lhd t_i)(a) = f(n_i a)$ and $(f\alpha) \lhd t_i = (f \lhd t_i)(\alpha \lhd t_i)$ for every $f \in \mathcal{Z}_0\big(\mathcal{L}_{0,1}^\epsilon\big)$, $a \in G$, $\alpha \in \mathcal{L}_{0,1}^\epsilon$.*

Proof. The first two claims follow from Proposition 2.30 and the definition $\mathcal{Z}_0\big(\mathcal{L}_{0,1}^\epsilon\big) = \mathcal{Z}_0(\mathcal{O}_\epsilon)$.

The last claim follows from the case $\mathfrak{g} = \mathfrak{sl}_2$, as in the proof of [41, Proposition 7.1]. In fact, it is enough to show that $t(fg) = t(f)t(g)$ for every $f \in \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon(\mathfrak{sl}_2)$, $g \in \mathcal{L}_{0,1}^\epsilon(\mathfrak{sl}_2)$; for completeness we explain this in Appendix C, see (C.3). A word of caution is needed: the proof of (C.3) uses that $\Delta\colon \mathcal{O}_\epsilon \to \mathcal{O}_\epsilon \otimes \mathcal{O}_\epsilon$ is a morphism of algebras. The analogous property for $\mathcal{L}_{0,1}^\epsilon$ is that Δ yields a morphism of algebras $\Delta\colon \mathcal{L}_{0,1}^\epsilon \to \mathcal{L}_{0,2}^\epsilon$. Since the algebra structure of $\mathcal{L}_{0,2}^\epsilon$ is not the product one on $\mathcal{L}_{0,1}^\epsilon \otimes \mathcal{L}_{0,1}^\epsilon$, it is not true in general that

$$\sum_{(f),(g)} (f_{(1)} \otimes f_{(2)})(g_{(1)} \otimes g_{(2)}) = \sum_{(f),(g)} f_{(1)} g_{(1)} \otimes f_{(2)} g_{(2)}$$

for every $f, g \in \mathcal{L}_{0,1}^{\epsilon}$. However, it holds whenever $f \in \mathcal{Z}_0(\mathcal{L}_{0,1}^{\epsilon})$, since $\Delta(\mathcal{Z}_0(\mathcal{L}_{0,1}^{\epsilon})) \subset \mathcal{Z}_0(\mathcal{L}_{0,1}^{\epsilon}) \otimes \mathcal{Z}_0(\mathcal{L}_{0,1}^{\epsilon})$ and therefore $f_{(2)} \in \mathcal{Z}_0(\mathcal{L}_{0,1}^{\epsilon}) = \mathcal{Z}_0(\mathcal{O}_\epsilon)$ commutes in $\mathcal{L}_{0,2}^{\epsilon}$ with any $g_{(1)} \in \mathcal{L}_{0,1}^{\epsilon} = \mathcal{O}_\epsilon$.

It is enough to prove the identity $t(fg) = t(f)t(g)$ when f ranges in a set of generators of the algebra $\mathcal{Z}_0(\mathcal{L}_{0,1}^{\epsilon}(\mathfrak{sl}_2))$. So one can take f among, say, $T_l(\omega) - d^l = a^{\star l}$ and $x^l = x^{\star l}$, $x \in \{b, c, d\}$ (using Lemma 4.5). By (2.9) and Proposition C.1, we have

$$t(fg) = \sum_{(R),(R)} t\left(R_{(2')}S(R_{(2)}) \rhd f\right) t\left(R_{(1')} \rhd g \lhd R_{(1)}\right).$$

Expanding coproducts and using that $R^{-1} = (S \otimes \mathrm{id})(R)$, we deduce

$$\begin{aligned}
t(fg) &= \sum_{(f),(R),(R)} t(f_{(1)}) \langle f_{(2)}, R_{(2')}S(R_{(2)})\rangle t\left(R_{(1')} \rhd g \lhd R_{(1)}\right) \\
&= \sum_{(f),(R),(R)} t(f_{(1)}) t(\langle f_{(2)}, R_{(2')}\rangle R_{(1')} \rhd g \lhd \langle f_{(3)}, S(R_{(2)})\rangle R_{(1)}) \\
&= \sum_{(f)} t(f_{(1)}) t\left(S^{-1}(\Phi^-(f_{(2)})) \rhd g \lhd S^{-2}(\Phi^-(f_{(3)}))\right) \\
&= \sum_{(f)} t(f_{(1)}) \langle g, S^{-2}(\Phi^-(f_{(3)}))\underline{w}S^{-1}(\Phi^-(f_{(2)}))\rangle \\
&= \sum_{(f)} t(f_{(1)}) \varepsilon\left(S^{-2}(\Phi^-(f_{(3)}))\right)\varepsilon\left(S^{-1}(\Phi^-(f_{(2)}))\right) t(g),
\end{aligned}$$

where $\underline{w} \in \mathbb{U}_\Gamma$ is the quantum Weyl group element dual to t (see Appendix B), and in the last equality we used that Φ^- maps $\mathcal{Z}_0(\mathcal{O}_\epsilon)$ into $\mathcal{Z}_0(U_\epsilon)$ (see Theorem 2.29 (2)), which acts on specializations of Γ-modules by the trivial character (the counit) $\varepsilon \colon U_\epsilon \to \mathbb{C}$. By (B.6)–(B.7), we have $t(a^{\star l}) = t(d^{\star l}) = 0$ and $t(b^{\star l}) = 1$, $t(c^{\star l}) = -1$. Now the computation of $t(fg)$ follows easily. For instance, taking $f = b^l = b^{\star l}$, by using $\Delta(b^{\star l}) = a^{\star l} \otimes b^{\star l} + b^{\star l} \otimes d^{\star l}$ and $\Delta(d^{\star l}) = c^{\star l} \otimes b^{\star l} + d^{\star l} \otimes d^{\star l}$, we get

$$t(b^l g) = \varepsilon\left(S^{-2}\left(\Phi^-\left(b^{\star l}\right)\right)\right)\varepsilon\left(S^{-1}\left(\Phi^-\left(c^{\star l}\right)\right)\right)t(g) + \varepsilon\left(S^{-2}\left(\Phi^-\left(d^{\star l}\right)\right)\right)\varepsilon\left(S^{-1}\left(\Phi^-\left(d^{\star l}\right)\right)\right)t(g).$$

Since $b^{\star l} \in \mathcal{O}_\epsilon(U_+)$, $\Phi^-(b^{\star l}) = 0$. Also, it is immediate from the definition of Φ^- that $\Phi^-(d^{\star l}) = \Phi^-(d)^l = L^l$; alternatively, one can bypass this computation by observing that Φ^- sets an isomorphism from $\mathcal{O}_\epsilon(T_G) = \mathcal{O}_\epsilon(B_+) \cap \mathcal{O}_\epsilon(B_-)$ to $\mathbb{C}[L^{\pm 1}] = U_\epsilon(\mathfrak{b}_+) \cap U_\epsilon(\mathfrak{b}_-)$, mapping a generator d to L or L^{-1}. We have $\varepsilon(L^l) = 1$, and therefore

$$t(b^l g) = t(g) = t(b^l)t(g).$$

The other cases $f = T_l(\omega) - d^l, c^l, d^l$ are similar. ■

Theorem 4.9. *$\mathcal{L}_{0,n}^{\epsilon}$ is a free $\mathcal{Z}_0(\mathcal{L}_{0,n}^{\epsilon})$-module of rank $l^{n.\dim \mathfrak{g}}$, and $(\mathcal{L}_{0,n}^{\epsilon})^{U_\epsilon}$ is a Noetherian ring and a finite, whence Noetherian, $\mathcal{Z}_0(\mathcal{L}_{0,n}^{\epsilon})$-module.*

Proof. We already proved the first claim in Proposition 4.1, and that $\mathcal{L}_{0,n}^{\epsilon}$ is a Noetherian $\mathcal{Z}_0(\mathcal{L}_{0,n}^{\epsilon})$-module. For the second claim, it follows that the $\mathcal{Z}_0(\mathcal{L}_{0,n}^{\epsilon})$-submodule $(\mathcal{L}_{0,n}^{\epsilon})^{U_\epsilon}$ is necessarily finitely generated. But $\mathcal{Z}_0(\mathcal{L}_{0,n}^{\epsilon})$ being Noetherian, $(\mathcal{L}_{0,n}^{\epsilon})^{U_\epsilon}$ is then a Noetherian $\mathcal{Z}_0(\mathcal{L}_{0,n}^{\epsilon})$-module and a Noetherian ring.

For the sake of clarity, let us provide a self-contained proof of the first claim, not invoking directly [28, 41] or [6, 25], but applying the same arguments directly to $\mathcal{L}_{0,n}^{\epsilon}$. Since $\mathcal{L}_{0,n}^{\epsilon}$ and $\mathcal{L}_{0,1}^{\epsilon \otimes n}$ coincide as modules over $\mathcal{Z}_0(\mathcal{L}_{0,n}^{\epsilon}) = \mathcal{Z}_0(\mathcal{L}_{0,1}^{\epsilon})^{\otimes n}$ by Proposition 4.1, the result will follow from the case $n=1$. Then we argue in four steps. First, using Theorem 2.1 we show that a certain localization of $\mathcal{L}_{0,1}^{\epsilon}$ is a free module of rank $l^{\dim\mathfrak{g}}$. Then, assuming that $\mathcal{L}_{0,1}^{\epsilon}$ is finitely generated and projective, we explain why it has constant rank $l^{\dim\mathfrak{g}}$ (this is very classical). Thirdly, we prove that $\mathcal{L}_{0,1}^{\epsilon}$ is finitely generated and projective as in [41, Theorem 7.2]. Finally, we obtain that it is a free module as in [28].

Recall Proposition 4.2: U_ϵ is a free $\Phi_1\big(\mathcal{L}_{0,1}^{\epsilon}\big[d^{-l}\big]\big)$-module of rank 2^m $\big($note that $\mathcal{L}_{0,1}^{\epsilon}\big[d^{-l}\big]=\mathcal{L}_{0,1}^{\epsilon}\big[d^{-1}\big]\big)$, $\mathcal{Z}_0(U_\epsilon)$ is free over

$$T_{2-}^{(l)-1}\mathcal{Z}_0\big(U_\epsilon^{\mathrm{lf}}\big)=\Phi_1\big(\mathcal{Z}_0\big(\mathcal{L}_{0,1}^{\epsilon}\big)\big[d^{-l}\big]\big)$$

of rank 2^m. Since U_ϵ is also free of rank $l^{\dim\mathfrak{g}}$ over $\mathcal{Z}_0(U_\epsilon)$ (see Theorem 2.27 (1)), it is free over $\Phi_1\big(\mathcal{Z}_0\big(\mathcal{L}_{0,1}^{\epsilon}\big)\big[d^{-l}\big]\big)$ of rank $2^m l^{\dim\mathfrak{g}}$. The decomposition being unique, $\Phi_1\big(\mathcal{L}_{0,1}^{\epsilon}\big[d^{-l}\big]\big)$ is free of rank $l^{\dim\mathfrak{g}}$ over $\Phi_1\big(\mathcal{Z}_0\big(\mathcal{L}_{0,1}^{\epsilon}\big)\big[d^{-l}\big]\big)$, and injectivity of Φ_1 implies that $\mathcal{L}_{0,1}^{\epsilon}\big[d^{-l}\big]$ is free of rank $l^{\dim\mathfrak{g}}$ over $\mathcal{Z}_0\big(\mathcal{L}_{0,1}^{\epsilon}\big)\big[d^{-l}\big]$.

Assume now that $\mathcal{L}_{0,1}^{\epsilon}$ is finitely generated and projective. Let us show that its rank is $l^{\dim\mathfrak{g}}$. The localization $\big(\mathcal{L}_{0,1}^{\epsilon}\big)_P$ of $\mathcal{L}_{0,1}^{\epsilon}$ at any prime ideal P of $\mathcal{Z}_0\big(\mathcal{L}_{0,1}^{\epsilon}\big)$ is a free module over $\mathcal{Z}_0\big(\mathcal{L}_{0,1}^{\epsilon}\big)_P$ [96, Proposition 2.12.15]; the ranks of such modules are finite in number [96, Proposition 2.12.20]. If these ranks are all equal, then, by definition, it is the rank of $\mathcal{L}_{0,1}^{\epsilon}$ over $\mathcal{Z}_0\big(\mathcal{L}_{0,1}^{\epsilon}\big)$. This happens if $\mathcal{Z}_0\big(\mathcal{L}_{0,1}^{\epsilon}\big)$ has no nontrivial (i.e., $\neq 1$) idempotent [96, Corollary 2.12.23], which is the case since it has no nontrivial zero divisors. To compute the rank, suppose P does not contain $d^l=\psi_{-\rho}^l$. Such ideals P are in 1-1 correspondence with the prime ideals of $\mathcal{Z}_0\big(\mathcal{L}_{0,1}^{\epsilon}\big)\big[d^{-l}\big]$ by the natural ring monomorphism $\mathcal{Z}_0\big(\mathcal{L}_{0,1}^{\epsilon}\big)\to\mathcal{Z}_0\big(\mathcal{L}_{0,1}^{\epsilon}\big)\big[d^{-l}\big]$. The set $S=\mathcal{Z}_0\big(\mathcal{L}_{0,1}^{\epsilon}\big)\setminus P$ is multiplicatively closed, and we have also a ring morphism $\mathcal{Z}_0\big(\mathcal{L}_{0,1}^{\epsilon}\big)\big[d^{-l}\big]\to S^{-1}\mathcal{Z}_0\big(\mathcal{L}_{0,1}^{\epsilon}\big)$, which is also an injection (there are no zero divisors in $\mathcal{Z}_0\big(\mathcal{L}_{0,1}^{\epsilon}\big)$, whence in S). Then

$$\big(\mathcal{L}_{0,1}^{\epsilon}\big)_P=S^{-1}\mathcal{L}_{0,1}^{\epsilon}=\mathcal{L}_{0,1}^{\epsilon}\big[d^{-l}\big]\bigotimes_{\mathcal{Z}_0(\mathcal{L}_{0,1}^{\epsilon})[d^{-l}]}S^{-1}\mathcal{Z}_0\big(\mathcal{L}_{0,1}^{\epsilon}\big)$$

shows that $\big(\mathcal{L}_{0,1}^{\epsilon}\big)_P$ has over $\mathcal{Z}_0\big(\mathcal{L}_{0,1}^{\epsilon}\big)_P=S^{-1}\mathcal{Z}_0\big(\mathcal{L}_{0,1}^{\epsilon}\big)$ the same rank $l^{\dim\mathfrak{g}}$ as $\mathcal{L}_{0,1}^{\epsilon}\big[d^{-l}\big]$ over $\mathcal{Z}_0\big(\mathcal{L}_{0,1}^{\epsilon}\big)\big[d^{-l}\big]$. This proves our claim.

In order to show that $\mathcal{L}_{0,1}^{\epsilon}$ is finitely generated and projective over $\mathcal{Z}_0\big(\mathcal{L}_{0,1}^{\epsilon}\big)$, it is enough to show it is finite locally free, i.e., there are elements $d_i\in\mathcal{Z}_0\big(\mathcal{L}_{0,1}^{\epsilon}\big)$ such that the localizations $\mathcal{L}_{0,1}^{\epsilon}\big[d_i^{-1}\big]$ are finite free $\mathcal{Z}_0\big(\mathcal{L}_{0,1}^{\epsilon}\big)\big[d_i^{-1}\big]$-modules, and $\mathrm{Maxspec}\big(\mathcal{Z}_0\big(\mathcal{L}_{0,1}^{\epsilon}\big)\big)$ is covered by the open sets $U(d_i)$ made of the ideals not containing d_i (see [100, Lemma 77.2]).

We have seen above that $\mathcal{L}_{0,1}^{\epsilon}\big[d^{-l}\big]$ is free of rank $l^{\dim\mathfrak{g}}$ over $\mathcal{Z}_0\big(\mathcal{L}_{0,1}^{\epsilon}\big)\big[d^{-l}\big]$. By Remark 4.3, $\mathcal{Z}_0\big(\mathcal{L}_{0,1}^{\epsilon}\big)\big[d^{-l}\big]\cong\mathcal{Z}_0\big(U_\epsilon^{\mathrm{lf}}\big)\big[\ell^l\big]$ is isomorphic to $\mathcal{O}\big(G^0\big)$, and $\mathcal{O}\big(G^0\big)=\mathcal{O}(G)\big[\delta^{-1}\big]$. Now, given $w\in W$ with a reduced expression $s_{i_1}\cdots s_{i_k}$, put $t_w=t_{i_1}\cdots t_{i_k}$. Let w be represented by $n_w=n_{i_1}\cdots n_{i_k}$ in $N(T_G)$. By Proposition 4.8, we have $(f\lhd t_w)(x)=f(n_wx)$ for every $f\in\mathcal{Z}_0\big(\mathcal{L}_{0,1}^{\epsilon}\big)$, $x\in G$. Then

$$\mathcal{Z}_0\big(\mathcal{L}_{0,1}^{\epsilon}\big)\big[d^{-l}\big]\lhd t_w\cong\mathcal{O}\big(n_w^{-1}G^0\big)\cong\mathcal{O}(G)\big[(\delta\lhd t_w)^{-1}\big]. \tag{4.6}$$

If $b_1,\dots,b_r$ $\big(r:=l^{\dim\mathfrak{g}}\big)$ is a basis of $\mathcal{L}_{0,1}^{\epsilon}\big[d^{-l}\big]$ over $\mathcal{Z}_0\big(\mathcal{L}_{0,1}^{\epsilon}\big)\big[d^{-l}\big]$, then $\mathcal{L}_{0,1}^{\epsilon}\big[d^{-l}\big]\lhd t_w$ is free over $\mathcal{Z}_0\big(\mathcal{L}_{0,1}^{\epsilon}\big)\big[(d\lhd t_w)^{-l}\big]\cong\mathcal{O}\big(n_w^{-1}G^0\big)$ with basis $b_1\lhd t_w,\dots,b_r\lhd t_w$. Consider the Bruhat decomposition of G: any $g\in G$ can be written in the form $g=b_1nb_2$, where $b_1,b_2\in B_-$, $n\in W$. Hence $g=nn^{-1}b_1nb_2\in nB_+B_-=nG^0$, and therefore

$$G=\bigcup_{w\in W}(B_-n_wB_-)=\bigcup_{w\in W}\big(n_wG^0\big).$$

For every $w \in W$, put $d^l_w := d^l \lhd t_w$. Under the isomorphism of $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,1})$ with $\mathcal{O}(G)$, we thus get that $\mathrm{Maxspec}(\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,1}))$ is covered by the open sets $U(d^l_w) \cong n_w G^0$, and $\mathcal{L}^\epsilon_{0,1}[d^{-l}_w]$ is finite free over $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,1})[d^{-l}_w]$. Therefore, $\mathcal{L}^\epsilon_{0,1}$ is finitely generated and projective over $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,1})$.

Finally, let us explain why $\mathcal{L}^\epsilon_{0,1}$ is free over $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,1})$, following the arguments of [28]. Let R be a commutative Noetherian ring, put $X = \mathrm{Maxspec}(R)$, and let P be an R-module. Denote by R_I, P_I the localizations of R, P at a maximal ideal $I \in X$. Define the *f-rank* of P as $\text{f-rank}(P) = \inf_{I\in X}\{\text{ f-rank}_{R_I}(P_I)\}$, where $\text{f-rank}_{R_I}(P_I) = \sup\{r \in \mathbb{N}, R_I^{\otimes r} \subset P_I\} \in \mathbb{N} \cup \{+\infty\}$ (i.e., the maximal dimension of a free summand of P_I). Bass' Cancellation theorem asserts that if P is projective and $\text{f-rank}(P) > \dim(X)$, and $P \oplus Q \cong M \oplus Q$ for some R-modules Q and M such that Q is finitely generated and projective, then $P \cong M$ (see [19, Section IV.3.5, p. 167 and p. 170], taking $A = R$, or [88, Section 11.7.13]). Let us apply this to $R = \mathcal{O}(G)$ and $P = \mathcal{L}^\epsilon_{0,1}$. We proved above that $\text{f-rank}_{R_I}(P_I) = l^{\dim \mathfrak{g}}$, a constant, and we have $l^{\dim \mathfrak{g}} > \dim \mathfrak{g} = \dim(G)$. By a result of Marlin [87], G being semisimple and simply connected the Grothendieck ring $K_0(\mathcal{O}(G))$ is isomorphic to $\mathbb{Z}$. Therefore, $\mathcal{L}^\epsilon_{0,1} \oplus Q \cong \mathcal{O}(G)^r$ for some free $\mathcal{O}(G)$-module Q and $r \in \mathbb{N}$. Then Bass' cancellation implies $\mathcal{L}^\epsilon_{0,1}$ is free over $\mathcal{Z}_0(\mathcal{L}_{0,1}) \cong \mathcal{O}(G)$. ■

5 Proof of Theorem 1.3

We begin with the following lemma, interesting by itself.

Lemma 5.1. *$\mathcal{Z}(\mathcal{L}^\epsilon_{0,n})$ is a finite $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n})$-module and a Noetherian ring. Therefore, the ring $\mathcal{Z}(\mathcal{L}^\epsilon_{0,n})$ is integral over $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n})$.*

Proof. We know by Proposition 4.1 that $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n})$ is a Noetherian ring, and $\mathcal{L}^\epsilon_{0,n}$ is a finite Noetherian $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n})$-module. Therefore, the submodule $\mathcal{Z}(\mathcal{L}^\epsilon_{0,n})$ is finitely generated. Being finite over $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n})$, it is necessarily a Noetherian ring (e.g., by [7, Proposition 7.2]).

Let $x \in \mathcal{Z}(\mathcal{L}^\epsilon_{0,n})$. The $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n})$-submodule $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n})[x]$ of $\mathcal{L}^\epsilon_{0,n}$ is finitely generated by the same argument. Using the fact that an element x is integral over $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n})$ if and only if $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n})[x]$ is a finitely generated $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n})$-module (e.g., by [7, Proposition 5.1]), this proves the last claim. ■

We will use the following notations. Let A be a ring with no nontrivial zero divisors. The center $Z = Z(A)$ is a commutative integral domain. We denote by $Q(Z)$ its field of fractions, and put

$$Q(A) := Q(Z) \bigotimes_Z A.$$

It is an algebra over its center $Q(Z)$. Since $\mathcal{L}^\epsilon_{0,n}$ has no nontrivial zero divisors [18, Proposition 6.30], we can take $A = \mathcal{L}^\epsilon_{0,n}$, or $A = (\mathcal{L}^\epsilon_{0,n})^{U_\epsilon}$.

By the lemma, $\mathcal{Z}(\mathcal{L}^\epsilon_{0,n})$ is finite over $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n})$, so the ring $\mathcal{Z}(\mathcal{L}^\epsilon_{0,n}) \otimes_{\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n})} Q(\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n}))$ is a field. Necessarily it coincides with $Q(\mathcal{Z}(\mathcal{L}^\epsilon_{0,n}))$, and therefore

$$Q(\mathcal{L}^\epsilon_{0,n}) = Q(\mathcal{Z}(\mathcal{L}^\epsilon_{0,n})) \bigotimes_{\mathcal{Z}(\mathcal{L}^\epsilon_{0,n})} \mathcal{L}^\epsilon_{0,n} = Q(\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n})) \bigotimes_{\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n})} \mathcal{L}^\epsilon_{0,n}. \tag{5.1}$$

Recall that we denote by N the number of positive roots of $\mathfrak{g}$.

Theorem 5.2. *$Q(\mathcal{L}^\epsilon_{0,n})$ is a division algebra and a central simple algebra of PI degree l^{Nn}.*

Proof. It follows from (5.1) and Theorem 4.9 that $Q(\mathcal{L}_{0,n}^{\epsilon})$ is a vector space of dimension $l^{n.\dim\mathfrak{g}}$ over $Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^{\epsilon}))$, and therefore has finite dimension over its center $Q(\mathcal{Z}(\mathcal{L}_{0,n}^{\epsilon}))$. Because $\mathcal{L}_{0,n}^{\epsilon}$ has no nontrivial divisors [18, Proposition 6.30] and $Q(\mathcal{L}_{0,n}^{\epsilon})$ is finite-dimensional over $Q(\mathcal{Z}(\mathcal{L}_{0,n}^{\epsilon}))$, $Q(\mathcal{L}_{0,n}^{\epsilon})$ is a division algebra, whence a central simple algebra. By classical theory (see, e.g., [88, Section 13.3.5], or [96, Corollary 2.3.25]), there is a finite extension $\mathbb{F}$ of $Q(\mathcal{Z}(\mathcal{L}_{0,n}^{\epsilon}))$, a *splitting field*, such that

$$\mathbb{F} \bigotimes_{Q(\mathcal{Z}(\mathcal{L}_{0,n}^{\epsilon}))} Q(\mathcal{L}_{0,n}^{\epsilon}) = M_d(\mathbb{F}),$$

where $d \in \mathbb{N}$, the PI degree of $Q(\mathcal{L}_{0,n}^{\epsilon})$, satisfies

$$d^2 = \left[Q(\mathcal{L}_{0,n}^{\epsilon}) : Q(\mathcal{Z}(\mathcal{L}_{0,n}^{\epsilon}))\right] = \frac{\left[Q(\mathcal{L}_{0,n}^{\epsilon}) : Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^{\epsilon}))\right]}{\left[Q(\mathcal{Z}(\mathcal{L}_{0,n}^{\epsilon})) : Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^{\epsilon}))\right]}. \tag{5.2}$$

We have to show $d^2 = l^{2nN}$. We will obtain this equality by proving firstly that $d^2 \geq l^{2nN}$, and then $d^2 \leq l^{2nN}$.

In order to show that $d^2 \geq l^{2nN}$, it is enough to exhibit an irreducible representation V of $\mathcal{L}_{0,n}^{\epsilon}$ of dimension $k := l^{nN}$. Indeed, the representation map $\rho_V \colon \mathcal{L}_{0,n}^{\epsilon} \to \mathrm{End}_{\mathbb{C}}(V)$ being surjective, given basis elements $v_1, \dots, v_{k^2} \in \mathrm{End}(V)$, and elements $\alpha_1, \dots, \alpha_{k^2} \in \mathcal{L}_{0,n}^{\epsilon}$ such that $\rho(\alpha_i) = v_i$ for every $i \in \{1, \dots, k^2\}$, necessarily $\alpha_1, \dots, \alpha_{k^2}$ form a free family of $Q(\mathcal{L}_{0,n}^{\epsilon})$. For, if there was a nontrivial relation $\sum_i z_i\alpha_i = 0$, with $z_i \in Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^{\epsilon}))$, by clearing denominators and then applying the representation map ρ_V, we would get a nontrivial relation in $\mathrm{End}_{\mathbb{C}}(V)$ between $v_1, \dots, v_{k^2}$.

Now, by Theorem 2.27 (1) (see [42, Section 20]), the dimension of a generic irreducible representation space of U_ϵ is l^N. Because $U_\epsilon = T_{2-}^{-1}U_\epsilon^{\mathrm{lf}}[T/T_2]$ by Proposition 4.2, an irreducible representation of U_ϵ yields an irreducible representation of U_ϵ^{lf}. Moreover, the tensor product of n irreducible representation spaces of U_ϵ^{lf} of dimension l^N is an irreducible representation space of $(U_\epsilon^{\mathrm{lf}})^{\otimes n}$ of dimension l^{nN} (see, e.g., [51, Theorem 3.10.2]). Applying the linear isomorphism $\psi_n = \Phi_n \circ (\Phi_1^{-1})^{\otimes n}$ in (2.21) thus provides an irreducible representation of $\mathcal{L}_{0,n}^{\epsilon}$ of dimension l^{nN}.

It remains to show $d^2 \leq l^{2nN}$, which by $\left[Q(\mathcal{L}_{0,n}^{\epsilon}) : Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^{\epsilon}))\right] = l^{n(2N+m)}$ is equivalent to $\left[Q(\mathcal{Z}(\mathcal{L}_{0,n}^{\epsilon})) : Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^{\epsilon}))\right] \geq l^{mn}$. For this, it is enough to exhibit an extension of $Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^{\epsilon}))$ contained in $Q(\mathcal{Z}(\mathcal{L}_{0,n}^{\epsilon}))$ and of degree l^{mn}. There is a very natural one, which we denote by $Q(\hat{\mathcal{Z}}_0(\mathcal{L}_{0,n}^{\epsilon}))$ and is constructed as follows. Consider for every $\lambda \in P_+$ the matrices

$$M_\lambda := \left({}_AV_\lambda\phi_{e_k}^{e_l}\right)_{k,l} \in \mathrm{End}({}_AV_\lambda) \otimes \mathcal{L}_{0,n}^A, \qquad M_\lambda^{(i)} := \left(\left({}_AV_\lambda\phi_{e_k}^{e_l}\right)^{(i)}\right)_{k,l} \in \mathrm{End}({}_AV_\lambda) \otimes \mathcal{L}_{0,n}^A,$$

where $i = 1, \dots, n$, and as usual ${}_AV_\lambda\phi_{e_k}^{e_l}$ is a matrix coefficient of ${}_AV_\lambda$, $\{e_k\}$ the canonical basis of ${}_AV_\lambda$, and $\left({}_{V_\lambda}\phi_{e_k}^{e_l}\right)^{(i)} := 1^{\otimes(i-1)} \otimes {}_{V_\lambda}\phi_{e_k}^{e_l} \otimes 1^{\otimes(n-i)}$. Set

$${}_\lambda\omega := Tr(\pi_{V_\lambda}(\ell)M_\lambda), \qquad {}_\lambda\omega^{(i)} := Tr\left(\pi_{V_\lambda}(\ell)M_\lambda^{(i)}\right),$$

where Tr is the standard trace on $\mathrm{End}(V_\lambda)$. Clearly, ${}_\lambda\omega \in \mathcal{L}_{0,1}^A$, ${}_\lambda\omega^{(i)} \in \mathcal{L}_{0,n}^A$. By [18, Propositions 4.8 and 6.24], the family of elements $\prod_{i=1}^n {}_{\lambda_i}\omega^{(i)}$, where $\lambda_1, \dots, \lambda_n \in P_+$, is a basis of $\mathcal{Z}(\mathcal{L}_{0,n})$; moreover the Alekseev map Φ_n affords an isomorphism from $\mathcal{Z}(\mathcal{L}_{0,n})$ to $\mathcal{Z}(U_q)^{\otimes n}$, and $\Phi_n\left({}_\lambda\omega^{(i)}\right) = (\Phi_1({}_\lambda\omega))^{(i)}$. For $n = 1$, specializing q to ϵ it follows

$$\mathcal{Z}_1(U_\epsilon) = \mathrm{Vect}\{\Phi_1({}_\lambda\omega),\, \lambda \in P_+\}, \tag{5.3}$$

where $\mathcal{Z}_1(U_\epsilon)$ is defined before Theorem 2.27. Then, for every $i = 1, \dots, n$ define

$$\mathcal{Z}_{0,(i)}(\mathcal{L}_{0,n}^{\epsilon}) := \mathcal{Z}_0(\mathcal{L}_{0,n}^{\epsilon})\left[\left\{{}_\lambda\omega^{(i)}, \lambda \in P_+\right\}\right]$$

and let $\hat{\mathcal{Z}}_0(\mathcal{L}_{0,n}^\epsilon) \subset \mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)$ be the algebra generated by $\mathcal{Z}_{0,(1)}(\mathcal{L}_{0,n}^\epsilon), \dots, \mathcal{Z}_{0,(n)}(\mathcal{L}_{0,n}^\epsilon)$. The fields $Q(\mathcal{Z}_{0,(i)}(\mathcal{L}_{0,n}^\epsilon))$ are n linearly disjoint extensions of $Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))$, so

$$\big[Q\big(\hat{\mathcal{Z}}_0\big(\mathcal{L}_{0,n}^\epsilon\big)\big) : Q\big(\mathcal{Z}_0\big(\mathcal{L}_{0,n}^\epsilon\big)\big)\big] = \prod_{i=1}^n \big[Q\big(\mathcal{Z}_{0,(i)}\big(\mathcal{L}_{0,n}^\epsilon\big)\big) : Q\big(\mathcal{Z}_0\big(\mathcal{L}_{0,n}^\epsilon\big)\big)\big].$$

Now, by Proposition 4.2, we know that Φ_1 affords isomorphisms $Q\big(\mathcal{Z}_0\big(\mathcal{L}_{0,1}^\epsilon\big)\big) \cong Q\big(\mathcal{Z}_0\big(U_\epsilon^{\mathrm{lf}}\big)\big)$ and $Q(\mathcal{Z}(\mathcal{L}_{0,1}^\epsilon)) \cong Q\big(\mathcal{Z}\big(U_\epsilon^{\mathrm{lf}}\big)\big)$, and moreover

$$Q(\mathcal{Z}_0(U_\epsilon)) = Q\big(\mathcal{Z}_0\big(U_\epsilon^{\mathrm{lf}}\big)\big)\big(T^{(l)}/T_2^{(l)}\big), \qquad Q(\mathcal{Z}(U_\epsilon)) = Q\big(\mathcal{Z}\big(U_\epsilon^{\mathrm{lf}}\big)\big)\big(T^{(l)}/T_2^{(l)}\big). \tag{5.4}$$

Computing via the field embedding $\Phi_1^{\otimes n} \colon Q\big(\hat{\mathcal{Z}}_0\big(\mathcal{L}_{0,n}^\epsilon\big)\big) \to Q\big(\mathcal{Z}\big(U_\epsilon^{\otimes n}\big)\big)$, we deduce

$$\begin{aligned}
&\big[Q\big(\mathcal{Z}_{0,(i)}\big(\mathcal{L}_{0,n}^\epsilon\big)\big) : Q\big(\mathcal{Z}_0\big(\mathcal{L}_{0,n}^\epsilon\big)\big)\big] \\
&\quad = \big[\Phi_1^{\otimes n}\big(Q\big(\mathcal{Z}_{0,(i)}\big(\mathcal{L}_{0,n}^\epsilon\big)\big)\big) : \Phi_1^{\otimes n}\big(Q\big(\mathcal{Z}_0\big(\mathcal{L}_{0,n}^\epsilon\big)\big)\big)\big] \\
&\quad = \big[Q\big(\mathcal{Z}_0\big(U_\epsilon^{\mathrm{lf}}\big)^{\otimes n}\big)\big[\big\{(\Phi_1({}_\lambda\omega))^{(i)},\, \lambda \in P_+,\, i = 1, \dots, n\big\}\big] : Q\big(\mathcal{Z}_0\big(U_\epsilon^{\mathrm{lf}}\big)^{\otimes n}\big)\big] \\
&\quad = \big[Q\big(\mathcal{Z}_0(U_\epsilon)^{\otimes n}\big)\big[\big\{(\Phi_1({}_\lambda\omega))^{(i)},\, \lambda \in P_+,\, i = 1, \dots, n\big\}\big] : Q\big(\mathcal{Z}_0(U_\epsilon)^{\otimes n}\big)\big] = l^m.
\end{aligned}$$

The second and third equalities follow from (5.4) and the properties of Φ_1 recalled before it, and the last equality follows from Theorem 2.29 (2) and (5.3). As a result, we have

$$\big[Q\big(\hat{\mathcal{Z}}_0\big(\mathcal{L}_{0,n}^\epsilon\big)\big) : Q\big(\mathcal{Z}_0\big(\mathcal{L}_{0,n}^\epsilon\big)\big)\big] = l^{mn},$$

whence

$$\big[Q\big(\mathcal{Z}\big(\mathcal{L}_{0,n}^\epsilon\big)\big) : Q\big(\mathcal{Z}_0\big(\mathcal{L}_{0,n}^\epsilon\big)\big)\big] \geq l^{mn}.$$

Since $\big[Q\big(\mathcal{L}_{0,n}^\epsilon\big) : Q\big(\mathcal{Z}_0\big(\mathcal{L}_{0,n}^\epsilon\big)\big)\big] = l^{n(m+2N)}$, by (5.2) we obtain $d^2 \leq l^{2nN}$, which concludes the proof. ■

Remark 5.3. It follows $\big[Q\big(\mathcal{Z}\big(\mathcal{L}_{0,n}^\epsilon\big)\big) : Q\big(\mathcal{Z}_0\big(\mathcal{L}_{0,n}^\epsilon\big)\big)\big] = l^{mn}$ by the degree computation above, whence $Q\big(\mathcal{Z}\big(\mathcal{L}_{0,n}^\epsilon\big)\big) = Q\big(\hat{\mathcal{Z}}_0\big(\mathcal{L}_{0,n}^\epsilon\big)\big)$. In [17], we prove that $\mathcal{Z}\big(\mathcal{L}_{0,n}^\epsilon\big) = \hat{\mathcal{Z}}_0\big(\mathcal{L}_{0,n}^\epsilon\big)$.

Theorem 5.4. *$Q\big(\big(\mathcal{L}_{0,n}^\epsilon\big)^{U_\epsilon}\big)$, $n \geq 2$, is a division algebra and a central simple algebra of PI degree $l^{N(n-1)-m}$.*

Proof. The center of $\big(\mathcal{L}_{0,n}^\epsilon\big)^{U_\epsilon}$ contains $\mathcal{Z}\big(\mathcal{L}_{0,n}^\epsilon\big)$, so the finite-dimensionality of $Q\big(\mathcal{L}_{0,n}^\epsilon\big)$ over $Q\big(\mathcal{Z}\big(\mathcal{L}_{0,n}^\epsilon\big)\big)$ implies the finite-dimensionality of $Q\big(\big(\mathcal{L}_{0,n}^\epsilon\big)^{U_\epsilon}\big)$ over its center. Since it has no non-zero divisors, this proves $Q\big(\big(\mathcal{L}_{0,n}^\epsilon\big)^{U_\epsilon}\big)$ is a division algebra.

Now denote by $\Delta^{(n)} \colon \mathcal{O}_\epsilon \to \mathcal{O}_\epsilon^{\otimes n}$, $n \geq 2$, the n-fold coproduct, i.e., $\Delta^{(2)} := \Delta$, the standard coproduct of $\mathcal{O}_\epsilon$, and $\Delta^{(n)} := (\mathrm{id} \otimes \Delta) \circ \Delta^{(n-1)}$ for $n \geq 3$. Identifying $\mathcal{L}_{0,n}^\epsilon$ with $\mathcal{O}_\epsilon^{\otimes n}$ as a vector space, we consider $\Delta^{(n)}$ as a map $\Delta^{(n)} \colon \mathcal{L}_{0,1}^\epsilon \to \mathcal{L}_{0,n}^\epsilon$. It is an algebra morphism [18, Proposition 6.18], injective because $\big(\varepsilon^{\otimes(n-1)} \otimes \mathrm{id}\big)\Delta^{(n)} = \mathrm{id}$. Then it extends uniquely to the fraction algebra $Q\big(\mathcal{L}_{0,1}^\epsilon\big)$. As noted above, $Q\big(\mathcal{L}_{0,1}^\epsilon\big) = Q\big(\mathcal{Z}_0\big(\mathcal{L}_{0,1}^\epsilon\big)\big) \bigotimes_{\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)} \mathcal{L}_{0,1}^\epsilon$. Since $\mathcal{Z}_0\big(\mathcal{L}_{0,1}^\epsilon\big) = \mathcal{Z}_0(\mathcal{O}_\epsilon)$ is a Hopf subalgebra of $\mathcal{O}_\epsilon$ [41, Proposition 6.4], $\Delta^{(n)}$ maps $\mathcal{Z}_0\big(\mathcal{L}_{0,1}^\epsilon\big)$ to $\mathcal{Z}_0\big(\mathcal{L}_{0,1}^\epsilon\big)^{\otimes n}$. Then, extending the scalars of $\Delta^{(n)}\big(Q\big(\mathcal{L}_{0,1}^\epsilon\big)\big)$ by the field $Q\big(\mathcal{Z}\big(\mathcal{L}_{0,n}^\epsilon\big)\big)$, consider the algebra

$$\begin{aligned}
Q_{\mathcal{Z}}\big(\Delta^{(n)}\big(\mathcal{L}_{0,1}^\epsilon\big)\big) &:= Q\big(\mathcal{Z}\big(\mathcal{L}_{0,n}^\epsilon\big)\big) \bigotimes_{\Delta^{(n)}(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon))} \Delta^{(n)}\big(\mathcal{L}_{0,1}^\epsilon\big) \\
&= Q\big(\mathcal{Z}\big(\mathcal{L}_{0,n}^\epsilon\big)\big) \bigotimes_{\Delta^{(n)}(Q(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)))} \Delta^{(n)}\big(Q\big(\mathcal{L}_{0,1}^\epsilon\big)\big)
\end{aligned}$$

$$= Q\big(\mathcal{Z}\big(\mathcal{L}_{0,n}^{\epsilon}\big)\big) \bigotimes_{\Delta^{(n)}(Q(\mathcal{Z}_0(\mathcal{L}_{0,1}^{\epsilon})))} \Delta^{(n)}\big(Q\big(\mathcal{Z}\big(\mathcal{L}_{0,1}^{\epsilon}\big)\big)\big)$$
$$\bigotimes_{\Delta^{(n)}(Q(\mathcal{Z}(\mathcal{L}_{0,1}^{\epsilon})))} \Delta^{(n)}\big(Q\big(\mathcal{L}_{0,1}^{\epsilon}\big)\big).$$

By Proposition 5.2, $\Delta^{(n)}\big(Q\big(\mathcal{L}_{0,1}^{\epsilon}\big)\big)$ is a $\Delta^{(n)}\big(Q\big(\mathcal{Z}\big(\mathcal{L}_{0,1}^{\epsilon}\big)\big)\big)$-central simple algebra. The left factor is a field, so $Q_{\mathcal{Z}}\big(\Delta^{(n)}\big(\mathcal{L}_{0,1}^{\epsilon}\big)\big)$ is a central simple algebra over it (see, e.g., [96, Theorem 1.7.27], or [101, Lemma 4.9]). Note that the left factor can also be written as

$$\tilde{Q}\big(\mathcal{Z}\big(\mathcal{L}_{0,n}^{\epsilon}\big)\big) := Q\big(\mathcal{Z}\big(\mathcal{L}_{0,n}^{\epsilon}\big)\big) \bigotimes_{\Delta^{(n)}(\mathcal{Z}_0(\mathcal{L}_{0,1}^{\epsilon}))} \Delta^{(n)}\big(\mathcal{Z}\big(\mathcal{L}_{0,1}^{\epsilon}\big)\big)$$

for it contains $\tilde{Q}\big(\mathcal{Z}\big(\mathcal{L}_{0,n}^{\epsilon}\big)\big)$, it is contained in its fraction field, and $\tilde{Q}\big(\mathcal{Z}\big(\mathcal{L}_{0,n}^{\epsilon}\big)\big)$ is a field because $\mathcal{Z}\big(\mathcal{L}_{0,1}^{\epsilon}\big)$ is finite over $\mathcal{Z}_0\big(\mathcal{L}_{0,1}^{\epsilon}\big)$ and has no nontrivial zero divisors. Note that

$$\big[\tilde{Q}(\mathcal{Z}\big(\mathcal{L}_{0,n}^{\epsilon}\big)) : Q\big(\mathcal{Z}\big(\mathcal{L}_{0,n}^{\epsilon}\big)\big)\big] = l^m.$$

We proved in [18, Proposition 6.19] that the ring $\big(\mathcal{L}_{0,n}^{A}\big)^{U_A}$ is the centralizer of $\Delta^{(n)}\big(\mathcal{L}_{0,1}^{A}\big)$ in $\mathcal{L}_{0,n}^{A}$; the same arguments show that $\big(\mathcal{L}_{0,n}^{\epsilon}\big)^{U_\epsilon}$ is the centralizer of $\Delta^{(n)}\big(\mathcal{L}_{0,1}^{\epsilon}\big)$ in $\mathcal{L}_{0,n}^{\epsilon}$. So the algebra

$$Q\big(\big(\mathcal{L}_{0,n}^{\epsilon}\big)^{U_\epsilon}\big) := Q\big(\mathcal{Z}\big(\mathcal{L}_{0,n}^{\epsilon}\big)\big) \bigotimes_{\mathcal{Z}(\mathcal{L}_{0,n}^{\epsilon})} \big(\mathcal{L}_{0,n}^{\epsilon}\big)^{U_\epsilon}$$

is the centralizer of $Q_{\mathcal{Z}}\big(\Delta^{(n)}\big(\mathcal{L}_{0,1}^{\epsilon}\big)\big)$ in $Q\big(\mathcal{L}_{0,n}^{\epsilon}\big)$. Since the latter is simple, we can apply the double centralizer theorem (see, e.g., [96, Theorem 7.1.9], or [101, Theorem 7.1]): $Q\big(\big(\mathcal{L}_{0,n}^{\epsilon}\big)^{U_\epsilon}\big)$ is a simple algebra, we have

$$\big[Q\big(\big(\mathcal{L}_{0,n}^{\epsilon}\big)^{U_\epsilon}\big) : Q\big(\mathcal{Z}\big(\mathcal{L}_{0,n}^{\epsilon}\big)\big)\big] = \frac{\big[Q\big(\mathcal{L}_{0,n}^{\epsilon}\big) : Q\big(\mathcal{Z}\big(\mathcal{L}_{0,n}^{\epsilon}\big)\big)\big]}{\big[Q_{\mathcal{Z}}\big(\Delta^{(n)}\big(\mathcal{L}_{0,1}^{\epsilon}\big)\big) : Q\big(\mathcal{Z}\big(\mathcal{L}_{0,n}^{\epsilon}\big)\big)\big]} = l^{2nN-(2N+m)},$$

and the centralizer of $Q\big(\big(\mathcal{L}_{0,n}^{\epsilon}\big)^{U_\epsilon}\big)$ is $Q_{\mathcal{Z}}\big(\Delta^{(n)}\big(\mathcal{L}_{0,1}^{\epsilon}\big)\big)$. In particular, $Q\big(\big(\mathcal{L}_{0,n}^{\epsilon}\big)^{U_\epsilon}\big)$ has center $Q\big(\big(\mathcal{L}_{0,n}^{\epsilon}\big)^{U_\epsilon}\big) \cap Q_{\mathcal{Z}}\big(\Delta^{(n)}\big(\mathcal{L}_{0,1}^{\epsilon}\big)\big)$, which is easily shown to be $\tilde{Q}\big(\mathcal{Z}\big(\mathcal{L}_{0,n}^{\epsilon}\big)\big)$. It then follows

$$\begin{aligned}\big[Q\big(\big(\mathcal{L}_{0,n}^{\epsilon}\big)^{U_\epsilon}\big) : \tilde{Q}\big(\mathcal{Z}\big(\mathcal{L}_{0,n}^{\epsilon}\big)\big)\big] &= \frac{\big[Q\big(\big(\mathcal{L}_{0,n}^{\epsilon}\big)^{U_\epsilon}\big) : Q\big(\mathcal{Z}\big(\mathcal{L}_{0,n}^{\epsilon}\big)\big)\big]}{\big[\tilde{Q}\big(\mathcal{Z}\big(\mathcal{L}_{0,n}^{\epsilon}\big)\big) : Q\big(\mathcal{Z}\big(\mathcal{L}_{0,n}^{\epsilon}\big)\big)\big]} \\ &= l^{2nN-(2N+m)}.l^{-m} = l^{2(N(n-1)-m)}.\end{aligned}$$

Therefore, $Q\big(\big(\mathcal{L}_{0,n}^{\epsilon}\big)^{U_\epsilon}\big)$ is a central simple algebra of PI degree $l^{N(n-1)-m}$. ■

A Low and up crystal structures in the $\mathfrak{sl}_2$ case

Let $k \in \mathbb{N}$, and denote by V_k the simple $U_q^{\mathrm{ad}}(\mathfrak{sl}_2)$ module of dimension $k+1$. It has a basis $v_0, \dots, v_k$ such that

$$\begin{aligned}&K.v_j = q^{k-2j} v_j, \qquad F.v_j = [j+1]_q v_{j+1} \quad \text{if} \quad j<k, \qquad F.v_k = 0,\\ &E.v_j = [k-j+1]_q v_{j-1} \quad \text{if} \quad j>0, \qquad E.v_0 = 0.\end{aligned}$$

This basis defines the full A-sublattice ${}_AV_k$, which is left invariant by U_A^{res}, and we have

$$F^{(a)}.v_j = \begin{bmatrix} j+a \\ a \end{bmatrix}_q v_{j+a}, \qquad E^{(a)}.v_j = \begin{bmatrix} k-j+a \\ a \end{bmatrix}_q v_{j-a}, \qquad a \geq 0.$$

The action of the Kashiwara operator $\tilde{e}$, $\tilde{f}$ on V_k are given by $\tilde{f}(v_j) = v_{j+1}$, $\tilde{e}(v_j) = v_{j-1}$.

The crystal basis $\left(\mathcal{L}^{\mathrm{low}}, \mathcal{B}^{\mathrm{low}}\right)$ at $q = 0$ is formed by the $\mathcal{A}_0$-sublattice $\mathcal{L}^{\mathrm{low}}$ generated by $v_0, \dots, v_k$, and $\mathcal{B}^{\mathrm{low}}$ by the images $\bar{v}_0, \dots, \bar{v}_k$ of these vectors in $\mathcal{L}^{\mathrm{low}}/q\mathcal{L}^{\mathrm{low}}$.

The bilinear form $\langle\ \rangle_k$ defined by (2.39) is easily computed

$$\langle v_i, v_j\rangle_k = \left\langle F^{(i)}.v_0, F^{(j)}.v_0\right\rangle_k = \left\langle v_0, E^{(i)}F^{(j)}.v_0\right\rangle_k = \begin{bmatrix} k \\ i \end{bmatrix}_q \delta_{i,j}.$$

By definition,

$${}_AV_k^{\mathrm{up}} = \{v \in V_k,\ \langle v, {}_AV_k\rangle_k \subset A\} = \bigoplus_{j=0}^{k} A v_j^{\mathrm{up}},$$

where

$$v_j^{\mathrm{up}} = \begin{bmatrix} k \\ j \end{bmatrix}_q^{-1} v_j.$$

The upper crystal basis $(\mathcal{L}^{\mathrm{up}}, \mathcal{B}^{\mathrm{up}})$ at $q = 0$ is formed by the $\mathcal{A}_0$-sublattice $\mathcal{L}^{\mathrm{up}}$ generated by $v_0^{\mathrm{up}}, \dots, v_k^{\mathrm{up}}$, and $\mathcal{B}^{\mathrm{up}}$ by the images $\bar{v}_0^{\mathrm{up}}, \dots, \bar{v}_k^{\mathrm{up}}$ of these vectors in $\mathcal{L}^{\mathrm{up}}/q\mathcal{L}^{\mathrm{up}}$.

Using that $[n]_q \in q^{1-n}(1 + q\mathcal{A}_0)$, we obtain

$$\begin{bmatrix} k \\ j \end{bmatrix}_q \in q^{j^2-kj}(1 + q\mathcal{A}_0).$$

As a result, we get $\bar{v}_j^{\mathrm{up}} = q^{kj-j^2}\bar{v}_j$, which is exactly the relation (2.41) relating the low and up crystal bases, with $\lambda = k\varpi_1$, $\mu = (k-2j)\varpi_1$.

B Quantum Weyl group

We recall some of the formulas of [31]. Let $e_q(z)$ be the formal power series in z with coefficients in $\mathbb{C}(q)$ defined by

$$e_q(z) = \sum_{n=0}^{+\infty} \frac{z^n}{(n)_{q^2}!}.$$

We first consider the case of $\mathfrak{g} = \mathfrak{sl}_2$. As explained in [18, Section 3], the Cartan element $H \in \mathfrak{g}$ defines an element of $\mathbb{U}_q(\mathfrak{sl}_2)$. Viewed as elements of $\mathbb{U}_q(\mathfrak{sl}_2)$ we have $L = q^{H/2}$. The series $\Theta = q^{H\otimes H/2}$ defines an element of $\mathbb{U}_q(\mathfrak{sl}_2)^{\hat{\otimes}2}$, its image under multiplication being $q^{H^2/2}$. The R-matrix can be expressed as $R = \Theta\hat{R}$ where $\hat{R} = e_{q^{-1}}\big(\big(q - q^{-1}\big)E \otimes F\big)$ is a well defined element of $\mathbb{U}_q(\mathfrak{sl}_2)^{\hat{\otimes}2}$. Consider the Lusztig [82] braid group automorphism of $U_q(\mathfrak{sl}_2)$, defined by

$$T(L) = L^{-1}, \qquad T(E) = -FK^{-1}, \qquad T(F) = -KE. \tag{B.1}$$

For every $x \in U_q(\mathfrak{sl}_2)$ it satisfies: $\Delta(T(x)) = \hat{R}^{-1}(T \otimes T)(\Delta(x))\hat{R}$. Define the quantum Weyl group element $\hat{w} \in \mathbb{U}_q(\mathfrak{sl}_2)$ by Saito's formula [97]:

$$\hat{w} = e_{q^{-1}}(F)q^{-H^2/4}e_{q^{-1}}(-E)q^{-H^2/4}e_{q^{-1}}(F)q^{-H/2}. \tag{B.2}$$

For every $x \in U_q(\mathfrak{sl}_2)$, it satisfies

$$T(x) = \hat{w}x\hat{w}^{-1}, \qquad \Delta(\hat{w}) = \hat{R}^{-1}(\hat{w} \otimes \hat{w}), \tag{B.3}$$

$$\hat{w}^2 = q^{H^2/2}\xi\theta, \tag{B.4}$$

where $\theta \in \mathbb{U}_q(\mathfrak{sl}_2)$ is the ribbon element, and $\xi \in \mathbb{U}_q(\mathfrak{sl}_2)$ is the central group element whose value on the type 1 simple module V_k of $U_q^{\mathrm{ad}}(\mathfrak{sl}_2)$ of dimension $k+1$ is the scalar endomorphism $(-1)^k id_{V_k}$.

In order to compare our setting to the one of [41], we need an explicit formula of $\hat{w}$. Using the basis v_j of V_k of Appendix A, (B.1), (B.3) and (B.4), we obtain

$$\hat{w}v_j = (-1)^j q^{-j(k-j-1)-k} v_{k-j}. \tag{B.5}$$

In [41], another quantum Weyl group element $\underline{w}$ is defined. It is dual to the Vaksman–Soibelman functional $t\colon \mathcal{O}_q(\mathrm{SL}_2) \to \mathbb{C}(q)$ of [98, 102], that is, $t(\alpha) = \langle \alpha, \underline{w}\rangle$ for all $\alpha \in \mathcal{O}_q(\mathrm{SL}_2)$. By comparing (B.5) with the formulas defining the action of t in [41, Section 1.7], we find $\underline{w} = \xi\hat{w}K$ and the basis vectors w_r^p of [41], where $p \in (1/2)\mathbb{N}$ and $r \in \{-p, -p+1, \dots, p-1, p\}$, are related to the vectors v_j above as follows: $v_j = \lambda_j w_r^p$, where $k = 2p$, $j = p - r$, $\lambda_0 = 1$, $\lambda_1 = [k]q^{-k}$, and

$$\lambda_j = \frac{[k]!}{[j]![k-(j-2)]!} q^{j(j+1)-j(k+2)}, \qquad j \geq 2.$$

Explicit formulas of the evaluation of t on basis vectors of $\mathcal{O}_q(\mathrm{SL}_2)$ can be computed. We get

$$t\big(\tilde{a}^{\star m} \star \tilde{b}^{\star n} \star \tilde{d}^{\star p}\big) = \delta_{m,p} q^{-np} \prod_{i=1}^{p} \big(1 - q^{-2i}\big), \tag{B.6}$$

$$t\big(\tilde{a}^{\star m} \star \tilde{c}^{\star n} \star \tilde{d}^{\star p}\big) = (-1)^n \delta_{m,p} q^{-n(p+1)} \prod_{i=1}^{p} \big(1 - q^{-2i}\big), \tag{B.7}$$

where $\tilde{a} = a$, $\tilde{b} = qb$, $\tilde{c} = q^{-1}c$, $\tilde{d} = d$ and as usual a, b, c, d are the standard generators of $\mathcal{O}_q(\mathrm{SL}_2)$, i.e., the matrix coefficients in the basis of weight vectors v_0, v_1 of the 2-dimensional irreducible representation V_1 of $U_q(\mathfrak{sl}_2)$ such that $K.v_0 = qv_0$ and $v_1 = F.v_0$. Here we have introduced the generators $\tilde{a}, \dots, \tilde{d}$ to facilitate the comparison with the formulas in [41]; these generators come naturally in their setup because they use different generators E_i and F_i of $U_q(\mathfrak{g})$, which in our notations can be written respectively as $K_i^{-1}E_i$ and F_iK_i.

The formulas (B.6)–(B.7) can be shown by two independent methods. The first uses a definition of t as a GNS state associated to an infinite-dimensional representation of $\mathcal{O}_q(\mathrm{SL}_2)$, as recalled in [41, Section 1.6]. The second is to write, e.g.,

$$t\big(\tilde{a}^{\star m} \star \tilde{b}^{\star n} \star \tilde{d}^{\star p}\big) = \big\langle \tilde{a}^{\otimes m} \otimes \tilde{b}^{\otimes n} \otimes \tilde{d}^{\otimes p}, \Delta^{(m+n+p)}(\underline{w})\big\rangle \tag{B.8}$$

and to use explicit expressions of $\Delta^{(m+n+p)}(\underline{w})$ when represented on $V_1^{\otimes(m+n+p)}$. In general, one can check that

$$\begin{aligned}\Delta^{(n)}(\hat{\omega}) = {}& \big(\Delta^{(n-1)} \otimes \mathrm{id}\big)\big(\widehat{R}^{-1}\big)\big(\big(\Delta^{(n-2)} \otimes \mathrm{id}\big)\big(\widehat{R}^{-1}\big) \otimes \mathrm{id}\big) \cdots \big(\big(\Delta \otimes \mathrm{id}\big)\big(\widehat{R}^{-1}\big) \otimes \mathrm{id}^{\otimes(n-3)}\big) \\ & \times \big(\widehat{R}^{-1} \otimes \mathrm{id}^{\otimes(n-2)}\big)\hat{\omega}^{\otimes n}.\end{aligned}$$

By (B.5) or (B.6)–(B.7), we see that $\hat{w}$ (or $\underline{w}$) and t are well defined on the integral forms,

$$\hat{w} \in \mathbb{U}_\Gamma, \qquad t\colon\ \mathcal{O}_A(\mathrm{SL}_2) \to A.$$

We now consider the case where $\mathfrak{g}$ is of rank $m \geq 2$. To each simple root α_i, $1 \leq i \leq m$, is associated the subalgebra of U_q generated by E_i, F_i, L_i, L_i^{-1}. It is a copy of $U_{q_i}(\mathfrak{sl}_2)$, where $q_i = q^{d_i}$. Let $\hat{w}_i$ be the corresponding quantum Weyl group element in $\mathbb{U}_q = \mathbb{U}_q(\mathfrak{g})$, defined by Saito's formula (B.2), replacing H, E, F by H_i, E_i and F_i. Also, denote by $\nu_i\colon \mathcal{O}_q \to \mathcal{O}_{q_i}(\mathrm{SL}_2)$ the projection map dual to the inclusion $U_{q_i}(\mathfrak{sl}_2)\bigotimes_{\mathbb{C}(q_i)}\mathbb{C}(q) \hookrightarrow U_q$, and put $t_i = t\circ\nu_i$. Let $\underline{w}_i$ be the corresponding quantum Weyl group element in $\mathbb{U}_q$, i.e., $t_i(\alpha) = \langle \alpha, \underline{w}_i\rangle$ for all $\alpha\in\mathcal{O}_q$. On integral forms they yield well-defined elements $\hat{w}_i, \underline{w}_i \in \mathbb{U}_\Gamma$ and $t_i\colon \mathcal{O}_A \to A$ (see [41, Proposition 5.1], and [84] for a different construction). They satisfy the defining relations of the braid group $\mathcal{B}(\mathfrak{g})$ of $\mathfrak{g}$ [70]:

$$\hat{w}_i\hat{w}_j\hat{w}_i = \hat{w}_j\hat{w}_i\hat{w}_j \qquad \text{if} \quad a_{ij}a_{ji} = 1,$$
$$(\hat{w}_i\hat{w}_j)^k = (\hat{w}_j\hat{w}_i)^k \qquad \text{for} \quad k = 1,2,3 \quad \text{if} \quad a_{ij}a_{ji} = 0,2,3,$$

and similarly by replacing $\hat{w}_i$ with $\underline{w}_i$, or with t_i (see [98] for the latter). The Weyl group $W = W(\mathfrak{g}) = N(T_G)/T_G$ is generated by the reflections s_i associated to the simple roots α_i. Denote by $n_i\in N(T_G)$ a representative of s_i. Let $w\in W$ and denote by $w = s_{i_1}\dots s_{i_k}$ a reduced expression. Because of the braid group relations the elements $\hat{w} = \hat{w}_{i_1}\cdots\hat{w}_{i_k}$, $\underline{w} = \underline{w}_{i_1}\cdots\underline{w}_{i_k}$ and the functional $t_w = t_{i_1}\cdots t_{i_k}$ do not depend on the choice of reduced expression. The Lusztig [82] braid group automorphism $T_w\colon\Gamma\to\Gamma$ associated to w satisfies (see [41])

$$T_w(x) = \hat{w}x\hat{w}^{-1}, \qquad x\in\Gamma.$$

Let w_0 be the longest element in W. We have

$$\Delta(\hat{w}_0) = \hat{R}^{-1}(\hat{w}_0\otimes\hat{w}_0), \tag{B.9}$$

where as usual $R = \Theta\hat{R}$.

C Regular action on $\mathcal{O}_\epsilon$

The following result is proved in [41, Section 1.10]. For completeness, let us give a (different) proof. Recall from (2.72) that we may identify $\mathcal{Z}_0(\mathcal{O}_\epsilon)$ with $\mathcal{O}(G)$.

Proposition C.1. *For every $f\in\mathcal{Z}_0(\mathcal{O}_\epsilon)$, $g\in\mathcal{O}_\epsilon$, we have*

$$t_i(f) = f(n_i), \tag{C.1}$$
$$t_i(f\star g) = t_i(f)t_i(g). \tag{C.2}$$

Proof. It is sufficient to prove the results for SL_2 because $\nu_i\colon\mathcal{O}_\epsilon\to\mathcal{O}_{\epsilon_i}(\mathrm{SL}_2)$ is a morphism of Hopf algebras and $\nu_i(\mathcal{Z}_0(\mathcal{O}_\epsilon))\subset\mathcal{Z}_0(\mathcal{O}_{\epsilon_i}(\mathrm{SL}_2))$. In this case, (C.1) can be proved by using (B.6)–(B.7), evaluating t on basis elements of $\mathcal{Z}_0(\mathcal{O}_\epsilon(\mathrm{SL}_2))$ as is done in [41, Lemma 1.5 (a)]. Such a basis is formed by monomials like in (B.6)–(B.7), with all exponents divisible by l; then for instance

$$t\big(\tilde{a}^{\star ml}\star\tilde{b}^{\star nl}\star\tilde{d}^{\star pl}\big) = \delta_{p,0}\delta_{m,0} = \underline{a}^m\underline{b}^n\underline{d}^p(n),$$

where $\underline{a},\dots,\underline{d}$ are the generators of $\mathcal{O}(G) = \mathcal{O}_1(G)$ corresponding to $a,\dots,d$, and we take

$$n = \begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}$$

as representative of the reflection s generating the Weyl group $W(\mathfrak{sl}_2)$. Here is an alternative proof of (C.1): (C.2) shows that t is a homomorphism on $\mathcal{Z}_0(\mathcal{O}_\epsilon(\mathrm{SL}_2))$, so by proving (C.2) at first one is reduced to check (C.1) on the generators $a^{\star l},\dots,d^{\star l}$, which is easy by means of (B.8) and (B.9).

We provide a proof of (C.2) that we find more conceptual than the one in [41, Lemma 1.5 (b)] (which uses again (B.6)–(B.7)). As above, let us denote $\underline{w} = \xi\hat{w}K$. For any $f, g \in \mathcal{O}_\epsilon$, we have

$$\begin{aligned} t(f \star g) &= (f \otimes g)(\Delta(\underline{w})) = (f \otimes g)\big(\widehat{R}^{-1}(\underline{w} \otimes \underline{w})\big) = \sum_{(\widehat{R}^{-1})} f\big(\big(\widehat{R}^{-1}\big)_{(1)}\underline{w}\big) g\big(\big(\widehat{R}^{-1}\big)_{(2)}\underline{w}\big) \\ &= \sum_{(\widehat{R}^{-1}),(f)} f_{(1)}\big(\big(\widehat{R}^{-1}\big)_{(1)}\big) f_{(2)}(\underline{w}) g\big(\big(\widehat{R}^{-1}\big)_{(2)}\underline{w}\big) = \sum_{(f)} f_{(2)}(\underline{w}) g\big((f_{(1)} \otimes \mathrm{id})\big(\widehat{R}^{-1}\big)\underline{w}\big). \end{aligned}$$

Assume now $f \in \mathcal{Z}_0(\mathcal{O}_\epsilon(\mathrm{SL}_2))$. Since $\mathcal{Z}_0(\mathcal{O}_\epsilon(\mathrm{SL}_2))$ is a Hopf subalgebra of $\mathcal{O}_\epsilon(\mathrm{SL}_2)$, we have $f_{(1)} \in \mathcal{Z}_0(\mathcal{O}_\epsilon(\mathrm{SL}_2))$. From Theorem 2.29 (2), we deduce

$$(f_{(1)} \otimes \mathrm{id})\big(\widehat{R}^{-1}\big) \in U_\epsilon(\mathfrak{n}_-) \cap \mathcal{Z}_0\big(U_\epsilon^{\mathrm{ad}}\big).$$

Denote by z this element. Note that from its expression we have $\epsilon(z) = \epsilon(f_{(1)})$. Now $g(z\underline{w}) = \sum_{(g)} g_{(1)}(z) g_{(2)}(\underline{w})$, but $g_{(1)}$ is a linear combination of matrix elements of Γ-modules, on which $\mathcal{Z}_0\big(U_\epsilon^{\mathrm{ad}}\big)$ acts by the trivial character. Therefore,

$$g(z\underline{w}) = \sum_{(g)} \epsilon(z) g_{(1)}(1) g_{(2)}(\underline{w}) = \epsilon(z) g(\underline{w}) = \epsilon(f_{(1)}) g(\underline{w}),$$

and eventually

$$t(f \star g) = \sum_{(f)} f_{(2)}(\underline{w}) \epsilon(f_{(1)}) g(\underline{w}) = t(f)t(g).$$

This concludes the proof. ■

For the sake of completeness, let us show how this result implies:

Proof of Proposition 2.30 (i.e., [41, Proposition 7.1]). We have $f \triangleleft t_i = \sum_{(f)} t_i(f_{(1)}) f_{(2)}$, $f \in \mathcal{Z}_0(\mathcal{O}_\epsilon)$. Since $\mathcal{Z}_0(\mathcal{O}_\epsilon)$ is a Hopf subalgebra of $\mathcal{O}_\epsilon$, $f_{(2)} \in \mathcal{Z}_0(\mathcal{O}_\epsilon)$ and therefore the maps $\triangleleft t_i \colon \mathcal{O}_\epsilon \to \mathcal{O}_\epsilon$ preserve $\mathcal{Z}_0(\mathcal{O}_\epsilon)$. Moreover, $(f \triangleleft t_i)(a) = \sum_{(f)} f_{(1)}(n_i) f_{(2)}(a) = f(n_i a)$, $a \in G$, by (C.1).

It remains to show that $(f \star \alpha) \triangleleft t_i = (f \triangleleft t_i)(\alpha \triangleleft t_i)$ for every $f \in \mathcal{Z}_0(\mathcal{O}_\epsilon)$, $\alpha \in \mathcal{O}_\epsilon$. We have

$$\begin{aligned} (f \star g) \triangleleft t_i &= \sum_{(f \star g)} t_i\left((f \star g)_{(1)}\right)(f \star g)_{(2)} = \sum_{(f),(g)} t_i\left(f_{(1)} \star g_{(1)}\right) f_{(2)} \star g_{(2)} \\ &= \sum_{(f),(g)} t\left(\nu_i(f_{(1)}) \nu_i(g_{(1)})\right) f_{(2)} \star g_{(2)} \\ &= \sum_{(f),(g)} t\left(\nu_i(f_{(1)})\right) t\left(\nu_i(g_{(1)})\right) f_{(2)} \star g_{(2)}, \end{aligned} \tag{C.3}$$

using that ν_i is a homomorphism in the third equality, and (C.2) in the last one. The result is just $(f \triangleleft t_i)(g \triangleleft t_i)$. ■

Acknowledgements

We are grateful to M. Faitg for many valuable discussions on the subject, especially concerning the filtration arguments in the proof of Theorem 1.1, and the use of the partial order $\preceq$ in the proof of Theorem 1.1. We also thank K.A. Brown for pointing out the references [6] and [25] (see the comments before Theorem 1.2). We thank also the referees for their comments and suggestions, which greatly improved the clarity of our work.

References

[1] Alekseev A.Yu., Integrability in the Hamiltonian Chern–Simons theory, *St. Petersburg Math. J.* **6** (1995), 241–253, arXiv:hep-th/9311074.

[2] Alekseev A.Yu., Grosse H., Schomerus V., Combinatorial quantization of the Hamiltonian Chern–Simons theory. I, *Comm. Math. Phys.* **172** (1995), 317–358, arXiv:hep-th/9403066.

[3] Alekseev A.Yu., Grosse H., Schomerus V., Combinatorial quantization of the Hamiltonian Chern–Simons theory. II, *Comm. Math. Phys.* **174** (1996), 561–604, arXiv:hep-th/9408097.

[4] Alekseev A.Yu., Schomerus V., Representation theory of Chern–Simons observables, *Duke Math. J.* **85** (1996), 447–510, arXiv:q-alg/9503016.

[5] Andersen H.H., Polo P., Wen K.X., Representations of quantum algebras, *Invent. Math.* **104** (1991), 1–59.

[6] Andruskiewitsch N., García G.A., Quantum subgroups of a simple quantum group at roots of 1, *Compos. Math.* **145** (2009), 476–500, arXiv:0707.0070.

[7] Atiyah M.F., Macdonald I.G., Introduction to commutative algebra, Vol. 64, Addison-Wesley Publishing Co., Reading, Mass., 1969.

[8] Baseilhac S., Quantum coadjoint action and the $6j$-symbols of $U_q\mathrm{sl}_2$, in Interactions between Hyperbolic Geometry, Quantum Topology and Number Theory, *Contemp. Math.*, Vol. 541, American Mathematical Society, Providence, RI, 2011, 103–143, arXiv:1101.3440.

[9] Baseilhac S., Benedetti R., Quantum hyperbolic invariants of 3-manifolds with $\mathrm{PSL}(2,\mathbb{C})$-characters, *Topology* **43** (2004), 1373–1423, arXiv:math.GT/0306280.

[10] Baseilhac S., Benedetti R., Classical and quantum dilogarithmic invariants of flat $\mathrm{PSL}(2,\mathbb{C})$-bundles over 3-manifolds, *Geom. Topol.* **9** (2005), 493–569, arXiv:math.GT/0306283.

[11] Baseilhac S., Benedetti R., Quantum hyperbolic geometry, *Algebr. Geom. Topol.* **7** (2007), 845–917, arXiv:math.GT/0611504.

[12] Baseilhac S., Benedetti R., The Kashaev and quantum hyperbolic link invariants, *J. Gökova Geom. Topol. GGT* **5** (2011), 31–85, arXiv:1101.1851.

[13] Baseilhac S., Benedetti R., Analytic families of quantum hyperbolic invariants, *Algebr. Geom. Topol.* **15** (2015), 1983–2063, arXiv:1212.4261.

[14] Baseilhac S., Benedetti R., Non ambiguous structures on 3-manifolds and quantum symmetry defects, *Quantum Topol.* **8** (2017), 749–846, arXiv:1506.01174.

[15] Baseilhac S., Benedetti R., On the quantum Teichmüller invariants of fibred cusped 3-manifolds, *Geom. Dedicata* **197** (2018), 1–32, arXiv:1704.05667.

[16] Baseilhac S., Faitg M., Roche P., Unrestricted quantum moduli algebras, III: surfaces of arbitrary genus and skein algebras, arXiv:2302.00396.

[17] Baseilhac S., Faitg M., Roche P., Structure and representations of quantum moduli and $\mathfrak{g}$-skein algebras at roots of unity, in preparation.

[18] Baseilhac S., Roche P., Unrestricted quantum moduli algebras. I. The case of punctured spheres, *SIGMA* **18** (2022), 025, 78 pages, arXiv:1912.02440.

[19] Bass H., Algebraic K-theory, *Math. Lect. Note Ser.*, W.A. Benjamin, Inc., New York, 1968.

[20] Baumann P., Another proof of Joseph and Letzter's separation of variables theorem for quantum groups, *Transform. Groups* **5** (2000), 3–20.

[21] Beliakova A., Blanchet C., Geer N., Logarithmic Hennings invariants for restricted quantum $\mathfrak{sl}(2)$, *Algebr. Geom. Topol.* **18** (2018), 4329–4358, arXiv:1705.03083.

[22] Ben-Zvi D., Brochier A., Jordan D., Quantum character varieties and braided module categories, *Selecta Math. (N.S.)* **24** (2018), 4711–4748, arXiv:1606.04769.

[23] Bonahon F., Wong H., Quantum traces for representations of surface groups in $\mathrm{SL}_2(\mathbb{C})$, *Geom. Topol.* **15** (2011), 1569–1615, arXiv:1003.5250.

[24] Bonahon F., Wong H., Representations of the Kauffman bracket skein algebra I: invariants and miraculous cancellations, *Invent. Math.* **204** (2016), 195–243, arXiv:1206.1638.

[25] Brown K.A., Couto M., Affine commutative-by-finite Hopf algebras, *J. Algebra* **573** (2021), 56–94, arXiv:1907.10527.

[26] Brown K.A., Goodearl K.R., Lectures on algebraic quantum groups, *Adv. Courses Math. CRM Barcelona*, Birkhäuser, Basel, 2002.

[27] Brown K.A., Gordon I., The ramifications of the centres: quantised function algebras at roots of unity, *Proc. London Math. Soc.* **84** (2002), 147–178, arXiv:math.RT/9912042.

[28] Brown K.A., Gordon I., Stafford J.T., $\mathcal{O}_\varepsilon[G]$ is a free module over $\mathcal{O}[G]$, arXiv:math.QA/0007179.

[29] Buffenoir E., Roche P., Two-dimensional lattice gauge theory based on a quantum group, *Comm. Math. Phys.* **170** (1995), 669–698, arXiv:hep-th/9405126.

[30] Buffenoir E., Roche P., Link invariants and combinatorial quantization of Hamiltonian Chern–Simons theory, *Comm. Math. Phys.* **181** (1996), 331–365, arXiv:q-alg/9507001.

[31] Buffenoir E., Roche P., Terras V., Quantum dynamical coboundary equation for finite dimensional simple Lie algebras, *Adv. Math.* **214** (2007), 181–229, arXiv:math.QA/0512500.

[32] Bullock D., A finite set of generators for the Kauffman bracket skein algebra, *Math. Z.* **231** (1999), 91–101.

[33] Bullock D., Frohman C., Kania-Bartoszyńska J., Topological interpretations of lattice gauge field theory, *Comm. Math. Phys.* **198** (1998), 47–81, arXiv:q-alg/9710003.

[34] Caldero P., Éléments ad-finis de certains groupes quantiques, *C. R. Acad. Sci. Paris Sér. I Math.* **316** (1993), 327–329.

[35] Chari V., Pressley A., A guide to quantum groups, Cambridge University Press, Cambridge, 1994.

[36] Costantino F., Lê T.T.Q., Stated skein algebras of surfaces, *J. Eur. Math. Soc.* **24** (2022), 4063–4142, arXiv:1907.11400.

[37] Cui W., Canonical bases of modified quantum algebras for type A_2, *J. Algebra Appl.* **17** (2018), 1850113, 27 pages, arXiv:1208.5531.

[38] Dabrowski L., Reina C., Zampa A., $A(\mathrm{SL}_q(2))$ at roots of unity is a free module over $A(\mathrm{SL}(2))$, *Lett. Math. Phys.* **52** (2000), 339–342, arXiv:math.QA/0004092.

[39] De Concini C., Kac V.G., Representations of quantum groups at roots of 1, in Operator Algebras, Unitary Representations, Enveloping Algebras, and Invariant Theory (Paris, 1989), *Progr. Math.*, Vol. 92, Birkhäuser, Boston, MA, 1990, 471–506.

[40] De Concini C., Kac V.G., Procesi C., Quantum coadjoint action, *J. Amer. Math. Soc.* **5** (1992), 151–189

[41] De Concini C., Lyubashenko V., Quantum function algebra at roots of 1, *Adv. Math.* **108** (1994), 205–262.

[42] De Concini C., Procesi C., Quantum groups, in D-modules, Representation Theory, and Quantum Qroups (Venice, 1992), *Lecture Notes in Math.*, Vol. 1565, Springer, Berlin, 1993, 31–140.

[43] de Graaf W.A., Constructing canonical bases of quantized enveloping algebras, *Experiment. Math.* **11** (2002), 161–170.

[44] De Renzi M., Gainutdinov A.M., Geer N., Patureau-Mirand B., Runkel I., 3-dimensional TQFTs from non-semisimple modular categories, *Selecta Math. (N.S.* **28** (2022), 42, 60 pages, arXiv:1912.02063.

[45] De Renzi M., Geer N., Patureau-Mirand B., Renormalized Hennings Invariants and $2+1$-TQFTs, *Comm. Math. Phys.* **362** (2020), 855–907, arXiv:1707.08044.

[46] Dieudonné J.A., Carrell J.B., Invariant theory, old and new, *Adv. Math.* **4** (1970), 1–80.

[47] Domokos M., Lenagan T.H., Quantized trace rings, *Q. J. Math.* **56** (2005), 507–523, arXiv:math.QA/0407053.

[48] Drinfeld V.G., On almost cocommutative Hopf algebras, *Leningrad Math. J.* **1** (1990), 321–342.

[49] Du J., Global IC bases for quantum linear groups, *J. Pure Appl. Algebra* **114** (1996), 25–37.

[50] Enriquez B., Le centre des algébres de coordonnées des groupes quantiques aux racines p^α-iémes de l'unité, *Bull. Soc. Math. France* **122** (1994), 443–485.

[51] Etingof P., Golberg O., Hensel S., Liu T., Schwendner A., Vaintrob D., Yudovina E., Introduction to representation theory, *Stud. Math. Libr.*, Vol. 59, American Mathematical Society, Providence, RI, 2011, arXiv:0901.0827.

[52] Faitg M., Mapping class groups, skein algebras and combinatorial quantization, Ph.D. Thesis, Montpellier de Université, 2019, arXiv:1910.04110.

[53] Faitg M., Projective representations of mapping class groups in combinatorial quantization, *Comm. Math. Phys.* **377** (2020), 161–198, arXiv:1812.00446.

[54] Faitg M., Holonomy and (stated) skein algebras in combinatorial quantization, *Quantum Topol.*, to appear, arXiv:2003.08992.

[55] Fock V.V., Rosly A.A., Poisson structure on moduli of flat connections on Riemann surfaces and the r-matrix, in Moscow Seminar in Mathematical Physics, *Amer. Math. Soc. Transl. Ser.*, Vol. 191, American Mathematical Society, Providence, RI, 1999, 67–86, arXiv:math.QA/9802054.

[56] Frenkel I.B., Khovanov M.G., Canonical bases in tensor products and graphical calculus for $U_q(\mathfrak{sl}_2)$, *Duke Math. J.* **87** (1997), 409–480.

[57] Frohman C., Kania-Bartoszynska J., Lê T., Unicity for representations of the Kauffman bracket skein algebra, *Invent. Math.* **215** (2019), 609–650, arXiv:1707.09234.

[58] Ganev I., Jordan D., Safronov P., The quantum Frobenius for character varieties and multiplicative quiver varieties, *J. Eur. Math. Soc.*, to appear, arXiv:1901.11450.

[59] Humphreys J.E., Linear algebraic groups, *Grad. Texts Math.*, Vol. 21, Springer, New York, 1975.

[60] Jordan D., White N., The center of the reflection equation algebra via quantum minors, *J. Algebra* **542** (2020), 308–342, arXiv:1709.09149.

[61] Joseph A., Quantum groups and their primitive ideals, *Ergeb. Math. Grenzgeb. (3)*, Vol. 29, Springer, Berlin, 1995.

[62] Joseph A., Letzter G., Local finiteness of the adjoint action for quantized enveloping algebras, *J. Algebra* **153** (1992), 289–318.

[63] Joseph A., Letzter G., Separation of variables for quantized enveloping algebras, *Amer. J. Math.* **116** (1994), 127–177.

[64] Karuo H., Korinman J., Classification of semi-weight representations of reduced stated skein algebras, arXiv:2303.09433.

[65] Kashiwara M., On crystal bases of the Q-analogue of universal enveloping algebras, *Duke Math. J.* **63** (1991), 465–516.

[66] Kashiwara M., Global crystal bases of quantum groups, *Duke Math. J.* **69** (1993), 455–485.

[67] Kashiwara M., Crystal bases of modified quantized enveloping algebra, *Duke Math. J.* **73** (1994), 383–413.

[68] Kashiwara M., On crystal bases, in Representations of Groups (Banff, AB, 1994), *CMS Conf. Proc.*, Vol. 16, American Mathematical Society, Providence, RI, 1995, 155–197.

[69] Khoroshkin S.M., Tolstoy V.N., Universal R-matrix for quantized (super)algebras, *Comm. Math. Phys.* **141** (1991), 599–617.

[70] Kirillov A.N., Reshetikhin N., q-Weyl group and a multiplicative formula for universal R-matrices, *Comm. Math. Phys.* **134** (1990), 421–431.

[71] Klimyk A., Schmüdgen K., Quantum groups and their representations, *Texts Monogr. Phys.*, Springer, Berlin, 1997.

[72] Kolb S., Lorenz M., Nguyen B., Yammine R., On the adjoint representation of a Hopf algebra, *Proc. Edinb. Math. Soc.* **63** (2020), 1092–1099, arXiv:1905.03020.

[73] Korinman J., Unicity for representations of reduced stated skein algebras, *Topology Appl.* **293** (2021), 107570, 28 pages, arXiv:2001.00969.

[74] Korinman J., Finite presentations for stated skein algebras and lattice gauge field theory, *Algebr. Geom. Topol.* **23** (2023), 1249–1302, arXiv:2012.03237.

[75] Korinman J., Quesney A., Classical shadows of stated skein representations at odd roots of unity, arXiv:1905.03441.

[76] Kostant B., Groups over Z, in Algebraic Groups and Discontinuous Subgroups (Proc. Sympos. Pure Math., Boulder, Colo., 1965), American Mathematical Society, Providence, RI, 1966, 90–98.

[77] Lang S., Algebraic structures, Addison-Wesley Publishing Co., Reading, Mass., 1967.

[78] Lê T.T.Q., Triangular decomposition of skein algebras, *Quantum Topol.* **9** (2018), 591–632, arXiv:1609.04987.

[79] Lê T.T.Q., Sikora A.S., Stated SL(n)-skein modules and algebras, arXiv:2201.00045.

[80] Lê T.T.Q., Yu T., Quantum traces and embeddings of stated skein algebras into quantum tori, *Selecta Math. (N.S.)* **28** (2022), 66, 48 pages, arXiv:2012.15272.

[81] Levendorskii S.Z., Soibelman Y.S., Some applications of the quantum Weyl groups, *J. Geom. Phys.* **7** (1990), 241–254.

[82] Lusztig G., Quantum groups at roots of 1, *Geom. Dedicata* **35** (1990), 89–113.

[83] Lusztig G., Introduction to quantum groups, *Mod. Birkhäuser Class.*, Vol. 110, Birkhäuser, Boston, MA, 1993.

[84] Lusztig G., Study of a $\mathbb{Z}$-form of the coordinate ring of a reductive group, *J. Amer. Math. Soc.* **22** (2009), 739–769.

[85] Lyubashenko V., Majid S., Braided groups and quantum Fourier transform, *J. Algebra* **166** (1994), 506–528.

[86] Majid S., Braided matrix structure of the Sklyanin algebra and of the quantum Lorentz group, *Comm. Math. Phys.* **156** (1993), 607–638, arXiv:hep-th/9208008.

[87] Marlin R., Anneaux de Grothendieck des variétés de drapeaux, *Bull. Soc. Math. France* **104** (1976), 337–348.

[88] McConnell J.C., Robson J.C., Noncommutative Noetherian rings, *Grad. Stud. Math.*, Vol. 30, American Mathematical Society, Providence, RI, 2001.

[89] Meusburger C., Kitaev lattice models as a Hopf algebra gauge theory, *Comm. Math. Phys.* **353** (2017), 413–468, arXiv:1607.01144.

[90] Murakami J., Generalized Kashaev invariants for knots in three manifolds, *Quantum Topol.* **8** (2017), 35–73, arXiv:1312.0330.

[91] Paradowski J., Filtrations of modules over the quantum algebra, in Algebraic Groups and their Generalizations: Quantum and Infinite-Dimensional Methods (University Park, PA, 1991), *Proc. Sympos. Pure Math.*, Vol. 56, American Mathematical Society, Providence, RI, 1994, 93–108.

[92] Parshall B., Wang J.P., Quantum linear groups, *Mem. Amer. Math. Soc.* **89** (1991), vi+157 pages.

[93] Przytycki J.H., Sikora A.S., Skein algebras of surfaces, *Trans. Amer. Math. Soc.* **371** (2019), 1309–1332, arXiv:1602.07402.

[94] Reshetikhin N.Yu., Semenov-Tian-Shansky M.A., Quantum R-matrices and factorization problems, *J. Geom. Phys.* **5** (1988), 533–550 (1989).

[95] Reshetikhin N.Yu., Turaev V.G., Invariants of 3-manifolds via link polynomials and quantum groups, *Invent. Math.* **103** (1991), 547–597.

[96] Rowen L.H., Ring theory, Academic Press, Inc., Boston, MA, 1991.

[97] Saito Y., PBW basis of quantized universal enveloping algebras, *Publ. Res. Inst. Math. Sci.* **30** (1994), 209–232.

[98] Soibelman Y.S., Algebra of functions on a compact quantum group and its representations, *Leningrad Math. J.* **2** (1990), 161–268.

[99] Springer T.A., Invariant theory, *Lect. Notes in Math.*, Vol. 585, Springer, Berlin, 1977.

[100] The Stacks Project Authors, Commutative Alg, The Stack Project, Chapter 10, available at `https://stacks.math.columbia.edu`.

[101] The Stacks Project Authors, Brauer groups, The Stack Project, Chapter 11, available at `https://stacks.math.columbia.edu`.

[102] Vaksman L.L., Soibelman Y.S., Algebra of functions on the quantum group SU(2), *Funct. Anal. Appl.* **22** (1988), 170–181.

[103] Varagnolo M., Vasserot E., Double affine Hecke algebras at roots of unity, *Represent. Theory* **14** (2010), 510–600, arXiv:math.RT/0603744.

[104] Voigt C., Yuncken R., Complex semisimple quantum groups and representation theory, *Lect. Notes in Math.*, Vol. 2264, Springer, Cham, 2020.

[105] Wang Z., On stated SL(n)-skein modules, arXiv:2307.10288.

[106] Witten E., Topological quantum field theory, *Comm. Math. Phys.* **117** (1988), 353–386.

[107] Xi N.H., Root vectors in quantum groups, *Comment. Math. Helv.* **69** (1994), 612–639.