

Computation of Weighted Bergman Inner Products on Bounded Symmetric Domains and Parseval–Plancherel-Type Formulas under Subgroups

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Abstract. Let $(G, G_1) = (G, (G^\sigma)_0)$ be a symmetric pair of holomorphic type, and we consider a pair of Hermitian symmetric spaces $D_1 = G_1/K_1 \subset D = G/K$, realized as bounded symmetric domains in complex vector spaces $\mathfrak{p}_1^+ := (\mathfrak{p}^+)^\sigma \subset \mathfrak{p}^+$ respectively. Then the universal covering group \tilde{G} of G acts unitarily on the weighted Bergman space $\mathcal{H}_\lambda(D) \subset \mathcal{O}(D) = \mathcal{O}_\lambda(D)$ on D for sufficiently large λ . Its restriction to the subgroup \tilde{G}_1 decomposes discretely and multiplicity-freely, and its branching law is given explicitly by Hua–Kostant–Schmid–Kobayashi’s formula in terms of the \tilde{K}_1 -decomposition of the space $\mathcal{P}(\mathfrak{p}_2^+)$ of polynomials on $\mathfrak{p}_2^+ := (\mathfrak{p}^+)^{-\sigma} \subset \mathfrak{p}^+$. The object of this article is to understand the decomposition of the restriction $\mathcal{H}_\lambda(D)|_{\tilde{G}_1}$ by studying the weighted Bergman inner product on each \tilde{K}_1 -type in $\mathcal{P}(\mathfrak{p}_2^+) \subset \mathcal{H}_\lambda(D)$. For example, by computing explicitly the norm $\|f\|_\lambda$ for $f = f(x_2) \in \mathcal{P}(\mathfrak{p}_2^+)$, we can determine the Parseval–Plancherel-type formula for the decomposition of $\mathcal{H}_\lambda(D)|_{\tilde{G}_1}$. Also, by computing the poles of $\langle f(x_2), e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\lambda, x}$ for $f(x_2) \in \mathcal{P}(\mathfrak{p}_2^+)$, $x = (x_1, x_2)$, $z \in \mathfrak{p}^+ = \mathfrak{p}_1^+ \oplus \mathfrak{p}_2^+$, we can get some information on branching of $\mathcal{O}_\lambda(D)|_{\tilde{G}_1}$ also for λ in non-unitary range. In this article we consider these problems for all \tilde{K}_1 -types in $\mathcal{P}(\mathfrak{p}_2^+)$.

Key words: weighted Bergman spaces; holomorphic discrete series representations; branching laws; Parseval–Plancherel-type formulas; highest weight modules

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1 Introduction

The purpose of this article is to study the restriction of holomorphic discrete series representations to some subgroups, e.g., to determine the Parseval–Plancherel-type formulas, by computing the weighted Bergman inner products on bounded symmetric domains. This article is a continuation of the author’s previous articles [43, 44].

We consider a Hermitian symmetric space $D \simeq G/K$, realized as a bounded symmetric domain $D \subset \mathfrak{p}^+$ in a complex vector space \mathfrak{p}^+ centered at the origin. We assume that G is connected, and K is the isotropy subgroup at $0 \in \mathfrak{p}^+$. Let \tilde{G}, \tilde{K} denote the universal covering groups of G, K , let (τ, V) be a finite-dimensional representation of \tilde{K} , and we consider the homogeneous vector bundle $\tilde{G} \times_{\tilde{K}} V \rightarrow \tilde{G}/\tilde{K} \simeq D$. Then we can trivialize this bundle, and the space of holomorphic sections is identified with the space $\mathcal{O}(D, V) = \mathcal{O}_\tau(D, V)$ of V -valued holomorphic functions on D , on which \tilde{G} acts by

$$(\hat{\tau}(g)f)(x) := \tau(\kappa(g^{-1}, x))^{-1} f(g^{-1}x), \quad g \in \tilde{G}, \quad x \in D, \quad f \in \mathcal{O}(D, V),$$

by using some function $\tau: \tilde{G} \times D \rightarrow \tilde{K}^{\mathbb{C}}$. According to the choice of the representation (τ, V) of \tilde{K} , there may or may not exist a Hilbert subspace $\mathcal{H}_\tau(D, V) \subset \mathcal{O}_\tau(D, V)$ on which \tilde{G} acts unitarily. The classification of (τ, V) such that $\mathcal{H}_\tau(D, V)$ exists is given by [10, 21]. Especially, if the \tilde{G} -invariant inner product of $\mathcal{H}_\tau(D, V)$ is given by an integral on D , then this is called a *weighted Bergman inner product* and $(\hat{\tau}, \mathcal{H}_\tau(D, V))$ is called a *holomorphic discrete series representation*. For example, suppose G is simple and $(\tau, V) = (\chi^{-\lambda}, \mathbb{C})$ is one-dimensional,

where χ is a suitably normalized character of K . Then for sufficiently large $\lambda \in \mathbb{R}$, the weighted Bergman inner product is given by the integral of the form

$$\langle f, g \rangle_\lambda := C_\lambda \int_D f(x) \overline{g(x)} h(x, \bar{x})^{\lambda-p} dx,$$

where $p \in \mathbb{Z}_{>0}$, $h(x, \bar{y})$ is a suitable polynomial on $\mathfrak{p}^+ \times \overline{\mathfrak{p}^+}$, and the corresponding reproducing kernel (*weighted Bergman kernel*) is given by $h(x, \bar{y})^{-\lambda}$, if we choose the normalizing constant C_λ suitably. In this case we write $(\hat{\tau}, \mathcal{H}_\tau(D, V)) =: (\tau_\lambda, \mathcal{H}_\lambda(D))$, and call it a holomorphic discrete series representation of *scalar type*.

Let us give a concrete example. We consider

$$G = \mathrm{U}(q, s) := \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(q+s, \mathbb{C}) \mid g^* \begin{pmatrix} I_q & 0 \\ 0 & -I_s \end{pmatrix} g = \begin{pmatrix} I_q & 0 \\ 0 & -I_s \end{pmatrix} \right\}.$$

Then G acts transitively on

$$D := \{x \in \mathrm{M}(q, s; \mathbb{C}) \mid I - xx^* \text{ is positive definite}\} \subset \mathfrak{p}^+ := \mathrm{M}(q, s; \mathbb{C})$$

by the linear fractional transform

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} .x := (ax + b)(cx + d)^{-1} = (a^* + xb^*)^{-1}(c^* + xd^*).$$

Next let $\lambda_1, \lambda_2 \in \mathbb{C}$. Then the universal covering group \tilde{G} acts on $\mathcal{O}(D)$ by

$$\left(\tau_{\lambda_1, \lambda_2} \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \right) f \right) (x) := \det(a^* + xb^*)^{-\lambda_1} \det(cx + d)^{-\lambda_2} f((ax + b)(cx + d)^{-1}).$$

We note that $\det(a^* + xb^*)^{-\lambda_1} \det(cx + d)^{-\lambda_2}$ is not well-defined on $G \times D$ unless $\lambda_1, \lambda_2 \in \mathbb{Z}$, but is well-defined on the universal covering space $\tilde{G} \times D$. If $\lambda_1, \lambda_2 \in \mathbb{R}$ satisfies $\lambda_1 + \lambda_2 > q + s - 1$, then this preserves the weighted Bergman inner product

$$\langle f, g \rangle_{\lambda_1 + \lambda_2} := C_\lambda \int_D f(x) \overline{g(x)} \det(I - xx^*)^{\lambda_1 + \lambda_2 - (q+s)} dx,$$

and the corresponding Hilbert space $\mathcal{H}_{\lambda_1 + \lambda_2}(D) \subset \mathcal{O}(D)$ gives a holomorphic discrete series representation of scalar type. The restriction of $\tau_{\lambda_1, \lambda_2}$ to the subgroup $\mathrm{SU}(q, s)$ depends only on the sum $\lambda_1 + \lambda_2$.

Next we consider an involution σ on G , and let $G_1 := (G^\sigma)_0$ be the identity component of the group of fixed points by σ . Without loss of generality we may assume σ stabilizes K , and let $K_1 := G_1 \cap K$. Then $D_1 := G_1 \cdot 0 \simeq G_1/K_1$ is either a complex submanifold or a totally real submanifold of $D \simeq G/K$. The pair (G, G_1) is called a symmetric pair of *holomorphic type* for the former case, and of *anti-holomorphic type* for the latter case (see [26, Section 3.4]). In the following we assume (G, G_1) is of holomorphic type. Then σ induces a holomorphic action on $\mathfrak{p}^+ \simeq T_0(G/K)$, and let $(\mathfrak{p}^+)^\sigma =: \mathfrak{p}_1^+$, $(\mathfrak{p}^+)^{-\sigma} =: \mathfrak{p}_2^+$, so that $\mathfrak{p}^+ = \mathfrak{p}_1^+ \oplus \mathfrak{p}_2^+$ holds. Now we consider the restriction of a holomorphic discrete series representation $\mathcal{H}_\tau(D, V)$ of \tilde{G} to the subgroup \tilde{G}_1 . Then the restriction $\mathcal{H}_\tau(D, V)|_{\tilde{G}_1}$ decomposes into a Hilbert direct sum of irreducible representations of \tilde{G}_1 , and this decomposition is given in terms of the decomposition of $\mathcal{P}(\mathfrak{p}_2^+) \otimes (V|_{\tilde{K}_1})$ under \tilde{K}_1 , where $\mathcal{P}(\mathfrak{p}_2^+)$ denotes the space of polynomials on \mathfrak{p}_2^+ . That is, if $\mathcal{P}(\mathfrak{p}_2^+) \otimes (V|_{\tilde{K}_1})$ is decomposed under \tilde{K}_1 as

$$\mathcal{P}(\mathfrak{p}_2^+) \otimes (V|_{\tilde{K}_1}) \simeq \bigoplus_j m(\tau, \rho_j)(\rho_j, W_j), \quad m(\tau, \rho_j) \in \mathbb{Z}_{\geq 0},$$

then $\mathcal{H}_\tau(D, V)|_{\tilde{G}_1}$ is decomposed abstractly into the Hilbert direct sum

$$\mathcal{H}_\tau(D, V)|_{\tilde{G}_1} \simeq \sum_j^\oplus m(\tau, \rho_j) \mathcal{H}_{\rho_j}(D_1, W_j)$$

(see Kobayashi [26, Lemma 8.8], [25, Section 8], for earlier results, see also [22, 39]). We note that if the unitary subrepresentation $\mathcal{H}_\tau(D, V)$ is not holomorphic discrete, then its K -finite part $\mathcal{H}_\tau(D, V)|_{\tilde{K}}$ may be strictly smaller than $\mathcal{O}_\tau(D, V)|_{\tilde{K}} = \mathcal{P}(\mathfrak{p}^+) \otimes V$, and the above decomposition may not hold in general.

Let us observe this decomposition when G is simple and $\mathcal{H}_\tau(D, V) = \mathcal{H}_\lambda(D)$ is a holomorphic discrete series representation of scalar type. In general, \mathfrak{p}^+ has a *Jordan triple system* structure, and if \mathfrak{p}^+ is simple, then for an involution σ on \mathfrak{p}^+ , $\mathfrak{p}_2^+ = (\mathfrak{p}^+)^{-\sigma}$ is a direct sum of at most 2 simple Jordan triple subsystems. Let χ, χ_1 be suitable characters of \tilde{K}, \tilde{K}_1 , respectively, and let $\chi|_{\tilde{K}_1} = \chi_1^{\varepsilon_1}$, where $\varepsilon_1 \in \{1, 2\}$. Then the decomposition of $\chi^{-\lambda} \otimes \mathcal{P}(\mathfrak{p}_2^+)$ under \tilde{K}_1 ,

$$(\chi^{-\lambda}|_{\tilde{K}_1}) \otimes \mathcal{P}(\mathfrak{p}_2^+) \simeq \chi^{-\varepsilon_1 \lambda} \otimes \bigoplus_{\tilde{\mathbf{k}}} \mathcal{P}_{\tilde{\mathbf{k}}}(\mathfrak{p}_2^+)$$

is given by using the parameter set

$$\begin{aligned} \tilde{\mathbf{k}} = \mathbf{k} &\in \mathbb{Z}_{++}^{r_2} := \{(k_1, \dots, k_{r_2}) \in \mathbb{Z}^{r_2} \mid k_1 \geq \dots \geq k_{r_2} \geq 0\}, & \mathfrak{p}_2^+ : \text{simple}, \\ \tilde{\mathbf{k}} &= (\mathbf{k}, \mathbf{l}) \in \mathbb{Z}_{++}^{r'} \times \mathbb{Z}_{++}^{r''}, & \mathfrak{p}_2^+ : \text{non-simple} \end{aligned}$$

for some $r_2, r', r'' \in \mathbb{Z}_{>0}$. According to this decomposition, $\mathcal{H}_\lambda(D)$ is decomposed under \tilde{G}_1 as

$$\mathcal{H}_\lambda(D)|_{\tilde{G}_1} \simeq \sum_{\tilde{\mathbf{k}}}^\oplus \mathcal{H}_{\varepsilon_1 \lambda}(D_1, \mathcal{P}_{\tilde{\mathbf{k}}}(\mathfrak{p}_2^+)) \quad (1.1)$$

(see Kobayashi [26, Theorem 8.3]), where $\mathcal{H}_{\varepsilon_1 \lambda}(D_1, \mathcal{P}_{\tilde{\mathbf{k}}}(\mathfrak{p}_2^+))$ denotes the holomorphic discrete series representation of \tilde{G}_1 with the fiber $\chi_1^{-\varepsilon_1 \lambda} \otimes \mathcal{P}_{\tilde{\mathbf{k}}}(\mathfrak{p}_2^+)$. Each $\mathcal{H}_{\varepsilon_1 \lambda}(D_1, \mathcal{P}_{\tilde{\mathbf{k}}}(\mathfrak{p}_2^+))$ -isotypic component in $\mathcal{H}_\lambda(D)$ is generated by $\mathcal{P}_{\tilde{\mathbf{k}}}(\mathfrak{p}_2^+) \subset \mathcal{P}(\mathfrak{p}^+) = \mathcal{H}_\lambda(D)|_{\tilde{K}}$ as a \tilde{G}_1 -module. Our aim is to understand this decomposition concretely by studying the weighted Bergman inner product

$$\langle f(x_2), e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\lambda, x}, \quad x = (x_1, x_2), \quad z \in \mathfrak{p}^+ = \mathfrak{p}_1^+ \oplus \mathfrak{p}_2^+, \quad f(x_2) \in \mathcal{P}_{\tilde{\mathbf{k}}}(\mathfrak{p}_2^+). \quad (1.2)$$

Here the subscript x stands for the variable of integration.

One of main problems on the restriction of representations is to determine the \tilde{G}_1 -intertwining operators

$$\begin{aligned} \mathcal{F}_{\tau \rho_j}^\downarrow : \mathcal{H}_\tau(D, V)|_{\tilde{G}_1} &\longrightarrow \mathcal{H}_{\rho_j}(D_1, W_j) \quad \text{or} \quad \mathcal{O}_\tau(D, V)|_{\tilde{G}_1} \longrightarrow \mathcal{O}_{\rho_j}(D_1, W_j), \\ \mathcal{F}_{\tau \rho_j}^\uparrow : \mathcal{H}_{\rho_j}(D_1, W_j) &\longrightarrow \mathcal{H}_\tau(D, V)|_{\tilde{G}_1} \quad \text{or} \quad \mathcal{O}_{\rho_j}(D_1, W_j) \longrightarrow \mathcal{O}_\tau(D, V)|_{\tilde{G}_1}. \end{aligned}$$

$\mathcal{F}_{\tau \rho_j}^\downarrow$ is called a *symmetry breaking operator* and $\mathcal{F}_{\tau \rho_j}^\uparrow$ is called a *holographic operator*, according to the terminology introduced in [32, 33] and [34] respectively. Such a problem is proposed by Kobayashi from the viewpoint of the representation theory (see [28]), and studied from various viewpoints, e.g., by [6, 7, 8, 19, 23, 27, 29, 31, 32, 33, 34, 35, 36, 42, 43, 44, 49, 50, 51, 53] for holomorphic discrete series, principal series, and complementary series representations. Especially, it is proved by [27, 32] that in the holomorphic setting, symmetry breaking operators $\mathcal{F}_{\tau \rho_j}^\downarrow : \mathcal{O}_\tau(D, V)|_{\tilde{G}_1} \rightarrow \mathcal{O}_{\rho_j}(D_1, W_j)$ are always given by differential operators, and their symbols

are characterized as polynomial solutions of certain systems of differential equations (F-method). Also, by [43] when $\mathcal{H}_\tau(D, V)$ is holomorphic discrete, the differential operator

$$\tilde{\mathcal{F}}_{\tau\rho_j}^\downarrow : \mathcal{H}_\tau(D, V) \longrightarrow \mathcal{H}_{\rho_j}(D_1, W_j), \quad f(x) = f(x_1, x_2) \longmapsto \tilde{F}_{\tau\rho_j}^\downarrow \left(\frac{\partial}{\partial x} \right) f(x) \Big|_{x_2=0}$$

defined by using the operator-valued polynomial $\tilde{F}_{\tau\rho_j}^\downarrow(z) \in \mathcal{P}(\mathfrak{p}^-) \otimes \text{Hom}_{\mathbb{C}}(V, W_j)$ on the dual space \mathfrak{p}^- of \mathfrak{p}^+ ,

$$\tilde{F}_{\tau\rho_j}^\downarrow(z) = \langle e^{(x|z)_{\mathfrak{p}^+}}, \mathbf{K}(x_2) \rangle_{\mathcal{H}_\tau(D, V), x}, \quad \mathbf{K}(x_2) \in (\mathcal{P}(\mathfrak{p}_2^+) \otimes \text{Hom}_{\mathbb{C}}(\overline{V}, \overline{W}_j))^{\tilde{K}_1}$$

becomes a symmetry breaking operator. Here $(\cdot|\cdot)_{\mathfrak{p}^+}$ is a suitable non-degenerate pairing on $\mathfrak{p}^+ \times \mathfrak{p}^-$. Especially when $\mathcal{H}_\tau(D, V) = \mathcal{H}_\lambda(D)$ is of scalar type, the construction of symmetry breaking operators

$$\mathcal{F}_{\lambda, \tilde{\mathbf{k}}}^\downarrow : \mathcal{H}_\lambda(D)|_{\tilde{G}_1} \longrightarrow \mathcal{H}_{\varepsilon_1\lambda}(D_1, \mathcal{P}_{\tilde{\mathbf{k}}}(\mathfrak{p}_2^+))$$

is reduced to the computation of (1.2). Since the restriction of a unitary highest weight representation of scalar type $\mathcal{H}_\lambda(D)|_{\tilde{G}_1}$ decomposes multiplicity-freely by [26, Theorem A], symmetry breaking operators from $\mathcal{H}_\lambda(D)|_{\tilde{G}_1}$ are unique up to constant multiple. On the other hand, if λ is not in the holomorphic discrete range, we do not know a priori whether symmetry breaking operators from $\mathcal{O}_\lambda(D)|_{\tilde{G}_1}$ are unique or not. Indeed, for tensor product case, the space of symmetry breaking operators from $\mathcal{O}_\lambda(D) \hat{\otimes} \mathcal{O}_\mu(D)$ may become greater than 1-dimensional for some $(\lambda, \mu) \in \mathbb{C}$ (see [33, Section 9], or Section 8 of this paper).

In [44], when \mathfrak{p}^+ is simple and when both \mathfrak{p}^+ , \mathfrak{p}_2^+ are of tube type, we explicitly computed the inner product (1.2) for $\tilde{\mathbf{k}}$ of the form one of $\mathbf{k} = \mathbf{k} = (k+l, k, \dots, k)$, $\tilde{\mathbf{k}} = \mathbf{k} = (k+1, \dots, k+1, k, \dots, k)$ or $\tilde{\mathbf{k}} = (\mathbf{k}, \mathbf{l}) = ((k, \dots, k), \mathbf{l})$ according to the choice of $(\mathfrak{p}^+, \mathfrak{p}_1^+, \mathfrak{p}_2^+)$, and determined the symmetry breaking operators for these $\tilde{\mathbf{k}}$. Especially if $\tilde{\mathbf{k}} = (k, \dots, k)$ or $\tilde{\mathbf{k}} = ((k, \dots, k), (l, \dots, l))$, then $\mathcal{H}_{\varepsilon_1\lambda}(D_1, \mathcal{P}_{\tilde{\mathbf{k}}}(\mathfrak{p}_2^+))$ becomes of scalar type, and the inner product (1.2) and the symmetry breaking operator are given by using Heckman–Opdam’s hypergeometric polynomials of type *BC*. Also in [43], we have constructed the holographic operators for some $\tilde{\mathbf{k}}$ containing the above cases, as infinite-order differential operators.

We continue to assume $\mathcal{H}_\tau(D, V) = \mathcal{H}_\lambda(D)$ is of scalar type. Then since the decomposition (1.1) is a Hilbert direct sum and is multiplicity-free, under a suitable normalization of $\mathcal{F}_{\lambda, \tilde{\mathbf{k}}}^\downarrow$, there exist constants $C(\lambda, \tilde{\mathbf{k}}) > 0$ such that

$$\|f\|_\lambda^2 = \sum_{\tilde{\mathbf{k}}} C(\lambda, \tilde{\mathbf{k}}) \|\mathcal{F}_{\lambda, \tilde{\mathbf{k}}}^\downarrow f\|_{\varepsilon_1\lambda, \tilde{\mathbf{k}}}^2, \quad f \in \mathcal{H}_\lambda(D)$$

holds, where $\|\cdot\|_{\varepsilon_1\lambda, \tilde{\mathbf{k}}}$ is a \tilde{G}_1 -invariant norm on $\mathcal{H}_{\varepsilon_1\lambda}(D_1, \mathcal{P}_{\tilde{\mathbf{k}}}(\mathfrak{p}_2^+))$. This formula is regarded as an analogue of the Parseval or the Plancherel theorems for the Fourier analysis. Such Parseval–Plancherel-type formulas for symmetric pairs of holomorphic type are studied, e.g., by [3, 4, 5, 16, 17, 34] under different realizations of $\mathcal{H}_\lambda(D)$, $\mathcal{H}_{\varepsilon_1\lambda}(D_1, \mathcal{P}_{\tilde{\mathbf{k}}}(\mathfrak{p}_2^+))$, and those for anti-holomorphic type cases are studied, e.g., by [40, 45, 46, 47, 48, 56, 57, 58, 60, 61, 62, 63]. In our setting, for (G, G_1) of holomorphic type, each $\mathcal{H}_{\varepsilon_1\lambda}(D_1, \mathcal{P}_{\tilde{\mathbf{k}}}(\mathfrak{p}_2^+))$ -isotypic component in $\mathcal{H}_\lambda(D)$ is generated by the minimal \tilde{K}_1 -type $\mathcal{P}_{\tilde{\mathbf{k}}}(\mathfrak{p}_2^+) \subset \mathcal{P}(\mathfrak{p}^+) = \mathcal{H}_\lambda(D)_{\tilde{K}}$, and we assume that $\mathcal{F}_{\lambda, \tilde{\mathbf{k}}}^\downarrow$ is normalized such that $\|\mathcal{F}_{\lambda, \tilde{\mathbf{k}}}^\downarrow f\|_{\varepsilon_1\lambda, \tilde{\mathbf{k}}}$ is independent of the parameter λ for $f = f(x_2) \in \mathcal{P}_{\tilde{\mathbf{k}}}(\mathfrak{p}_2^+)$. Under this setting we want to determine the λ -dependence of $C(\lambda, \tilde{\mathbf{k}})$ explicitly. To do this, it is enough to compute $\|f\|_\lambda$ for $f = f(x_2) \in \mathcal{P}_{\tilde{\mathbf{k}}}(\mathfrak{p}_2^+)$ for each $\tilde{\mathbf{k}}$, and this is deduced from the top term, i.e., the value at $z_1 = 0$, of the inner product (1.2) (see Proposition 2.6).

Next we consider $\mathcal{O}_\lambda(D)$ for λ not in the holomorphic discrete range. Following Faraut–Korányi [12, 13], the weighted Bergman inner product $\langle \cdot, \cdot \rangle_\lambda$ is explicitly computed on each \tilde{K} -type in $\mathcal{P}(\mathfrak{p}^+) = \mathcal{H}_\lambda(D)_{\tilde{K}}$ for sufficiently large λ , and this is meromorphically continued for all $\lambda \in \mathbb{C}$. For smaller λ , $\mathcal{O}_\lambda(D)_{\tilde{K}}$ becomes reducible as a $(\mathfrak{g}, \tilde{K})$ -module if λ is a pole of $\langle \cdot, \cdot \rangle_\lambda$, and for any $f \in \mathcal{P}(\mathfrak{p}^+)$, by computing the poles and their orders of the inner product $\langle f, e^{(\cdot|\bar{z})_{\mathfrak{p}^+}} \rangle_\lambda$ with respect to λ , we can determine which submodule f sits in. Similarly, for a symmetric subgroup $G_1 \subset G$ of holomorphic type, the decomposition (1.1) is restated as

$$\mathcal{O}_\lambda(D)_{\tilde{K}}|_{(\mathfrak{g}_1, \tilde{K}_1)} = \mathcal{P}(\mathfrak{p}^+)|_{(\mathfrak{g}_1, \tilde{K}_1)} = \bigoplus_{\tilde{\mathbf{k}}} d\tau_\lambda(\mathcal{U}(\mathfrak{g}_1))\mathcal{P}_{\tilde{\mathbf{k}}}(\mathfrak{p}_2^+)$$

for sufficiently large λ , where $\mathcal{U}(\mathfrak{g}_1)$ denotes the universal enveloping algebra of \mathfrak{g}_1 , but this does not hold for smaller λ in general. By computing the poles and their orders of the inner product (1.2) with respect to λ , we can determine which $(\mathfrak{g}, \tilde{K})$ -submodule of $\mathcal{O}_\lambda(D)_{\tilde{K}}$ contains the $(\mathfrak{g}_1, \tilde{K}_1)$ -module $d\tau_\lambda(\mathcal{U}(\mathfrak{g}_1))\mathcal{P}_{\tilde{\mathbf{k}}}(\mathfrak{p}_2^+)$ (see Proposition 2.7).

In this article we treat irreducible symmetric pairs $(G, G_1) = (G, (G^\sigma)_0)$ of holomorphic type, and treat holomorphic discrete series representations $\mathcal{H}_\lambda(D)$ of scalar type. According to the decomposition $\mathcal{P}(\mathfrak{p}_2^+) = \bigoplus_{\tilde{\mathbf{k}}} \mathcal{P}_{\tilde{\mathbf{k}}}(\mathfrak{p}_2^+)$ of the space of polynomials on $\mathfrak{p}_2^+ = (\mathfrak{p}^+)^{-\sigma}$, we compute the top terms (Theorems 5.1, 6.1, and 8.4) and the poles (Theorems 5.2, 6.2, and 8.1) of the weighted Bergman inner product (1.2) for all $\tilde{\mathbf{k}} = \mathbf{k} \in \mathbb{Z}_{++}^{r_2}$ or $\tilde{\mathbf{k}} = (\mathbf{k}, \mathbf{l}) \in \mathbb{Z}_{++}^{r'_2} \times \mathbb{Z}_{++}^{r''_2}$. Also, we apply these results for the determination of Parseval–Plancherel-type formulas (Corollaries 4.2, 5.7, 6.7, and 8.7) and $(\mathfrak{g}, \tilde{K})$ -modules (Corollaries 4.3, 5.8, 6.8, and 8.8).

This paper is organized as follows. In Section 2, we review Jordan triple systems, Jordan algebras and holomorphic discrete series representations. In Section 3, we introduce some lemmas needed in later sections. In Section 4, we treat (G, G_1) such that (1.2) is easily computable. For these (G, G_1) , the symmetry breaking operators are given by normal derivatives along \mathfrak{p}_2^+ independent of the parameter λ , and (1.2) is computed directly by using Faraut–Korányi’s result [12, 13]. In Section 5, we treat the cases such that \mathfrak{p}_2^+ is a direct sum of two simple Jordan triple systems, and in Section 6, we treat the cases such that \mathfrak{p}_2^+ is simple. However, for some exceptional $(\mathfrak{p}^+, \mathfrak{p}_1^+, \mathfrak{p}_2^+)$, we need some extra computation to determine the poles of (1.2). In Section 7, we compute the poles of (1.2) when $\mathfrak{p}^+, \mathfrak{p}_2^+$ are simple of rank 3. The above exceptional case is contained in these rank 3 cases. Also, in this section we give a conjecture on the full computation of (1.2) for rank 3 cases. In Section 8, we treat the cases $(\mathfrak{p}^+, \mathfrak{p}_1^+) = (\mathfrak{p}_0^+ \oplus \mathfrak{p}_0^+, \Delta(\mathfrak{p}_0^+))$, and the tensor product representation $\mathcal{H}_\lambda(D_0) \otimes \mathcal{H}_\mu(D_0)$. In this section we also discuss the higher-multiplicity phenomena, which is a generalization of the argument in [33, Section 9].

2 Preliminaries

In this section we review Jordan triple systems, Jordan algebras and holomorphic discrete series representations. We use almost the same notations as in [44], and readers who read [44] can skip Sections 2.1–2.6. For detail see also, e.g., [11, Parts III and V] and [13, 38, 54].

2.1 Hermitian positive Jordan triple systems

We consider a Hermitian positive Jordan triple system $(\mathfrak{p}^+, \mathfrak{p}^-, \{\cdot, \cdot, \cdot\}, \bar{\cdot})$, where \mathfrak{p}^\pm are finite-dimensional vector spaces over \mathbb{C} , with a non-degenerate bilinear form $(\cdot|\cdot)_{\mathfrak{p}^\pm}: \mathfrak{p}^\pm \times \mathfrak{p}^\mp \rightarrow \mathbb{C}$, $\{\cdot, \cdot, \cdot\}: \mathfrak{p}^\pm \times \mathfrak{p}^\mp \times \mathfrak{p}^\pm \rightarrow \mathfrak{p}^\pm$ is a \mathbb{C} -trilinear map satisfying

$$\begin{aligned} \{x, y, z\} &= \{z, y, x\}, \\ \{u, v, \{x, y, z\}\} &= \{\{u, v, x\}, y, z\} - \{x, \{v, u, y\}, z\} + \{x, y, \{u, v, z\}\}, \\ (\{u, v, x\}|y)_{\mathfrak{p}^\pm} &= (x|\{v, u, y\})_{\mathfrak{p}^\pm} \end{aligned}$$

for any $u, x, z \in \mathfrak{p}^\pm$, $v, y \in \mathfrak{p}^\mp$, and $\bar{\cdot}: \mathfrak{p}^\pm \rightarrow \mathfrak{p}^\mp$ is a \mathbb{C} -antilinear involutive isomorphism satisfying $(x|\bar{x})_{\mathfrak{p}^\pm} \geq 0$ for any $x \in \mathfrak{p}^\pm$. Let $D, B: \mathfrak{p}^\pm \times \mathfrak{p}^\mp \rightarrow \text{End}_{\mathbb{C}}(\mathfrak{p}^\pm)$, $Q: \mathfrak{p}^\pm \times \mathfrak{p}^\pm \rightarrow \text{Hom}_{\mathbb{C}}(\mathfrak{p}^\mp, \mathfrak{p}^\pm)$, $Q: \mathfrak{p}^\pm \rightarrow \text{Hom}_{\mathbb{C}}(\mathfrak{p}^\mp, \mathfrak{p}^\pm)$ be the maps given by

$$\begin{aligned} D(x, y)z &= Q(x, z)y := \{x, y, z\}, \\ Q(x) &:= \frac{1}{2}Q(x, x), \\ B(x, y) &:= I_{\mathfrak{p}^\pm} - D(x, y) + Q(x)Q(y) \end{aligned}$$

for $x, z \in \mathfrak{p}^\pm$, $y \in \mathfrak{p}^\mp$, let $h = h_{\mathfrak{p}^\pm}: \mathfrak{p}^\pm \times \mathfrak{p}^\mp \rightarrow \mathbb{C}$ be the *generic norm*, which is a polynomial on $\mathfrak{p}^+ \times \mathfrak{p}^-$ irreducible on each simple Jordan triple subsystem, and write $B(x) := B(x, \bar{x})$, $h(x) := h(x, \bar{x})$. If \mathfrak{p}^+ is simple, then $B(x, y)$ and $h(x, y)$ are related as

$$h(x, y)^p = \text{Det}_{\mathfrak{p}^+} B(x, y), \quad x \in \mathfrak{p}^+, \quad y \in \mathfrak{p}^- \quad (2.1)$$

for some $p \in \mathbb{Z}_{>0}$.

If $e \in \mathfrak{p}^+$ is a tripotent, i.e., $\{e, \bar{e}, e\} = 2e$, then $D(e, \bar{e}) \in \text{End}_{\mathbb{C}}(\mathfrak{p}^+)$ has the eigenvalues 0, 1, 2. For $j = 0, 1, 2$, let

$$\begin{aligned} \mathfrak{p}^+(e)_j &= \mathfrak{p}^+(\bar{e})_j := \{x \in \mathfrak{p}^+ \mid D(e, \bar{e})x = jx\} \subset \mathfrak{p}^+, \\ \mathfrak{p}^-(\bar{e})_j &= \mathfrak{p}^-(e)_j := \{x \in \mathfrak{p}^- \mid D(\bar{e}, e)x = jx\} \subset \mathfrak{p}^-, \end{aligned} \quad (2.2)$$

so that $\mathfrak{p}^\pm = \mathfrak{p}^\pm(e)_2 \oplus \mathfrak{p}^\pm(e)_1 \oplus \mathfrak{p}^\pm(e)_0$ holds (*Peirce decomposition*). A non-zero tripotent $e \in \mathfrak{p}^+$ is called *primitive* if $\mathfrak{p}^+(e)_2 = \mathbb{C}e$, and *maximal* if $\mathfrak{p}^+(e)_0 = \{0\}$. If $\mathfrak{p}^+(e)_2 = \mathfrak{p}^+$ holds for some (or equivalently any) maximal tripotent $e \in \mathfrak{p}^+$, then we say that \mathfrak{p}^+ is of *tube type*. Throughout the paper, we assume that the bilinear form $(\cdot|\cdot)_{\mathfrak{p}^\pm}: \mathfrak{p}^\pm \times \mathfrak{p}^\mp \rightarrow \mathbb{C}$ is normalized such that $(e|\bar{e})_{\mathfrak{p}^+} = (\bar{e}|e)_{\mathfrak{p}^-} = 1$ holds for any primitive tripotent $e \in \mathfrak{p}^+$. If \mathfrak{p}^+ is simple, then under this normalization $(x|y)_{\mathfrak{p}^+}$ and $D(x, y)$ are related as

$$p(x|y)_{\mathfrak{p}^+} = \text{Tr}_{\mathfrak{p}^+} D(x, y), \quad x \in \mathfrak{p}^+, \quad y \in \mathfrak{p}^- \quad (2.3)$$

with the same $p \in \mathbb{Z}_{>0}$ as in (2.1).

Next we fix a tripotent $e \in \mathfrak{p}^+$, and consider $\mathfrak{p}^+(e)_2 \subset \mathfrak{p}^+$ as in (2.2). Then

$$Q(\bar{e}): \mathfrak{p}^+(e)_2 \longrightarrow \mathfrak{p}^-(e)_2, \quad Q(e): \mathfrak{p}^-(e)_2 \longrightarrow \mathfrak{p}^+(e)_2$$

are mutually inverse. For $x \in \mathfrak{p}^+(e)_2$ let $P(x) := Q(x)Q(\bar{e}) \in \text{End}_{\mathbb{C}}(\mathfrak{p}^+(e)_2)$. Then $\mathfrak{p}^+(e)_2$ becomes a Jordan algebra with the product

$$x \cdot y := \frac{1}{2}\{x, \bar{e}, y\}, \quad x, y \in \mathfrak{p}^+(e)_2,$$

with the unit element e , and with the inverse

$$x^{-1} := P(x)^{-1}x, \quad x \in \mathfrak{p}^+(e)_2.$$

The Euclidean real form $\mathfrak{n}^+ \subset \mathfrak{p}^+(e)_2$ is given by

$$\mathfrak{n}^+ := \{x \in \mathfrak{p}^+(e)_2 \mid Q(e)\bar{x} = x\}.$$

Let $(\cdot|\cdot)_{\mathfrak{n}^+}: \mathfrak{p}^+(e)_2 \times \mathfrak{p}^+(e)_2 \rightarrow \mathbb{C}$ be the symmetric bilinear form on $\mathfrak{p}^+(e)_2 = \mathfrak{n}^{+\mathbb{C}}$ given by

$$(x|y)_{\mathfrak{n}^+} := (x|Q(\bar{e})y)_{\mathfrak{p}^+} = (y|Q(\bar{e})x)_{\mathfrak{p}^+}, \quad x, y \in \mathfrak{p}^+(e)_2.$$

This is positive definite on \mathfrak{n}^+ . Also, let $\det_{\mathfrak{n}^+}(x)$ be the *determinant polynomial* on $\mathfrak{p}^+(e)_2 = \mathfrak{n}^{+\mathbb{C}}$, and let $\det_{\mathfrak{n}^-}(y) := \det_{\mathfrak{n}^+}(Q(e)y)$ for $y \in \mathfrak{p}^-(e)_2$. Then the generic norm $h(x, y)$ and $\det_{\mathfrak{n}^+}(x)$ are related as

$$\begin{aligned} t^r h(t^{-1}x, y)|_{t=0} &= (-1)^r \det_{\mathfrak{n}^+}(x) \det_{\mathfrak{n}^-}(y), \\ h(x, \bar{e}) &= \det_{\mathfrak{n}^+}(e - x), \quad x \in \mathfrak{p}^+(e)_2, \quad y \in \mathfrak{p}^-(e)_2, \end{aligned}$$

where $r := \text{rank } \mathfrak{p}^+(e)_2 = \text{deg } \det_{\mathfrak{n}^+}(x)$. In addition, let

$$\begin{aligned} \Omega &:= (\text{connected component of } \{x \in \mathfrak{n}^+ \mid P(x) \text{ is positive definite}\} \text{ which contains } e) \\ &= (\text{connected component of } \{x \in \mathfrak{n}^+ \mid \det_{\mathfrak{n}^+}(x) > 0\} \text{ which contains } e) \end{aligned}$$

be the *symmetric cone*.

2.2 Structure groups and the Kantor–Koecher–Tits construction

In this subsection we consider some Lie algebras corresponding to Jordan triple systems \mathfrak{p}^\pm . For $l \in \text{End}_{\mathbb{C}}(\mathfrak{p}^+)$, let $\bar{l}, {}^t l \in \text{End}_{\mathbb{C}}(\mathfrak{p}^-)$, $l^* \in \text{End}_{\mathbb{C}}(\mathfrak{p}^+)$ be the elements given by $\bar{l}\bar{x} = \overline{lx}$, $(lx\bar{y})_{\mathfrak{p}^+} = (x{}^t l\bar{y})_{\mathfrak{p}^+} = (x\bar{l}^*y)_{\mathfrak{p}^+}$ for $x, y \in \mathfrak{p}^+$. Then the *Structure group* $\text{Str}(\mathfrak{p}^+)$ and the *automorphism group* $\text{Aut}(\mathfrak{p}^+)$ are defined as

$$\begin{aligned} \text{Str}(\mathfrak{p}^+) &:= \{l \in \text{GL}_{\mathbb{C}}(\mathfrak{p}^+) \mid \{lx, {}^t l^{-1}y, lz\} = l\{x, y, z\}, x, z \in \mathfrak{p}^+, y \in \mathfrak{p}^-\}, \\ \text{Aut}(\mathfrak{p}^+) &:= \{k \in \text{Str}(\mathfrak{p}^+) \mid k^{-1} = k^*\}. \end{aligned}$$

Let $\mathfrak{str}(\mathfrak{p}^+) = \mathfrak{k}^{\mathbb{C}}$ and $\mathfrak{der}(\mathfrak{p}^+) = \mathfrak{k}$ denote the Lie algebras of $\text{Str}(\mathfrak{p}^+)$ and $\text{Aut}(\mathfrak{p}^+)$ respectively. Then $D(\mathfrak{p}^+, \mathfrak{p}^-) = \mathfrak{str}(\mathfrak{p}^+)$ holds.

Next we recall the *Kantor–Koecher–Tits construction*. As vector spaces let

$$\begin{aligned} \mathfrak{g}^{\mathbb{C}} &:= \mathfrak{p}^+ \oplus \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{p}^-, \\ \mathfrak{g} &:= \{(x, k, \bar{x}) \mid x \in \mathfrak{p}^+, k \in \mathfrak{k}\} \subset \mathfrak{g}^{\mathbb{C}}, \end{aligned}$$

and give the Lie algebra structure on $\mathfrak{g}^{\mathbb{C}}$ by

$$[(x, k, y), (z, l, w)] := (kz - lx, [k, l] + D(x, w) - D(z, y), -{}^t kw + {}^t ly).$$

Then $\mathfrak{g}^{\mathbb{C}}$ becomes a Lie algebra and \mathfrak{g} becomes a real form of $\mathfrak{g}^{\mathbb{C}}$. Let $(\cdot|\cdot)_{\mathfrak{g}^{\mathbb{C}}}: \mathfrak{g}^{\mathbb{C}} \times \mathfrak{g}^{\mathbb{C}} \rightarrow \mathbb{C}$ be the $\mathfrak{g}^{\mathbb{C}}$ -invariant bilinear form normalized such that $(x|y)_{\mathfrak{g}^{\mathbb{C}}} = (x|y)_{\mathfrak{p}^+}$ holds for any $x \in \mathfrak{p}^+$, $y \in \mathfrak{p}^-$. We fix a connected complex Lie group $G^{\mathbb{C}}$ with the Lie algebra $\mathfrak{g}^{\mathbb{C}}$, and let $G, K^{\mathbb{C}}, K, P^+, P^- \subset G^{\mathbb{C}}$ be the connected closed subgroups corresponding to the Lie algebras $\mathfrak{g}, \mathfrak{k}^{\mathbb{C}}, \mathfrak{k}, \mathfrak{p}^+, \mathfrak{p}^- \subset \mathfrak{g}^{\mathbb{C}}$, respectively. Then $\text{Ad}|_{\mathfrak{p}^+}: K^{\mathbb{C}} \rightarrow \text{Str}(\mathfrak{p}^+)_0$ gives a covering map, and for $l \in K^{\mathbb{C}}$, $x \in \mathfrak{p}^+$, we abbreviate $\text{Ad}(l)x =: lx$.

Next we fix a tripotent $e \in \mathfrak{p}^+$, regard $\mathfrak{p}^+(e)_2 \subset \mathfrak{p}^+$ as a Jordan algebra, and let $\mathfrak{n}^+ \subset \mathfrak{p}_2^+(e)$ be the Euclidean real form. Also, let

$$\begin{aligned} \mathfrak{n}^- &:= \overline{\mathfrak{n}^+} = Q(\bar{e})\mathfrak{n}^+ \subset \mathfrak{p}^-(e)_2, \\ \mathfrak{k}^{\mathbb{C}}(e)_2 &:= D(\mathfrak{p}^+(e)_2, \mathfrak{p}^-(e)_2) = [\mathfrak{p}^+(e)_2, \mathfrak{p}^-(e)_2] \subset \mathfrak{k}^{\mathbb{C}}, \\ \mathfrak{k}(e)_2 &:= \mathfrak{k}^{\mathbb{C}}(e)_2 \cap \mathfrak{k} \subset \mathfrak{k}^{\mathbb{C}}(e)_2, \\ \mathfrak{l} &:= D(\mathfrak{n}^+, \mathfrak{n}^-) = [\mathfrak{n}^+, \mathfrak{n}^-] \subset \mathfrak{k}^{\mathbb{C}}(e)_2, \\ \mathfrak{g}^{\mathbb{C}}(e)_2 &:= \mathfrak{p}^+(e)_2 \oplus \mathfrak{k}^{\mathbb{C}}(e)_2 \oplus \mathfrak{p}^-(e)_2 \subset \mathfrak{g}^{\mathbb{C}}, \\ \mathfrak{g}(e)_2 &:= \mathfrak{g}^{\mathbb{C}}(e)_2 \cap \mathfrak{g} \subset \mathfrak{g}^{\mathbb{C}}(e)_2, \\ \mathfrak{c}\mathfrak{g} &:= \mathfrak{n}^+ \oplus \mathfrak{l} \oplus \mathfrak{n}^- \subset \mathfrak{g}^{\mathbb{C}}(e)_2. \end{aligned}$$

These become Lie subalgebras of the right-hand sides. Let $K^{\mathbb{C}}(e)_2, K(e)_2, G(e)_2, {}^cG \subset G^{\mathbb{C}}$ be the connected closed subgroups corresponding to the Lie algebras $\mathfrak{k}^{\mathbb{C}}(e)_2, \mathfrak{k}(e)_2, \mathfrak{g}(e)_2, {}^c\mathfrak{g} \subset \mathfrak{g}^{\mathbb{C}}$ respectively, and let $L := K^{\mathbb{C}} \cap {}^cG, K_L := L \cap K, \mathfrak{l}_1 := \mathfrak{l} \cap \mathfrak{k}$. Then $G(e)_2$ and cG are isomorphic via the Cayley transform in $G^{\mathbb{C}}(e)_2$, both $K(e)_2$ and L are real forms of $K^{\mathbb{C}}(e)_2$, L acts transitively on the symmetric cone $\Omega \subset \mathfrak{n}^+$, and K_L acts on \mathfrak{n}^+ as Jordan algebra automorphisms. Also, for $l \in \mathfrak{k}^{\mathbb{C}}(e)_2 = \mathfrak{l}^{\mathbb{C}} \subset \text{End}_{\mathbb{C}}(\mathfrak{p}^+(e)_2) = \text{End}_{\mathbb{C}}(\mathfrak{n}^{+\mathbb{C}})$, let $l^{\top} = Q(e)^t l Q(\bar{e}) \in \mathfrak{k}^{\mathbb{C}}(e)_2 = \mathfrak{l}^{\mathbb{C}}$, and extend to the anti-automorphism on $K^{\mathbb{C}}(e)_2 = L^{\mathbb{C}}$, so that $(lx|y)_{\mathfrak{n}^+} = (x|l^{\top}y)_{\mathfrak{n}^+}$ holds.

2.3 Simultaneous Peirce decomposition

In this subsection we assume \mathfrak{p}^+ is simple, or equivalently, the corresponding Lie algebra \mathfrak{g} is simple. We fix a Jordan frame $\{e_1, \dots, e_r\} \subset \mathfrak{p}^+$, i.e., a maximal set of primitive tripotents in \mathfrak{p}^+ satisfying $D(e_i, \bar{e}_j) = 0$ for $i \neq j$, where $r = \text{rank } \mathfrak{p}^+ = \text{rank}_{\mathbb{R}} \mathfrak{g}$. Then $e := e_1 + \dots + e_r \in \mathfrak{p}^+$ becomes a maximal tripotent, and we take $\mathfrak{n}^+ \subset \mathfrak{p}^+(e)_2 \subset \mathfrak{p}^+, \mathfrak{l} \subset \mathfrak{k}^{\mathbb{C}}(e)_2 \subset \mathfrak{k}^{\mathbb{C}}$ as in the previous subsection. Next let $h_j := D(e_j, \bar{e}_j) = [e_j, \bar{e}_j] \in \mathfrak{l} \subset \mathfrak{k}^{\mathbb{C}}$, and let

$$\begin{aligned} \mathfrak{a}_1 &:= \bigoplus_{j=1}^r \mathbb{R}h_j, & \mathfrak{a}_1^{\mathbb{C}} &:= \bigoplus_{j=1}^r \mathbb{C}h_j, \\ \mathfrak{p}_{ij}^{\pm} &:= \{x \in \mathfrak{p}^{\pm} \mid [h_l, x] = \pm(\delta_{il} + \delta_{jl})x, l = 1, \dots, r\}, & 1 \leq i \leq j \leq r, \\ \mathfrak{p}_{0j}^{\pm} &:= \{x \in \mathfrak{p}^{\pm} \mid [h_l, x] = \pm\delta_{jl}x, l = 1, \dots, r\}, & 1 \leq j \leq r, \\ \mathfrak{k}_{ij}^{\mathbb{C}} &:= \{x \in \mathfrak{k}^{\mathbb{C}} \mid [h_l, x] = (\delta_{il} - \delta_{jl})x, l = 1, \dots, r\}, & 1 \leq i, j \leq r, \quad i \neq j, \\ \mathfrak{k}_{i0}^{\mathbb{C}} &:= \{x \in \mathfrak{k}^{\mathbb{C}} \mid [h_l, x] = \delta_{il}x, l = 1, \dots, r\}, & 1 \leq i \leq r, \\ \mathfrak{k}_{0j}^{\mathbb{C}} &:= \{x \in \mathfrak{k}^{\mathbb{C}} \mid [h_l, x] = -\delta_{jl}x, l = 1, \dots, r\}, & 1 \leq j \leq r, \\ \mathfrak{m}^{\mathbb{C}} &:= \{x \in \mathfrak{k}^{\mathbb{C}} \mid [h_l, x] = 0, (h_l|x)_{\mathfrak{g}^{\mathbb{C}}} = 0, l = 1, \dots, r\}, \\ \mathfrak{l}_{ij} &:= \mathfrak{k}_{ij}^{\mathbb{C}} \cap \mathfrak{l}, & 1 \leq i, j \leq r, \quad i \neq j, \\ \mathfrak{m}_1 &:= \mathfrak{m}^{\mathbb{C}} \cap \mathfrak{l}, \end{aligned}$$

so that

$$\mathfrak{p}^{\pm} = \bigoplus_{\substack{0 \leq i < j \leq r \\ (i,j) \neq (0,0)}} \mathfrak{p}_{ij}^{\pm}, \quad \mathfrak{k}^{\mathbb{C}} = \mathfrak{a}_1^{\mathbb{C}} \oplus \mathfrak{m}^{\mathbb{C}} \oplus \bigoplus_{\substack{0 \leq i, j \leq r \\ i \neq j}} \mathfrak{k}_{ij}^{\mathbb{C}}, \quad \mathfrak{l} = \mathfrak{a}_1 \oplus \mathfrak{m}_1 \oplus \bigoplus_{\substack{1 \leq i, j \leq r \\ i \neq j}} \mathfrak{l}_{ij}$$

hold. The decomposition of \mathfrak{p}^{\pm} is called the *simultaneous Peirce decomposition*. Using this decomposition, we define integers (d, b, p, n) by

$$\begin{aligned} d &:= \dim \mathfrak{p}_{ij}^+, \quad 1 \leq i < j \leq r, & b &:= \dim \mathfrak{p}_{0j}^+, \quad 1 \leq j \leq r, \\ p &:= 2 + d(r-1) + b, & n &:= \dim \mathfrak{p}^+ = r + \frac{d}{2}r(r-1) + br. \end{aligned} \quad (2.4)$$

We note that if $r = 1$, then d is not determined uniquely, and any number is allowed. Then p coincides with the one in (2.1) and (2.3). Also we set

$$\mathfrak{n}_1 := \bigoplus_{1 \leq i < j \leq r} \mathfrak{l}_{ij}, \quad \mathfrak{n}_1^{\top} := \bigoplus_{1 \leq i < j \leq r} \mathfrak{l}_{ji}, \quad M_L := \{k \in K_L \mid \text{Ad}(k)h_l = 0, l = 1, \dots, r\},$$

and let $A_L, N_L, N_L^{\top} \subset L$ be the connected closed subgroups corresponding to the Lie subalgebras $\mathfrak{a}_1, \mathfrak{n}_1, \mathfrak{n}_1^{\top}$ respectively, so that $M_L A_L N_L, M_L A_L N_L^{\top} \subset L$ are minimal parabolic subgroups.

2.4 Space of polynomials on Jordan triple systems

In this subsection we consider the space $\mathcal{P}(\mathfrak{p}^\pm)$ of polynomials on the Jordan triple system \mathfrak{p}^\pm , on which $K^\mathbb{C}$ acts by

$$\begin{aligned} (\text{Ad}|_{\mathfrak{p}^+}(l))^\vee f(x) &= f(l^{-1}x), & l \in K^\mathbb{C}, & f \in \mathcal{P}(\mathfrak{p}^+), & x \in \mathfrak{p}^+, \\ (\text{Ad}|_{\mathfrak{p}^-}(l))^\vee f(y) &= f({}^tly), & l \in K^\mathbb{C}, & f \in \mathcal{P}(\mathfrak{p}^-), & y \in \mathfrak{p}^-. \end{aligned}$$

We assume \mathfrak{p}^+ is simple, fix a Jordan frame $\{e_1, \dots, e_r\} \subset \mathfrak{p}^+$, and consider the tripotents $e^k := \sum_{j=1}^k e_j$, $\check{e}^k := \sum_{j=r-k+1}^r e_j$ for $k = 1, 2, \dots, r$. We also write $e^r = \check{e}^r =: e$. Then the subalgebras

$$\mathfrak{p}^+(e^k)_2 = \bigoplus_{1 \leq i \leq j \leq k} \mathfrak{p}_{ij}^+, \quad \mathfrak{p}^+(\check{e}^k)_2 = \bigoplus_{r-k+1 \leq i \leq j \leq r} \mathfrak{p}_{ij}^+$$

have Jordan algebra structures. Let $\mathfrak{n}^+(e^k) \subset \mathfrak{p}^+(e^k)_2$, $\mathfrak{n}^+(\check{e}^k) \subset \mathfrak{p}^+(\check{e}^k)_2$ be the Euclidean real forms, and we extend the determinant polynomials $\det_{\mathfrak{n}^+(e^k)}$, $\det_{\mathfrak{n}^+(\check{e}^k)}$ on $\mathfrak{p}^+(e^k)_2$, $\mathfrak{p}^+(\check{e}^k)_2$ to polynomials on \mathfrak{p}^+ . Using these, for $\mathbf{m} \in \mathbb{C}^r$ we define the functions $\Delta_{\mathbf{m}}^{\mathfrak{n}^+}(x)$, $\check{\Delta}_{\mathbf{m}}^{\mathfrak{n}^+}(x)$ on the symmetric cone $\Omega \subset \mathfrak{n}^+ := \mathfrak{n}^+(e^r) = \mathfrak{n}^+(\check{e}^r)$ by

$$\Delta_{\mathbf{m}}^{\mathfrak{n}^+}(x) := \prod_{k=1}^r \det_{\mathfrak{n}^+(e^k)}(x)^{m_k - m_{k+1}}, \quad \check{\Delta}_{\mathbf{m}}^{\mathfrak{n}^+}(x) := \prod_{k=1}^r \det_{\mathfrak{n}^+(\check{e}^k)}(x)^{m_k - m_{k+1}}, \quad x \in \Omega, \quad (2.5)$$

where we set $m_{r+1} := 0$. Then these satisfy

$$\begin{aligned} \Delta_{\mathbf{m}}^{\mathfrak{n}^+}(man.x) &= \Delta_{\mathbf{m}}^{\mathfrak{n}^+}(man.e)\Delta_{\mathbf{m}}^{\mathfrak{n}^+}(x) = \Delta_{\mathbf{m}}^{\mathfrak{n}^+}(a.e)\Delta_{\mathbf{m}}^{\mathfrak{n}^+}(x) = e^{2t_1m_1+2t_2m_2+\dots+2t_r m_r} \Delta_{\mathbf{m}}^{\mathfrak{n}^+}(x), \\ & \quad m \in M_L, \quad a = e^{t_1h_1+\dots+t_r h_r} \in A_L, \quad n \in N_L^\top, \\ \check{\Delta}_{\mathbf{m}}^{\mathfrak{n}^+}(man.x) &= \check{\Delta}_{\mathbf{m}}^{\mathfrak{n}^+}(man.e)\check{\Delta}_{\mathbf{m}}^{\mathfrak{n}^+}(x) = \check{\Delta}_{\mathbf{m}}^{\mathfrak{n}^+}(a.e)\check{\Delta}_{\mathbf{m}}^{\mathfrak{n}^+}(x) = e^{2t_1m_r+t_2m_{r-1}+\dots+2t_r m_1} \check{\Delta}_{\mathbf{m}}^{\mathfrak{n}^+}(x), \\ & \quad m \in M_L, \quad a = e^{t_1h_1+\dots+t_r h_r} \in A_L, \quad n \in N_L. \end{aligned}$$

Especially if

$$\mathbf{m} \in \mathbb{Z}_{++}^r := \{\mathbf{m} = (m_1, \dots, m_r) \in \mathbb{Z}^r \mid m_1 \geq m_2 \geq \dots \geq m_r \geq 0\},$$

then $\Delta_{\mathbf{m}}^{\mathfrak{n}^+}(x)$, $\check{\Delta}_{\mathbf{m}}^{\mathfrak{n}^+}(x)$ are extended to polynomials on \mathfrak{p}^+ . For $\mathbf{m} = (m_1, \dots, m_r) \in \mathbb{C}^r$, we write $\mathbf{m}^\vee := (m_r, \dots, m_1) \in \mathbb{C}^r$. Then the following holds.

Lemma 2.1 ([13, Proposition VII.1.6]). *For $\mathbf{k} \in \mathbb{Z}_{++}^r$, $\mathbf{m} \in \mathbb{C}^r$, $x \in \Omega$, we have*

$$\check{\Delta}_{\mathbf{k}}^{\mathfrak{n}^+} \left(\frac{\partial}{\partial x} \right) \Delta_{\mathbf{m}}^{\mathfrak{n}^+}(x) = \prod_{j=1}^r \left(m_j - k_{r-j+1} + \frac{d}{2}(r-j) + 1 \right)_{k_{r-j+1}} \Delta_{\mathbf{m}-\mathbf{k}^\vee}^{\mathfrak{n}^+}(x).$$

Here we normalize the differential operator $\frac{\partial}{\partial x}$ on $\mathfrak{n}^{\mathbb{C}} = \mathfrak{p}^+(e)_2$ with respect to the bilinear form $(\cdot|\cdot)_{\mathfrak{n}^+} = (\cdot|Q(\bar{\cdot}))_{\mathfrak{p}^+}$. Especially, if $k_1 = \dots = k_r = k \in \mathbb{Z}_{\geq 0}$ and \mathbf{m} is of the form $\mathbf{m} = (\mu + l_1, \mu + l_2, \dots, \mu + l_r)$ with $\mu, l_j \in \mathbb{C}$, then we have

$$\begin{aligned} \det_{\mathfrak{n}^+} \left(\frac{\partial}{\partial x} \right)^k \det_{\mathfrak{n}^+}(x)^\mu \Delta_{\mathbf{1}}^{\mathfrak{n}^+}(x) &= \prod_{j=1}^r \left(\mu + l_j - k + \frac{d}{2}(r-j) + 1 \right)_k \det_{\mathfrak{n}^+}(x)^{\mu-k} \Delta_{\mathbf{1}}^{\mathfrak{n}^+}(x) \\ &= (-1)^{kr} \prod_{j=1}^r \left(-\mu - l_{r-j+1} - \frac{d}{2}(j-1) \right)_k \det_{\mathfrak{n}^+}(x)^{\mu-k} \Delta_{\mathbf{1}}^{\mathfrak{n}^+}(x). \end{aligned} \quad (2.6)$$

Next, for $\mathbf{m} \in \mathbb{Z}_{++}^r$, let

$$\begin{aligned}\mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+) &:= \text{span}_{\mathbb{C}} \{ \Delta_{\mathbf{m}}^{\mathfrak{n}^+}(l^{-1}x) \mid l \in K^{\mathbb{C}} \} \subset \mathcal{P}(\mathfrak{p}^+), \\ \mathcal{P}_{\mathbf{m}}(\mathfrak{p}^-) &:= \{ \overline{f(\bar{y})} \mid f(x) \in \mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+) \} \subset \mathcal{P}(\mathfrak{p}^-).\end{aligned}$$

Then the following holds.

Theorem 2.2 (Hua–Kostant–Schmid, [11, Part III, Theorem V.2.1]). *Under the $K^{\mathbb{C}}$ -action, $\mathcal{P}(\mathfrak{p}^{\pm})$ is decomposed into the sum of irreducible submodules as*

$$\mathcal{P}(\mathfrak{p}^{\pm}) = \bigoplus_{\mathbf{m} \in \mathbb{Z}_{++}^r} \mathcal{P}_{\mathbf{m}}(\mathfrak{p}^{\pm}).$$

In addition, if \mathfrak{p}^+ is of tube type, then for

$$\mathbf{m} \in \mathbb{Z}_+^r := \{ \mathbf{m} = (m_1, \dots, m_r) \in \mathbb{Z}^r \mid m_1 \geq m_2 \geq \dots \geq m_r \},$$

let

$$\mathcal{P}_{\mathbf{m}}(\mathfrak{p}^{\pm}) := \mathcal{P}_{(m_1-m_r, m_2-m_r, \dots, m_{r-1}-m_r, 0)}(\mathfrak{p}^{\pm}) \det_{\mathfrak{n}^{\pm}}(x)^{m_r} \subset \mathcal{P}(\mathfrak{p}^{\pm}) [\det_{\mathfrak{n}^{\pm}}(x)^{-1}],$$

so that $\mathcal{P}_{\mathbf{m}}(\mathfrak{p}^{\pm}) \simeq \mathcal{P}_{-\mathbf{m}^{\vee}}(\mathfrak{p}^{\mp})$ holds as a $K^{\mathbb{C}}$ -module.

2.5 Holomorphic discrete series representations

In this subsection we review holomorphic discrete series representations. First we recall the bounded symmetric domain realization (Harish-Chandra realization) of the Hermitian symmetric space G/K via the Borel embedding,

$$\begin{array}{ccc} G/K & \longrightarrow & G^{\mathbb{C}}/K^{\mathbb{C}}P^- \\ \downarrow \wr & & \uparrow \exp \\ D^{\mathbb{C}} & \longrightarrow & \mathfrak{p}^+, \end{array}$$

where

$$\begin{aligned} D &= (\text{connected component of } \{x \in \mathfrak{p}^+ \mid B(x) \text{ is positive definite}\} \text{ which contains } 0) \\ &= (\text{connected component of } \{x \in \mathfrak{p}^+ \mid h(x) > 0\} \text{ which contains } 0). \end{aligned}$$

Here we write $B(x) := B(x, \bar{x})$, $h(x) := h(x, \bar{x})$. For $g \in G^{\mathbb{C}}$, $x \in D$, if $g \exp(x) \in P^+K^{\mathbb{C}}P^-$ holds, then we write

$$g \exp(x) = \exp(\pi^+(g, x)) \kappa(g, x) \exp(\pi^-(g, x)),$$

where $\pi^{\pm}(g, x) \in \mathfrak{p}^{\pm}$ and $\kappa(g, x) \in K^{\mathbb{C}}$. Then the map $\pi^+ : G \times D \rightarrow D$ gives an action of G on D , and we abbreviate $\pi^+(g, x) =: gx$.

Next let (τ, V) be an irreducible holomorphic representation of the universal covering group $\tilde{K}^{\mathbb{C}}$ of $K^{\mathbb{C}}$, with the \tilde{K} -invariant inner product $(\cdot, \cdot)_{\tau}$. Then the universal covering group \tilde{G} of G acts on the space $\mathcal{O}(D, V) = \mathcal{O}_{\tau}(D, V)$ of V -valued holomorphic functions on D by

$$(\hat{\tau}(g)f)(x) := \tau(\kappa(g^{-1}, x))^{-1} f(g^{-1}x), \quad g \in \tilde{G}, \quad x \in D, \quad f \in \mathcal{O}(D, V),$$

where we lift the map $\kappa: G \times D \rightarrow K^{\mathbb{C}}$ to the universal covering spaces, and represent by the same symbol $\kappa: \tilde{G} \times D \rightarrow \tilde{K}^{\mathbb{C}}$. Its differential action is given by

$$(d\hat{\tau}(z, k, w)f)(x) = d\tau(k - D(x, w))f(x) + \left. \frac{d}{dt} \right|_{t=0} f(x - t(z + kx - Q(x)w)) \quad (2.7)$$

for $z \in \mathfrak{p}^+$, $k \in \mathfrak{k}^{\mathbb{C}}$, $w \in \mathfrak{p}^-$. This becomes a highest weight representation with the minimal \tilde{K} -type (τ, V) . If this contains a unitary subrepresentation $\mathcal{H}_{\tau}(D, V) \subset \mathcal{O}_{\tau}(D, V)$, then its reproducing kernel is proportional to $\tau(B(x, \bar{y}))$, and such unitary subrepresentation is unique. Here we lift the map $B: D \times \bar{D} \rightarrow \text{Str}(\mathfrak{p}^+)_0 \subset \text{End}_{\mathbb{C}}(\mathfrak{p}^+)$ to the universal covering space, and represent by the same symbol $B: D \times \bar{D} \rightarrow \tilde{K}^{\mathbb{C}}$. Especially, if the \tilde{G} -invariant inner product is given by the converging integral

$$\langle f, g \rangle_{\hat{\tau}} := C_{\tau} \int_D (\tau(B(x)^{-1})f(x), g(x))_{\tau} \text{Det}_{\mathfrak{p}^+}(B(x))^{-1} dx$$

(a *weighted Bergman inner product*), then $(\hat{\tau}, \mathcal{H}_{\tau}(D, V))$ is called a *holomorphic discrete series representation*. Here we normalize the Lebesgue measure dx on $D \subset \mathfrak{p}^+$ with respect to the inner product $(\cdot | \cdot)_{\mathfrak{p}^+}$, and determine the constant C_{τ} such that $\|v\|_{\hat{\tau}} = \|v\|_{\tau}$ holds for all constant functions $v \in V$.

Next, let $\chi: \tilde{K}^{\mathbb{C}} \rightarrow \mathbb{C}^{\times}$ be the character of $\tilde{K}^{\mathbb{C}}$ normalized such that

$$d\chi([x, y]) = (x|y)_{\mathfrak{p}^+}, \quad x \in \mathfrak{p}^+, \quad y \in \mathfrak{p}^-, \quad (2.8)$$

so that $h(x, y) = \chi(B(x, y))$ holds, and we fix a representation (τ_0, V) of $K^{\mathbb{C}}$. When (τ, V) is of the form $(\tau, V) = (\chi^{-\lambda} \otimes \tau_0, V)$, we write $\mathcal{H}_{\tau}(D, V) = \mathcal{H}_{\lambda}(D, V) \subset \mathcal{O}_{\tau}(D, V) = \mathcal{O}_{\lambda}(D, V)$. In addition, if $(\tau, V) = (\chi^{-\lambda}, \mathbb{C})$, then we write $\mathcal{H}_{\tau}(D, V) = \mathcal{H}_{\lambda}(D) \subset \mathcal{O}_{\tau}(D, V) = \mathcal{O}_{\lambda}(D)$ and $\hat{\tau} = \tau_{\lambda}$.

In the rest of this subsection we assume \mathfrak{p}^+ is simple and $(\tau, V) = (\chi^{-\lambda}, \mathbb{C})$. Then $\mathcal{H}_{\lambda}(D)$ is holomorphic discrete if $\lambda > p - 1$, and then the inner product is given by

$$\begin{aligned} \langle f, g \rangle_{\lambda} &= \langle f, g \rangle_{\lambda, \mathfrak{p}^+} := C_{\lambda} \int_D f(x) \overline{g(x)} h(x)^{\lambda-p} dx, \\ C_{\lambda} &:= \frac{\prod_{j=1}^r \Gamma(\lambda - \frac{d}{2}(j-1))}{\pi^n \prod_{j=1}^r \Gamma(\lambda - \frac{n}{r} - \frac{d}{2}(j-1))}. \end{aligned} \quad (2.9)$$

We consider another inner product on $\mathcal{P}(\mathfrak{p}^+)$, called the *Fischer inner product* (see, e.g., [13, Section XI.1]), defined by

$$\langle f, g \rangle_F = \langle f, g \rangle_{F, \mathfrak{p}^+} := \frac{1}{\pi^n} \int_{\mathfrak{p}^+} f(x) \overline{g(x)} e^{-(x|\bar{x})_{\mathfrak{p}^+}} dx = g \left(\overline{\left(\frac{\partial}{\partial x} \right)} f(x) \right) \Big|_{x=0}. \quad (2.10)$$

Here $\overline{g(\cdot)}$ is a holomorphic polynomial on \mathfrak{p}^- , and we normalize $\frac{\partial}{\partial x}$ with respect to the bilinear form $(\cdot | \cdot)_{\mathfrak{p}^+}: \mathfrak{p}^+ \times \mathfrak{p}^- \rightarrow \mathbb{C}$. Then the following holds.

Theorem 2.3 (Faraut–Korányi, [12] and [11, Part III, Corollary V.3.9]). *For $\lambda > p - 1$, $\mathfrak{m} \in \mathbb{Z}_{++}^r$, $f \in \mathcal{P}_{\mathfrak{m}}(\mathfrak{p}^+)$, $g \in \mathcal{P}(\mathfrak{p}^+)$, we have*

$$\langle f, g \rangle_{\lambda} = \frac{1}{(\lambda)_{\mathfrak{m}, d}} \langle f, g \rangle_F,$$

where

$$(\lambda)_{\mathfrak{m}, d} := \prod_{j=1}^r \left(\lambda - \frac{d}{2}(j-1) \right)_{m_j}, \quad (2.11)$$

and $(\lambda)_m := \lambda(\lambda+1)(\lambda+2) \cdots (\lambda+m-1)$.

Since the reproducing kernel on $\mathcal{P}(\mathfrak{p}^+)$ with respect to $\langle \cdot, \cdot \rangle_F$ is given by $e^{(x|\bar{y})_{\mathfrak{p}^+}}$, the following holds.

Corollary 2.4. *For $\lambda > p - 1$, $\mathbf{m} \in \mathbb{Z}_{++}^r$, $f(x) \in \mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+)$, we have*

$$\langle f(x), e^{(x|\bar{y})_{\mathfrak{p}^+}} \rangle_{\lambda, x} = \frac{1}{(\lambda)_{\mathbf{m}, d}} f(y).$$

Here the subscript x stands for the variable of integration. By this theorem, $\langle \cdot, \cdot \rangle_{\lambda}$ is meromorphically continued for all $\lambda \in \mathbb{C}$, and the \tilde{K} -finite part $\mathcal{O}_{\lambda}(D)_{\tilde{K}} = \chi^{-\lambda} \otimes \mathcal{P}(\mathfrak{p}^+)$ is reducible as a $(\mathfrak{g}, \tilde{K})$ -module if and only if λ is a pole of $\langle \cdot, \cdot \rangle_{\lambda}$. Especially, for $j = 1, 2, \dots, r$, $\lambda \in \frac{d}{2}(j-1) - \mathbb{Z}_{\geq 0}$,

$$M_j(\lambda) = M_j^{\mathfrak{g}}(\lambda) := \bigoplus_{\substack{\mathbf{m} \in \mathbb{Z}_{++}^r \\ m_j \leq \frac{d}{2}(j-1) - \lambda}} \chi^{-\lambda} \otimes \mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+) \subset \chi^{-\lambda} \otimes \mathcal{P}(\mathfrak{p}^+) = \mathcal{O}_{\lambda}(D)_{\tilde{K}} \quad (2.12)$$

gives a $(\mathfrak{g}, \tilde{K})$ -submodule. Moreover, $\langle \cdot, \cdot \rangle_{\lambda}$ is positive definite on $\mathcal{P}(\mathfrak{p}^+)$ for $\lambda > \frac{d}{2}(r-1)$, and on $M_j^{\mathfrak{g}}(\frac{d}{2}(j-1))$ for $\lambda = \frac{d}{2}(j-1)$, $j = 1, \dots, r$, that is, $\mathcal{O}_{\lambda}(D)_{\tilde{K}}$ contains a unitary submodule $\mathcal{H}_{\lambda}(D)$ if λ sits in the *Wallach set*

$$\lambda \in \left\{ 0, \frac{d}{2}, d, \dots, \frac{d}{2}(r-1) \right\} \cup \left(\frac{d}{2}(r-1), \infty \right), \quad (2.13)$$

and its \tilde{K} -finite part is given by

$$\mathcal{H}_{\lambda}(D)_{\tilde{K}} = \begin{cases} \chi^{-\lambda} \otimes \mathcal{P}(\mathfrak{p}^+), & \lambda > \frac{d}{2}(r-1), \\ M_j^{\mathfrak{g}}(\frac{d}{2}(j-1)), & \lambda = \frac{d}{2}(j-1), \quad j = 1, 2, \dots, r. \end{cases}$$

In addition, the quotient module $\mathcal{O}_{\lambda}(D)_{\tilde{K}}/M_r(\lambda)$, $\lambda \in \frac{d}{2}(r-1) - \mathbb{Z}_{\geq 0}$, also gives an infinitesimally unitary module (see [12]). Also by the corollary, for $\mathbf{k} \in \mathbb{Z}_{++}^r$, $f(x) \in \mathcal{P}(\mathfrak{p}^+)$, $(\lambda)_{\mathbf{k}, d} \langle f(x), e^{(x|\bar{y})_{\mathfrak{p}^+}} \rangle_{\lambda, x}$ is holomorphically continued for all $\lambda \in \mathbb{C}$ if and only if

$$f(x) \in \bigoplus_{\substack{\mathbf{m} \in \mathbb{Z}_{++}^r \\ m_j \leq k_j, j=1, \dots, r}} \mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+)$$

holds, and if this is satisfied, then for $j = 1, \dots, r$,

$$f(x) \in M_j(\lambda) \quad \text{holds if} \quad \lambda \in \frac{d}{2}(j-1) - k_j - \mathbb{Z}_{\geq 0}.$$

2.6 Classification

A simple Hermitian positive Jordan triple system \mathfrak{p}^{\pm} is isomorphic to one of the following:

$$\begin{array}{lll} \mathfrak{p}^{\pm} = \mathbb{C}^n, & n \neq 2, & \text{Sym}(r, \mathbb{C}), & \text{M}(q, s; \mathbb{C}), \\ & & \text{Alt}(s, \mathbb{C}), & \text{Herm}(3, \mathbb{O})^{\mathbb{C}}, & \text{M}(1, 2; \mathbb{O})^{\mathbb{C}}. \end{array}$$

Here $\text{Sym}(r, \mathbb{C})$ and $\text{Alt}(s, \mathbb{C})$ denote the spaces of symmetric and alternating matrices over \mathbb{C} , respectively, and $\text{Herm}(3, \mathbb{O})^{\mathbb{C}}$ denotes the complexification of the space of 3×3 Hermitian

matrices over the octonions \mathbb{O} . Then the corresponding Lie groups G and their maximal compact subgroups K are given by

$$(G, K) = \begin{cases} (\mathrm{SO}_0(2, n), \mathrm{SO}(2) \times \mathrm{SO}(n)), & \mathfrak{p}^\pm = \mathbb{C}^n, \\ (\mathrm{Sp}(r, \mathbb{R}), \mathrm{U}(r)), & \mathfrak{p}^\pm = \mathrm{Sym}(r, \mathbb{C}), \\ (\mathrm{SU}(q, s), \mathrm{S}(\mathrm{U}(q) \times \mathrm{U}(s))), & \mathfrak{p}^\pm = \mathrm{M}(q, s; \mathbb{C}), \\ (\mathrm{SO}^*(2s), \mathrm{U}(s)), & \mathfrak{p}^\pm = \mathrm{Alt}(s, \mathbb{C}), \\ (E_{7(-25)}, \mathrm{U}(1) \times E_6), & \mathfrak{p}^\pm = \mathrm{Herm}(3, \mathbb{O})^\mathbb{C}, \\ (E_{6(-14)}, \mathrm{U}(1) \times \mathrm{Spin}(10)), & \mathfrak{p}^\pm = \mathrm{M}(1, 2; \mathbb{O})^\mathbb{C} \end{cases}$$

(up to covering), and the numbers (n, r, d, b, p) (see (2.4)) are given by

$$(n, r, d, b, p) = \begin{cases} (n, 2, n-2, 0, n), & \mathfrak{p}^\pm = \mathbb{C}^n, \quad n \geq 3, \\ (1, 1, -, 0, 2), & \mathfrak{p}^\pm = \mathbb{C}, \\ (\frac{1}{2}r(r+1), r, 1, 0, r+1), & \mathfrak{p}^\pm = \mathrm{Sym}(r, \mathbb{C}), \\ (qs, \min\{q, s\}, 2, |q-s|, q+s), & \mathfrak{p}^\pm = \mathrm{M}(q, s; \mathbb{C}), \\ (\frac{1}{2}s(s-1), \frac{s}{2}, 4, 0, 2(s-1)), & \mathfrak{p}^\pm = \mathrm{Alt}(s, \mathbb{C}), \quad s: \text{even}, \\ (\frac{1}{2}s(s-1), \lfloor \frac{s}{2} \rfloor, 4, 2, 2(s-1)), & \mathfrak{p}^\pm = \mathrm{Alt}(s, \mathbb{C}), \quad s: \text{odd}, \\ (27, 3, 8, 0, 18), & \mathfrak{p}^\pm = \mathrm{Herm}(3, \mathbb{O})^\mathbb{C}, \\ (16, 2, 6, 4, 12), & \mathfrak{p}^\pm = \mathrm{M}(1, 2; \mathbb{O})^\mathbb{C}. \end{cases}$$

Here, if $r = 1$, then d is not determined uniquely, and any number is allowed. When $b = 0$, \mathfrak{p}^\pm is of tube type, and has a Jordan algebra structure. That is, $\mathfrak{p}^\pm = \mathbb{C}^n, \mathrm{Sym}(r, \mathbb{C}), \mathrm{M}(r, \mathbb{C}), \mathrm{Alt}(2r, \mathbb{C})$ and $\mathrm{Herm}(3, \mathbb{O})^\mathbb{C}$ are of tube type. For these cases $p = \frac{2n}{r} = d(r-1) + 2$ holds. When $\mathfrak{p}^\pm = \mathrm{M}(q, s; \mathbb{C})$ we also consider $(G, K) = (\mathrm{U}(q, s), \mathrm{U}(q) \times \mathrm{U}(s))$ instead of $(\mathrm{SU}(q, s), \mathrm{S}(\mathrm{U}(q) \times \mathrm{U}(s)))$.

Next we fix the parametrization of finite-dimensional irreducible representations of $\widetilde{\mathrm{GL}}(s, \mathbb{C})$ and $\mathrm{Spin}(n, \mathbb{C})$, $n \geq 3$. We take a basis $\{\epsilon_j\}_{j=1}^s \subset \mathfrak{h}^{\mathrm{CV}}$ of the dual space of a Cartan subalgebra $\mathfrak{h}^\mathbb{C} \subset \mathfrak{gl}(s, \mathbb{C})$ such that the positive root system is given by $\{\epsilon_i - \epsilon_j \mid 1 \leq i < j \leq s\}$. For $\mathbf{m} \in \mathbb{C}^s$ with $m_j - m_{j+1} \in \mathbb{Z}_{\geq 0}$, let $V_{\mathbf{m}}^{(s)}$ be the irreducible representation of $\widetilde{\mathrm{GL}}(s, \mathbb{C})$ with the highest weight $\sum_j m_j \epsilon_j$, and let $V_{\mathbf{m}}^{(s)\vee}$ be that with the lowest weight $-\sum_j m_j \epsilon_j$. If $\mathbf{m} \in \mathbb{Z}_+^r$, then $V_{\mathbf{m}}^{(s)}, V_{\mathbf{m}}^{(s)\vee}$ are reduced to the representations of $\mathrm{GL}(s, \mathbb{C})$. Similarly, we take a basis $\{\epsilon_j\}_{j=1}^{\lfloor n/2 \rfloor} \subset \mathfrak{h}^{\mathrm{CV}}$ of the dual space of a Cartan subalgebra $\mathfrak{h}^\mathbb{C} \subset \mathfrak{so}(n, \mathbb{C})$ such that the positive root system is given by

$$\begin{aligned} & \{\epsilon_i \pm \epsilon_j \mid 1 \leq i < j \leq n/2\}, & n: \text{even}, \\ & \{\epsilon_i \pm \epsilon_j \mid 1 \leq i < j \leq \lfloor n/2 \rfloor\} \cup \{\epsilon_j \mid 1 \leq j \leq \lfloor n/2 \rfloor\}, & n: \text{odd}. \end{aligned}$$

For $\mathbf{m} \in \mathbb{Z}^{\lfloor n/2 \rfloor} \cup (\mathbb{Z} + \frac{1}{2})^{\lfloor n/2 \rfloor}$ with $m_1 \geq \dots \geq m_{n/2-1} \geq |m_{n/2}|$ for even n , $m_1 \geq \dots \geq m_{\lfloor n/2 \rfloor} \geq 0$ for odd n , let $V_{\mathbf{m}}^{[n]}$ be the irreducible representation of $\mathrm{Spin}(n, \mathbb{C})$ with the highest weight $\sum_j m_j \epsilon_j$, and let $V_{\mathbf{m}}^{[n]\vee}$ be that with the lowest weight $-\sum_j m_j \epsilon_j$. If $\mathbf{m} \in \mathbb{Z}^{\lfloor n/2 \rfloor}$, then $V_{\mathbf{m}}^{[n]}, V_{\mathbf{m}}^{[n]\vee}$ are reduced to the representations of $\mathrm{SO}(n, \mathbb{C})$. Under this notation, the \widetilde{K} -type decompositions of the holomorphic discrete series representations of scalar type

$$\mathcal{O}_\lambda(D)_{\widetilde{K}} = \chi^{-\lambda} \otimes \mathcal{P}(\mathfrak{p}^+) = \chi^{-\lambda} \otimes \bigoplus_{\mathbf{m} \in \mathbb{Z}_{++}^r} \mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+)$$

are given as

$$\chi^{-\lambda} \simeq \begin{cases} \mathbb{C}_{-\lambda} \boxtimes V_{(0,\dots,0)}^{[n]\vee}, & \mathfrak{p}^+ = \mathbb{C}^n, \ n \geq 3, \\ V_{(\lambda,\dots,\lambda)}^{(r)\vee}, & \mathfrak{p}^+ = \text{Sym}(r, \mathbb{C}), \\ V_{\left(\frac{s\lambda}{q+s}, \dots, \frac{s\lambda}{q+s}\right)}^{(q)\vee} \boxtimes V_{\left(\frac{q\lambda}{q+s}, \dots, \frac{q\lambda}{q+s}\right)}^{(s)}, & \mathfrak{p}^+ = M(q, s; \mathbb{C}), \\ V_{\left(\frac{\lambda}{2}, \dots, \frac{\lambda}{2}\right)}^{(s)\vee}, & \mathfrak{p}^+ = \text{Alt}(s, \mathbb{C}), \\ \mathbb{C}_{-\lambda} \boxtimes V_{(0,\dots,0)}^{[10]\vee}, & \mathfrak{p}^+ = M(1, 2; \mathbb{O})^{\mathbb{O}}, \end{cases} \quad (2.14)$$

$$\mathcal{P}_{\mathfrak{m}}(\mathfrak{p}^+) \simeq \begin{cases} \mathbb{C}_{-m_1-m_2} \boxtimes V_{(m_1-m_2, 0, \dots, 0)}^{[n]\vee}, & \mathfrak{p}^+ = \mathbb{C}^n, \ n \geq 3, \\ V_{(2m_1, \dots, 2m_r)}^{(r)\vee} =: V_{2\mathfrak{m}}^{(r)\vee}, & \mathfrak{p}^+ = \text{Sym}(r, \mathbb{C}), \\ V_{\mathfrak{m}}^{(q)\vee} \boxtimes V_{(m_1, \dots, m_q, 0, \dots, 0)}^{(s)} =: V_{\mathfrak{m}}^{(q)\vee} \boxtimes V_{\mathfrak{m}}^{(s)}, & \mathfrak{p}^+ = M(q, s; \mathbb{C}), \ q \leq s, \\ V_{(m_1, \dots, m_s, 0, \dots, 0)}^{(q)\vee} \boxtimes V_{\mathfrak{m}}^{(s)} =: V_{\mathfrak{m}}^{(q)\vee} \boxtimes V_{\mathfrak{m}}^{(s)}, & \mathfrak{p}^+ = M(q, s; \mathbb{C}), \ q \geq s, \\ V_{(m_1, m_1, m_2, m_2, \dots, m_{\lfloor s/2 \rfloor}, m_{\lfloor s/2 \rfloor}, 0)}^{(s)\vee} =: V_{\mathfrak{m}^2}^{(s)\vee}, & \mathfrak{p}^+ = \text{Alt}(s, \mathbb{C}), \\ \mathbb{C}_{-\frac{3}{4}|\mathfrak{m}|} \boxtimes V_{\left(\frac{m_1+m_2}{2}, \frac{m_1-m_2}{2}, \frac{m_1-m_2}{2}, \frac{m_1-m_2}{2}, \frac{m_1-m_2}{2}\right)}^{[10]\vee}, & \mathfrak{p}^+ = M(1, 2; \mathbb{O})^{\mathbb{C}}, \end{cases}$$

when $K^{\mathbb{C}}$ is classical, if we normalize the representations $\mathbb{C}_{-\lambda}$ of $\widetilde{\text{SO}}(2) \simeq \widetilde{\text{U}}(1)$ for the first and the last cases suitably. When $\mathfrak{p}^+ = \mathbb{C}^n$ with $n = 1, 2$, we have isomorphisms

$$\begin{aligned} \widetilde{\text{SO}}_0(2, 1) &\simeq \widetilde{\text{SL}}(2, \mathbb{R}) = \widetilde{\text{Sp}}(1, \mathbb{R}), \\ \widetilde{\text{SO}}_0(2, 2) &\simeq \widetilde{\text{SL}}(2, \mathbb{R}) \times \widetilde{\text{SL}}(2, \mathbb{R}) = \widetilde{\text{Sp}}(1, \mathbb{R}) \times \widetilde{\text{Sp}}(1, \mathbb{R}), \end{aligned}$$

and we write

$$\mathcal{H}_{\lambda}(D_{\text{SO}_0(2,1)}) := \mathcal{H}_{2\lambda}(D_{\text{SL}(2, \mathbb{R})}), \quad \mathcal{H}_{\lambda}(D_{\text{SO}_0(2,2)}) := \mathcal{H}_{\lambda}(D_{\text{SL}(2, \mathbb{R})}) \boxtimes \mathcal{H}_{\lambda}(D_{\text{SL}(2, \mathbb{R})}),$$

and similar for $\mathcal{O}_{\lambda}(D)$. When we consider $\mathfrak{p}^+ = M(q, s; \mathbb{C})$, $G = \text{U}(q, s)$, for $\lambda_1, \lambda_2 \in \mathbb{C}$, let $\chi^{-\lambda_1, -\lambda_2}$ be the character of $\widetilde{K}^{\mathbb{C}} = \widetilde{\text{GL}}(q, \mathbb{C}) \times \widetilde{\text{GL}}(s, \mathbb{C})$ given by

$$\chi^{-\lambda_1, -\lambda_2} \simeq V_{(\lambda_1, \dots, \lambda_1)}^{(q)\vee} \boxtimes V_{(\lambda_2, \dots, \lambda_2)}^{(s)},$$

and for a fixed representation $V' \boxtimes V''$ of $K^{\mathbb{C}}$, let $\mathcal{H}_{\lambda_1+\lambda_2}(D, V' \boxtimes V'') \subset \mathcal{O}_{\lambda_1+\lambda_2}(D, V' \boxtimes V'')$ be the representations of \widetilde{G} with the minimal \widetilde{K} -type $\chi^{-\lambda_1, -\lambda_2} \otimes (V' \boxtimes V'')$.

The inner product $\langle \cdot, \cdot \rangle_{\lambda}$ of the holomorphic discrete series representation $\mathcal{H}_{\lambda}(D)$ of scalar type originally converges for $\lambda > p - 1$, and by Theorem 2.3, this is meromorphically continued for all $\lambda \in \mathbb{C}$. This has poles at $\lambda \in \frac{d}{2}(j-1) - \mathbb{Z}_{\geq 0}$ for $j = 1, 2, \dots, r$, and then $M_j(\lambda) = M_j^{\mathfrak{g}}(\lambda) \subset \mathcal{O}_{\lambda}(D)_{\widetilde{K}}$ defined in (2.12) becomes a $(\mathfrak{g}, \widetilde{K})$ -submodule. Especially, when $\mathfrak{p}^+ = \mathbb{C}^n$ with $n \geq 3$, $\mathcal{O}_{\lambda}(D)_{\widetilde{K}}$ is reducible if and only if $\lambda \in -\mathbb{Z}_{\geq 0} \cap \left(\frac{n-2}{2} - \mathbb{Z}_{\geq 0}\right)$, and then we have the $(\mathfrak{so}(2, n), \widetilde{\text{SO}}(2) \times \text{SO}(n))$ -submodules

$$\begin{aligned} \mathcal{O}_{\lambda}(D)_{\widetilde{K}} \supset M_2(\lambda) \supset M_1(\lambda) \supset \{0\}, & \quad n: \text{even}, \ \lambda \in \mathbb{Z}, \ \lambda \leq 0, \\ \mathcal{O}_{\lambda}(D)_{\widetilde{K}} \supset M_2(\lambda) \supset \{0\}, & \quad n: \text{even}, \ \lambda \in \mathbb{Z}, \ 1 \leq \lambda \leq \frac{n-2}{2}, \\ \mathcal{O}_{\lambda}(D)_{\widetilde{K}} \supset M_1(\lambda) \supset \{0\}, & \quad n: \text{odd}, \ \lambda \in \mathbb{Z}, \ \lambda \leq 0, \\ \mathcal{O}_{\lambda}(D)_{\widetilde{K}} \supset M_2(\lambda) \supset \{0\}, & \quad n: \text{odd}, \ \lambda \in \mathbb{Z} + \frac{1}{2}, \ \lambda \leq \frac{n-2}{2}. \end{aligned}$$

When $\mathfrak{p}^+ = \text{Sym}(r, \mathbb{C})$ with $r \geq 2$, $\mathcal{O}_\lambda(D)_{\tilde{K}}$ is reducible if and only if $\lambda \in \frac{1}{2}(r-1) - \frac{1}{2}\mathbb{Z}_{\geq 0}$, and then we have the $(\mathfrak{sp}(r, \mathbb{R}), \tilde{U}(r))$ -submodules

$$\begin{aligned}\mathcal{O}_\lambda(D)_{\tilde{K}} &\supset M_{2\lceil \frac{r}{2} \rceil - 1}(\lambda) \supset M_{2\lceil \frac{r}{2} \rceil - 3}(\lambda) \supset \cdots \supset M_{\max\{2\lambda, 0\} + 1}(\lambda) \supset \{0\}, & \lambda \in \mathbb{Z}, \\ \mathcal{O}_\lambda(D)_{\tilde{K}} &\supset M_{2\lfloor \frac{r}{2} \rfloor}(\lambda) \supset M_{2\lfloor \frac{r}{2} \rfloor - 2}(\lambda) \supset \cdots \supset M_{\max\{2\lambda, 1\} + 1}(\lambda) \supset \{0\}, & \lambda \in \mathbb{Z} + \frac{1}{2}.\end{aligned}$$

When $\mathfrak{p}^+ = \text{M}(q, s; \mathbb{C})$, $\mathcal{O}_\lambda(D)_{\tilde{K}}$ is reducible if and only if $\lambda \in \min\{q, s\} - 1 - \mathbb{Z}_{\geq 0}$, and then we have the $(\mathfrak{su}(q, s), \text{S}(\text{U}(q) \times \text{U}(s))^\sim)$ -submodules

$$\mathcal{O}_\lambda(D)_{\tilde{K}} \supset M_{\min\{q, s\}}(\lambda) \supset M_{\min\{q, s\} - 1}(\lambda) \supset \cdots \supset M_{\max\{\lambda, 0\} + 1}(\lambda) \supset \{0\}.$$

When $\mathfrak{p}^+ = \text{Alt}(s, \mathbb{C})$, $\mathcal{O}_\lambda(D)_{\tilde{K}}$ is reducible if and only if $\lambda \in 2\left(\lfloor \frac{s}{2} \rfloor - 1\right) - \mathbb{Z}_{\geq 0}$, and then we have the $(\mathfrak{so}^*(2s), \tilde{U}(s))$ -submodules

$$\mathcal{O}_\lambda(D)_{\tilde{K}} \supset M_{\lfloor \frac{s}{2} \rfloor}(\lambda) \supset M_{\lfloor \frac{s}{2} \rfloor - 1}(\lambda) \supset \cdots \supset M_{\max\{\lfloor \frac{s}{2} \rfloor, 0\} + 1}(\lambda) \supset \{0\}.$$

When $\mathfrak{p}^+ = \text{Herm}(3, \mathbb{O})^\mathbb{C}$, $\mathcal{O}_\lambda(D)_{\tilde{K}}$ is reducible if and only if $\lambda \in 8 - \mathbb{Z}_{\geq 0}$, and then we have the $(\mathfrak{e}_{7(-25)}, \tilde{U}(1) \times E_6)$ -submodules

$$\begin{aligned}\mathcal{O}_\lambda(D)_{\tilde{K}} &\supset M_3(\lambda) \supset M_2(\lambda) \supset M_1(\lambda) \supset \{0\}, & \lambda \in \mathbb{Z}, \lambda \leq 0, \\ \mathcal{O}_\lambda(D)_{\tilde{K}} &\supset M_3(\lambda) \supset M_2(\lambda) \supset \{0\}, & \lambda \in \mathbb{Z}, 1 \leq \lambda \leq 4, \\ \mathcal{O}_\lambda(D)_{\tilde{K}} &\supset M_3(\lambda) \supset \{0\}, & \lambda \in \mathbb{Z}, 5 \leq \lambda \leq 8.\end{aligned}$$

When $\mathfrak{p}^+ = \text{M}(1, 2; \mathbb{O})^\mathbb{C}$, $\mathcal{O}_\lambda(D)_{\tilde{K}}$ is reducible if and only if $\lambda \in 3 - \mathbb{Z}_{\geq 0}$, and then we have the $(\mathfrak{e}_{6(-14)}, \tilde{U}(1) \times \text{Spin}(10))$ -submodules

$$\begin{aligned}\mathcal{O}_\lambda(D)_{\tilde{K}} &\supset M_2(\lambda) \supset M_1(\lambda) \supset \{0\}, & \lambda \in \mathbb{Z}, \lambda \leq 0, \\ \mathcal{O}_\lambda(D)_{\tilde{K}} &\supset M_2(\lambda) \supset \{0\}, & \lambda \in \mathbb{Z}, 1 \leq \lambda \leq 3.\end{aligned}$$

2.7 Restriction to symmetric subgroups

In this subsection we consider a \mathbb{C} -linear involution σ on a Hermitian positive Jordan triple system \mathfrak{p}^\pm , i.e., a Jordan triple system automorphism $\sigma: \mathfrak{p}^\pm \rightarrow \mathfrak{p}^\pm$ of order 2 which commutes with the \mathbb{C} -antilinear map $\bar{\cdot}: \mathfrak{p}^\pm \rightarrow \mathfrak{p}^\mp$, and extend to the involution of the Lie algebra $\mathfrak{g}^\mathbb{C} = \mathfrak{p}^+ \oplus \mathfrak{k}^\mathbb{C} \oplus \mathfrak{p}^-$ by letting σ act on $\mathfrak{k}^\mathbb{C} = \mathfrak{str}(\mathfrak{p}^+) \subset \text{End}_\mathbb{C}(\mathfrak{p}^+)$ by $\sigma(l) := \sigma l \sigma$. Also let $\vartheta := -I_{\mathfrak{p}^+} + I_{\mathfrak{k}^\mathbb{C}} - I_{\mathfrak{p}^-}$. Using these, we set

$$\begin{aligned}\mathfrak{p}_1^\pm &:= (\mathfrak{p}^\pm)^\sigma = \{x \in \mathfrak{p}^\pm \mid \sigma(x) = x\}, \\ \mathfrak{p}_2^\pm &:= (\mathfrak{p}^\pm)^{-\sigma} = \{x \in \mathfrak{p}^\pm \mid \sigma(x) = -x\}, \\ \mathfrak{k}_1^\mathbb{C} &:= (\mathfrak{k}^\mathbb{C})^\sigma = \{l \in \mathfrak{k}^\mathbb{C} \mid \sigma(l) = l\}, \\ \mathfrak{k}_1 &:= \mathfrak{k}^\sigma = \mathfrak{k}_1^\mathbb{C} \cap \mathfrak{k}, \\ \mathfrak{g}_1^\mathbb{C} &:= (\mathfrak{g}^\mathbb{C})^\sigma = \mathfrak{p}_1^+ \oplus \mathfrak{k}_1^\mathbb{C} \oplus \mathfrak{p}_1^-, \\ \mathfrak{g}_2^\mathbb{C} &:= (\mathfrak{g}^\mathbb{C})^{\sigma\vartheta} = \mathfrak{p}_2^+ \oplus \mathfrak{k}_1^\mathbb{C} \oplus \mathfrak{p}_2^-, \\ \mathfrak{g}_1 &:= \mathfrak{g}^\sigma = \mathfrak{g}_1^\mathbb{C} \cap \mathfrak{g}, \\ \mathfrak{g}_2 &:= \mathfrak{g}^{\sigma\vartheta} = \mathfrak{g}_2^\mathbb{C} \cap \mathfrak{g},\end{aligned}$$

and let $G_1, G_1^\mathbb{C}, G_2, G_2^\mathbb{C}, K_1, K_1^\mathbb{C} \subset G^\mathbb{C}$ be the connected closed subgroups corresponding to $\mathfrak{g}_1, \mathfrak{g}_1^\mathbb{C}, \mathfrak{g}_2, \mathfrak{g}_2^\mathbb{C}, \mathfrak{k}_1, \mathfrak{k}_1^\mathbb{C}$, respectively. Such (G, G_1) is called a symmetric pair of holomorphic type

(see [26, Section 3.4]). Also let $\tilde{G}_1 \subset \tilde{G}$ and $\tilde{K}_1^{\mathbb{C}} \subset \tilde{K}^{\mathbb{C}}$ be the connected closed subgroups of the universal covering groups of G and $K^{\mathbb{C}}$ corresponding to \mathfrak{g}_1 and $\mathfrak{k}_1^{\mathbb{C}}$ respectively, and let $D \subset \mathfrak{p}^+$, $D_1 \subset \mathfrak{p}_1^+$ be the corresponding bounded symmetric domains, so that $D \simeq G/K$, $D_1 \simeq G_1/K_1$ hold. Let χ be the character of $K^{\mathbb{C}}$ given in (2.8). Similarly, we normalize the bilinear form $(\cdot|\cdot)_{\mathfrak{p}_j^+} : \mathfrak{p}_j^+ \times \mathfrak{p}_j^- \rightarrow \mathbb{C}$ ($j = 1, 2$) such that $(e|\bar{e})_{\mathfrak{p}_j^+} = 1$ holds for any primitive tripotent $e \in \mathfrak{p}_j^+$, and define the characters χ_j ($j = 1, 2$) of $K_1^{\mathbb{C}}$ by

$$\begin{aligned} d\chi_j([x, y]) &= (x|y)_{\mathfrak{p}_j^+}, & x \in \mathfrak{p}_j^+, & y \in \mathfrak{p}_j^-, \\ d\chi_j(l) &= 0, & l \in [\mathfrak{p}_j^+, \mathfrak{p}_j^-]^\perp &\subset \mathfrak{k}_1^{\mathbb{C}}. \end{aligned}$$

As in Section 2.5, for $\lambda \in \mathbb{R}$ and for an irreducible representation V of $K_1^{\mathbb{C}}$, let $\mathcal{H}_\lambda(D) \subset \mathcal{O}_\lambda(D)$ and $\mathcal{H}_\lambda(D_1, V) \subset \mathcal{O}_\lambda(D_1, V)$ be the unitary representations of \tilde{G} and \tilde{G}_1 with the minimal \tilde{K} -type $\chi^{-\lambda}$ and with the minimal \tilde{K}_1 -type $\chi_1^{-\lambda} \otimes V$ respectively, if they exist.

In the following we assume (G, G_1) is an irreducible symmetric pair. Then \mathfrak{p}_2^+ is a direct sum of at most two simple Jordan triple systems. Let $\text{rank } \mathfrak{p}^+ =: r$, $\text{rank } \mathfrak{p}_2^+ =: r_2$, and define $\varepsilon_1, \varepsilon_2 \in \{1, 2\}$ by $d\chi|_{\mathfrak{k}_1^{\mathbb{C}}} = \varepsilon_j d\chi_j$, or equivalently, by

$$(x|y)_{\mathfrak{p}^+} = \varepsilon_j (x|y)_{\mathfrak{p}_j^+}, \quad j = 1, 2, \quad x \in \mathfrak{p}_j^+, \quad y \in \mathfrak{p}_j^-. \quad (2.15)$$

When \mathfrak{p}_2^+ is not simple, we write $\mathfrak{p}_2^+ =: \mathfrak{p}_{11}^+ \oplus \mathfrak{p}_{22}^+$, $\mathfrak{p}_1^+ =: \mathfrak{p}_{12}^+$, and let $\text{rank } \mathfrak{p}_{11}^+ =: r'$, $\text{rank } \mathfrak{p}_{22}^+ =: r''$. Now we consider the restriction of the representation $\mathcal{H}_\lambda(D)$ of \tilde{G} to the subgroup \tilde{G}_1 . If λ satisfies (2.13), then the unitary representation $\mathcal{H}_\lambda(D)$ exists, $\mathcal{H}_\lambda(D)|_{\tilde{G}_1}$ is discretely decomposable, and every \tilde{G}_1 -submodule in $\mathcal{H}_\lambda(D)|_{\tilde{G}_1}$ contains a \mathfrak{p}_1^+ -null vector, that is, has an intersection with $(\mathcal{H}_\lambda(D)_{\tilde{K}})^{\mathfrak{p}_1^+} = (\chi_1^{-\varepsilon_1 \lambda} \otimes \mathcal{P}(\mathfrak{p}_2^+)) \cap \mathcal{H}_\lambda(D)_{\tilde{K}}$. Especially, $\mathcal{H}_\lambda(D)_{\tilde{K}} = \chi^{-\lambda} \otimes \mathcal{P}(\mathfrak{p}^+)$ holds if $\lambda > \frac{d}{2}(r-1)$, and $\mathcal{H}_\lambda(D)$ is holomorphic discrete if $\lambda > p-1$. For such λ , according to the decomposition (Theorem 2.2)

$$\begin{aligned} \mathcal{P}(\mathfrak{p}_2^+) &= \bigoplus_{\mathfrak{k} \in \mathbb{Z}_{++}^{r_2}} \mathcal{P}_{\mathfrak{k}}(\mathfrak{p}_2^+), & \mathfrak{p}_2^+ : \text{ simple}, \\ \mathcal{P}(\mathfrak{p}_2^+) &= \bigoplus_{\mathfrak{k} \in \mathbb{Z}_{++}^{r'}} \bigoplus_{\mathfrak{l} \in \mathbb{Z}_{++}^{r''}} \mathcal{P}_{\mathfrak{k}}(\mathfrak{p}_{11}^+) \boxtimes \mathcal{P}_{\mathfrak{l}}(\mathfrak{p}_{22}^+), & \mathfrak{p}_2^+ : \text{ non-simple}, \end{aligned}$$

the following holds.

Theorem 2.5 (Kobayashi, [26, Theorems 8.3 and 8.4]).

- (1) *Suppose \mathfrak{p}^+ is simple. For $\lambda > \frac{d}{2}(r-1)$, the restriction of $\mathcal{H}_\lambda(D)$ to the subgroup \tilde{G}_1 is decomposed into the Hilbert direct sum of irreducible representations as*

$$\begin{aligned} \mathcal{H}_\lambda(D)|_{\tilde{G}_1} &\simeq \sum_{\mathfrak{k} \in \mathbb{Z}_{++}^{r_2}}^\oplus \mathcal{H}_{\varepsilon_1 \lambda}(D_1, \mathcal{P}_{\mathfrak{k}}(\mathfrak{p}_2^+)), & \mathfrak{p}_2^+ : \text{ simple}, \\ \mathcal{H}_\lambda(D)|_{\tilde{G}_1} &\simeq \sum_{\mathfrak{k} \in \mathbb{Z}_{++}^{r'}}^\oplus \sum_{\mathfrak{l} \in \mathbb{Z}_{++}^{r''}}^\oplus \mathcal{H}_{\varepsilon_1 \lambda}(D_1, \mathcal{P}_{\mathfrak{k}}(\mathfrak{p}_{11}^+) \boxtimes \mathcal{P}_{\mathfrak{l}}(\mathfrak{p}_{22}^+)), & \mathfrak{p}_2^+ : \text{ non-simple}. \end{aligned}$$

- (2) *Suppose $\mathfrak{p}^+ = \mathfrak{p}_0^+ \oplus \mathfrak{p}_0^+$ with \mathfrak{p}_0^+ simple of rank r_0 , and $\sigma : (x, y) \mapsto (y, x)$, so that (G, G_1) is of the form $(G_0 \times G_0, \Delta(G_0))$. For $\lambda, \mu > \frac{d}{2}(r_0-1)$, the tensor product representation $\mathcal{H}_\lambda(D_0) \hat{\otimes} \mathcal{H}_\mu(D_0)$ is decomposed under the diagonal subgroup $\Delta(\tilde{G}_0)$ into the Hilbert direct sum of irreducible representations as*

$$\mathcal{H}_\lambda(D_0) \hat{\otimes} \mathcal{H}_\mu(D_0) \simeq \sum_{\mathfrak{k} \in \mathbb{Z}_{++}^{r_0}}^\oplus \mathcal{H}_{\lambda+\mu}(D_0, \mathcal{P}_{\mathfrak{k}}(\mathfrak{p}_0^+)).$$

In the following, suppose \mathfrak{p}^+ is simple. Similar results also hold for the tensor product case. Let $\langle \cdot, \cdot \rangle_\lambda = \langle \cdot, \cdot \rangle_{\lambda, \mathfrak{p}^+}$ be the \tilde{G} -invariant inner product on $\mathcal{H}_\lambda(D)$, which is originally defined for $\lambda > p - 1$ by (2.9). The purpose of this article is to study the above decomposition by observing the inner product

$$\begin{aligned} \langle f(x_2), e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\lambda, x}, & \quad f(x_2) \in \mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+), \\ \langle f(x_{11})g(x_{22}), e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\lambda, x}, & \quad f(x_{11}) \in \mathcal{P}_{\mathbf{k}}(\mathfrak{p}_{11}^+), \quad g(x_{22}) \in \mathcal{P}_1(\mathfrak{p}_{22}^+), \end{aligned} \quad (2.16)$$

where $z \in \mathfrak{p}^+$, $x = x_1 + x_2 \in \mathfrak{p}^+ = \mathfrak{p}_1^+ \oplus \mathfrak{p}_2^+$ or $x = x_{11} + x_{12} + x_{22} \in \mathfrak{p}^+ = \mathfrak{p}_{11}^+ \oplus \mathfrak{p}_{12}^+ \oplus \mathfrak{p}_{22}^+$, and the subscript x stands for the variable of integration.

In the author's previous article [44], when \mathfrak{p}^+ and \mathfrak{p}_2^+ are of tube type, we computed the above inner products explicitly for some special $\mathbf{k} \in \mathbb{Z}_{++}^{r_2}$ or $(\mathbf{k}, \mathbf{l}) \in \mathbb{Z}_{++}^{r'_+} \times \mathbb{Z}_{++}^{r''_+}$, that is, for

$$\begin{aligned} \mathbf{k} = (k + l, k, \dots, k), & \quad \mathfrak{p}_2^+ : \text{simple}, \quad \varepsilon_2 = 1, \\ \mathbf{k} = (k + 1, \dots, k + 1, k \dots, k), & \quad \mathfrak{p}_2^+ : \text{simple}, \quad \varepsilon_2 = 2, \\ (\mathbf{k}, \mathbf{l}) = ((k, \dots, k), \mathbf{l}), & \quad \mathfrak{p}_2^+ : \text{non-simple}, \end{aligned}$$

and applied these results for determination of the \tilde{G}_1 -intertwining operators (symmetry breaking operators)

$$\begin{aligned} \mathcal{F}_{\lambda, \mathbf{k}}^\downarrow : \mathcal{H}_\lambda(D)|_{\tilde{G}_1} &\longrightarrow \mathcal{H}_{\varepsilon_1 \lambda}(D_1, \mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+)) & \mathfrak{p}_2^+ : \text{simple}, \\ \mathcal{F}_{\lambda, \mathbf{k}, \mathbf{l}}^\downarrow : \mathcal{H}_\lambda(D)|_{\tilde{G}_1} &\longrightarrow \mathcal{H}_{\varepsilon_1 \lambda}(D_1, \mathcal{P}_{\mathbf{k}}(\mathfrak{p}_{11}^+) \boxtimes \mathcal{P}_1(\mathfrak{p}_{22}^+)) & \mathfrak{p}_2^+ : \text{non-simple} \end{aligned}$$

for the above special \mathbf{k} or (\mathbf{k}, \mathbf{l}) . In this article we treat general $\mathbf{k} \in \mathbb{Z}_{++}^{r_2}$ or $(\mathbf{k}, \mathbf{l}) \in \mathbb{Z}_{++}^{r'_+} \times \mathbb{Z}_{++}^{r''_+}$, compute the top term of (2.16), i.e., the value of (2.16) at $z_1 = 0$ or $z_{12} = 0$, and also compute the poles of (2.16) as a function of $\lambda \in \mathbb{C}$.

In the following, when \mathfrak{p}_2^+ is not simple, we write $\tilde{\mathbf{k}} = (\mathbf{k}, \mathbf{l}) \in \mathbb{Z}_{++}^{r'_+} \times \mathbb{Z}_{++}^{r''_+}$, and write $\mathcal{P}_{\tilde{\mathbf{k}}}(\mathfrak{p}_2^+) := \mathcal{P}_{\mathbf{k}}(\mathfrak{p}_{11}^+) \boxtimes \mathcal{P}_1(\mathfrak{p}_{22}^+)$. Then since the map

$$\mathcal{P}(\mathfrak{p}_2^+) \longrightarrow \mathcal{P}(\mathfrak{p}_2^+), \quad f(x_2) \longmapsto \langle f(x_2), e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\lambda, x} \Big|_{z_1=0}$$

is K_1 -equivariant and since $\mathcal{P}(\mathfrak{p}_2^+)$ decomposes multiplicity-freely under K_1 by Theorem 2.2, there exist constants $C_{\mathfrak{p}^+, \mathfrak{p}_2^+}(\lambda, \tilde{\mathbf{k}}) \in \mathbb{C}$ such that

$$\langle f(x_2), e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\lambda, x} \Big|_{z_1=0} = C_{\mathfrak{p}^+, \mathfrak{p}_2^+}(\lambda, \tilde{\mathbf{k}}) f(z_2), \quad f(x_2) \in \mathcal{P}_{\tilde{\mathbf{k}}}(\mathfrak{p}_2^+) \quad (2.17)$$

holds for every $\tilde{\mathbf{k}} \in \mathbb{Z}_{++}^{r_2}$ or $\tilde{\mathbf{k}} \in \mathbb{Z}_{++}^{r'_+} \times \mathbb{Z}_{++}^{r''_+}$. Then the Parseval–Plancherel-type formula is given by using these constants. Let $V_{\tilde{\mathbf{k}}}$ be an abstract K_1 -module isomorphic to $\mathcal{P}_{\tilde{\mathbf{k}}}(\mathfrak{p}_2^+)$, let $\|\cdot\|_{\varepsilon_1 \lambda, \tilde{\mathbf{k}}, \mathfrak{p}_1^+}$ be the \tilde{G}_1 -invariant norm on $\mathcal{H}_{\varepsilon_1 \lambda}(D_1, V_{\tilde{\mathbf{k}}})$ normalized such that $\|v\|_{\varepsilon_1 \lambda, \tilde{\mathbf{k}}, \mathfrak{p}_1^+} = |v|_{V_{\tilde{\mathbf{k}}}}$ holds for all constant functions $v \in V_{\tilde{\mathbf{k}}}$, and let $\|\cdot\|_{F, \mathfrak{p}^+}$ be the Fischer norm on \mathfrak{p}^+ given in (2.10).

Proposition 2.6. *For $\lambda > p - 1$ and for $\tilde{\mathbf{k}} \in \mathbb{Z}_{++}^{r_2}$ (when \mathfrak{p}_2^+ is simple) or $\tilde{\mathbf{k}} \in \mathbb{Z}_{++}^{r'_+} \times \mathbb{Z}_{++}^{r''_+}$ (when \mathfrak{p}_2^+ is not simple), let $C_{\mathfrak{p}^+, \mathfrak{p}_2^+}(\lambda, \tilde{\mathbf{k}}) \in \mathbb{C}$ be as in (2.17).*

(1) *For $f(x_2) \in \mathcal{P}_{\tilde{\mathbf{k}}}(\mathfrak{p}_2^+)$, we have $\|f\|_{\lambda, \mathfrak{p}^+}^2 = C_{\mathfrak{p}^+, \mathfrak{p}_2^+}(\lambda, \tilde{\mathbf{k}}) \|f\|_{F, \mathfrak{p}^+}^2$.*

(2) *We take a vector-valued polynomial $K_{\tilde{\mathbf{k}}}(x_2) \in \mathcal{P}(\mathfrak{p}_2^+, \overline{V_{\tilde{\mathbf{k}}}})^{K_1}$ normalized such that*

$$|\langle f(x_2), K_{\tilde{\mathbf{k}}}(x_2) \rangle_{F, \mathfrak{p}^+}|_{V_{\tilde{\mathbf{k}}}}^2 = \|f\|_{F, \mathfrak{p}^+}^2, \quad f(x_2) \in \mathcal{P}_{\tilde{\mathbf{k}}}(\mathfrak{p}_2^+),$$

and define the vector-valued polynomial $F_{\lambda, \tilde{\mathbf{k}}}^\downarrow(z) \in \mathcal{P}(\mathfrak{p}^-, V_{\tilde{\mathbf{k}}})$ by

$$F_{\lambda, \tilde{\mathbf{k}}}^\downarrow(z) := \frac{1}{C_{\mathfrak{p}^+, \mathfrak{p}_2^+}(\lambda, \tilde{\mathbf{k}})} \langle e^{(x|z)_{\mathfrak{p}^+}}, K_{\tilde{\mathbf{k}}}(x_2) \rangle_{\lambda, x}. \quad (2.18)$$

Then the differential operator

$$\mathcal{F}_{\lambda, \tilde{\mathbf{k}}}^\downarrow: \mathcal{H}_\lambda(D)|_{\tilde{G}_1} \longrightarrow \mathcal{H}_{\varepsilon_1 \lambda}(D_1, V_{\tilde{\mathbf{k}}}), \quad (\mathcal{F}_{\lambda, \tilde{\mathbf{k}}}^\downarrow f)(x_1) := F_{\lambda, \tilde{\mathbf{k}}}^\downarrow \left(\frac{\partial}{\partial x} \right) f(x) \Big|_{x_2=0} \quad (2.19)$$

becomes a symmetry breaking operator satisfying

$$\|\mathcal{F}_{\lambda, \tilde{\mathbf{k}}}^\downarrow f\|_{\varepsilon_1 \lambda, \tilde{\mathbf{k}}, \mathfrak{p}_1^+}^2 = \|f\|_{F, \mathfrak{p}^+}^2, \quad f(x_2) \in \mathcal{P}_{\tilde{\mathbf{k}}}(\mathfrak{p}_2^+).$$

$$(3) \text{ For } f \in \mathcal{H}_\lambda(D), \text{ we have } \|f\|_{\lambda, \mathfrak{p}^+}^2 = \sum_{\tilde{\mathbf{k}}} C_{\mathfrak{p}^+, \mathfrak{p}_2^+}(\lambda, \tilde{\mathbf{k}}) \|\mathcal{F}_{\lambda, \tilde{\mathbf{k}}}^\downarrow f\|_{\varepsilon_1 \lambda, \tilde{\mathbf{k}}, \mathfrak{p}_1^+}^2.$$

Proof. (1) Since the reproducing kernel of $\langle \cdot, \cdot \rangle_{F, \mathfrak{p}^+}$ is given by $e^{(z|\bar{x})_{\mathfrak{p}^+}}$ (see [13, Proposition XI.1.1]), for $f, g \in \mathcal{P}(\mathfrak{p}^+)$ we have

$$\langle f(x), g(x) \rangle_{\lambda, x} = \langle f(x), \langle g(z), e^{(z|\bar{x})_{\mathfrak{p}^+}} \rangle_{F, z} \rangle_{\lambda, x} = \langle \langle f(x), e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\lambda, x}, g(z) \rangle_{F, z}.$$

Then by putting $f = g \in \mathcal{P}_{\tilde{\mathbf{k}}}(\mathfrak{p}_2^+) \subset \mathcal{P}(\mathfrak{p}^+)$, we get the desired formula.

(2) The symmetry breaking property follows from [43, Theorem 3.10 (1)]. Let $f(x_2) \in \mathcal{P}_{\tilde{\mathbf{k}}}(\mathfrak{p}_2^+)$. Then since $\exp(x|\frac{\partial}{\partial y})_{\mathfrak{p}^+} f(y)|_{y=0} = f(x)$ holds, by (1), we have

$$\begin{aligned} (\mathcal{F}_{\lambda, \tilde{\mathbf{k}}}^\downarrow f)(y_1) &= \frac{1}{C_{\mathfrak{p}^+, \mathfrak{p}_2^+}(\lambda, \tilde{\mathbf{k}})} \left\langle \exp \left(x \left| \frac{\partial}{\partial y} \right)_{\mathfrak{p}^+}, K_{\tilde{\mathbf{k}}}(x_2) \right\rangle_{\lambda, x} f(y_2) \Big|_{y_2=0} \\ &= \frac{1}{C_{\mathfrak{p}^+, \mathfrak{p}_2^+}(\lambda, \tilde{\mathbf{k}})} \langle f(x_2), K_{\tilde{\mathbf{k}}}(x_2) \rangle_{\lambda, x} \Big|_{y_2=0} = \langle f(x_2), K_{\tilde{\mathbf{k}}}(x_2) \rangle_{F, x} \in V_{\tilde{\mathbf{k}}}. \end{aligned}$$

Then by the normalization of $\|\cdot\|_{\varepsilon_1 \lambda, \tilde{\mathbf{k}}, \mathfrak{p}_1^+}$ and $K_{\tilde{\mathbf{k}}}(x_2)$, we get

$$\|\mathcal{F}_{\lambda, \tilde{\mathbf{k}}}^\downarrow f\|_{\varepsilon_1 \lambda, \tilde{\mathbf{k}}, \mathfrak{p}_1^+}^2 = |\langle f(x_2), K_{\tilde{\mathbf{k}}}(x_2) \rangle_{F, \mathfrak{p}^+}|_{V_{\tilde{\mathbf{k}}}}^2 = \|f\|_{F, \mathfrak{p}^+}^2, \quad f(x_2) \in \mathcal{P}_{\tilde{\mathbf{k}}}(\mathfrak{p}_2^+).$$

(3) Let

$$\mathcal{F}_{\lambda, \tilde{\mathbf{k}}}^\uparrow: \mathcal{H}_{\varepsilon_1 \lambda}(D_1, V_{\tilde{\mathbf{k}}}) \longrightarrow \mathcal{H}_\lambda(D)|_{\tilde{G}_1}$$

be the \tilde{G}_1 -intertwining operator (holographic operator) normalized such that

$$\mathcal{F}_{\lambda, \tilde{\mathbf{k}}}^\downarrow \circ \mathcal{F}_{\lambda, \tilde{\mathbf{k}}}^\uparrow = I_{\mathcal{H}_{\varepsilon_1 \lambda}(D_1, V_{\tilde{\mathbf{k}}})}$$

holds. Then since $\mathcal{H}_\lambda(D)$ decomposes multiplicity-freely under \tilde{G}_1 , for $f \in \mathcal{H}_\lambda(D)$, we have

$$f = \sum_{\tilde{\mathbf{k}}} \mathcal{F}_{\lambda, \tilde{\mathbf{k}}}^\uparrow \mathcal{F}_{\lambda, \tilde{\mathbf{k}}}^\downarrow f, \quad \|f\|_{\lambda, \mathfrak{p}^+}^2 = \sum_{\tilde{\mathbf{k}}} \|\mathcal{F}_{\lambda, \tilde{\mathbf{k}}}^\uparrow \mathcal{F}_{\lambda, \tilde{\mathbf{k}}}^\downarrow f\|_{\lambda, \mathfrak{p}^+}^2.$$

If $f(x_2) \in \mathcal{P}_{\tilde{\mathbf{k}}}(\mathfrak{p}_2^+)$, then since $\mathcal{F}_{\lambda, \tilde{\mathbf{k}}}^\uparrow \mathcal{F}_{\lambda, \tilde{\mathbf{k}}}^\downarrow f = f$ holds, by (1) and (2), we have

$$\|\mathcal{F}_{\lambda, \tilde{\mathbf{k}}}^\uparrow \mathcal{F}_{\lambda, \tilde{\mathbf{k}}}^\downarrow f\|_{\lambda, \mathfrak{p}^+}^2 = \|f\|_{\lambda, \mathfrak{p}^+}^2 = C_{\mathfrak{p}^+, \mathfrak{p}_2^+}(\lambda, \tilde{\mathbf{k}}) \|f\|_{F, \mathfrak{p}^+}^2 = C_{\mathfrak{p}^+, \mathfrak{p}_2^+}(\lambda, \tilde{\mathbf{k}}) \|\mathcal{F}_{\lambda, \tilde{\mathbf{k}}}^\downarrow f\|_{\varepsilon_1 \lambda, \tilde{\mathbf{k}}, \mathfrak{p}_1^+}^2,$$

and since $\mathcal{F}_{\lambda, \tilde{\mathbf{k}}}^\uparrow$ is an isometry up to scalar multiple,

$$\|\mathcal{F}_{\lambda, \tilde{\mathbf{k}}}^\uparrow \mathcal{F}_{\lambda, \tilde{\mathbf{k}}}^\downarrow f\|_{\lambda, \mathfrak{p}^+}^2 = C_{\mathfrak{p}^+, \mathfrak{p}_2^+}(\lambda, \tilde{\mathbf{k}}) \|\mathcal{F}_{\lambda, \tilde{\mathbf{k}}}^\downarrow f\|_{\varepsilon_1 \lambda, \tilde{\mathbf{k}}, \mathfrak{p}_1^+}^2$$

holds for all $f \in \mathcal{H}_\lambda(D)$. Hence we get the desired formula. ■

Next we consider the meromorphic continuation of (2.16), (2.18), and (2.19). The differential operator $\mathcal{F}_{\lambda, \tilde{\mathbf{k}}}^\downarrow$ in (2.19) is extended to the map

$$\mathcal{F}_{\lambda, \tilde{\mathbf{k}}}^\downarrow: \mathcal{O}_\lambda(D)|_{\tilde{G}_1} \longrightarrow \mathcal{O}_{\varepsilon_1 \lambda}(D_1, V_{\tilde{\mathbf{k}}}),$$

and this is meromorphically continued for all $\lambda \in \mathbb{C}$. For generic λ , we have an abstract $(\mathfrak{g}_1, \tilde{K}_1)$ -isomorphism

$$\mathcal{O}_{\varepsilon_1 \lambda}(D_1, V_{\tilde{\mathbf{k}}})_{\tilde{K}_1} \simeq d\tau_\lambda(\mathcal{U}(\mathfrak{g}_1))\mathcal{P}_{\tilde{\mathbf{k}}}(\mathfrak{p}_2^+) \subset \mathcal{P}(\mathfrak{p}^+) = \mathcal{O}_\lambda(D)_{\tilde{K}},$$

where $\mathcal{U}(\mathfrak{g}_1)$ is the universal enveloping algebra of $\mathfrak{g}_1^{\mathbb{C}}$, and $d\tau_\lambda$ is the differential of $(\tau_\lambda, \mathcal{O}_\lambda(D))$. The restriction of $\mathcal{F}_{\lambda, \tilde{\mathbf{k}}}^\downarrow$ gives the explicit isomorphism from the right to the left. On the other hand, this does not hold for all $\lambda \in \mathbb{C}$. By computing the poles of (2.16) with respect to λ , we can get some information on $d\tau_\lambda(\mathcal{U}(\mathfrak{g}_1))\mathcal{P}_{\tilde{\mathbf{k}}}(\mathfrak{p}_2^+)$ for such singular λ .

Proposition 2.7. *Let $\mathbf{l} = (l_1, \dots, l_r) \in \mathbb{Z}_{++}^r$. Suppose*

$$(\lambda)_{\mathbf{l}, d} \langle f(x_2), e^{(x|\tilde{\mathbf{z}})_{\mathfrak{p}^+}} \rangle_{\lambda, x}$$

is holomorphically continued for all $\lambda \in \mathbb{C}$ for some non-zero $f(x_2) \in \mathcal{P}_{\tilde{\mathbf{k}}}(\mathfrak{p}_2^+)$.

(1) *We have*

$$\mathcal{P}_{\tilde{\mathbf{k}}}(\mathfrak{p}_2^+) \subset \bigoplus_{\substack{\mathbf{m} \in \mathbb{Z}_{++}^r \\ m_j \leq l_j, j=1, \dots, r}} \mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+),$$

and especially, for $j = 1, \dots, r$,

$$d\tau_\lambda(\mathcal{U}(\mathfrak{g}_1))\mathcal{P}_{\tilde{\mathbf{k}}}(\mathfrak{p}_2^+) \subset M_j^{\mathfrak{g}}(\lambda) \quad \text{holds if } \lambda \in \frac{d}{2}(j-1) - l_j - \mathbb{Z}_{\geq 0},$$

where $M_j^{\mathfrak{g}}(\lambda) \subset \mathcal{O}_\lambda(D)_{\tilde{K}}$ is as in (2.12).

(2) *For $a = 0, 1, \dots, r-1$, if $l_{a+1} = 0$ and $C_{\mathfrak{p}^+, \mathfrak{p}_2^+}(\frac{d}{2}a, \tilde{\mathbf{k}}) \neq 0$, then $\mathcal{F}_{\lambda, \tilde{\mathbf{k}}}^\downarrow$ is holomorphic at $\lambda = \frac{d}{2}a$, and its restriction gives the symmetry breaking operator*

$$\mathcal{F}_{\frac{d}{2}a, \tilde{\mathbf{k}}}^\downarrow: \mathcal{H}_{\frac{d}{2}a}(D)|_{\tilde{G}_1} \longrightarrow \mathcal{H}_{\varepsilon_1 \frac{d}{2}a}(D_1, V_{\tilde{\mathbf{k}}}).$$

Proof. (1) Follows from the last paragraph of Section 2.5.

(2) The holomorphy of $\mathcal{F}_{\lambda, \tilde{\mathbf{k}}}^\downarrow$ at $\lambda = \frac{d}{2}a$ is clear from the assumption. For the latter claim, $\mathcal{P}_{\tilde{\mathbf{k}}}(\mathfrak{p}_2^+) \subset M_{a+1}^{\mathfrak{g}}(\frac{d}{2}a) = \mathcal{H}_{\frac{d}{2}a}(D)_{\tilde{K}}$ holds by (1), and the image of $\mathcal{P}_{\tilde{\mathbf{k}}}(\mathfrak{p}_2^+)$ by $\mathcal{F}_{\frac{d}{2}a, \tilde{\mathbf{k}}}^\downarrow$ is non-zero by the normalization assumption of $\mathcal{F}_{\lambda, \tilde{\mathbf{k}}}^\downarrow$. Therefore, the image of $\mathcal{H}_{\frac{d}{2}a}(D)$ by $\mathcal{F}_{\frac{d}{2}a, \tilde{\mathbf{k}}}^\downarrow$ becomes a non-zero unitary submodule of $\mathcal{O}_{\varepsilon_1 \frac{d}{2}a}(D_1, V_{\tilde{\mathbf{k}}})$, which is unique and denoted by $\mathcal{H}_{\varepsilon_1 \frac{d}{2}a}(D_1, V_{\tilde{\mathbf{k}}})$. ■

Since $\mathcal{H}_\lambda(D)|_{\tilde{G}_1}$ decomposes multiplicity-freely [26, Theorem A], symmetry breaking operators on $\mathcal{H}_\lambda(D)$ are unique up to constant multiple for all λ in (2.13). On the other hand, we do not know a priori whether symmetry breaking operators on $\mathcal{O}_\lambda(D)$ are unique or not for $\lambda \leq \frac{d}{2}(r-1)$. For non-unique case see [33, Section 9] or Section 8 of this paper. It is known that symmetry breaking operators on $\mathcal{O}_\lambda(D)$ are always given by differential operators (localness theorem, [32, Theorem 5.3]), and their symbols are characterized as polynomial solutions

of certain differential equations (F-method, [33, Theorem 3.1]). That is, for $\lambda \in \mathbb{C}$ let $\mathcal{B}_\lambda^{\mathfrak{p}^\pm}$ be the vector-valued differential operator

$$\begin{aligned} \mathcal{B}_\lambda^{\mathfrak{p}^\pm} &: \mathcal{P}(\mathfrak{p}^\pm) \longrightarrow \mathcal{P}(\mathfrak{p}^\pm) \otimes \mathfrak{p}^\mp, \\ \mathcal{B}_\lambda^{\mathfrak{p}^\pm} f(z) &:= \sum_{\alpha, \beta} \frac{1}{2} Q(e_\alpha^\mp, e_\beta^\mp) z \frac{\partial^2 f}{\partial z_\alpha^\pm \partial z_\beta^\pm}(z) + \lambda \sum_\alpha e_\alpha^\mp \frac{\partial f}{\partial z_\alpha^\pm}(z), \end{aligned}$$

where $\{e_\alpha^\mp\} \subset \mathfrak{p}^\mp$ is a basis of \mathfrak{p}^\mp , and $\{z_\alpha^\pm\}$ is the coordinate of \mathfrak{p}^\pm dual to $\{e_\alpha^\mp\}$ (*Bessel operator* [9] and [13, Section XV.2]), let

$$(\mathcal{B}_\lambda^{\mathfrak{p}^\pm})_1: \mathcal{P}(\mathfrak{p}^\pm) \longrightarrow \mathcal{P}(\mathfrak{p}^\pm) \otimes \mathfrak{p}_1^\mp$$

be the orthogonal projection of $\mathcal{B}_\lambda^{\mathfrak{p}^\pm}$ onto \mathfrak{p}_1^\mp , and let

$$\text{Sol}_{\mathcal{P}(\mathfrak{p}^\pm)}((\mathcal{B}_\lambda^{\mathfrak{p}^\pm})_1) := \{f(z) \in \mathcal{P}(\mathfrak{p}^\pm) \mid (\mathcal{B}_\lambda^{\mathfrak{p}^\pm})_1 f = 0\}.$$

Then we have

$$\begin{aligned} \text{Hom}_{\tilde{G}_1}(\mathcal{O}_\lambda(D), \mathcal{O}_{\varepsilon_1 \lambda}(D_1, V)) &\simeq \text{Hom}_{\mathfrak{t}_1^{\mathbb{C}} \oplus \mathfrak{p}_1^-}(\chi_1^{\varepsilon_1 \lambda} \otimes V^\vee, \text{ind}_{\mathfrak{t}_1^{\mathbb{C}} \oplus \mathfrak{p}_1^-}^{\mathfrak{g}^{\mathbb{C}}}(\chi^\lambda)) \\ &\simeq (\text{Sol}_{\mathcal{P}(\mathfrak{p}^-)}((\mathcal{B}_\lambda^{\mathfrak{p}^-})_1) \otimes V)^{K_1}. \end{aligned} \quad (2.20)$$

By this F-method, we can prove the following, which gives an analogue of [35, Corollary 12.8 (3)].

Proposition 2.8. *Suppose $\mathfrak{p}^+ = \mathfrak{n}^{+\mathbb{C}}$, $\mathfrak{p}_2^+ = \mathfrak{n}_2^{+\mathbb{C}}$ are of tube type.*

(1) *For $a \in \mathbb{Z}_{>0}$, the following differential operator intertwines the \tilde{G} -action*

$$\det_{\mathfrak{n}^-} \left(\frac{\partial}{\partial x} \right)^a : \mathcal{O}_{\frac{n}{r}-a}(D) \longrightarrow \mathcal{O}_{\frac{n}{r}+a}(D).$$

(See [2, Theorem 6.4], [20, Propositions 1.2 and 2.2], [59, Lemma 7.1, formulas (8.6)–(8.8)].)

(2) *We fix $\tilde{\mathbf{k}} \in \mathbb{Z}_{++}^2$ (when \mathfrak{p}_2^+ is simple) or $\tilde{\mathbf{k}} \in \mathbb{Z}_{++}^{r'} \times \mathbb{Z}_{++}^{r''}$ (when \mathfrak{p}_2^+ is not simple), and let $\mathbf{l} \in \mathbb{Z}_{++}^r$, $a \in \mathbb{Z}_{>0}$. Suppose that $(\lambda)_{1,d} \langle \det_{\mathfrak{n}_2^+}(x_2)^{\varepsilon_2 a} f(x_2), e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\lambda, x}$ is holomorphic at $\lambda = \frac{n}{r} - a$, and*

$$\det_{\mathfrak{n}^+}(z)^{-a} (\lambda)_{1,d} \langle \det_{\mathfrak{n}_2^+}(x_2)^{\varepsilon_2 a} f(x_2), e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\lambda, x} \Big|_{\lambda = \frac{n}{r} - a} \in \mathcal{P}(\mathfrak{p}^+)$$

holds for some non-zero $f(x_2) \in \mathcal{P}_{\tilde{\mathbf{k}}}(\mathfrak{p}_2^+)$. Then there exists $C \in \mathbb{C}$ such that

$$(\lambda)_{1,d} \langle \det_{\mathfrak{n}_2^+}(x_2)^{\varepsilon_2 a} f(x_2), e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\lambda, x} \Big|_{\lambda = \frac{n}{r} - a} = C \det_{\mathfrak{n}^+}(z)^a \langle f(x_2), e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\frac{n}{r}+a, x}$$

holds for all $f(x_2) \in \mathcal{P}_{\tilde{\mathbf{k}}}(\mathfrak{p}_2^+)$.

We note that $\det_{\mathfrak{n}_2^+}(x_2) = \det_{\mathfrak{n}_{11}^+}(x_{11}) \det_{\mathfrak{n}_{22}^+}(x_{22})$ holds when \mathfrak{p}_2^+ is not simple.

Proof. (1) By the proof of [13, Proposition XV.2.4], as a differential operator we have

$$\mathcal{B}_{\frac{n}{r}-a}^{\mathfrak{p}^\pm} \det_{\mathfrak{n}^\pm}(z)^a = \det_{\mathfrak{n}^\pm}(z)^a \mathcal{B}_{\frac{n}{r}+a}^{\mathfrak{p}^\pm}, \quad a \in \mathbb{C}, \quad (2.21)$$

and hence for $a \in \mathbb{Z}_{>0}$, as a polynomial we have $\det_{\mathfrak{n}^\pm}(z)^a \in \text{Sol}_{\mathcal{P}(\mathfrak{p}^\pm)}(\mathcal{B}_{\frac{n}{r}-a}^{\mathfrak{p}^\pm})$. Now since $\mathbb{C} \det_{\mathfrak{n}^\pm}(z)^a \simeq \chi^{\mp 2a}$ holds as a K -module, by applying the F-method for $\sigma = I_{\mathfrak{g}^{\mathbb{C}}}$, $\mathfrak{p}_1^+ = \mathfrak{p}^+$ case, we get the intertwining property.

(2) As in [44, formula (2.21)], we have $(\mathcal{B}_\lambda^{\mathfrak{p}^+})_1 \langle g(x_2), e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\lambda, x} = 0$ for all $g(x_2) \in \mathcal{P}(\mathfrak{p}_2^+)$, and by (2.21), we have

$$(\mathcal{B}_{\frac{n}{r}+a}^{\mathfrak{p}^+})_1 \det_{\mathfrak{n}^+}(z)^{-a} (\lambda)_{1,d} \langle \det_{\mathfrak{n}_2^+}(x_2)^{\varepsilon_{2a}} f(x_2), e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\lambda, x} \Big|_{\lambda = \frac{n}{r} - a} = 0.$$

Hence under the polynomiality assumption, we have

$$\begin{aligned} (f(x_2) \mapsto \det_{\mathfrak{n}^+}(z)^{-a} (\lambda)_{1,d} \langle \det_{\mathfrak{n}_2^+}(x_2)^{\varepsilon_{2a}} f(x_2), e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\lambda, x} \Big|_{\lambda = \frac{n}{r} - a}) \\ \in \text{Hom}_{K_1}(\mathcal{P}_{\mathfrak{k}}(\mathfrak{p}_2^+), \text{Sol}_{\mathcal{P}(\mathfrak{p}^+)}((\mathcal{B}_{\frac{n}{r}+a}^{\mathfrak{p}^+})_1)) = \mathbb{C}(f(x_2) \mapsto \langle f(x_2), e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\frac{n}{r}+a, x}), \end{aligned}$$

where the last equality holds since the space (2.20) is 1-dimensional for $\lambda > \frac{n}{r} - 1 = \frac{d}{2}(r-1)$, and this completes the proof. \blacksquare

3 Key lemmas

In this section, we introduce some equations on inner products and results on finite-dimensional representations of $U(\mathfrak{s})$ used later.

3.1 Reduction to smaller algebras

We consider a simple Jordan triple system \mathfrak{p}^+ . We fix a tripotent $e \in \mathfrak{p}^+$, and let $\mathfrak{p}^{+'} := \mathfrak{p}^+(e)_2 \subset \mathfrak{p}^+$ denote the eigenspace of $D(e, \bar{e})$ with the eigenvalue 2, as in (2.2). This $\mathfrak{p}^{+'}$ becomes of tube type. For $x \in \mathfrak{p}^+$, let $x' \in \mathfrak{p}^{+'}$ be the orthogonal projection. Also, let $D \subset \mathfrak{p}^+$, $D' \subset \mathfrak{p}^{+'}$ be the corresponding bounded symmetric domains, and let $\langle \cdot, \cdot \rangle_{\lambda, \mathfrak{p}^+}$, $\langle \cdot, \cdot \rangle_{\lambda, \mathfrak{p}^{+'}}$ be the weighted Bergman inner products on D , D' respectively, as in (2.9).

Proposition 3.1. *We fix a tripotent $e \in \mathfrak{p}^+$, and let $\mathfrak{p}^{+'} := \mathfrak{p}^+(e)_2$. Then for $\text{Re } \lambda > p - 1$, for $f(x') \in \mathcal{P}(\mathfrak{p}^{+'}) \subset \mathcal{P}(\mathfrak{p}^+)$, we have*

$$\langle f(x'), e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\lambda, x, \mathfrak{p}^+} = \langle f(x'), e^{(x'|\bar{z}')_{\mathfrak{p}^{+'}}} \rangle_{\lambda, x', \mathfrak{p}^{+'}}.$$

Here the subscripts x, x' denote the variables of integration. Note that $(x'|\bar{z}')_{\mathfrak{p}^{+'}} = (x'|\bar{z}')_{\mathfrak{p}^{+'}}$ holds for any $x', z' \in \mathfrak{p}^{+'}$, since a primitive tripotent $\tilde{e} \in \mathfrak{p}^{+'}$ is also primitive in \mathfrak{p}^+ , and both inner products are normalized such that $(\tilde{e}|\bar{\tilde{e}})_{\mathfrak{p}^+} = (\tilde{e}|\bar{\tilde{e}})_{\mathfrak{p}^{+'}} = 1$ holds. This lemma is proved easily by using Corollary 2.4 and $\mathcal{P}_{\mathfrak{m}}(\mathfrak{p}^{+'}) \subset \mathcal{P}_{\mathfrak{m}}(\mathfrak{p}^+)$. Here we give a proof that does not rely on Corollary 2.4.

Proof. We write $x = x_2 + x_1 + x_0 \in \mathfrak{p}^+ = \mathfrak{p}^+(e)_2 \oplus \mathfrak{p}^+(e)_1 \oplus \mathfrak{p}^+(e)_0$, so that $x' = x_2$. First we prove

$$\langle f(x_2), e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\lambda, x, \mathfrak{p}^+} = \langle f(x_2), e^{(x_2|\bar{z}_2)_{\mathfrak{p}^+}} \rangle_{\lambda, x, \mathfrak{p}^+} \quad (3.1)$$

for $f(x_2) \in \mathcal{P}(\mathfrak{p}^+(e)_2)$. Since

$$e^{(x|\bar{z})_{\mathfrak{p}^+}} = e^{(x_2|\bar{z}_2)_{\mathfrak{p}^+}} \sum_{k_1, k_0=0}^{\infty} \frac{1}{k_1! k_0!} (x_1|\bar{z}_1)_{\mathfrak{p}^+}^{k_1} (x_0|\bar{z}_0)_{\mathfrak{p}^+}^{k_0}$$

holds, it is enough to prove $\langle f(x_2), g(x_2, x_1, x_0) \rangle_{\lambda, x, \mathfrak{p}^+} = 0$ for $g(x_2, x_1, x_0) \in \mathcal{O}(\mathfrak{p}^+(e)_2) \otimes \mathcal{P}(\mathfrak{p}^+(e)_1) \otimes \mathcal{P}(\mathfrak{p}^+(e)_0)$ satisfying

$$g(x_2, sx_1, tx_0) = s^{k_1} t^{k_0} g(x_2, x_1, x_0), \quad s, t \in \mathbb{C}$$

for some $k_1, k_0 \in \mathbb{Z}_{\geq 0}$ with $(k_1, k_0) \neq (0, 0)$. For $t \in \sqrt{-1}\mathbb{R}$, let

$$l := \exp(t(2I_{\mathfrak{p}^+} - D(e, \bar{e}))) = I_{\mathfrak{p}^+(e)_2} + e^t I_{\mathfrak{p}^+(e)_1} + e^{2t} I_{\mathfrak{p}^+(e)_0} \in K,$$

and let $\text{Proj}_2: \mathfrak{p}^+ \rightarrow \mathfrak{p}^+(e)_2$ be the orthogonal projection. Then we have

$$\begin{aligned} \langle f(x_2), g(x_2, x_1, x_0) \rangle_{\lambda, x, \mathfrak{p}^+} &= \langle f(\text{Proj}_2(l^{-1}(x_2, x_1, x_0))), g(x_2, x_1, x_0) \rangle_{\lambda, x, \mathfrak{p}^+} \\ &= \langle f(\text{Proj}_2(x_2, x_1, x_0)), g(l(x_2, x_1, x_0)) \rangle_{\lambda, x, \mathfrak{p}^+} \\ &= \langle f(x_2), g(x_2, e^t x_1, e^{2t} x_0) \rangle_{\lambda, x, \mathfrak{p}^+} \\ &= \overline{e^{(k_1+2k_0)t}} \langle f(x_2), g(x_2, x_1, x_0) \rangle_{\lambda, x, \mathfrak{p}^+} \end{aligned}$$

for all $t \in \sqrt{-1}\mathbb{R}$, and hence this vanishes if $(k_1, k_0) \neq (0, 0)$. Therefore, (3.1) holds. Next, to prove

$$\langle f(x_2), e^{(x_2|\bar{z}_2)_{\mathfrak{p}^+}} \rangle_{\lambda, x, \mathfrak{p}^+} = \langle f(x_2), e^{(x_2|\bar{z}_2)_{\mathfrak{p}^+}} \rangle_{\lambda, x_2, \mathfrak{p}^+(e)_2}, \quad (3.2)$$

we check that the natural inclusion $\mathcal{H}_\lambda(D')_{\tilde{K}(e)_2} = \mathcal{P}(\mathfrak{p}^+(e)_2) \hookrightarrow \mathcal{H}_\lambda(D)_{\tilde{K}} = \mathcal{P}(\mathfrak{p}^+)$ intertwines the $(\mathfrak{g}(e)_2, \tilde{K}(e)_2)$ -action, where $\mathfrak{g}(e)_2, K(e)_2$ are as in the last paragraph of Section 2.2. For example, by (2.7), the action of \mathfrak{p}^- on $\mathcal{H}_\lambda(D)_{\tilde{K}} = \mathcal{P}(\mathfrak{p}^+)$ is given by

$$(d\tau_\lambda(0, 0, w)f)(x) = \lambda(x|w)_{\mathfrak{p}^+} f(x) + \left. \frac{d}{dt} \right|_{t=0} f(x + tQ(x)w), \quad w \in \mathfrak{p}^-.$$

If $w = w_2 \in \mathfrak{p}^-(e)_2$, then we have

$$\begin{aligned} (x|w_2)_{\mathfrak{p}^+} &= (x_2|w_2)_{\mathfrak{p}^+}, \\ x + tQ(x)w_2 &= (x_2 + tQ(x_2)w_2) + (x_1 + tQ(x_2, x_1)w_2) + (x_0 + tQ(x_1)w_2) \\ &\in \mathfrak{p}^+(e)_2 \oplus \mathfrak{p}^+(e)_1 \oplus \mathfrak{p}^+(e)_0, \end{aligned}$$

and hence if $f(x) = f(x_2) \in \mathcal{P}(\mathfrak{p}^+(e)_2)$, we have

$$(d\tau_\lambda(0, 0, w_2)f)(x) = \lambda(x_2|w_2)_{\mathfrak{p}^+} f(x_2) + \left. \frac{d}{dt} \right|_{t=0} f(x_2 + tQ(x_2)w_2).$$

Thus, the natural inclusion is $\mathfrak{p}^-(e)_2$ -equivariant. The $\tilde{K}(e)_2$ - and $\mathfrak{p}^+(e)_2$ -equivariance are also proved similarly. Hence, this is $(\mathfrak{g}(e)_2, \tilde{K}(e)_2)$ -equivariant, and therefore is an isometry up to scalar multiple. Moreover, by the normalization assumption $\|1\|_{\lambda, \mathfrak{p}^+} = \|1\|_{\lambda, \mathfrak{p}^+(e)_2} = 1$, this is exactly an isometry. Hence (3.2) holds, and this proves the proposition. \blacksquare

Now we consider an involution σ on \mathfrak{p}^+ , and let $\mathfrak{p}_1^+ := (\mathfrak{p}^+)^\sigma$, $\mathfrak{p}_2^+ := (\mathfrak{p}^+)^{-\sigma}$, $K_1 := K^\sigma$ as in Section 2.7. Then by Proposition 3.1, the computation of

$$\langle f(x_2), e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\lambda, x}, \quad f(x_2) \in \mathcal{P}_{\tilde{\mathbf{k}}}(\mathfrak{p}_2^+)$$

is reduced to the cases that both \mathfrak{p}^+ and \mathfrak{p}_2^+ are of tube type. We take a maximal tripotent $e \in \mathfrak{p}_2^+ \subset \mathfrak{p}^+$ of \mathfrak{p}_2^+ , and let $\mathfrak{p}^{+'} := \mathfrak{p}^+(e)_2$, $\mathfrak{p}_2^{+'} := \mathfrak{p}_2^+(e)_2 = \mathfrak{p}_2^+ \cap \mathfrak{p}^{+'}$, $\mathfrak{p}_1^{+'} := \mathfrak{p}_1^+ \cap \mathfrak{p}^{+'}$. If

$$\langle f(x'_2), e^{(x'|\bar{z}')_{\mathfrak{p}^{+'}}} \rangle_{\lambda, x', \mathfrak{p}^{+'}} \Big|_{z'_1=0} = C_{\mathfrak{p}^{+'}, \mathfrak{p}_2^{+'}}(\lambda, \tilde{\mathbf{k}}) f(z'_2), \quad f(x'_2) \in \mathcal{P}_{\tilde{\mathbf{k}}}(\mathfrak{p}_2^{+'})$$

holds for some $C_{\mathfrak{p}^{+'}, \mathfrak{p}_2^{+'}}(\lambda, \tilde{\mathbf{k}}) \in \mathbb{C}$, then by Proposition 3.1,

$$\langle f(x_2), e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\lambda, x, \mathfrak{p}^+} \Big|_{z_1=0} = C_{\mathfrak{p}^{+'}, \mathfrak{p}_2^{+'}}(\lambda, \tilde{\mathbf{k}}) f(z_2)$$

holds for all $f(x_2) = f(x'_2) \in \mathcal{P}_{\bar{\mathbf{k}}}(\mathfrak{p}_2^{+'}) \subset \mathcal{P}_{\bar{\mathbf{k}}}(\mathfrak{p}_2^+)$, and by the K_1 -equivariance, this also holds for all $f(x_2) \in \mathcal{P}_{\bar{\mathbf{k}}}(\mathfrak{p}_2^+)$. Similarly, let $b(\lambda) \in \mathbb{C}[\lambda]$. If for all $f(x'_2) \in \mathcal{P}_{\bar{\mathbf{k}}}(\mathfrak{p}_2^{+'})$,

$$b(\lambda) \langle f(x'_2), e^{(x'|z')} \rangle_{\lambda, x', \mathfrak{p}^{+'}}$$

is holomorphically continued for all $\lambda \in \mathbb{C}$, then for all $f(x_2) \in \mathcal{P}_{\bar{\mathbf{k}}}(\mathfrak{p}_2^+)$,

$$b(\lambda) \langle f(x_2), e^{(x|\bar{z})} \rangle_{\lambda, x, \mathfrak{p}^+}$$

is also holomorphically continued for all $\lambda \in \mathbb{C}$. That is, these computation for $(\mathfrak{p}^+, \mathfrak{p}_2^+)$ is reduced to those for $(\mathfrak{p}^{+'}, \mathfrak{p}_2^{+'})$, and both $\mathfrak{p}^{+'}, \mathfrak{p}_2^{+'}$ are of tube type, with a common maximal tripotent e .

3.2 Rodrigues-type formulas

In this subsection let \mathfrak{p}^+ be simple and of tube type. Let $r := \text{rank } \mathfrak{p}^+$, $n := \dim \mathfrak{p}^+ = r + \frac{d}{2}r(r-1)$ as in (2.4). In this case $p = \frac{2n}{r}$ holds. We fix a maximal tripotent $e \in \mathfrak{p}^+$, let $\Omega \subset \mathfrak{n}^+ \subset \mathfrak{p}^+$ be the corresponding symmetric cone and the Euclidean real form, and let $\det_{\mathfrak{n}^+}(x)$ be the determinant polynomial. We normalize the differential operator $\frac{\partial}{\partial z}$ with respect to the bilinear form $(\cdot|\cdot)_{\mathfrak{n}^+} = (\cdot|Q(\bar{\cdot}))_{\mathfrak{p}^+}$ on $\mathfrak{p}^+ = \mathfrak{n}^{+\mathbb{C}}$. Also, for $k \in \mathbb{Z}$, we write

$$\underline{k}_r := \underbrace{(k, \dots, k)}_r \in \mathbb{Z}^r,$$

and for $\lambda \in \mathbb{C}$, $\mathbf{m} \in \mathbb{Z}_{++}^r$, let $(\lambda)_{\mathbf{m}, d}$ be as in (2.11). First we recall the following theorem from the author's previous article [44].

Proposition 3.2. *Let $\text{Re } \lambda > \frac{2n}{r} - 1$, $z, a \in \Omega \subset \mathfrak{n}^+ \subset \mathfrak{p}^+$.*

(1) *For $f \in \mathcal{P}(\mathfrak{p}^+)$, we have*

$$\begin{aligned} & \langle f(x), e^{(x|\bar{z})} \rangle_{\lambda, x} \\ &= \det_{\mathfrak{n}^+}(z)^{-\lambda + \frac{n}{r}} \frac{\Gamma_r^d(\lambda)}{(2\pi\sqrt{-1})^n} \int_{a+\sqrt{-1}\mathfrak{n}^+} e^{(z|w)_{\mathfrak{n}^+}} f(w^{-1}) \det_{\mathfrak{n}^+}(w)^{-\lambda} dw, \end{aligned}$$

where $\Gamma_r^d(\lambda) := (2\pi)^{dr(r-1)/4} \prod_{j=1}^r \Gamma(\lambda - \frac{d}{2}(j-1))$. (See [44, Corollary 4.3].)

(2) *Let $f_1, f_2 \in \mathcal{P}(\mathfrak{p}^+)$. We take $k \in \mathbb{Z}_{\geq 0}$ such that*

$$f_1^{\sharp(k)}(x) := \det_{\mathfrak{n}^+}(x)^k f_1(x^{-1})$$

is a polynomial. Then we have

$$\begin{aligned} & \langle f_1(x) f_2(x), e^{(x|\bar{z})} \rangle_{\lambda, x} \\ &= \frac{1}{(\lambda)_{\underline{k}_r, d}} \det_{\mathfrak{n}^+}(z)^{-\lambda + \frac{n}{r}} f_1^{\sharp(k)} \left(\frac{\partial}{\partial z} \right) \det_{\mathfrak{n}^+}(z)^{\lambda + k - \frac{n}{r}} \langle f_2(x), e^{(x|\bar{z})} \rangle_{\lambda + k, x}. \end{aligned}$$

(See [44, Theorem 4.1].)

Next we consider an involution σ on \mathfrak{p}^+ , let $\mathfrak{p}_1^+ := (\mathfrak{p}^+)^\sigma$, $\mathfrak{p}_2^+ := (\mathfrak{p}^+)^{-\sigma}$, and assume \mathfrak{p}_2^+ is also of tube type. We take a common maximal tripotent $e \in \mathfrak{p}_2^+ \subset \mathfrak{p}^+$ of \mathfrak{p}_2^+ and \mathfrak{p}^+ , and let $\mathfrak{n}^+ \subset \mathfrak{p}^+$, $\mathfrak{n}_2^+ \subset \mathfrak{p}_2^+$ be the corresponding Euclidean real forms. When \mathfrak{p}_2^+ is not simple, then we

also write $\mathfrak{p}_1^+ =: \mathfrak{p}_{12}^+, \mathfrak{p}_2^+ =: \mathfrak{p}_{11}^+ \oplus \mathfrak{p}_{22}^+, \mathfrak{n}_2^+ =: \mathfrak{n}_{11}^+ \oplus \mathfrak{n}_{22}^+$. Then as in the proof of [44, Theorem 4.4], when $f_1(x) = \det_{\mathfrak{n}_2^+}(x_2)^k$, we have

$$(\det_{\mathfrak{n}_2^+}^k)^{\sharp(2k/\varepsilon_2)}(x) = \det_{\mathfrak{n}^+}(x)^{2k/\varepsilon_2} \det_{\mathfrak{n}_2^+}(\text{Proj}_2(x^{-1}))^k = \det_{\mathfrak{n}_2^+}(x_2)^k,$$

where ε_2 is as in (2.15), and $\text{Proj}_2: \mathfrak{p}^+ \rightarrow \mathfrak{p}_2^+$ denotes the orthogonal projection. Similarly, when $\mathfrak{p}_2^+ = \mathfrak{p}_{11}^+ \oplus \mathfrak{p}_{22}^+$ and $f_1(x) = \det_{\mathfrak{n}_{11}^+}(x_{11})^k$ or $f_1(x) = \det_{\mathfrak{n}_{22}^+}(x_{22})^k$, we have

$$(\det_{\mathfrak{n}_{11}^+}^k)^{\sharp(k)}(x) = \det_{\mathfrak{n}^+}(x)^k \det_{\mathfrak{n}_{11}^+}(\text{Proj}_{11}(x^{-1}))^k = \det_{\mathfrak{n}_{22}^+}(x_{22})^k,$$

$$(\det_{\mathfrak{n}_{22}^+}^k)^{\sharp(k)}(x) = \det_{\mathfrak{n}^+}(x)^k \det_{\mathfrak{n}_{22}^+}(\text{Proj}_{22}(x^{-1}))^k = \det_{\mathfrak{n}_{11}^+}(x_{11})^k,$$

where $\text{Proj}_{jj}: \mathfrak{p}^+ \rightarrow \mathfrak{p}_{jj}^+$ denotes the orthogonal projection. Thus the following holds. Here we normalize the differential operator $\frac{\partial}{\partial z_2}$ on \mathfrak{p}_2^+ with respect to the bilinear form $(\cdot|\cdot)_{\mathfrak{n}_2^+} = \varepsilon_2^{-1}(\cdot|\cdot)_{\mathfrak{n}^+}$ on $\mathfrak{p}_2^+ = \mathfrak{n}_2^{+\mathbb{C}}$.

Theorem 3.3.

(1) Let $\text{Re } \lambda > \frac{2n}{r} - 1$ and let $f \in \mathcal{P}(\mathfrak{p}^+)$. Then for $z = z_1 + z_2 \in \Omega \subset \mathfrak{n}^+ \subset \mathfrak{p}^+$, we have

$$\begin{aligned} & \langle \det_{\mathfrak{n}_2^+}(x_2)^k f(x), e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\lambda, x} \\ &= \frac{1}{(\lambda)_{\underline{2k/\varepsilon_2}, d}} \det_{\mathfrak{n}^+}(z)^{-\lambda + \frac{n}{r}} \det_{\mathfrak{n}_2^+} \left(\frac{1}{\varepsilon_2} \frac{\partial}{\partial z_2} \right)^k \det_{\mathfrak{n}^+}(z)^{\lambda + \frac{2k}{\varepsilon_2} - \frac{n}{r}} \langle f(x), e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\lambda + \frac{2k}{\varepsilon_2}, x}. \end{aligned}$$

(2) Suppose $\mathfrak{p}_2^+ = \mathfrak{p}_{11}^+ \oplus \mathfrak{p}_{22}^+$, let $\text{Re } \lambda > \frac{2n}{r} - 1$, $k \in \mathbb{Z}_{\geq 0}$ and let $f \in \mathcal{P}(\mathfrak{p}^+)$. Then for $z = z_{11} + z_{12} + z_{22} \in \Omega \subset \mathfrak{n}^+ \subset \mathfrak{p}^+$, we have

$$\begin{aligned} & \langle \det_{\mathfrak{n}_{11}^+}(x_{11})^k f(x), e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\lambda, x} \\ &= \frac{1}{(\lambda)_{\underline{k}, d}} \det_{\mathfrak{n}^+}(z)^{-\lambda + \frac{n}{r}} \det_{\mathfrak{n}_{22}^+} \left(\frac{\partial}{\partial z_{22}} \right)^k \det_{\mathfrak{n}^+}(z)^{\lambda + k - \frac{n}{r}} \langle f(x), e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\lambda + k, x}, \\ & \langle \det_{\mathfrak{n}_{22}^+}(x_{22})^k f(x), e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\lambda, x} \\ &= \frac{1}{(\lambda)_{\underline{k}, d}} \det_{\mathfrak{n}^+}(z)^{-\lambda + \frac{n}{r}} \det_{\mathfrak{n}_{11}^+} \left(\frac{\partial}{\partial z_{11}} \right)^k \det_{\mathfrak{n}^+}(z)^{\lambda + k - \frac{n}{r}} \langle f(x), e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\lambda + k, x}. \end{aligned}$$

Similarly, we consider the outer tensor product $\mathcal{H}_\lambda(D) \hat{\boxtimes} \mathcal{H}_\mu(D)$ of Hilbert spaces, and let $\langle \cdot, \cdot \rangle_{\lambda \otimes \mu}$ denote its inner product. Then the following holds.

Theorem 3.4. Let $\text{Re } \lambda, \text{Re } \mu > \frac{2n}{r} - 1$ and let $f(x, y) \in \mathcal{P}(\mathfrak{p}^+ \oplus \mathfrak{p}^+)$. Then for $z, -w \in \Omega \subset \mathfrak{n}^+ \subset \mathfrak{p}^+$, we have

$$\begin{aligned} & \langle f(x, y) \det_{\mathfrak{n}^+}(x - y)^k, e^{(x|\bar{z})_{\mathfrak{p}^+} + (y|\bar{w})_{\mathfrak{p}^+}} \rangle_{\lambda \otimes \mu, (x, y)} \\ &= \frac{1}{(\lambda)_{\underline{k}, d} (\mu)_{\underline{k}, d}} \det_{\mathfrak{n}^+}(z)^{-\lambda + \frac{n}{r}} \det_{\mathfrak{n}^+}(-w)^{-\mu + \frac{n}{r}} \det_{\mathfrak{n}^+} \left(\frac{\partial}{\partial z} - \frac{\partial}{\partial w} \right)^k \\ & \quad \times \det_{\mathfrak{n}^+}(z)^{\lambda + k - \frac{n}{r}} \det_{\mathfrak{n}^+}(-w)^{\mu + k - \frac{n}{r}} \langle f(x, y), e^{(x|\bar{z})_{\mathfrak{p}^+} + (y|\bar{w})_{\mathfrak{p}^+}} \rangle_{(\lambda + k) \otimes (\mu + k), (x, y)}, \\ & \langle f(x, y) \det_{\mathfrak{n}^+}(x - y)^k, e^{(x - y|\bar{z})_{\mathfrak{p}^+}} \rangle_{\lambda \otimes \mu, (x, y)} \\ &= \frac{\det_{\mathfrak{n}^+}(z)^{-\lambda - \mu + \frac{2n}{r}}}{(\lambda)_{\underline{k}, d} (\mu)_{\underline{k}, d}} \det_{\mathfrak{n}^+} \left(\frac{\partial}{\partial z} \right)^k \det_{\mathfrak{n}^+}(z)^{\lambda + \mu + 2k - \frac{2n}{r}} \langle f(x, y), e^{(x - y|\bar{z})_{\mathfrak{p}^+}} \rangle_{(\lambda + k) \otimes (\mu + k), (x, y)}. \end{aligned}$$

Proof. By Proposition 3.2 (1), for $a, z, -w \in \Omega$, we have

$$\begin{aligned} & \langle f(x, y) \det_{\mathfrak{n}^+}(x - y)^k, e^{(x|\bar{z})_{\mathfrak{p}^+} + (y|\bar{w})_{\mathfrak{p}^+}} \rangle_{\lambda \otimes \mu, (x, y)} \\ &= \langle f(x, -y) \det_{\mathfrak{n}^+}(x + y)^k, e^{(x|\bar{z})_{\mathfrak{p}^+} - (y|\bar{w})_{\mathfrak{p}^+}} \rangle_{\lambda \otimes \mu, (x, y)} \\ &= \det_{\mathfrak{n}^+}(z)^{-\lambda + \frac{n}{r}} \det_{\mathfrak{n}^+}(-w)^{-\mu + \frac{n}{r}} \frac{\Gamma_r^d(\lambda) \Gamma_r^d(\mu)}{(2\pi\sqrt{-1})^{2n}} \iint_{(a+\sqrt{-1}\mathfrak{n}^+)^2} e^{(x|z)_{\mathfrak{n}^+} - (y|w)_{\mathfrak{n}^+}} \\ & \quad \times f(x^{-1}, -y^{-1}) \det_{\mathfrak{n}^+}(x^{-1} + y^{-1})^k \det_{\mathfrak{n}^+}(x)^{-\lambda} \det_{\mathfrak{n}^+}(y)^{-\mu} dx dy, \end{aligned}$$

and since

$$\det_{\mathfrak{n}^+}(x^{-1} + y^{-1}) = \det_{\mathfrak{n}^+}(x + y) \det_{\mathfrak{n}^+}(x)^{-1} \det_{\mathfrak{n}^+}(y)^{-1}$$

holds (see [13, Lemma X.4.4 (i)]), by putting $w = -z$, we get

$$\begin{aligned} & \langle f(x, y) \det_{\mathfrak{n}^+}(x - y)^k, e^{(x-y|\bar{z})_{\mathfrak{p}^+}} \rangle_{\lambda \otimes \mu, (x, y)} \\ &= \det_{\mathfrak{n}^+}(z)^{-\lambda - \mu + \frac{2n}{r}} \frac{\Gamma_r^d(\lambda) \Gamma_r^d(\mu)}{(2\pi\sqrt{-1})^{2n}} \iint_{(a+\sqrt{-1}\mathfrak{n}^+)^2} e^{(x+y|z)_{\mathfrak{n}^+}} \\ & \quad \times f(x^{-1}, -y^{-1}) \det_{\mathfrak{n}^+}(x + y)^k \det_{\mathfrak{n}^+}(x)^{-\lambda - k} \det_{\mathfrak{n}^+}(y)^{-\mu - k} dx dy \\ &= \det_{\mathfrak{n}^+}(z)^{-\lambda - \mu + \frac{2n}{r}} \det_{\mathfrak{n}^+} \left(\frac{\partial}{\partial z} \right)^k \frac{\Gamma_r^d(\lambda) \Gamma_r^d(\mu)}{(2\pi\sqrt{-1})^{2n}} \iint_{(a+\sqrt{-1}\mathfrak{n}^+)^2} e^{(x+y|z)_{\mathfrak{n}^+}} \\ & \quad \times f(x^{-1}, -y^{-1}) \det_{\mathfrak{n}^+}(x)^{-\lambda - k} \det_{\mathfrak{n}^+}(y)^{-\mu - k} dx dy \\ &= \det_{\mathfrak{n}^+}(z)^{-\lambda - \mu + \frac{2n}{r}} \det_{\mathfrak{n}^+} \left(\frac{\partial}{\partial z} \right)^k \frac{\Gamma_r^d(\lambda) \Gamma_r^d(\mu)}{\Gamma_r^d(\lambda + k) \Gamma_r^d(\mu + k)} \\ & \quad \times \det_{\mathfrak{n}^+}(z)^{(\lambda+k) + (\mu+k) - \frac{2n}{r}} \langle f(x, -y), e^{(x+y|\bar{z})_{\mathfrak{p}^+}} \rangle_{(\lambda+k) \otimes (\mu+k), (x, y)} \\ &= \frac{\det_{\mathfrak{n}^+}(z)^{-\lambda - \mu + \frac{2n}{r}}}{(\lambda)_{\underline{k}_r, d} (\mu)_{\underline{k}_r, d}} \det_{\mathfrak{n}^+} \left(\frac{\partial}{\partial z} \right)^k \det_{\mathfrak{n}^+}(z)^{\lambda + \mu + 2k - \frac{2n}{r}} \langle f(x, y), e^{(x-y|\bar{z})_{\mathfrak{p}^+}} \rangle_{(\lambda+k) \otimes (\mu+k), (x, y)}. \end{aligned}$$

This proves the 2nd formula. The 1st formula is also proved similarly. \blacksquare

These theorems give analogues of the Rodrigues formulas for Jacobi polynomials. Especially, when \mathfrak{p}_2^+ is simple, let $r_2 := \text{rank } \mathfrak{p}_2^+$, fix a Jordan frame $\{e_1, \dots, e_{r_2}\} \subset \mathfrak{p}_2^+$, and for $\mathbf{k} \in \mathbb{Z}_{++}^{r_2}$, let $\Delta_{\mathbf{k}}^{\mathfrak{n}_2^+}(x_2) \in \mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+)$ be as in (2.5). Then by putting $k = k_{r_2}$, $f(x) = \Delta_{\mathbf{k} - \underline{k}_{r_2, r_2}}^{\mathfrak{n}_2^+}(x_2)$ in Theorem 3.3 (1), we have

$$\begin{aligned} & \langle \Delta_{\mathbf{k}}^{\mathfrak{n}_2^+}(x_2), e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\lambda, x} = \left\langle \det_{\mathfrak{n}_2^+}(x_2)^{k_{r_2}} \Delta_{\mathbf{k} - \underline{k}_{r_2, r_2}}^{\mathfrak{n}_2^+}(x_2), e^{(x|\bar{z})_{\mathfrak{p}^+}} \right\rangle_{\lambda, x} \quad (3.3) \\ &= \frac{\det_{\mathfrak{n}^+}(z)^{-\lambda + \frac{n}{r}}}{(\lambda)_{\underline{2k}_{r_2}/\varepsilon_2, d}} \det_{\mathfrak{n}_2^+} \left(\frac{1}{\varepsilon_2} \frac{\partial}{\partial z_2} \right)^{k_{r_2}} \det_{\mathfrak{n}^+}(z)^{\lambda + \frac{2k_{r_2}}{\varepsilon_2} - \frac{n}{r}} \left\langle \Delta_{\mathbf{k} - \underline{k}_{r_2, r_2}}^{\mathfrak{n}_2^+}(x_2), e^{(x|\bar{z})_{\mathfrak{p}^+}} \right\rangle_{\lambda + \frac{2k_{r_2}}{\varepsilon_2}, x}. \end{aligned}$$

Then $\Delta_{\mathbf{k} - \underline{k}_{r_2, r_2}}^{\mathfrak{n}_2^+}(x_2)$ depends only on the smaller algebra $\mathfrak{p}^+(e')_2$, with $e' = e_1 + \dots + e_{r_2-1}$, and by Proposition 3.1, computation of the inner product on \mathfrak{p}^+ is reduced to that on $\mathfrak{p}^+(e')_2$. Similarly, when $\mathfrak{p}_2^+ = \mathfrak{p}_{11}^+ \oplus \mathfrak{p}_{22}^+$ is not simple, let $r' := \text{rank } \mathfrak{p}_{11}^+$, $r'' := \text{rank } \mathfrak{p}_{22}^+$, fix Jordan frames in \mathfrak{p}_{11}^+ , \mathfrak{p}_{22}^+ , and for $\mathbf{k} \in \mathbb{Z}_{++}^{r'}$, $\mathbf{l} \in \mathbb{Z}_{++}^{r''}$, let $\Delta_{\mathbf{k}}^{\mathfrak{n}_{11}^+}(x_{11}) \in \mathcal{P}_{\mathbf{k}}(\mathfrak{p}_{11}^+)$, $\Delta_{\mathbf{l}}^{\mathfrak{n}_{22}^+}(x_{22}) \in \mathcal{P}_{\mathbf{l}}(\mathfrak{p}_{22}^+)$ be as in (2.5). Then

by putting $k = k_{r'}$, $f(x) = \Delta_{\mathbf{k}-\underline{k}_{r'}}^{\mathbf{n}_{11}^+}(x_{11})\Delta_{\mathbf{1}}^{\mathbf{n}_{22}^+}(x_{22})$ or $k = l_{r''}$, $f(x) = \Delta_{\mathbf{k}}^{\mathbf{n}_{11}^+}(x_{11})\Delta_{\mathbf{1}-\underline{l}_{r''}}^{\mathbf{n}_{22}^+}(x_{22})$ in Theorem 3.3 (2), we have

$$\begin{aligned} & \left\langle \Delta_{\mathbf{k}}^{\mathbf{n}_{11}^+}(x_{11})\Delta_{\mathbf{1}}^{\mathbf{n}_{22}^+}(x_{22}), e^{(x|\bar{z})_{\mathfrak{p}^+}} \right\rangle_{\lambda, x} \\ &= \frac{\det_{\mathfrak{n}^+}(z)^{-\lambda + \frac{n}{r}}}{(\lambda)_{\underline{k}_{r'}, d}} \det_{\mathfrak{n}_{22}^+} \left(\frac{\partial}{\partial z_{22}} \right)^{k_{r'}} \\ & \quad \times \det_{\mathfrak{n}^+}(z)^{\lambda + k_{r'} - \frac{n}{r}} \left\langle \Delta_{\mathbf{k}-\underline{k}_{r'}}^{\mathbf{n}_{11}^+}(x_{11})\Delta_{\mathbf{1}}^{\mathbf{n}_{22}^+}(x_{22}), e^{(x|\bar{z})_{\mathfrak{p}^+}} \right\rangle_{\lambda + k_{r'}, x} \end{aligned} \quad (3.4)$$

$$\begin{aligned} &= \frac{\det_{\mathfrak{n}^+}(z)^{-\lambda + \frac{n}{r}}}{(\lambda)_{\underline{l}_{r''}, d}} \det_{\mathfrak{n}_{11}^+} \left(\frac{\partial}{\partial z_{11}} \right)^{l_{r''}} \\ & \quad \times \det_{\mathfrak{n}^+}(z)^{\lambda + l_{r''} - \frac{n}{r}} \left\langle \Delta_{\mathbf{k}}^{\mathbf{n}_{11}^+}(x_{11})\Delta_{\mathbf{1}-\underline{l}_{r''}}^{\mathbf{n}_{22}^+}(x_{22}), e^{(x|\bar{z})_{\mathfrak{p}^+}} \right\rangle_{\lambda + l_{r''}, x}, \end{aligned} \quad (3.5)$$

and again by Proposition 3.1, computation of the inner product on \mathfrak{p}^+ is reduced to that on a smaller algebra of rank $r' + r'' - 1$. Similarly, for $\mathfrak{p}^+ \oplus \mathfrak{p}^+$ case, for $\mathbf{k} \in \mathbb{Z}_{++}^r$, by putting $k = k_r$, $f(x, y) = \Delta_{\mathbf{k}-\underline{k}_r}^{\mathbf{n}^+}(x - y)$ in Theorem 3.4, we have

$$\begin{aligned} & \left\langle \Delta_{\mathbf{k}}^{\mathbf{n}^+}(x - y), e^{(x-y|\bar{z})_{\mathfrak{p}^+}} \right\rangle_{\lambda \otimes \mu, (x, y)} \\ &= \frac{\det_{\mathfrak{n}^+}(z)^{-\lambda - \mu + \frac{2n}{r}}}{(\lambda)_{\underline{k}_r, d}(\mu)_{\underline{k}_r, d}} \det_{\mathfrak{n}^+} \left(\frac{\partial}{\partial z} \right)^{k_r} \det_{\mathfrak{n}^+}(z)^{\lambda + \mu + 2k_r - \frac{2n}{r}} \left\langle \Delta_{\mathbf{k}-\underline{k}_r}^{\mathbf{n}^+}(x - y), e^{(x-y|\bar{z})_{\mathfrak{p}^+}} \right\rangle_{(\lambda + k_r) \otimes (\mu + k_r), (x, y)}, \end{aligned} \quad (3.6)$$

and again by Proposition 3.1, computation of the inner product on \mathfrak{p}^+ is reduced to that on a smaller algebra of rank $r - 1$. For $\mathbf{k} = \underline{k}_r$ case, see also [6, 7].

In addition, by Proposition 2.8 (2) and Theorem 3.3, we easily get the following, which is an analogue of [35, Corollary 12.8 (3)]. We note that $\det_{\mathfrak{n}_2^+}(x_2) = \det_{\mathfrak{n}_{11}^+}(x_{11})\det_{\mathfrak{n}_{22}^+}(x_{22})$ holds, when \mathfrak{p}_2^+ is not simple.

Theorem 3.5. *We fix $\tilde{\mathbf{k}} \in \mathbb{Z}_{++}^{r^2}$ (when \mathfrak{p}_2^+ is simple) or $\tilde{\mathbf{k}} \in \mathbb{Z}_{++}^{r'} \times \mathbb{Z}_{++}^{r''}$ (when \mathfrak{p}_2^+ is not simple), and $a \in \mathbb{Z}_{>0}$. Then $(\lambda)_{\underline{2a}, d} \langle \det_{\mathfrak{n}_2^+}(x_2)^{\varepsilon_2 a} f(x), e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\lambda, x}$ is holomorphically continued for $\operatorname{Re} \lambda > \frac{n}{r} - 2a - 1$, and there exist $C_{\tilde{\mathbf{k}}, a}^{\mathbf{n}^+, \mathfrak{n}_2^+} \in \mathbb{C}$ such that*

$$(\lambda)_{\underline{2a}, d} \langle \det_{\mathfrak{n}_2^+}(x_2)^{\varepsilon_2 a} f(x_2), e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\lambda, x} \Big|_{\lambda = \frac{n}{r} - a} = C_{\tilde{\mathbf{k}}, a}^{\mathbf{n}^+, \mathfrak{n}_2^+} \det_{\mathfrak{n}^+}(z)^a \langle f(x_2), e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\frac{n}{r} + a, x}$$

holds for all $f(x_2) \in \mathcal{P}_{\tilde{\mathbf{k}}}(\mathfrak{p}_2^+)$.

Proof. Since $\langle \cdot, \cdot \rangle_{\lambda}$ is holomorphically continued for $\operatorname{Re} \lambda > \frac{n}{r} - 1 = \frac{d}{2}(r - 1)$, by Theorem 3.3,

$$\begin{aligned} & \det_{\mathfrak{n}^+}(z)^{\lambda - \frac{n}{r}} (\lambda)_{\underline{2a}, d} \langle \det_{\mathfrak{n}_2^+}(x_2)^{\varepsilon_2 a} f(x_2), e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\lambda, x} \\ &= \det_{\mathfrak{n}_2^+} \left(\frac{1}{\varepsilon_2} \frac{\partial}{\partial z_2} \right)^{\varepsilon_2 a} \det_{\mathfrak{n}^+}(z)^{\lambda + 2a - \frac{n}{r}} \langle f(x_2), e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\lambda + 2a, x} \end{aligned}$$

is holomorphically continued for $\operatorname{Re} \lambda + 2a > \frac{n}{r} - 1$, and when $\lambda = \frac{n}{r} - a$,

$$\begin{aligned} & \det_{\mathfrak{n}^+}(z)^{-a} (\lambda)_{\underline{2a}, d} \langle \det_{\mathfrak{n}_2^+}(x_2)^{\varepsilon_2 a} f(x_2), e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\lambda, x} \Big|_{\lambda = \frac{n}{r} - a} \\ &= \det_{\mathfrak{n}_2^+} \left(\frac{1}{\varepsilon_2} \frac{\partial}{\partial z_2} \right)^{\varepsilon_2 a} \det_{\mathfrak{n}^+}(z)^a \langle f(x_2), e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\frac{n}{r} + a, x} \end{aligned}$$

becomes a polynomial. Hence the theorem follows from Proposition 2.8 (2). \blacksquare

3.3 Tensor product of finite-dimensional representations of $U(s)$

In this subsection we treat some results on finite-dimensional representations of $U(s)$ used later, which easily follows from the Littlewood–Richardson rule. As in Section 2.6, for $\mathbf{k} \in \mathbb{Z}_{++}^s$, let $V_{\mathbf{k}}^{(s)}$ be the irreducible representation of $U(s)$ with the highest weight \mathbf{k} , and we write $\mathbf{k}^\vee := (k_s, \dots, k_1)$.

Lemma 3.6. *Let $\mathbf{k}, \mathbf{l}, \mathbf{n} \in \mathbb{Z}_{++}^s$. If $\text{Hom}_{U(s)}(V_{\mathbf{n}}^{(s)}, V_{\mathbf{k}}^{(s)} \otimes V_{\mathbf{l}}^{(s)}) \neq \{0\}$, then we have*

- (1) $n_{s-i-j} \geq k_{s-i} + l_{s-j}$ for $0 \leq i, j < s, i+j < s$.
- (2) $n_{i+j-1} \leq k_i + l_j$ for $1 \leq i, j \leq s, i+j \leq s+1$.

Proof. (1) Suppose $\text{Hom}_{U(s)}(V_{\mathbf{n}}^{(s)}, V_{\mathbf{k}}^{(s)} \otimes V_{\mathbf{l}}^{(s)}) \neq \{0\}$. Then by the Littlewood–Richardson rule, there exists a skew semistandard tableau Y of shape \mathbf{n}/\mathbf{k} , weight \mathbf{l} with the lattice word condition. Let $Y(a, b)$ ($1 \leq a \leq s, k_a + 1 \leq b \leq n_a$) be the (a, b) -entry of Y , so that $\#\{(a, b) \mid Y(a, b) = j\} = l_j$ holds. Then by the lattice word condition, we have

$$\begin{aligned} n_{s-i-j} - k_{s-i} &\geq \#\{(a, b) \mid Y(a, b) = s-i-j, a \leq s-i\} \\ &\geq \#\{(a, b) \mid Y(a, b) = s-i-j+1, a \leq s-i+1\} \\ &\geq \#\{(a, b) \mid Y(a, b) = s-i-j+2, a \leq s-i+2\} \geq \dots \\ &\geq \#\{(a, b) \mid Y(a, b) = s-j, a \leq s\} = l_{s-j}. \end{aligned}$$

(2) Since $\text{Hom}_{U(s)}(V_{\mathbf{n}}^{(s)}, V_{\mathbf{k}}^{(s)} \otimes V_{\mathbf{l}}^{(s)}) \simeq \text{Hom}_{U(s)}(V_{\underline{k_1+l_1-s-\mathbf{n}^\vee}}^{(s)}, V_{\underline{k_1-s-\mathbf{k}^\vee}}^{(s)} \otimes V_{\underline{l_1-s-\mathbf{l}^\vee}}^{(s)})$ holds, by (1) we have $k_1 + l_1 - n_{i+j+1} \geq (k_1 - k_{i+1}) + (l_1 - l_{j+1})$, that is, $n_{i+j+1} \leq k_{i+1} + l_{j+1}$ holds. \blacksquare

Using this lemma, we give necessary conditions on $(\tilde{\mathbf{k}}, \mathbf{m})$ for existence of non-zero K_1 -homomorphisms from $\mathcal{P}_{\tilde{\mathbf{k}}}(\mathfrak{p}_2^+)$ to $\mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+)$, when $(\mathfrak{p}^+, \mathfrak{p}_2^+)$ are of tube type and K_1 is of type A. First we consider the cases $(\mathfrak{p}^+, \mathfrak{p}_2^+) = (M(s, \mathbb{C}), \text{Sym}(s, \mathbb{C})), (M(2s, \mathbb{C}), \text{Alt}(2s, \mathbb{C}))$. Then for $(\mathbf{m}, \mathbf{n}) \in \mathbb{Z}_{++}^s \times \mathbb{Z}_{++}^s$ or $\mathbb{Z}_{++}^{2s} \times \mathbb{Z}_{++}^s$, by (2.14), we have

$$\begin{aligned} \text{Hom}_{U(s)}(\mathcal{P}_{\mathbf{n}}(\text{Sym}(s, \mathbb{C})), \mathcal{P}_{\mathbf{m}}(M(s, \mathbb{C}))) &\simeq \text{Hom}_{U(s)}(V_{2\mathbf{n}}^{(s)}, V_{\mathbf{m}}^{(s)} \otimes V_{\mathbf{m}}^{(s)}), \\ \text{Hom}_{U(2s)}(\mathcal{P}_{\mathbf{n}}(\text{Alt}(2s, \mathbb{C})), \mathcal{P}_{\mathbf{m}}(M(2s, \mathbb{C}))) &\simeq \text{Hom}_{U(2s)}(V_{\mathbf{n}^2}^{(2s)}, V_{\mathbf{m}}^{(2s)} \otimes V_{\mathbf{m}}^{(2s)}). \end{aligned}$$

Hence by Lemma 3.6 (1),

$$\begin{aligned} \text{Hom}_{U(s)}(\mathcal{P}_{\mathbf{n}}(\text{Sym}(s, \mathbb{C})), \mathcal{P}_{\mathbf{m}}(M(s, \mathbb{C}))) &\neq \{0\} \\ \text{implies } m_{s-i} + m_{s-j} &\leq 2n_{s-i-j}, \end{aligned} \tag{3.7}$$

$$\begin{aligned} \text{Hom}_{U(2s)}(\mathcal{P}_{\mathbf{n}}(\text{Alt}(2s, \mathbb{C})), \mathcal{P}_{\mathbf{m}}(M(2s, \mathbb{C}))) &\neq \{0\} \\ \text{implies } m_{2s-i} + m_{2s-j} &\leq n_{\lceil (2s-i-j)/2 \rceil}. \end{aligned} \tag{3.8}$$

Next we consider the cases $(\mathfrak{p}^+, \mathfrak{p}_2^+) = (\text{Sym}(2s, \mathbb{C}), M(s, \mathbb{C})), (\text{Alt}(2s, \mathbb{C}), M(s, \mathbb{C}))$. In general, for $\mathbf{k}, \mathbf{l}, \mathbf{m} \in \mathbb{Z}_{++}^s$, if $s = s' + s'', k_{s'+1} = \dots = k_s = 0$ and $l_{s''+1} = \dots = l_s = 0$, then by [15, Theorem 9.2.3], we have

$$\dim \text{Hom}_{U(s)}(V_{\mathbf{m}}^{(s)}, V_{\mathbf{k}}^{(s)} \otimes V_{\mathbf{l}}^{(s)}) = \dim \text{Hom}_{U(s') \times U(s'')}(V_{\mathbf{k}}^{(s')} \boxtimes V_{\mathbf{l}}^{(s'')}, V_{\mathbf{m}}^{(s)}).$$

Especially, for $(\mathbf{m}, \mathbf{k}) \in \mathbb{Z}_{++}^{2s} \times \mathbb{Z}_{++}^s$ or $\mathbb{Z}_{++}^s \times \mathbb{Z}_{++}^s$, since

$$\begin{aligned} \text{Hom}_{U(s) \times U(s)}(\mathcal{P}_{\mathbf{k}}(M(s, \mathbb{C})), \mathcal{P}_{\mathbf{m}}(\text{Sym}(2s, \mathbb{C}))) &\simeq \text{Hom}_{U(s) \times U(s)}(V_{\mathbf{k}}^{(s)} \boxtimes V_{\mathbf{k}}^{(s)}, V_{2\mathbf{m}}^{(2s)}), \\ \text{Hom}_{U(s) \times U(s)}(\mathcal{P}_{\mathbf{k}}(M(s, \mathbb{C})), \mathcal{P}_{\mathbf{m}}(\text{Alt}(2s, \mathbb{C}))) &\simeq \text{Hom}_{U(s) \times U(s)}(V_{\mathbf{k}}^{(s)} \boxtimes V_{\mathbf{k}}^{(s)}, V_{\mathbf{m}^2}^{(2s)}) \end{aligned}$$

hold by (2.14), by Lemma 3.6 (2),

$$\mathrm{Hom}_{U(s) \times U(s)}(\mathcal{P}_{\mathbf{k}}(M(s, \mathbb{C})), \mathcal{P}_{\mathbf{m}}(\mathrm{Sym}(2s, \mathbb{C}))) \neq \{0\} \quad \text{implies } 2m_{i+j-1} \leq k_i + k_j, \quad (3.9)$$

$$\mathrm{Hom}_{U(s) \times U(s)}(\mathcal{P}_{\mathbf{k}}(M(s, \mathbb{C})), \mathcal{P}_{\mathbf{m}}(\mathrm{Alt}(2s, \mathbb{C}))) \neq \{0\} \quad \text{implies } m_{\lfloor (i+j)/2 \rfloor} \leq k_i + k_j \quad (3.10)$$

for $i + j \leq 2s + 1$, where we set $k_{s+1} = \dots = k_{2s} = 0$. Similarly, for the cases

$$(\mathfrak{p}^+, \mathfrak{p}_{11}^+, \mathfrak{p}_{22}^+, K_1) = \begin{cases} (\mathrm{Sym}(r, \mathbb{C}), \mathrm{Sym}(r', \mathbb{C}), \mathrm{Sym}(r'', \mathbb{C}), U(r') \times U(r'')), \\ (M(r, \mathbb{C}), M(r', \mathbb{C}), M(r'', \mathbb{C}), U(r') \times U(r') \times U(r'') \times U(r'')), \\ (\mathrm{Alt}(2r, \mathbb{C}), \mathrm{Alt}(2r', \mathbb{C}), \mathrm{Alt}(2r'', \mathbb{C}), U(2r') \times U(2r'')), \end{cases}$$

by Lemma 3.6 (2), for $\mathbf{k} \in \mathbb{Z}_{++}^{r'}$, $\mathbf{l} \in \mathbb{Z}_{++}^{r''}$, $\mathbf{m} \in \mathbb{Z}_{++}^r$,

$$\mathrm{Hom}_{K_1}(\mathcal{P}_{\mathbf{k}}(\mathfrak{p}_{11}^+) \boxtimes \mathcal{P}_{\mathbf{l}}(\mathfrak{p}_{22}^+), \mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+)) \neq \{0\} \quad \text{implies } m_{i+j-1} \leq k_i + l_j. \quad (3.11)$$

4 Easy case

In this section we treat $\mathfrak{p}^+ = (\mathfrak{p}^+)^{\sigma} \oplus (\mathfrak{p}^+)^{-\sigma} = \mathfrak{p}_1^+ \oplus \mathfrak{p}_2^+$ such that $\mathfrak{p}_2^+(e)_2 = \mathfrak{p}^+(e)_2$ holds for some (or equivalently any) maximal tripotent $e \in \mathfrak{p}_2^+$, where $\mathfrak{p}^+(e)_2 \subset \mathfrak{p}^+$, $\mathfrak{p}_2^+(e)_2 \subset \mathfrak{p}_2^+$ are as in (2.2). This section contains some overlap with [44, Sections 6.1 and 7], but for the sake of completeness, we give the results for this case. Such $(\mathfrak{p}^+, \mathfrak{p}_1^+, \mathfrak{p}_2^+)$ are one of

$$(\mathfrak{p}^+, \mathfrak{p}_1^+, \mathfrak{p}_2^+) = \begin{cases} (M(q, s; \mathbb{C}), M(q, s'; \mathbb{C}), M(q, s''; \mathbb{C})) & \text{(Case 1),} \\ (\mathrm{Alt}(s, \mathbb{C}), \mathrm{Alt}(s-1, \mathbb{C}), M(s-1, 1; \mathbb{C})) & \text{(Case 2),} \\ (\mathrm{Alt}(s, \mathbb{C}), M(s-1, 1; \mathbb{C}), \mathrm{Alt}(s-1, \mathbb{C})) & \text{(Case 3),} \\ (\mathbb{C}^{2s}, ((1, \sqrt{-1})\mathbb{C})^s, ((1, -\sqrt{-1})\mathbb{C})^s) & \\ \simeq (\mathbb{C}^{2s}, M(1, s; \mathbb{C}), M(1, s; \mathbb{C})) & \text{(Case 4),} \\ (M(1, 2; \mathbb{O})^{\mathbb{C}}, \mathbb{O}^{\mathbb{C}}, \mathbb{O}^{\mathbb{C}}) \simeq (M(1, 2; \mathbb{O})^{\mathbb{C}}, \mathbb{C}^8, \mathbb{C}^8) & \text{(Case 5)} \end{cases}$$

($s' + s'' = s$ for Case 1). Then the corresponding symmetric pairs are

$$(G, G_1) = \begin{cases} (\mathrm{SU}(q, s), \mathrm{S}(U(q, s') \times U(s''))) & \text{(Case 1),} \\ (\mathrm{SO}^*(2s), \mathrm{SO}^*(2(s-1)) \times \mathrm{SO}(2)) & \text{(Case 2),} \\ (\mathrm{SO}^*(2s), U(s-1, 1)) & \text{(Case 3),} \\ (\mathrm{SO}_0(2, 2s), U(1, s)) & \text{(Case 4),} \\ (E_{6(-14)}, U(1) \times \mathrm{Spin}_0(2, 8)) & \text{(Case 5).} \end{cases}$$

Let $\mathrm{rank} \mathfrak{p}^+ =: r$, $\mathrm{rank} \mathfrak{p}_2^+ =: r_2$, and let d be the number defined in (2.4), so that

$$(r, r_2, d) = \begin{cases} (\min\{q, s\}, \min\{q, s''\}, 2) & \text{(Case 1),} \\ (\lfloor s/2 \rfloor, 1, 4) & \text{(Case 2),} \\ (\lfloor s/2 \rfloor, \lfloor (s-1)/2 \rfloor, 4) & \text{(Case 3),} \\ (2, 1, 2s-2) & \text{(Case 4),} \\ (2, 2, 6) & \text{(Case 5).} \end{cases}$$

For these cases, for $\mathbf{k} \in \mathbb{Z}_{++}^{r_2}$, since $\mathcal{P}_{\mathbf{k}}(\mathfrak{p}^+)$ and $\mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+)$ are generated by $\mathcal{P}_{\mathbf{k}}(\mathfrak{p}^+(e)_2) = \mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+(e)_2)$ as $K^{\mathbb{C}}$ - and $K_1^{\mathbb{C}}$ -modules respectively, we have $\mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+) \subset \mathcal{P}_{\mathbf{k}}(\mathfrak{p}^+)$, and hence by Corollary 2.4 and Theorem 2.3, the following holds. Here $(\lambda)_{\mathbf{k}, d}$ is as in (2.11).

Theorem 4.1. *Let $\operatorname{Re} \lambda > p - 1$, $\mathbf{k} \in \mathbb{Z}_{++}^{r_2}$, and let $f(x_2) \in \mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+)$. Then we have*

$$\langle f(x_2), e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\lambda, x} = \frac{1}{(\lambda)_{\mathbf{k}, d}} f(z_2), \quad \|f(x_2)\|_{\lambda, x}^2 = \frac{1}{(\lambda)_{\mathbf{k}, d}} \|f(z_2)\|_{F, z}^2.$$

Next we consider the decomposition of the holomorphic discrete series representation of scalar type $\mathcal{H}_\lambda(D)$ under the subgroup $\tilde{G}_1 \subset \tilde{G}$. By Theorem 2.5, we have

$$\mathcal{H}_\lambda(D)|_{\tilde{G}_1} \simeq \sum_{\mathbf{k} \in \mathbb{Z}_{++}^{r_2}}^{\oplus} \mathcal{H}_\lambda(D_1, \mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+)),$$

where

$$\mathcal{H}_\lambda(D_1, \mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+)) \simeq \begin{cases} \mathcal{H}_{\lambda_1 + \lambda_2}(D_{U(q, s')}, V_{\mathbf{k}}^{(q)\vee} \boxtimes \mathbb{C}) \boxtimes V_{(\lambda_2, \dots, \lambda_2) + \mathbf{k}}^{(s'')} & \text{(Case 1),} \\ \mathcal{H}_\lambda(D_{\text{SO}^*(2(s-1))}, V_{(k, 0, \dots, 0)}^{(s-1)\vee}) \boxtimes \chi_{\text{SO}(2)}^{-\lambda - 2k} & \text{(Case 2),} \\ \mathcal{H}_{\frac{\lambda}{2} + \frac{\lambda}{2}}(D_{U(s-1, 1)}, V_{\mathbf{k}^2}^{(s-1)\vee} \boxtimes \mathbb{C}) & \text{(Case 3),} \\ \mathcal{H}_{(\lambda+k)+0}(D_{U(1, s)}, \mathbb{C} \boxtimes V_{(0, \dots, 0, -k)}^{(s)}) & \text{(Case 4),} \\ \mathcal{H}_{\lambda + \frac{|\mathbf{k}|}{2}}(D_{\text{Spin}_0(2, 8)}, V_{\left(\frac{k_1+k_2}{2}, \frac{k_1-k_2}{2}, \frac{k_1-k_2}{2}, -\frac{k_1-k_2}{2}\right)}^{[8]\vee}) \boxtimes \chi_{U(1)}^{-\lambda - \frac{3}{2}|\mathbf{k}|} & \text{(Case 5),} \end{cases}$$

where we set $\lambda = \lambda_1 + \lambda_2$ for Case 1. For each $\mathbf{k} \in \mathbb{Z}_{++}^{r_2}$, let $V_{\mathbf{k}}$ be an abstract K_1 -module isomorphic to $\mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+)$, and take non-zero K_1 -invariant vector-valued polynomials $K_{\mathbf{k}}^\vee(z_2) \in \mathcal{P}(\mathfrak{p}_2^-, V_{\mathbf{k}})^{K_1}$ and $K_{\mathbf{k}}(x_2) \in \mathcal{P}(\mathfrak{p}_2^+, \text{Hom}_{\mathbb{C}}(V_{\mathbf{k}}, \mathbb{C}))^{K_1}$. Such polynomials are unique up to constant multiple. Then by [43, Section 5.1], [33, Section 5], the normal derivative and the multiplication operators

$$\begin{aligned} \mathcal{F}_{\mathbf{k}}^\downarrow: \mathcal{O}_\lambda(D)|_{\tilde{G}_1} &\longrightarrow \mathcal{O}_\lambda(D_1, V_{\mathbf{k}}), & (\mathcal{F}_{\mathbf{k}}^\downarrow f)(x_1) &:= K_{\mathbf{k}}^\vee \left(\frac{\partial}{\partial x_2} \right) f(x) \Big|_{x_2=0}, \\ \mathcal{F}_{\mathbf{k}}^\uparrow: \mathcal{O}_\lambda(D_1, V_{\mathbf{k}}) &\longrightarrow \mathcal{O}_\lambda(D)|_{\tilde{G}_1}, & (\mathcal{F}_{\mathbf{k}}^\uparrow g)(x) &:= K_{\mathbf{k}}(x_2) g(x_1) \end{aligned}$$

give the \tilde{G}_1 -intertwining operators (symmetry breaking operator, holographic operator) for all $\lambda \in \mathbb{C}$. Let $\|\cdot\|_{\lambda, \mathbf{k}}$ be the \tilde{G}_1 -invariant norm on $\mathcal{H}_\lambda(D_1, V_{\mathbf{k}})$ normalized such that $\|v\|_{\lambda, \mathbf{k}} = |v|_{V_{\mathbf{k}}}$ holds for all constant functions $v \in V_{\mathbf{k}}$. Then by Proposition 2.6 (3), the following Parseval–Plancherel-type formula holds.

Corollary 4.2. *Let $\lambda > p - 1$. For each $\mathbf{k} \in \mathbb{Z}_{++}^{r_2}$, we take a vector-valued polynomial $K_{\mathbf{k}}^\vee(z_2) \in \mathcal{P}(\mathfrak{p}_2^-, V_{\mathbf{k}})^{K_1}$ normalized such that*

$$|\langle f(x_2), \overline{K_{\mathbf{k}}^\vee(x_2)} \rangle_{F, \mathfrak{p}^+}|_{V_{\mathbf{k}}}^2 = \|f\|_{F, \mathfrak{p}^+}^2, \quad f(x_2) \in \mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+).$$

Then for $f \in \mathcal{H}_\lambda(D)$, we have

$$\|f\|_\lambda^2 = \sum_{\mathbf{k} \in \mathbb{Z}_{++}^{r_2}} \frac{1}{(\lambda)_{\mathbf{k}, d}} \left\| K_{\mathbf{k}}^\vee \left(\frac{\partial}{\partial x_2} \right) f(x) \Big|_{x_2=0} \right\|_{\lambda, \mathbf{k}, x_1}^2.$$

Also, by Proposition 2.7, we have the following.

Corollary 4.3. *Let $\mathbf{k} \in \mathbb{Z}_{++}^{r_2}$, and set $k_{r_2+1} = \dots = k_r := 0$. Then for $a = 1, 2, \dots, r$,*

$$\text{d}\tau_\lambda(\mathcal{U}(\mathfrak{g}_1))\mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+) \subset M_a^{\mathfrak{g}}(\lambda)$$

holds if and only if

$$\lambda \in \frac{d}{2}(a-1) - k_a - \mathbb{Z}_{\geq 0}.$$

Especially, for $a = 0, 1, \dots, r-1$, we have

$$\mathcal{H}_{\frac{d}{2}a}(D)|_{\tilde{G}_1} \simeq \sum_{\substack{\mathbf{k} \in \mathbb{Z}_{++}^{r_2} \\ k_{a+1}=0}}^{\oplus} \mathcal{H}_{\frac{d}{2}a}(D_1, \mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+)).$$

Here $M_a^{\mathfrak{g}}(\lambda) \subset \mathcal{O}_\lambda(D)_{\tilde{K}}$ is the $(\mathfrak{g}, \tilde{K})$ -submodule given in (2.12), so that $\mathcal{H}_{\frac{d}{2}a}(D)_{\tilde{K}} \simeq M_{a+1}^{\mathfrak{g}}(\frac{d}{2}a)$ holds for $a = 0, 1, \dots, r-1$. The decomposition of $\mathcal{H}_{\frac{d}{2}a}(D)$ for $a = 1$ is earlier given in [41], and we can also prove those for $a = 2, \dots, r-1$ for Cases 1–3 by using the seesaw dual pair theory (see, e.g., [18, Section 3], [37]) as in [41].

5 Case \mathfrak{p}_2^+ is non-simple

In this section we treat $\mathfrak{p}^+ = (\mathfrak{p}^+)^{\sigma} \oplus (\mathfrak{p}^+)^{-\sigma} = \mathfrak{p}_1^+ \oplus \mathfrak{p}_2^+$ such that \mathfrak{p}_2^+ is non-simple, and write $\mathfrak{p}_2^+ =: \mathfrak{p}_{11}^+ \oplus \mathfrak{p}_{22}^+$, $\mathfrak{p}_1^+ =: \mathfrak{p}_{12}^+$, so that

$$\begin{aligned} & (\mathfrak{p}^+, \mathfrak{p}_{11}^+, \mathfrak{p}_{12}^+, \mathfrak{p}_{22}^+) \\ &= \begin{cases} (\mathbb{C}^{d+2}, \mathbb{C}, \mathbb{C}^d, \mathbb{C}) & \text{(Case 1),} \\ (\text{Sym}(r, \mathbb{C}), \text{Sym}(r', \mathbb{C}), \text{M}(r', r''; \mathbb{C}), \text{Sym}(r'', \mathbb{C})) & \text{(Case 2),} \\ (\text{M}(q, s; \mathbb{C}), \text{M}(q', s'; \mathbb{C}), \text{M}(q', s''; \mathbb{C}) \oplus \text{M}(q'', s'; \mathbb{C}), \text{M}(q'', s''; \mathbb{C})) & \text{(Case 3),} \\ (\text{Alt}(s, \mathbb{C}), \text{Alt}(s', \mathbb{C}), \text{M}(s', s''; \mathbb{C}), \text{Alt}(s'', \mathbb{C})) & \text{(Case 4),} \\ (\text{Herm}(3, \mathbb{O})^{\mathbb{C}}, \mathbb{C}, \text{M}(1, 2; \mathbb{O})^{\mathbb{C}}, \text{Herm}(2, \mathbb{O})^{\mathbb{C}}) & \text{(Case 5),} \\ (\text{M}(1, 2; \mathbb{O})^{\mathbb{C}}, \mathbb{C}, \text{Alt}(5, \mathbb{C}), \text{M}(1, 5; \mathbb{C})) & \text{(Case 6)} \end{cases} \end{aligned}$$

with $r = r' + r''$, $q = q' + q''$, $s = s' + s''$. Then the corresponding symmetric pairs are

$$(G, G_1) = \begin{cases} (\text{SO}_0(2, d+2), \text{SO}_0(2, d) \times \text{SO}(2)) & \text{(Case 1),} \\ (\text{Sp}(r, \mathbb{R}), \text{U}(r', r'')) & \text{(Case 2),} \\ (\text{SU}(q, s), \text{S}(\text{U}(q', s'') \times \text{U}(q'', s'))) & \text{(Case 3),} \\ (\text{SO}^*(2s), \text{U}(s', s'')) & \text{(Case 4),} \\ (E_{7(-25)}, \text{U}(1) \times E_{6(-14)}) & \text{(Case 5),} \\ (E_{6(-14)}, \text{U}(1) \times \text{SO}^*(10)) & \text{(Case 6)} \end{cases}$$

(up to covering). Let $\dim \mathfrak{p}^+ =: n$, $\dim \mathfrak{p}_{11}^+ =: n'$, $\dim \mathfrak{p}_{22}^+ =: n''$, $\text{rank } \mathfrak{p}^+ =: r$, $\text{rank } \mathfrak{p}_{11}^+ =: r'$, $\text{rank } \mathfrak{p}_{22}^+ =: r''$, and let d, d', d'' be the numbers defined in (2.4) for \mathfrak{p}^+ , \mathfrak{p}_{11}^+ , \mathfrak{p}_{22}^+ , respectively. Then the numbers (r, r', r'', d) are given by

$$(r, r', r'', d) = \begin{cases} (2, 1, 1, d) & \text{(Case 1),} \\ (r, r', r'', 1) & \text{(Case 2),} \\ (\min\{q, s\}, \min\{q', s'\}, \min\{q'', s''\}, 2) & \text{(Case 3),} \\ (\lfloor s/2 \rfloor, \lfloor s'/2 \rfloor, \lfloor s''/2 \rfloor, 4) & \text{(Case 4),} \\ (3, 1, 2, 8) & \text{(Case 5),} \\ (2, 1, 1, 6) & \text{(Case 6),} \end{cases}$$

and we have $d = d' = d''$ if $r', r'' \neq 1$. Even when r' or $r'' = 1$, since d', d'' are not determined uniquely and any numbers are allowed, we may assume $d = d' = d''$.

5.1 Results on weighted Bergman inner products

First, for $f(x_{11}) \in \mathcal{P}_{\mathbf{k}}(\mathfrak{p}_{11}^+)$, $g(x_{22}) \in \mathcal{P}_1(\mathfrak{p}_{22}^+)$, we want to compute the top terms and the poles of $\langle f(x_{11})g(x_{22}), e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\lambda, x}$, where $\langle \cdot, \cdot \rangle_{\lambda} = \langle \cdot, \cdot \rangle_{\lambda, x}$ is the weighted Bergman inner product given in (2.9), with the variable of integration $x = x_{11} + x_{12} + x_{22}$.

Theorem 5.1. *Let $\operatorname{Re} \lambda > p-1$, $\mathbf{k} \in \mathbb{Z}_{++}^{r'}$, $\mathbf{l} \in \mathbb{Z}_{++}^{r''}$, and let $f(x_{11}) \in \mathcal{P}_{\mathbf{k}}(\mathfrak{p}_{11}^+)$, $g(x_{22}) \in \mathcal{P}_1(\mathfrak{p}_{22}^+)$. Then we have*

$$\langle f(x_{11})g(x_{22}), e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\lambda, x} \Big|_{z_{12}=0} = C_0^d(\lambda, \mathbf{k}, \mathbf{l})f(z_{11})g(z_{22}),$$

where

$$\begin{aligned} C_0^d(\lambda, \mathbf{k}, \mathbf{l}) &= \frac{\prod_{i=1}^{r'} \prod_{j=1}^{r''} (\lambda - \frac{d}{2}(i+j-1))_{k_i+l_j}}{\prod_{i=1}^{r'+1} \prod_{j=1}^{r''+1} (\lambda - \frac{d}{2}(i+j-2))_{k_i+l_j}} \\ &= \frac{\prod_{a=2}^r \prod_{i=\max\{1, a-r''\}}^{\min\{a-1, r'\}} (\lambda - \frac{d}{2}(a-1))_{k_i+l_{a-i}}}{\prod_{a=1}^r \prod_{i=\max\{1, a-r''\}}^{\min\{a, r'+1\}} (\lambda - \frac{d}{2}(a-1))_{k_i+l_{a+1-i}}}. \end{aligned} \quad (5.1)$$

Here we put $k_{r'+1} = l_{r''+1} := 0$.

Theorem 5.2. *For $\mathbf{k} \in \mathbb{Z}_{++}^{r'}$, $\mathbf{l} \in \mathbb{Z}_{++}^{r''}$, we define $\phi_0(\mathbf{k}, \mathbf{l}) \in \mathbb{Z}_{++}^{r'+r''}$ by*

$$\phi_0(\mathbf{k}, \mathbf{l})_a := \min\{k_i + l_j \mid 1 \leq i \leq r' + 1, 1 \leq j \leq r'' + 1, i + j = a + 1\}, \quad (5.2)$$

where $1 \leq a \leq r' + r''$, $k_{r'+1} = l_{r''+1} := 0$. Then for $f(x_{11}) \in \mathcal{P}_{\mathbf{k}}(\mathfrak{p}_{11}^+)$, $g(x_{22}) \in \mathcal{P}_1(\mathfrak{p}_{22}^+)$, as a function of λ ,

$$(\lambda)_{\phi_0(\mathbf{k}, \mathbf{l}), d} \langle f(x_{11})g(x_{22}), e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\lambda, x} \quad (5.3)$$

is holomorphically continued for all \mathbb{C} .

Remark 5.3.

- (1) By [44, Corollary 5.7 (3)], if $f(x_{11})g(x_{22}) \neq 0$ and if $\mathbf{k} \in \mathbb{Z}_{++}^{r'}$, $\mathbf{l} \in \mathbb{Z}_{++}^{r''}$ satisfy “ $\mathbf{k} = \underline{0}_{r'}$ ” or “ $\mathbf{l} = \underline{0}_{r''}$ ” or “ $\mathbf{k} = (\underline{k}_a, \underline{0}_{r'-a})$, $1 \leq a \leq r'$ and $l_{a+1} = 0$ ” or “ $\mathbf{l} = (\underline{l}_a, \underline{0}_{r''-a})$, $1 \leq a \leq r''$ and $k_{a+1} = 0$ ”, then (5.3) gives a non-zero polynomial in $z \in \mathfrak{p}^+$ for all $\lambda \in \mathbb{C}$.
- (2) The restriction of Theorem 5.2 to $z_{12} = 0$, the holomorphy of

$$(\lambda)_{\phi_0(\mathbf{k}, \mathbf{l}), d} \langle f(x_{11})g(x_{22}), e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\lambda, x} \Big|_{z_{12}=0}$$

follows from Theorem 5.1, since it is easily proved that $(\lambda)_{\phi_0(\mathbf{k}, \mathbf{l}), d} C_0^d(\lambda, \mathbf{k}, \mathbf{l})$ is holomorphic for all $\lambda \in \mathbb{C}$.

- (3) Under Corollary 2.4, Theorem 5.2 is equivalent to the inclusion

$$\mathcal{P}_{\mathbf{k}}(\mathfrak{p}_{11}^+) \mathcal{P}_1(\mathfrak{p}_{22}^+) \subset \bigoplus_{\substack{\mathbf{m} \in \mathbb{Z}_{++}^{r'+r''} \\ m_a \leq \phi_0(\mathbf{k}, \mathbf{l})_a}} \mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+). \quad (5.4)$$

When \mathfrak{p}^+ is classical, this follows from (3.11), but in the following we prove the theorem by another way, which is available also for the exceptional case.

Here $(\lambda)_{\mathbf{m}, d}$ is as in (2.11), and $\underline{k}_a := \underbrace{(k, \dots, k)}_a$.

Proof of Theorem 5.1. The 2nd equality of (5.1) is easy. For the 1st equality, by the last paragraph of Section 3.1, we may assume \mathfrak{p}^+ , \mathfrak{p}_{11}^+ , \mathfrak{p}_{22}^+ are of tube type, so that $r = r' + r''$ holds, and by the K_1 -equivariance, may assume $f(x_{11}) = \Delta_{\mathbf{k}}^{n_{11}^+}(x_{11})$, $g(x_{22}) = \Delta_{\mathbf{l}}^{n_{22}^+}(x_{22})$. We prove the theorem by induction on r' . When $r' = 0$, this follows from Corollary 2.4, where we set $\prod_{i=1}^0(\cdots) =: 1$. Next we assume the theorem for $r' - 1$, and prove it for r' . Since we have defined $k_{r'+1} = l_{r''+1} = 0$, by (3.4), we have

$$\begin{aligned}
& \langle \Delta_{\mathbf{k}}^{n_{11}^+}(x_{11}) \Delta_{\mathbf{l}}^{n_{22}^+}(x_{22}), e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\lambda, x} \Big|_{z_{12}=0} \\
&= \frac{1}{(\lambda)_{\underline{k}_{r'}, d}} \det_{\mathfrak{n}^+}(z_{11} + z_{22})^{-\lambda + \frac{n}{r}} \det_{\mathfrak{n}_{22}^+} \left(\frac{\partial}{\partial z_{22}} \right)^{k_{r'}} \det_{\mathfrak{n}^+}(z_{11} + z_{22})^{\lambda + k_{r'} - \frac{n}{r}} \\
&\quad \times \langle \Delta_{\mathbf{k} - \underline{k}_{r'}}^{n_{11}^+}(x_{11}) \Delta_{\mathbf{l}}^{n_{22}^+}(x_{22}), e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\lambda + k_{r'}, x} \Big|_{z_{12}=0} \\
&= \frac{\det_{\mathfrak{n}_{11}^+}(z_{11})^{k_{r'}}}{(\lambda)_{\underline{k}_{r'}, d}} \det_{\mathfrak{n}_{22}^+}(z_{22})^{-\lambda + \frac{d}{2}(r-1)+1} \det_{\mathfrak{n}_{22}^+} \left(\frac{\partial}{\partial z_{22}} \right)^{k_{r'}} \det_{\mathfrak{n}_{22}^+}(z_{22})^{\lambda + k_{r'} - \frac{d}{2}(r-1)-1} \\
&\quad \times \frac{\prod_{i=1}^{r'-1} \prod_{j=1}^{r''} (\lambda + k_{r'} - \frac{d}{2}(i+j-1))_{k_i - k_{r'} + l_j}}{\prod_{i=1}^{r'} \prod_{j=1}^{r''+1} (\lambda + k_{r'} - \frac{d}{2}(i+j-2))_{k_i - k_{r'} + l_j}} \Delta_{\mathbf{k} - \underline{k}_{r'}}^{n_{11}^+}(z_{11}) \Delta_{\mathbf{l}}^{n_{22}^+}(z_{22}) \\
&= \frac{1}{(\lambda)_{\underline{k}_{r'}, d}} \frac{\prod_{i=1}^{r'-1} \prod_{j=1}^{r''} (\lambda + k_{r'} - \frac{d}{2}(i+j-1))_{k_i - k_{r'} + l_j}}{\prod_{i=1}^{r'} \prod_{j=1}^{r''+1} (\lambda + k_{r'} - \frac{d}{2}(i+j-2))_{k_i - k_{r'} + l_j}} \prod_{j=1}^{r''} \left(\lambda + l_j - \frac{d}{2}(r' + j - 1) \right)_{k_{r'}} \\
&\quad \times \Delta_{\mathbf{k}}^{n_{11}^+}(z_{11}) \Delta_{\mathbf{l}}^{n_{22}^+}(z_{22}) \\
&= \frac{1}{(\lambda)_{\underline{k}_{r'}, d}} \frac{\prod_{i=1}^{r'-1} \prod_{j=1}^{r''} (\lambda - \frac{d}{2}(i+j-1))_{k_i + l_j}}{\prod_{i=1}^{r'} \prod_{j=1}^{r''+1} (\lambda - \frac{d}{2}(i+j-2))_{k_i + l_j}} \frac{\prod_{i=1}^{r'} \prod_{j=1}^{r''+1} (\lambda - \frac{d}{2}(i+j-2))_{k_{r'}}}{\prod_{i=1}^{r'-1} \prod_{j=1}^{r''} (\lambda - \frac{d}{2}(i+j-1))_{k_{r'}}} \\
&\quad \times \frac{\prod_{j=1}^{r''} (\lambda - \frac{d}{2}(r' + j - 1))_{k_{r'} + l_j}}{\prod_{j=1}^{r''} (\lambda - \frac{d}{2}(r' + j - 1))_{l_j}} \Delta_{\mathbf{k}}^{n_{11}^+}(z_{11}) \Delta_{\mathbf{l}}^{n_{22}^+}(z_{22}) \\
&= \frac{1}{(\lambda)_{\underline{k}_{r'}, d}} \frac{\prod_{i=1}^{r'-1} \prod_{j=1}^{r''} (\lambda - \frac{d}{2}(i+j-1))_{k_i + l_j}}{\prod_{i=1}^{r'} \prod_{j=1}^{r''+1} (\lambda - \frac{d}{2}(i+j-2))_{k_i + l_j}} \frac{\prod_{i=1}^{r'} \prod_{j=1}^{r''+1} (\lambda - \frac{d}{2}(i+j-2))_{k_{r'}}}{\prod_{i=2}^{r'} \prod_{j=1}^{r''} (\lambda - \frac{d}{2}(i+j-2))_{k_{r'}}} \\
&\quad \times \frac{\prod_{j=1}^{r''} (\lambda - \frac{d}{2}(r' + j - 1))_{k_{r'} + l_j}}{\prod_{j=1}^{r''+1} (\lambda - \frac{d}{2}((r' + 1) + j - 2))_{k_{r'+1} + l_j}} \Delta_{\mathbf{k}}^{n_{11}^+}(z_{11}) \Delta_{\mathbf{l}}^{n_{22}^+}(z_{22}) \\
&= \frac{1}{\prod_{j=1}^{r'} (\lambda - \frac{d}{2}(j-1))_{k_{r'}}} \frac{\prod_{i=1}^{r'} \prod_{j=1}^{r''} (\lambda - \frac{d}{2}(i+j-1))_{k_i + l_j}}{\prod_{i=1}^{r'+1} \prod_{j=1}^{r''+1} (\lambda - \frac{d}{2}(i+j-2))_{k_i + l_j}} \\
&\quad \times \prod_{j=1}^{r''} \left(\lambda - \frac{d}{2}(j-1) \right)_{k_{r'}} \prod_{i=1}^{r'} \left(\lambda - \frac{d}{2}(i+r''-1) \right)_{k_{r'}} \Delta_{\mathbf{k}}^{n_{11}^+}(z_{11}) \Delta_{\mathbf{l}}^{n_{22}^+}(z_{22}) \\
&= \frac{\prod_{i=1}^{r'} \prod_{j=1}^{r''} (\lambda - \frac{d}{2}(i+j-1))_{k_i + l_j}}{\prod_{i=1}^{r'+1} \prod_{j=1}^{r''+1} (\lambda - \frac{d}{2}(i+j-2))_{k_i + l_j}} \Delta_{\mathbf{k}}^{n_{11}^+}(z_{11}) \Delta_{\mathbf{l}}^{n_{22}^+}(z_{22}),
\end{aligned}$$

where we have used the induction hypothesis and Proposition 3.1 at the 2nd equality, and (2.6) at the 3rd equality. Therefore, the theorem holds for all r' . \blacksquare

Proof of Theorem 5.2. Again, we may assume \mathfrak{p}^+ , \mathfrak{p}_{11}^+ , \mathfrak{p}_{22}^+ are of tube type, and $f(x_{11}) = \Delta_{\mathbf{k}}^{n_{11}^+}(x_{11})$, $g(x_{22}) = \Delta_{\mathbf{l}}^{n_{22}^+}(x_{22})$. We prove the theorem by induction on (r', r'') . If $r' = 0$ or $r'' = 0$,

then this is true by Corollary 2.4. Next we assume that the theorem holds for $(r' - 1, r'')$ and $(r', r'' - 1)$, and prove for (r', r'') . First, by (3.4), we have

$$\begin{aligned} & \det_{\mathfrak{n}^+}(z)^{-\lambda + \frac{n}{r}} \det_{\mathfrak{n}_{22}^+} \left(\frac{\partial}{\partial z_{22}} \right)^{k_{r'}} \det_{\mathfrak{n}^+}(z)^{\lambda + k_{r'} - \frac{n}{r}} \\ & \quad \times (\lambda + k_{r'})_{\phi_0(\mathbf{k} - \underline{k}_{r'}, \mathbf{l}), d} \langle \Delta_{\mathbf{k} - \underline{k}_{r'}}^{\mathfrak{n}_{11}^+}(x_{11}) \Delta_{\mathbf{l}}^{\mathfrak{n}_{22}^+}(x_{22}), e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\lambda + k_{r'}, x} \\ & = (\lambda)_{\underline{k}_{r'}, d} (\lambda + k_{r'})_{\phi_0(\mathbf{k} - \underline{k}_{r'}, \mathbf{l}), d} \langle \Delta_{\mathbf{k}}^{\mathfrak{n}_{11}^+}(x_{11}) \Delta_{\mathbf{l}}^{\mathfrak{n}_{22}^+}(x_{22}), e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\lambda, x} \\ & = (\lambda)_{\phi_0(\mathbf{k} - \underline{k}_{r'}, \mathbf{l}) + \underline{k}_{r'}, d} \langle \Delta_{\mathbf{k}}^{\mathfrak{n}_{11}^+}(x_{11}) \Delta_{\mathbf{l}}^{\mathfrak{n}_{22}^+}(x_{22}), e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\lambda, x}, \end{aligned}$$

and by Proposition 3.1 and the induction hypothesis for $(r' - 1, r'')$, this is holomorphically continued for all $\lambda \in \mathbb{C}$. Similarly, by (3.5), we have

$$\begin{aligned} & \det_{\mathfrak{n}^+}(z)^{-\lambda + \frac{n}{r}} \det_{\mathfrak{n}_{11}^+} \left(\frac{\partial}{\partial z_{11}} \right)^{l_{r''}} \det_{\mathfrak{n}^+}(z)^{\lambda + l_{r''} - \frac{n}{r}} \\ & \quad \times (\lambda + l_{r''})_{\phi_0(\mathbf{k}, \mathbf{l} - \underline{l}_{r''}), d} \langle \Delta_{\mathbf{k}}^{\mathfrak{n}_{11}^+}(x_{11}) \Delta_{\mathbf{l} - \underline{l}_{r''}}^{\mathfrak{n}_{22}^+}(x_{22}), e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\lambda + l_{r''}, x} \\ & = (\lambda)_{\underline{l}_{r''}, d} (\lambda + l_{r''})_{\phi_0(\mathbf{k}, \mathbf{l} - \underline{l}_{r''}), d} \langle \Delta_{\mathbf{k}}^{\mathfrak{n}_{11}^+}(x_{11}) \Delta_{\mathbf{l}}^{\mathfrak{n}_{22}^+}(x_{22}), e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\lambda, x} \\ & = (\lambda)_{\phi_0(\mathbf{k}, \mathbf{l} - \underline{l}_{r''}) + \underline{l}_{r''}, d} \langle \Delta_{\mathbf{k}}^{\mathfrak{n}_{11}^+}(x_{11}) \Delta_{\mathbf{l}}^{\mathfrak{n}_{22}^+}(x_{22}), e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\lambda, x}, \end{aligned}$$

and again by Proposition 3.1 and the induction hypothesis for $(r', r'' - 1)$, this is holomorphically continued for all $\lambda \in \mathbb{C}$. Now, the greatest common divisor of

$$\begin{aligned} & (\lambda)_{\phi_0(\mathbf{k} - \underline{k}_{r'}, \mathbf{l}) + \underline{k}_{r'}, d} = \prod_{a=1}^r \left(\lambda - \frac{d}{2}(a-1) \right)_{\min\{k_i + l_j \mid 1 \leq i \leq r', 1 \leq j \leq r'' + 1, i+j=a+1\}}, \\ & (\lambda)_{\phi_0(\mathbf{k}, \mathbf{l} - \underline{l}_{r''}) + \underline{l}_{r''}, d} = \prod_{a=1}^r \left(\lambda - \frac{d}{2}(a-1) \right)_{\min\{k_i + l_j \mid 1 \leq i \leq r' + 1, 1 \leq j \leq r'', i+j=a+1\}} \end{aligned}$$

is

$$(\lambda)_{\phi_0(\mathbf{k}, \mathbf{l}), d} = \prod_{a=1}^r \left(\lambda - \frac{d}{2}(a-1) \right)_{\min\{k_i + l_j \mid 1 \leq i \leq r' + 1, 1 \leq j \leq r'' + 1, i+j=a+1\}},$$

and hence $(\lambda)_{\phi_0(\mathbf{k}, \mathbf{l}), d} \langle \Delta_{\mathbf{k}}^{\mathfrak{n}_{11}^+}(x_{11}) \Delta_{\mathbf{l}}^{\mathfrak{n}_{22}^+}(x_{22}), e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\lambda, x}$ is holomorphically continued for all $\lambda \in \mathbb{C}$. This completes the proof of Theorem 5.2. \blacksquare

By Theorem 5.1 and Proposition 2.6(1), we get the following.

Corollary 5.4. *Let $\operatorname{Re} \lambda > p-1$, $\mathbf{k} \in \mathbb{Z}_{++}^{r'}$, $\mathbf{l} \in \mathbb{Z}_{++}^{r''}$, and let $f(x_{11}) \in \mathcal{P}_{\mathbf{k}}(\mathfrak{p}_{11}^+)$, $g(x_{22}) \in \mathcal{P}_{\mathbf{l}}(\mathfrak{p}_{22}^+)$. Then we have*

$$\|f(x_{11})g(x_{22})\|_{\lambda, x}^2 = C_0^d(\lambda, \mathbf{k}, \mathbf{l}) \|f(z_{11})g(z_{22})\|_{F, z}^2,$$

where $C_0^d(\lambda, \mathbf{k}, \mathbf{l})$ is as in (5.1).

Next we give a rough estimate of zeroes of $(\lambda)_{\phi_0(\mathbf{k}, \mathbf{l}), d} C_0^d(\lambda, \mathbf{k}, \mathbf{l})$.

Proposition 5.5. For $\mathbf{k} \in \mathbb{Z}_{++}^{r'}$, $\mathbf{l} \in \mathbb{Z}_{++}^{r''}$, let $C_0^d(\lambda, \mathbf{k}, \mathbf{l})$, $\phi_0(\mathbf{k}, \mathbf{l})$ be as in (5.1), (5.2). For $1 \leq m'_1 \leq m'_2 \leq r'$, $1 \leq m''_1 \leq m''_2 \leq r''$, if $k_1 = \cdots = k_{m'_1}$, $l_1 = \cdots = l_{m''_1}$ and $k_{m'_2+1} = l_{m''_2+1} = 0$, then we have

$$\begin{aligned} & \{\lambda \in \mathbb{C} \mid (\lambda)_{\phi_0(\mathbf{k}, \mathbf{l}), d} C_0^d(\lambda, \mathbf{k}, \mathbf{l}) = 0\} \\ & \subset \begin{cases} \left[\frac{d}{2} \max\{m'_1, m''_1\} - (k_1 + l_1) + 1, \frac{d}{2}(m'_2 + m''_2 - 1) - \max\{k_{m'_2}, l_{m''_2}\} \right], & k_1 l_1 \geq 1, \\ \emptyset, & k_1 l_1 = 0. \end{cases} \end{aligned}$$

Epecially, for $m = 1, 2, \dots, r' + r'' - 1$, if $\phi_0(\mathbf{k}, \mathbf{l})_{m+1} = 0$ and $\phi_0(\mathbf{k}, \mathbf{l})_m \neq 0$, then $C_0^d(\lambda, \mathbf{k}, \mathbf{l})$ is non-zero holomorphic at $\lambda = \frac{d}{2}m$, and has a pole at $\lambda = \frac{d}{2}(m - 1)$.

Proof. By direct computation, we have

$$\begin{aligned} \phi_0(\mathbf{k}, \mathbf{l})_a &= \begin{cases} k_1 + l_a, & 1 \leq a \leq \min\{m'_1, m''_2\}, \\ k_1, & m'_2 < a \leq m'_1 \end{cases} = \begin{cases} k_a + l_1, & 1 \leq a \leq \min\{m''_1, m'_2\}, \\ l_1, & m'_2 < a \leq m''_1, \end{cases} \\ \phi_0(\mathbf{k}, \mathbf{l})_{m'_2+m''_2+1} &= 0, \end{aligned}$$

and

$$\begin{aligned} C_0^d(\lambda, \mathbf{k}, \mathbf{l}) &= C_0^d(\lambda, (k_1, \dots, k_{m'_2}), (l_1, \dots, l_{m''_2})) \\ &= \frac{1}{\prod_{a=1}^{\max\{m'_1, m''_1\}} (\lambda - \frac{d}{2}(a-1))_{\phi_0(\mathbf{k}, \mathbf{l})_a}} \\ &\quad \times \prod_{a=\max\{m'_1, m''_1\}+1}^{m'_2+m''_2} \frac{\prod_{i=\max\{1, a-m''_2\}}^{\min\{a-1, m'_2\}} (\lambda - \frac{d}{2}(a-1))_{k_i+l_{a-i}}}{\prod_{i=\max\{1, a-m''_2\}}^{\min\{a, m'_2+1\}} (\lambda - \frac{d}{2}(a-1))_{k_i+l_{a+1-i}}}. \end{aligned}$$

For $2 \leq a \leq m'_2 + m''_2$, let

$$\begin{aligned} \phi(\mathbf{k}, \mathbf{l})_a &:= \min_2\{k_i + l_j \mid 1 \leq i \leq m'_2 + 1, 1 \leq j \leq m''_2 + 1, i + j = a + 1\}, \\ \psi(\mathbf{k}, \mathbf{l})_a &:= \max\{k_i + l_j \mid 1 \leq i \leq m'_2, 1 \leq j \leq m''_2, i + j = a\}, \end{aligned}$$

where \min_2 denotes the second smallest element, so that

$$\max\{k_{m'_2}, l_{m''_2}\} \leq \phi(\mathbf{k}, \mathbf{l})_a \leq \psi(\mathbf{k}, \mathbf{l})_a \leq k_1 + l_1$$

holds for every a . Then we have

$$\begin{aligned} & \{\lambda \in \mathbb{C} \mid (\lambda)_{\phi_0(\mathbf{k}, \mathbf{l}), d} C_0^d(\lambda, \mathbf{k}, \mathbf{l}) = 0\} \\ &= \bigcup_{a=\max\{m'_1, m''_1\}+1}^{m'_2+m''_2} \left\{ \lambda \in \mathbb{C} \mid \begin{array}{l} (\lambda - \frac{d}{2}(a-1))_{\phi_0(\mathbf{k}, \mathbf{l})_a} \\ \times \frac{\prod_{i=\max\{1, a-m''_2\}}^{\min\{a-1, m'_2\}} (\lambda - \frac{d}{2}(a-1))_{k_i+l_{a-i}}}{\prod_{i=\max\{1, a-m''_2\}}^{\min\{a, m'_2+1\}} (\lambda - \frac{d}{2}(a-1))_{k_i+l_{a+1-i}}} = 0 \end{array} \right\} \\ &\subset \bigcup_{a=\max\{m'_1, m''_1\}+1}^{m'_2+m''_2} \left\{ \lambda \in \mathbb{C} \mid \frac{(\lambda - \frac{d}{2}(a-1))_{\psi(\mathbf{k}, \mathbf{l})_a}}{(\lambda - \frac{d}{2}(a-1))_{\phi(\mathbf{k}, \mathbf{l})_a}} = 0 \right\} \\ &= \bigcup_{a=\max\{m'_1, m''_1\}+1}^{m'_2+m''_2} \left\{ \frac{d}{2}(a-1) - j \mid j \in \mathbb{Z}, \phi(\mathbf{k}, \mathbf{l})_a \leq j \leq \psi(\mathbf{k}, \mathbf{l})_a - 1 \right\} \\ &\subset \left[\frac{d}{2} \max\{m'_1, m''_1\} - (k_1 + l_1) + 1, \frac{d}{2}(m'_2 + m''_2 - 1) - \max\{k_{m'_2}, l_{m''_2}\} \right]. \end{aligned}$$

When $k_1 = 0$, i.e., $\mathbf{k} = (0, \dots, 0)$, by direct computation, we have

$$(\lambda)_{\phi_0(\mathbf{k}, \mathbf{l}), d} C_0^d(\lambda, \mathbf{k}, \mathbf{l}) = \frac{(\lambda)_{\mathbf{l}, d}}{(\lambda)_{\mathbf{l}, d}} = 1,$$

and this is non-zero everywhere. The $l_1 = 0$, i.e., $\mathbf{l} = (0, \dots, 0)$ case is similar.

If $\phi_0(\mathbf{k}, \mathbf{l})_{m+1} = 0$ and $\phi_0(\mathbf{k}, \mathbf{l})_m \neq 0$, then $k_{m'_2+1} = l_{m''_2+1} = 0$ and $\max\{k_{m'_2}, l_{m''_2}\} \neq 0$ hold for some m'_2, m''_2 with $m'_2 + m''_2 = m$, and by the above formula $(\lambda)_{\phi_0(\mathbf{k}, \mathbf{l}), d} C_0^d(\lambda, \mathbf{k}, \mathbf{l})$ is non-zero at $\lambda = \frac{d}{2}m, \frac{d}{2}(m-1)$. Hence, we get the last claim. \blacksquare

Especially, if $\phi_0(\mathbf{k}, \mathbf{l})_{a+1} = 0$ and $\phi_0(\mathbf{k}, \mathbf{l})_a \neq 0$, then for non-zero polynomials $f(x_{11}) \in \mathcal{P}_{\mathbf{k}}(\mathfrak{p}_{11}^+)$, $g(x_{22}) \in \mathcal{P}_1(\mathfrak{p}_{22}^+)$, $\langle f(x_{11})g(x_{22}), e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\lambda, x}$ has a pole at $\lambda = \frac{d}{2}(a-1)$, and combining with Corollary 2.4 and (5.4) we have

$$\mathcal{P}_{\mathbf{k}}(\mathfrak{p}_{11}^+) \mathcal{P}_1(\mathfrak{p}_{22}^+) \subset \bigoplus_{\substack{\mathbf{m} \in \mathbb{Z}_{++}^{r'+r''} \\ m_{a+1}=0}} \mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+), \quad \mathcal{P}_{\mathbf{k}}(\mathfrak{p}_{11}^+) \mathcal{P}_1(\mathfrak{p}_{22}^+) \not\subset \bigoplus_{\substack{\mathbf{m} \in \mathbb{Z}_{++}^{r'+r''} \\ m_a=0}} \mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+),$$

that is, for $a = 0, 1, \dots, r-1$, we have

$$\begin{aligned} \mathcal{P}(\mathfrak{p}_{11}^+ \oplus \mathfrak{p}_{22}^+) \cap \bigoplus_{\substack{\mathbf{m} \in \mathbb{Z}_{++}^r \\ m_{a+1}=0}} \mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+) &= \bigoplus_{\substack{(\mathbf{k}, \mathbf{l}) \in \mathbb{Z}_{++}^{r'} \times \mathbb{Z}_{++}^{r''} \\ \phi_0(\mathbf{k}, \mathbf{l})_{a+1}=0}} \mathcal{P}_{\mathbf{k}}(\mathfrak{p}_{11}^+) \mathcal{P}_1(\mathfrak{p}_{22}^+) \\ &= \bigoplus_{\substack{0 \leq a' \leq r' \\ 0 \leq a'' \leq r'' \\ a'+a'' \leq a}} \bigoplus_{\substack{(\mathbf{k}, \mathbf{l}) \in \mathbb{Z}_{++}^{a'} \times \mathbb{Z}_{++}^{a''} \\ k_{a'}, l_{a''} \geq 1}} \mathcal{P}_{(\mathbf{k}, \mathbf{0}_{r'-a'})}(\mathfrak{p}_{11}^+) \mathcal{P}_{(\mathbf{0}, \mathbf{0}_{r''-a''})}(\mathfrak{p}_{22}^+), \end{aligned} \quad (5.5)$$

where we set $\phi_0(\mathbf{k}, \mathbf{l})_{r'+r''+1} = \dots = \phi_0(\mathbf{k}, \mathbf{l})_r := 0$ and $k_0 = l_0 := +\infty$. This equality is earlier given in [14, Section 6].

Next we consider general $(\mathbf{k}, \mathbf{l}) \in \mathbb{Z}_{++}^{r'} \times \mathbb{Z}_{++}^{r''}$. Then again by the above proposition, $\frac{1}{C_0^d(\lambda, \mathbf{k}, \mathbf{l})} \langle f(x_{11})g(x_{22}), e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\lambda, x}$ is holomorphic for $\operatorname{Re} \lambda > \frac{d}{2}(r'+r''-1) - \max\{k_{r'}, l_{r''}\}$. Then by Theorem 3.5, comparing the top terms, we get the following.

Theorem 5.6. *Suppose $\mathfrak{p}^+, \mathfrak{p}_{11}^+, \mathfrak{p}_{22}^+$ are of tube type, and consider the meromorphic continuation of $\langle \cdot, \cdot \rangle_{\lambda}$. Then for $\mathbf{k} \in \mathbb{Z}_{++}^{r'}$, $\mathbf{l} \in \mathbb{Z}_{++}^{r''}$, $a = 1, 2, \dots, \min\{k_{r'}, l_{r''}\}$ and for $f(x_{11}) \in \mathcal{P}_{\mathbf{k}}(\mathfrak{p}_{11}^+)$, $g(x_{22}) \in \mathcal{P}_1(\mathfrak{p}_{22}^+)$, we have*

$$\begin{aligned} &\frac{1}{C_0^d(\lambda, \mathbf{k}, \mathbf{l})} \langle f(x_{11})g(x_{22}), e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\lambda, x} \Big|_{\lambda = \frac{n}{r} - a} \\ &= \frac{\det_{\mathfrak{n}^+}(z)^a}{C_0^d\left(\frac{n}{r} + a, \mathbf{k} - \underline{a}_{r'}, \mathbf{l} - \underline{a}_{r''}\right)} \langle \det_{\mathfrak{n}_{11}^+}(x_{11})^{-a} \det_{\mathfrak{n}_{22}^+}(x_{22})^{-a} f(x_{11})g(x_{22}), e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\frac{n}{r} + a, x}. \end{aligned}$$

5.2 Results on restriction of $\mathcal{H}_{\lambda}(D)$ to subgroups

Next we consider the decomposition of the holomorphic discrete series representation of scalar type $\mathcal{H}_{\lambda}(D)$ under the subgroup $\tilde{G}_1 \subset \tilde{G}$. By Theorem 2.5, we have

$$\mathcal{H}_{\lambda}(D)|_{\tilde{G}_1} \simeq \sum_{\mathbf{k} \in \mathbb{Z}_{++}^{r'}}^{\oplus} \sum_{\mathbf{l} \in \mathbb{Z}_{++}^{r''}}^{\oplus} \mathcal{H}_{\varepsilon_1 \lambda}(D_1, \mathcal{P}_{\mathbf{k}}(\mathfrak{p}_{11}^+) \boxtimes \mathcal{P}_1(\mathfrak{p}_{22}^+)),$$

where

$$\mathcal{H}_{\varepsilon_1 \lambda}(D_1, \mathcal{P}_{\mathbf{k}}(\mathfrak{p}_{11}^+) \boxtimes \mathcal{P}_1(\mathfrak{p}_{22}^+))$$

$$\simeq \begin{cases} \mathcal{H}_{\lambda+k+l}(D_{\text{SO}_0(2,d)}) \boxtimes \chi_{\text{SO}(2)}^{-k+l} & \text{(Case 1),} \\ \mathcal{H}_{\lambda+\lambda}(D_{\text{U}(r',r'')}, V_{2\mathbf{k}}^{(r')\vee} \boxtimes V_{2\mathbf{l}}^{(r'')\vee}) & \text{(Case 2),} \\ \mathcal{H}_{\lambda_1+\lambda_2}(D_{\text{U}(q',s'')}, V_{\mathbf{k}}^{(q')\vee} \boxtimes V_{\mathbf{l}}^{(s'')\vee}) \hat{\boxtimes} \mathcal{H}_{\lambda_1+\lambda_2}(D_{\text{U}(q'',s')}, V_{\mathbf{l}}^{(q'')\vee} \boxtimes V_{\mathbf{k}}^{(s')\vee}) & \text{(Case 3),} \\ \mathcal{H}_{\frac{\lambda}{2}+\frac{\lambda}{2}}(D_{\text{U}(s',s'')}, V_{\mathbf{k}^2}^{(s')\vee} \boxtimes V_{\mathbf{l}^2}^{(s'')\vee}) & \text{(Case 4),} \\ \mathcal{H}_{\lambda+k+\frac{\|\mathbf{l}\|}{2}}(D_{E_{6(-14)}}, V_{(l_1-l_2,0,0,0)}^{[10]\vee}) \boxtimes \chi_{\text{U}(1)}^{-\lambda+2k-2\|\mathbf{l}\|} & \text{(Case 5),} \\ \mathcal{H}_{\lambda+k}(D_{\text{SO}^*(10)}, V_{(\frac{l}{2}, \frac{l}{2}, \frac{l}{2}, -\frac{l}{2})}^{(5)\vee}) \boxtimes \chi_{\text{U}(1)}^{-\lambda+3k-3l} & \text{(Case 6),} \end{cases}$$

where we set $\lambda = \lambda_1 + \lambda_2$ for Case 3. For each $\mathbf{k} \in \mathbb{Z}_{++}^{r'}$, $\mathbf{l} \in \mathbb{Z}_{++}^{r''}$, let $V_{\mathbf{k}} \boxtimes W_{\mathbf{l}}$ be an abstract K_1 -module isomorphic to $\mathcal{P}_{\mathbf{k}}(\mathfrak{p}_{11}^+) \boxtimes \mathcal{P}_{\mathbf{l}}(\mathfrak{p}_{22}^+)$, let $\|\cdot\|_{\varepsilon_1\lambda, \mathbf{k}, \mathbf{l}}$ be the \tilde{G}_1 -invariant norm on $\mathcal{H}_{\varepsilon_1\lambda}(D_1, V_{\mathbf{k}} \boxtimes W_{\mathbf{l}})$ normalized such that $\|v\|_{\varepsilon_1\lambda, \mathbf{k}, \mathbf{l}} = |v|_{V_{\mathbf{k}} \boxtimes W_{\mathbf{l}}}$ holds for all constant functions $v \in V_{\mathbf{k}} \boxtimes W_{\mathbf{l}}$, and for $\lambda > p - 1$ let

$$\mathcal{F}_{\lambda, \mathbf{k}, \mathbf{l}}^\downarrow: \mathcal{H}_\lambda(D)|_{\tilde{G}_1} \longrightarrow \mathcal{H}_{\varepsilon_1\lambda}(D_1, V_{\mathbf{k}} \boxtimes W_{\mathbf{l}})$$

be the symmetry breaking operator given in (2.18) and (2.19), using a vector-valued polynomial $\mathbf{K}_{\mathbf{k}, \mathbf{l}}(x_{11}, x_{22}) \in ((\mathcal{P}_{\mathbf{k}}(\mathfrak{p}_{11}^+) \boxtimes \mathcal{P}_{\mathbf{l}}(\mathfrak{p}_{22}^+)) \otimes (\overline{V_{\mathbf{k}}} \boxtimes \overline{W_{\mathbf{l}}}))^{K_1}$ satisfying

$$|\langle f(x_{11})g(x_{22}), \mathbf{K}_{\mathbf{k}, \mathbf{l}}(x_{11}, x_{22}) \rangle_{F, x}|_{V_{\mathbf{k}} \boxtimes W_{\mathbf{l}}} = \|f(z_{11})g(z_{22})\|_{F, z}$$

for $f(x_{11}) \in \mathcal{P}_{\mathbf{k}}(\mathfrak{p}_{11}^+)$, $g(x_{22}) \in \mathcal{P}_{\mathbf{l}}(\mathfrak{p}_{22}^+)$, so that

$$\|\mathcal{F}_{\lambda, \mathbf{k}, \mathbf{l}}^\downarrow(f(x_{11})g(x_{22}))\|_{\varepsilon_1\lambda, \mathbf{k}, \mathbf{l}} = \|f(z_{11})g(z_{22})\|_{F, z}, \quad f(x_{11}) \in \mathcal{P}_{\mathbf{k}}(\mathfrak{p}_{11}^+), \quad g(x_{22}) \in \mathcal{P}_{\mathbf{l}}(\mathfrak{p}_{22}^+)$$

holds. Also, when \mathfrak{p}_{11}^+ , \mathfrak{p}_{22}^+ are of tube type, we fix $[K_1, K_1]$ -isomorphisms $V_{\mathbf{k}+a_{r'}} \boxtimes W_{\mathbf{l}+a_{r''}} \simeq V_{\mathbf{k}} \boxtimes W_{\mathbf{l}}$ for each $a \in \mathbb{Z}_{>0}$. Then by Proposition 2.6 (3) and Theorem 5.1, the following Parseval-Plancherel-type formula holds.

Corollary 5.7. *For $\lambda > p - 1$ and for $f \in \mathcal{H}_\lambda(D)$ we have*

$$\|f\|_\lambda^2 = \sum_{\mathbf{k} \in \mathbb{Z}_{++}^{r'}} \sum_{\mathbf{l} \in \mathbb{Z}_{++}^{r''}} C_0^d(\lambda, \mathbf{k}, \mathbf{l}) \|\mathcal{F}_{\lambda, \mathbf{k}, \mathbf{l}}^\downarrow f\|_{\varepsilon_1\lambda, \mathbf{k}, \mathbf{l}}^2$$

where $C_0^d(\lambda, \mathbf{k}, \mathbf{l})$ is as in (5.1).

Next we consider the meromorphic continuation for smaller λ . Then by Propositions 2.7, 5.5, Theorems 5.2, 5.6 and the formula (5.5), we have the following.

Corollary 5.8. *For $\mathbf{k} \in \mathbb{Z}_{++}^{r'}$, $\mathbf{l} \in \mathbb{Z}_{++}^{r''}$, let $\phi_0(\mathbf{k}, \mathbf{l}) \in \mathbb{Z}_{++}^{r'+r''}$ be as in (5.2), and set $\phi_0(\mathbf{k}, \mathbf{l})_{r'+r''+1} = \dots = \phi_0(\mathbf{k}, \mathbf{l})_r := 0$.*

(1) *For $a = 1, 2, \dots, r$,*

$$d\tau_\lambda(\mathcal{U}(\mathfrak{g}_1))\mathcal{P}_{\mathbf{k}}(\mathfrak{p}_{11}^+)\mathcal{P}_{\mathbf{l}}(\mathfrak{p}_{22}^+) \subset M_a^{\mathfrak{g}}(\lambda)$$

holds if

$$\lambda \in \frac{d}{2}(a-1) - \phi_0(\mathbf{k}, \mathbf{l})_a - \mathbb{Z}_{\geq 0},$$

where $M_a^{\mathfrak{g}}(\lambda) \subset \mathcal{O}_\lambda(D)_{\tilde{K}}$ is the $(\mathfrak{g}, \tilde{K})$ -submodule given in (2.12).

(2) For $a = 0, 1, \dots, r-1$ we have

$$\begin{aligned} \mathcal{H}_{\frac{d}{2}a}(D)|_{\tilde{G}_1} &\simeq \sum_{\substack{(\mathbf{k}, \mathbf{l}) \in \mathbb{Z}_{++}^{r'} \times \mathbb{Z}_{++}^{r''} \\ \phi_0(\mathbf{k}, \mathbf{l})_{a+1} = 0}}^{\oplus} \mathcal{H}_{\varepsilon_1 \frac{d}{2}a}(D_1, \mathcal{P}_{\mathbf{k}}(\mathfrak{p}_{11}^+) \boxtimes \mathcal{P}_{\mathbf{l}}(\mathfrak{p}_{22}^+)) \\ &= \sum_{\substack{0 \leq a' \leq r' \\ 0 \leq a'' \leq r'' \\ a' + a'' \leq a}}^{\oplus} \sum_{\substack{(\mathbf{k}, \mathbf{l}) \in \mathbb{Z}_{++}^{a'} \times \mathbb{Z}_{++}^{a''} \\ k_{a'}, l_{a''} \geq 1}}^{\oplus} \mathcal{H}_{\varepsilon_1 \frac{d}{2}a}(D_1, \mathcal{P}_{(\mathbf{k}, \mathbf{l})_{r'-a'}}(\mathfrak{p}_{11}^+) \boxtimes \mathcal{P}_{(\mathbf{l}, \mathbf{0})_{r''-a''}}(\mathfrak{p}_{22}^+)), \end{aligned}$$

where we set $k_0 = l_0 := +\infty$.

(3) For $a = 0, 1, \dots, r-1$, if $\phi_0(\mathbf{k}, \mathbf{l})_{a+1} = 0$, then $\mathcal{F}_{\lambda, \mathbf{k}, \mathbf{l}}^\downarrow$ is holomorphic at $\lambda = \frac{d}{2}a$, and its restriction gives the symmetry breaking operator

$$\mathcal{F}_{\frac{d}{2}a, \mathbf{k}, \mathbf{l}}^\downarrow: \mathcal{H}_{\frac{d}{2}a}(D)|_{\tilde{G}_1} \longrightarrow \mathcal{H}_{\varepsilon_1 \frac{d}{2}a}(D_1, V_{\mathbf{k}} \boxtimes W_{\mathbf{l}}).$$

(4) Suppose \mathfrak{p}^+ , \mathfrak{p}_{11}^+ , \mathfrak{p}_{22}^+ are of tube type. For $a = 1, 2, \dots, \min\{k_{r'}, l_{r''}\}$, if

$$K_{\mathbf{k}, \mathbf{l}}(x_{11}, x_{22}) = c \det_{\mathfrak{n}_{11}^+}(x_{11})^a \det_{\mathfrak{n}_{22}^+}(x_{22})^a K_{\mathbf{k}-\mathbf{a}_{r'}, \mathbf{l}-\mathbf{a}_{r''}}(x_{11}, x_{22}),$$

then we have

$$\begin{aligned} \mathcal{F}_{\frac{n}{r}-a, \mathbf{k}, \mathbf{l}}^\downarrow &= c \mathcal{F}_{\frac{n}{r}+a, \mathbf{k}-\mathbf{a}_{r'}, \mathbf{l}-\mathbf{a}_{r''}}^\downarrow \circ \det_{\mathfrak{n}^-} \left(\frac{\partial}{\partial x} \right)^a : \\ \mathcal{O}_{\frac{n}{r}-a}(D)|_{\tilde{G}_1} &\longrightarrow \mathcal{O}_{\varepsilon_1(\frac{n}{r}-a)}(D_1, V_{\mathbf{k}} \boxtimes W_{\mathbf{l}}) \simeq \mathcal{O}_{\varepsilon_1(\frac{n}{r}+a)}(D_1, V_{\mathbf{k}-\mathbf{a}_{r'}} \boxtimes W_{\mathbf{l}-\mathbf{a}_{r''}}). \end{aligned}$$

If \mathbf{k}, \mathbf{l} satisfy the condition in Remark 5.3 (1), then ‘‘only if’’ in Corollary 5.8 (1) also holds. Also, Corollary 5.8 (2) for $\lambda = \frac{d}{2}$ ($a = 1$ case) is earlier given in [41]. The parameter set in Corollary 5.8 (2) also appears in Howe’s correspondence for the dual pair $(U(r', r''), U(a))$ (see, e.g., [1, 24, 52]), and especially, we can prove (2) for Cases 2–4 by using the seesaw dual pair theory (see, e.g., [18, Section 3], [37]) as in [41].

6 Case \mathfrak{p}_2^+ is simple

In this section we treat $\mathfrak{p}^+ = (\mathfrak{p}^+)^{\sigma} \oplus (\mathfrak{p}^+)^{-\sigma} = \mathfrak{p}_1^+ \oplus \mathfrak{p}_2^+$ such that \mathfrak{p}_2^+ is simple, and $\mathfrak{p}_2^+(e)_2 \subsetneq \mathfrak{p}^+(e)_2$ holds for some (or equivalently any) maximal tripotent $e \in \mathfrak{p}_2^+$, so that

$$(\mathfrak{p}^+, \mathfrak{p}_1^+, \mathfrak{p}_2^+) = \begin{cases} (\mathbb{C}^n, \mathbb{C}^{n'}, \mathbb{C}^{n''}) & \text{(Case 1),} \\ (\text{Sym}(r, \mathbb{C}), \text{Sym}(r', \mathbb{C}) \oplus \text{Sym}(r'', \mathbb{C}), \text{M}(r', r''; \mathbb{C})) & \text{(Case 2),} \\ (\text{Alt}(s, \mathbb{C}), \text{Alt}(s', \mathbb{C}) \oplus \text{Alt}(s'', \mathbb{C}), \text{M}(s', s''; \mathbb{C})) & \text{(Case 3),} \\ (\text{M}(r, \mathbb{C}), \text{Alt}(r, \mathbb{C}), \text{Sym}(r, \mathbb{C})) & \text{(Case 4),} \\ (\text{M}(r, \mathbb{C}), \text{Sym}(r, \mathbb{C}), \text{Alt}(r, \mathbb{C})) & \text{(Case 5),} \\ (\text{Herm}(3, \mathbb{O})^{\mathbb{C}}, \text{M}(2, 6; \mathbb{C}), \text{Alt}(6, \mathbb{C})) & \text{(Case 6),} \\ (\text{Herm}(3, \mathbb{O})^{\mathbb{C}}, \text{Alt}(6, \mathbb{C}), \text{M}(2, 6; \mathbb{C})) & \text{(Case 7),} \\ (\text{Herm}(3, \mathbb{O})^{\mathbb{C}}, \mathbb{C} \oplus \text{Herm}(2, \mathbb{O})^{\mathbb{C}}, \text{M}(1, 2; \mathbb{O})^{\mathbb{C}}) & \text{(Case 8),} \\ (\text{M}(1, 2; \mathbb{O})^{\mathbb{C}}, \text{M}(2, 4; \mathbb{C}), \text{M}(4, 2; \mathbb{C})) & \text{(Case 9),} \\ (\text{M}(1, 2; \mathbb{O})^{\mathbb{C}}, \mathbb{C} \oplus \text{M}(1, 5; \mathbb{C}), \text{Alt}(5, \mathbb{C})) & \text{(Case 10)} \end{cases}$$

($n = n' + n''$, $n \geq 3$, $n'' \neq 2$ for Case 1, $r = r' + r''$ for Case 2, and $s = s' + s''$, $s', s'' \geq 2$ for Case 3). Then the corresponding symmetric pairs are

$$(G, G_1) = \begin{cases} (\mathrm{SO}_0(2, n), \mathrm{SO}_0(2, n') \times \mathrm{SO}(n'')) & \text{(Case 1),} \\ (\mathrm{Sp}(r, \mathbb{R}), \mathrm{Sp}(r', \mathbb{R}) \times \mathrm{Sp}(r'', \mathbb{R})) & \text{(Case 2),} \\ (\mathrm{SO}^*(2s), \mathrm{SO}^*(2s') \times \mathrm{SO}^*(2s'')) & \text{(Case 3),} \\ (\mathrm{SU}(r, r), \mathrm{SO}^*(2r)) & \text{(Case 4),} \\ (\mathrm{SU}(r, r), \mathrm{Sp}(r, \mathbb{R})) & \text{(Case 5),} \\ (E_{7(-25)}, \mathrm{SU}(2, 6)) & \text{(Case 6),} \\ (E_{7(-25)}, \mathrm{SU}(2) \times \mathrm{SO}^*(12)) & \text{(Case 7),} \\ (E_{7(-25)}, \mathrm{SL}(2, \mathbb{R}) \times \mathrm{Spin}_0(2, 10)) & \text{(Case 8),} \\ (E_{6(-14)}, \mathrm{SU}(2, 4) \times \mathrm{SU}(2)) & \text{(Case 9),} \\ (E_{6(-14)}, \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SU}(1, 5)) & \text{(Case 10)} \end{cases}$$

(up to covering). Let $\dim \mathfrak{p}^+ =: n$, $\dim \mathfrak{p}_2^+ =: n_2$, $\mathrm{rank} \mathfrak{p}^+ =: r$, $\mathrm{rank} \mathfrak{p}_2^+ =: r_2$, let d, d_2 be the numbers defined in (2.4) for $\mathfrak{p}^+, \mathfrak{p}_2^+$ respectively, and let $\varepsilon_2 \in \{1, 2\}$ be as in (2.15). Then the numbers $(r, r_2, d, d_2, \varepsilon_2)$ are given by

$$(r, r_2, d, d_2, \varepsilon_2) = \begin{cases} (2, 2, n - 2, n'' - 2, 1) & \text{(Case 1, } n'' \geq 3), \\ (2, 1, n - 2, -, 2) & \text{(Case 1, } n'' = 1), \\ (r, \min\{r', r''\}, 1, 2, 2) & \text{(Case 2),} \\ (\lfloor s/2 \rfloor, \min\{s', s''\}, 4, 2, 1) & \text{(Case 3),} \\ (r, r, 2, 1, 1) & \text{(Case 4),} \\ (r, \lfloor r/2 \rfloor, 2, 4, 2) & \text{(Case 5),} \\ (3, 3, 8, 4, 1) & \text{(Case 6),} \\ (3, 2, 8, 2, 1) & \text{(Case 7),} \\ (3, 2, 8, 6, 1) & \text{(Case 8),} \\ (2, 2, 6, 2, 1) & \text{(Case 9),} \\ (2, 2, 6, 4, 1) & \text{(Case 10).} \end{cases}$$

When $\varepsilon_2 = 1$, we have $d_2 = d/2$ or $r_2 = 2$. Similarly, when $\varepsilon_2 = 2$, we have $d_2 = 2d$ or $r_2 = 1$. For $r_2 = 1$ case, since d_2 is not determined uniquely and any number is allowed, we may assume $d_2 = 2d$.

6.1 Results on weighted Bergman inner products

First, for $f(x_2) \in \mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+)$, we want to compute the top terms and the poles of $\langle f(x_2), e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\lambda, x}$, where $\langle \cdot, \cdot \rangle_{\lambda} = \langle \cdot, \cdot \rangle_{\lambda, x}$ is the weighted Bergman inner product given in (2.9), with the variable of integration $x = x_1 + x_2$.

Theorem 6.1. *Let $\mathrm{Re} \lambda > p - 1$, $\mathbf{k} \in \mathbb{Z}_{++}^{r_2}$, and let $f(x_2) \in \mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+)$. Then we have*

$$\langle f(x_2), e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\lambda, x} \Big|_{z_1=0} = C_{\varepsilon_2}^{d, d_2}(\lambda, \mathbf{k}) f(z_2),$$

where, for $\varepsilon_2 = 1$,

$$C_1^{d, d_2}(\lambda, \mathbf{k}) = \frac{(\lambda + k_1 - \frac{d-d_2}{2})_{k_2}}{(\lambda)_{k_1+k_2} (\lambda - \frac{d}{2})_{k_2}}, \quad r_2 = 2,$$

$$\begin{aligned}
C_1^{d,d_2}(\lambda, \mathbf{k}) &= \frac{\prod_{1 \leq i < j \leq r_2} (\lambda - \frac{d}{4}(i+j-2))_{k_i+k_j}}{\prod_{1 \leq i < j \leq r_2+1} (\lambda - \frac{d}{4}(i+j-3))_{k_i+k_j}} \\
&= \frac{\prod_{a=2}^{2r_2-2} \prod_{i=\max\{1, a+1-r_2\}}^{\lfloor a/2 \rfloor} (\lambda - \frac{d}{4}(a-1))_{k_i+k_{a+1-i}}}{\prod_{a=1}^{2r_2-1} \prod_{i=\max\{1, a+1-r_2\}}^{\lfloor a/2 \rfloor} (\lambda - \frac{d}{4}(a-1))_{k_i+k_{a+2-i}}}, \quad d_2 = d/2, \quad (6.1)
\end{aligned}$$

and for $\varepsilon_2 = 2$,

$$\begin{aligned}
C_2^{d,d_2}(\lambda, \mathbf{k}) &= \frac{2^{|\mathbf{k}|} \prod_{1 \leq i < j \leq r_2} (2\lambda - 1 - d(i+j-1))_{k_i+k_j} \prod_{i=1}^{r_2} (\lambda - \frac{1}{2} - \frac{d}{2}(2i-1))_{k_i}}{\prod_{1 \leq i < j \leq r_2+1} (2\lambda - 1 - d(i+j-2))_{k_i+k_j} \prod_{i=1}^{r_2} (\lambda - d(i-1))_{k_i}} \\
&= \frac{\prod_{1 \leq i < j \leq r_2} (\lambda - \frac{d}{2}(i+j-1))_{\lfloor \frac{k_i+k_j}{2} \rfloor} \prod_{1 \leq i \leq j \leq r_2} (\lambda - \frac{1}{2} - \frac{d}{2}(i+j-1))_{\lfloor \frac{k_i+k_j}{2} \rfloor}}{\prod_{1 \leq i \leq j \leq r_2+1} (\lambda - \frac{d}{2}(i+j-2))_{\lfloor \frac{k_i+k_j}{2} \rfloor} \prod_{1 \leq i < j \leq r_2+1} (\lambda - \frac{1}{2} - \frac{d}{2}(i+j-2))_{\lfloor \frac{k_i+k_j}{2} \rfloor}} \\
&= \frac{\prod_{a=3}^{2r_2-1} \prod_{i=\max\{1, a-r_2\}}^{\lfloor a/2 \rfloor - 1} (\lambda - \frac{d}{2}(a-1))_{\lfloor \frac{k_i+k_{a-i}}{2} \rfloor}}{\prod_{a=1}^{2r_2} \prod_{i=\max\{1, a-r_2\}}^{\lfloor a/2 \rfloor} (\lambda - \frac{d}{2}(a-1))_{\lfloor \frac{k_i+k_{a+1-i}}{2} \rfloor}} \\
&\quad \times \frac{\prod_{a=2}^{2r_2} \prod_{i=\max\{1, a-r_2\}}^{\lfloor a/2 \rfloor} (\lambda - \frac{1}{2} - \frac{d}{2}(a-1))_{\lfloor \frac{k_i+k_{a-i}}{2} \rfloor}}{\prod_{a=2}^{2r_2} \prod_{i=\max\{1, a-r_2\}}^{\lfloor a/2 \rfloor} (\lambda - \frac{1}{2} - \frac{d}{2}(a-1))_{\lfloor \frac{k_i+k_{a+1-i}}{2} \rfloor}}. \quad (6.2)
\end{aligned}$$

Here we put $k_{r_2+1} := 0$.

Theorem 6.2. For $\mathbf{k} \in \mathbb{Z}_{++}^{r_2}$, $\varepsilon_2 = 1, 2$, we define $\phi_{\varepsilon_2}(\mathbf{k}) \in \mathbb{Z}_{++}^{\varepsilon_2 r_2}$ by

$$\phi_1(\mathbf{k})_a := \min\{k_i + k_j \mid 1 \leq i < j \leq r_2 + 1, i + j = 2a + 1\}, \quad 1 \leq a \leq r_2, \quad (6.3)$$

$$\phi_2(\mathbf{k})_a := \min\left\{\left\lfloor \frac{k_i + k_j}{2} \right\rfloor \mid 1 \leq i \leq j \leq r_2 + 1, i + j = a + 1\right\}, \quad 1 \leq a \leq 2r_2, \quad (6.4)$$

where $k_{r_2+1} := 0$. Then for $f(x_2) \in \mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+)$, as a function of λ ,

$$(\lambda)_{\phi_{\varepsilon_2}(\mathbf{k}), d} \langle f(x_2), e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\lambda, x} \quad (6.5)$$

is holomorphically continued for all \mathbb{C} .

Remark 6.3.

(1) By [44, Corollary 6.6], if $f(x_2) \neq 0$ and if $\mathbf{k} \in \mathbb{Z}_{++}^{r_2}$ satisfies “ $\varepsilon_2 = 1$, $\mathbf{k} = (k_a, 0_{r_2-a})$, $1 \leq a \leq r_2$ ” or “ $\varepsilon_2 = 1$, $\mathbf{k} = (k_1, k_{2a-1}, 0_{r_2-a})$, $2 \leq a \leq r_2$, a : even” or “ $\varepsilon_2 = 2$, $\mathbf{k} = (k+1_l, k_{a-l}, 0_{r_2-a})$, $0 \leq l < a \leq r_2$ ”, then (6.5) gives a non-zero polynomial in $z \in \mathfrak{p}^+$ for all $\lambda \in \mathbb{C}$.

(2) The restriction of Theorem 6.2 to $z_1 = 0$, the holomorphy of

$$(\lambda)_{\phi_{\varepsilon_2}(\mathbf{k}), d} \langle f(x_2), e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\lambda, x} \Big|_{z_1=0}$$

follows from Theorem 6.1, since it is easily proved that $(\lambda)_{\phi_{\varepsilon_2}(\mathbf{k}), d} C_{\varepsilon_2}^{d, d_2}(\lambda, \mathbf{k})$ is holomorphic for all $\lambda \in \mathbb{C}$.

(3) Under Corollary 2.4, Theorem 6.2 is equivalent to the inclusion

$$\mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+) \subset \bigoplus_{\substack{\mathbf{m} \in \mathbb{Z}_{++}^{\varepsilon_2 r_2} \\ m_a \leq \phi_{\varepsilon_2}(\mathbf{k})_a}} \mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+). \quad (6.6)$$

Here $(\lambda)_{\mathbf{m}, d}$ is as in (2.11), and $k_a := \underbrace{(k, \dots, k)}_a$.

Proof of Theorem 6.1. By the last paragraph of Section 3.1, we may assume \mathfrak{p}^+ , \mathfrak{p}_2^+ are of tube type, so that $r = \varepsilon_2 r_2$ holds, and by the K_1 -equivariance, may assume $f(x_2) = \Delta_{\mathbf{k}}^{\mathfrak{n}_2^+}(x_2)$.

First suppose $\varepsilon_2 = 1$, $r_2 = 2$. Then the theorem follows from [44, Theorems 6.3 (1), Corollary 6.5 (2)],

$$\begin{aligned}
\langle \Delta_{\mathbf{k}}^{\mathfrak{n}_2^+}(x_2), e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\lambda, x} &= \frac{\det_{\mathfrak{n}^+}(z)^{-\lambda + \frac{d}{2} + 1}}{(\lambda)_{(k_1+k_2, 2k_2), d}} \det_{\mathfrak{n}_2^+} \left(\frac{\partial}{\partial z_2} \right)^{k_2} \det_{\mathfrak{n}^+}(z)^{\lambda + 2k_2 - \frac{d}{2} - 1} \Delta_{(k_1-k_2, 0)}^{\mathfrak{n}_2^+}(z_2) \\
&= \frac{1}{(\lambda)_{(k_1+k_2, 2k_2), d}} \det_{\mathfrak{n}^+}(z)^{-\lambda + \frac{d}{2} + 1} \det_{\mathfrak{n}_2^+} \left(\frac{\partial}{\partial z_2} \right)^{k_2} \det_{\mathfrak{n}_2^+}(z_2)^{\lambda + k_2 - \frac{d}{2} - 1} \\
&\quad \times {}_1F_0 \left(-\lambda - 2k_2 + \frac{d}{2} + 1; -\frac{\det_{\mathfrak{n}^+}(z_1)}{\det_{\mathfrak{n}_2^+}(z_2)} \right) \Delta_{\mathbf{k}}^{\mathfrak{n}_2^+}(z_2) \\
&= \frac{(\lambda + k_1 - \frac{d-d_2}{2})_{k_2}}{(\lambda)_{(k_1+k_2, k_2), d}} \det_{\mathfrak{n}^+}(z)^{-\lambda + \frac{d}{2} + 1} \det_{\mathfrak{n}_2^+}(z_2)^{\lambda - \frac{d}{2} - 1} \\
&\quad \times {}_2F_1 \left(\begin{matrix} -\lambda - k_2 + \frac{d}{2} + 1, -\lambda - k_1 + \frac{d-d_2}{2} + 1 \\ -\lambda - k_1 - k_2 + \frac{d-d_2}{2} + 1 \end{matrix}; -\frac{\det_{\mathfrak{n}^+}(z_1)}{\det_{\mathfrak{n}_2^+}(z_2)} \right) \Delta_{\mathbf{k}}^{\mathfrak{n}_2^+}(z_2) \\
&= \frac{(\lambda + k_1 - \frac{d-d_2}{2})_{k_2}}{(\lambda)_{(k_1+k_2, k_2), d}} {}_2F_1 \left(\begin{matrix} -k_2, -k_1 - \frac{d_2}{2} \\ -\lambda - k_1 - k_2 + \frac{d-d_2}{2} + 1 \end{matrix}; -\frac{\det_{\mathfrak{n}^+}(z_1)}{\det_{\mathfrak{n}_2^+}(z_2)} \right) \Delta_{\mathbf{k}}^{\mathfrak{n}_2^+}(z_2), \quad (6.7)
\end{aligned}$$

where ${}_1F_0(\alpha; t) := \sum_{m=0}^{\infty} \frac{(\alpha)_m}{m!} t^m$, ${}_2F_1(\alpha, \beta; \gamma; t) := \sum_{m=0}^{\infty} \frac{(\alpha)_m (\beta)_m}{(\gamma)_m m!} t^m$.

Next suppose $\varepsilon_2 = 1$, $d_2 = d/2$. The 2nd equality of (6.1) is easy. We prove the 1st equality by induction on r_2 . If $r_2 = 1$, then we have $\mathfrak{p}^+ = \mathfrak{p}_2^+$ under the tube type assumption, and this follows from Corollary 2.4. Next we assume the theorem for $r_2 - 1$, and prove it for r_2 . Then since $r = r_2$ holds, by (3.3) we have

$$\begin{aligned}
&\langle \Delta_{\mathbf{k}}^{\mathfrak{n}_2^+}(x_2), e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\lambda, x} \Big|_{z_1=0} \\
&= \frac{\det_{\mathfrak{n}^+}(z_2)^{-\lambda + \frac{n}{r}}}{(\lambda)_{\underline{2k_{r_2}, r_2}, d}} \det_{\mathfrak{n}_2^+} \left(\frac{\partial}{\partial z_2} \right)^{k_{r_2}} \det_{\mathfrak{n}^+}(z_2)^{\lambda + 2k_{r_2} - \frac{n}{r}} \langle \Delta_{\mathbf{k} - \underline{k_{r_2}, r_2}}^{\mathfrak{n}_2^+}(x_2), e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\lambda + 2k_{r_2}, x} \Big|_{z_1=0} \\
&= \frac{1}{(\lambda)_{\underline{2k_{r_2}, r_2}, d}} \det_{\mathfrak{n}_2^+}(z_2)^{-\lambda + \frac{d}{2}(r_2-1) + 1} \det_{\mathfrak{n}_2^+} \left(\frac{\partial}{\partial z_2} \right)^{k_{r_2}} \det_{\mathfrak{n}_2^+}(z_2)^{\lambda + 2k_{r_2} - \frac{d}{2}(r_2-1) - 1} \\
&\quad \times \frac{\prod_{1 \leq i < j \leq r_2-1} (\lambda + 2k_{r_2} - \frac{d}{4}(i+j-2))_{(k_i-k_{r_2})+(k_j-k_{r_2})}}{\prod_{1 \leq i < j \leq r_2} (\lambda + 2k_{r_2} - \frac{d}{4}(i+j-3))_{(k_i-k_{r_2})+(k_j-k_{r_2})}} \Delta_{\mathbf{k} - \underline{k_{r_2}, r_2}}^{\mathfrak{n}_2^+}(z_2) \\
&= \frac{1}{(\lambda)_{\underline{2k_{r_2}, r_2}, d}} \prod_{i=1}^{r_2} \left(\lambda + k_i - \frac{d}{2}(r_2-1) + \frac{d}{4}(r_2-i) \right)_{k_{r_2}} \\
&\quad \times \frac{\prod_{1 \leq i < j \leq r_2-1} (\lambda + 2k_{r_2} - \frac{d}{4}(i+j-2))_{k_i+k_j-2k_{r_2}}}{\prod_{1 \leq i < j \leq r_2} (\lambda + 2k_{r_2} - \frac{d}{4}(i+j-3))_{k_i+k_j-2k_{r_2}}} \det_{\mathfrak{n}_2^+}(z_2)^{k_{r_2}} \Delta_{\mathbf{k} - \underline{k_{r_2}, r_2}}^{\mathfrak{n}_2^+}(z_2) \\
&= \frac{1}{(\lambda)_{\underline{2k_{r_2}, r_2}, d}} \frac{\prod_{i=1}^{r_2} (\lambda - \frac{d}{4}(i+r_2-2))_{k_i+k_{r_2}}}{\prod_{i=1}^{r_2} (\lambda - \frac{d}{4}(i+(r_2+1)-3))_{k_i}} \\
&\quad \times \frac{\prod_{1 \leq i < j \leq r_2-1} (\lambda - \frac{d}{4}(i+j-2))_{k_i+k_j}}{\prod_{1 \leq i < j \leq r_2} (\lambda - \frac{d}{4}(i+j-3))_{k_i+k_j}} \frac{\prod_{1 \leq i < j \leq r_2} (\lambda - \frac{d}{4}(i+j-3))_{2k_{r_2}}}{\prod_{1 \leq i < j \leq r_2-1} (\lambda - \frac{d}{4}(i+j-2))_{2k_{r_2}}} \Delta_{\mathbf{k}}^{\mathfrak{n}_2^+}(z_2)
\end{aligned}$$

$$\begin{aligned}
&= \frac{(\lambda - \frac{d}{4}(2r_2 - 2))_{2k_{r_2}}}{\prod_{i=1}^{r_2} (\lambda - \frac{d}{2}(i - 1))_{2k_{r_2}}} \\
&\quad \times \frac{\prod_{1 \leq i < j \leq r_2} (\lambda - \frac{d}{4}(i + j - 2))_{k_i + k_j}}{\prod_{1 \leq i < j \leq r_2 + 1} (\lambda - \frac{d}{4}(i + j - 3))_{k_i + k_j}} \frac{\prod_{1 \leq i \leq j \leq r_2 - 1} (\lambda - \frac{d}{4}(i + j - 2))_{2k_{r_2}}}{\prod_{1 \leq i < j \leq r_2 - 1} (\lambda - \frac{d}{4}(i + j - 2))_{2k_{r_2}}} \Delta_{\mathbf{k}}^{n_2^+}(z_2) \\
&= \frac{\prod_{1 \leq i < j \leq r_2} (\lambda - \frac{d}{4}(i + j - 2))_{k_i + k_j}}{\prod_{1 \leq i < j \leq r_2 + 1} (\lambda - \frac{d}{4}(i + j - 3))_{k_i + k_j}} \Delta_{\mathbf{k}}^{n_2^+}(z_2),
\end{aligned}$$

where we have used the induction hypothesis and Proposition 3.1 at the 2nd equality, and (2.6) at the 3rd equality. Hence the theorem follows.

Next suppose $\varepsilon_2 = 2$. The 2nd equality of (6.2) follows from the formula $(2\mu - 1)_k = 2^k(\mu)_{\lfloor k/2 \rfloor} (\mu - \frac{1}{2})_{\lceil k/2 \rceil}$, and the 3rd equality is easy. Next we prove the 1st equality by induction on r_2 . When $r_2 = 1$, the theorem follows from [44, Theorem 6.3 (2), Corollary 6.5 (3)],

$$\begin{aligned}
\langle \Delta_{k_1}^{n_2^+}(x_2), e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\lambda, x} &= \frac{\det_{\mathfrak{n}^+}(z)^{-\lambda + \frac{d}{2} + 1}}{(\lambda)_{(k_1, k_1), d}} \det_{\mathfrak{n}_2^+} \left(\frac{1}{2} \frac{\partial}{\partial z_2} \right)^{k_1} \det_{\mathfrak{n}^+}(z)^{\lambda + k_1 - \frac{d}{2} - 1} \\
&= \frac{1}{(\lambda)_{(k_1, k_1), d}} \det_{\mathfrak{n}^+}(z)^{-\lambda + \frac{d}{2} + 1} \det_{\mathfrak{n}_2^+} \left(\frac{1}{2} \frac{\partial}{\partial z_2} \right)^{k_1} \det_{\mathfrak{n}_2^+}(z_2)^{2(\lambda + k_1 - \frac{d}{2} - 1)} \\
&\quad \times {}_1F_0 \left(-\lambda - k_1 + \frac{d}{2} + 1; -\frac{\det_{\mathfrak{n}^+}(z_1)}{\det_{\mathfrak{n}_2^+}(z_2)^2} \right) \\
&= \frac{(\lambda + \lceil \frac{k_1}{2} \rceil - \frac{d+1}{2})_{\lfloor k_1/2 \rfloor}}{(\lambda)_{(k_1, \lfloor k_1/2 \rfloor), d}} \det_{\mathfrak{n}^+}(z)^{-\lambda + \frac{d}{2} + 1} \det_{\mathfrak{n}_2^+}(z_2)^{2(\lambda - \frac{d}{2} - 1) + k_1} \\
&\quad \times {}_2F_1 \left(-\lambda - \frac{k_1}{2} + \frac{d}{2} + 1, -\lambda - \frac{k_1}{2} + \frac{d}{2} + \frac{3}{2}; -\frac{\det_{\mathfrak{n}^+}(z_1)}{\det_{\mathfrak{n}_2^+}(z_2)^2} \right) \\
&= \frac{(\lambda + \lceil \frac{k_1}{2} \rceil - \frac{d+1}{2})_{\lfloor k_1/2 \rfloor}}{(\lambda)_{(k_1, \lfloor k_1/2 \rfloor), d}} {}_2F_1 \left(-\frac{k_1}{2}, -\frac{k_1}{2} + \frac{1}{2}; -\frac{\det_{\mathfrak{n}^+}(z_1)}{\det_{\mathfrak{n}_2^+}(z_2)^2} \right) \det_{\mathfrak{n}_2^+}(z_2)^{k_1}. \quad (6.8)
\end{aligned}$$

Next we assume the theorem for $r_2 - 1$, and prove it for r_2 . Then since $r = 2r_2$, $d_2 = 2d$ hold, by (3.3) we have

$$\begin{aligned}
&\langle \Delta_{\mathbf{k}}^{n_2^+}(x_2), e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\lambda, x} \Big|_{z_1=0} \\
&= \frac{\det_{\mathfrak{n}^+}(z_2)^{-\lambda + \frac{n}{r}}}{(\lambda)_{\underline{k}_{r_2, 2r_2}, d}} \det_{\mathfrak{n}_2^+} \left(\frac{1}{2} \frac{\partial}{\partial z_2} \right)^{k_{r_2}} \det_{\mathfrak{n}^+}(z_2)^{\lambda + k_{r_2} - \frac{n}{r}} \langle \Delta_{\mathbf{k} - \underline{k}_{r_2, r_2}}^{n_2^+}(x_2), e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\lambda + k_{r_2}, x} \Big|_{z_1=0} \\
&= \frac{1}{(\lambda)_{\underline{k}_{r_2, 2r_2}, d}} \det_{\mathfrak{n}_2^+}(z_2)^{2(-\lambda + \frac{d}{2}(2r_2 - 1) + 1)} \det_{\mathfrak{n}_2^+} \left(\frac{1}{2} \frac{\partial}{\partial z_2} \right)^{k_{r_2}} \det_{\mathfrak{n}_2^+}(z_2)^{2(\lambda + k_{r_2} - \frac{d}{2}(2r_2 - 1) - 1)} \\
&\quad \times \frac{2^{|\mathbf{k}| - k_{r_2} r_2} \prod_{1 \leq i < j \leq r_2 - 1} (2(\lambda + k_{r_2}) - 1 - d(i + j - 1))_{(k_i - k_{r_2}) + (k_j - k_{r_2})}}{\prod_{1 \leq i < j \leq r_2} (2(\lambda + k_{r_2}) - 1 - d(i + j - 2))_{(k_i - k_{r_2}) + (k_j - k_{r_2})}} \\
&\quad \times \frac{\prod_{i=1}^{r_2 - 1} (\lambda + k_{r_2} - \frac{1}{2} - \frac{d}{2}(2i - 1))_{k_i - k_{r_2}}}{\prod_{i=1}^{r_2 - 1} (\lambda + k_{r_2} - d(i - 1))_{k_i - k_{r_2}}} \Delta_{\mathbf{k} - \underline{k}_{r_2, r_2}}^{n_2^+}(z_2) \\
&= \frac{2^{|\mathbf{k}| - 2k_{r_2} r_2}}{(\lambda)_{\underline{k}_{r_2, 2r_2}, d}} \prod_{i=1}^{r_2} (2\lambda + k_i - d(2r_2 - 1) - 2 + d(r_2 - i) + 1)_{k_{r_2}}
\end{aligned}$$

$$\begin{aligned}
& \times \frac{\prod_{1 \leq i < j \leq r_2-1} (2\lambda + 2k_{r_2} - 1 - d(i+j-1))_{k_i+k_j-2k_{r_2}}}{\prod_{1 \leq i < j \leq r_2} (2\lambda + 2k_{r_2} - 1 - d(i+j-2))_{k_i+k_j-2k_{r_2}}} \\
& \times \frac{\prod_{i=1}^{r_2-1} (\lambda + k_{r_2} - \frac{1}{2} - \frac{d}{2}(2i-1))_{k_i-k_{r_2}}}{\prod_{i=1}^{r_2-1} (\lambda + k_{r_2} - d(i-1))_{k_i-k_{r_2}}} \det_{\mathfrak{n}_2^+}(z_2)^{k_{r_2}} \Delta_{\mathbf{k}-\underline{k}_{r_2, r_2}}^{\mathfrak{n}_2^+}(z_2) \\
& = \frac{2^{|\mathbf{k}|-2k_{r_2}r_2} \prod_{i=1}^{r_2} (2\lambda - 1 - d(i+r_2-1))_{k_i+k_{r_2}}}{(\lambda)_{k_{r_2}, 2r_2, d} \prod_{i=1}^{r_2} (2\lambda - 1 - d(i+(r_2+1)-2))_{k_i}} \\
& \times \frac{\prod_{1 \leq i < j \leq r_2-1} (2\lambda - 1 - d(i+j-1))_{k_i+k_j}}{\prod_{1 \leq i < j \leq r_2} (2\lambda - 1 - d(i+j-2))_{k_i+k_j}} \frac{\prod_{1 \leq i < j \leq r_2-1} (2\lambda - 1 - d(i+j-2))_{2k_{r_2}}}{\prod_{1 \leq i < j \leq r_2-1} (2\lambda - 1 - d(i+j-1))_{2k_{r_2}}} \\
& \times \frac{\prod_{i=1}^{r_2} (\lambda - \frac{1}{2} - \frac{d}{2}(2i-1))_{k_i}}{\prod_{i=1}^{r_2} (\lambda - d(i-1))_{k_i}} \frac{\prod_{i=1}^{r_2} (\lambda - d(i-1))_{k_{r_2}}}{\prod_{i=1}^{r_2} (\lambda - \frac{1}{2} - \frac{d}{2}(2i-1))_{k_{r_2}}} \Delta_{\mathbf{k}}^{\mathfrak{n}_2^+}(z_2) \\
& = \frac{2^{|\mathbf{k}|-2k_{r_2}r_2} (2\lambda - 1 - d(2r_2-1))_{2k_{r_2}}}{\prod_{i=1}^{r_2} (\lambda - \frac{d}{2}(2i-2))_{k_{r_2}} \prod_{i=1}^{r_2} (\lambda - \frac{d}{2}(2i-1))_{k_{r_2}}} \\
& \times \frac{\prod_{1 \leq i < j \leq r_2} (2\lambda - 1 - d(i+j-1))_{k_i+k_j}}{\prod_{1 \leq i < j \leq r_2+1} (2\lambda - 1 - d(i+j-2))_{k_i+k_j}} \frac{\prod_{1 \leq i < j \leq r_2-1} (2\lambda - 1 - d(i+j-1))_{2k_{r_2}}}{\prod_{1 \leq i < j \leq r_2-1} (2\lambda - 1 - d(i+j-1))_{2k_{r_2}}} \\
& \times \frac{\prod_{i=1}^{r_2} (\lambda - \frac{1}{2} - \frac{d}{2}(2i-1))_{k_i}}{\prod_{i=1}^{r_2} (\lambda - d(i-1))_{k_i}} \frac{\prod_{i=1}^{r_2} (\lambda - d(i-1))_{k_{r_2}}}{\prod_{i=1}^{r_2} (\lambda - \frac{1}{2} - \frac{d}{2}(2i-1))_{k_{r_2}}} \Delta_{\mathbf{k}}^{\mathfrak{n}_2^+}(z_2) \\
& = \frac{2^{|\mathbf{k}|-2k_{r_2}r_2} \prod_{i=1}^{r_2} (2\lambda - 1 - d(2i-1))_{2k_{r_2}}}{\prod_{i=1}^{r_2} (\lambda - \frac{1}{2} - \frac{d}{2}(2i-1))_{k_{r_2}} \prod_{i=1}^{r_2} (\lambda - \frac{d}{2}(2i-1))_{k_{r_2}}} \\
& \times \frac{\prod_{1 \leq i < j \leq r_2} (2\lambda - 1 - d(i+j-1))_{k_i+k_j}}{\prod_{1 \leq i < j \leq r_2+1} (2\lambda - 1 - d(i+j-2))_{k_i+k_j}} \frac{\prod_{i=1}^{r_2} (\lambda - \frac{1}{2} - \frac{d}{2}(2i-1))_{k_i}}{\prod_{i=1}^{r_2} (\lambda - d(i-1))_{k_i}} \Delta_{\mathbf{k}}^{\mathfrak{n}_2^+}(z_2) \\
& = \frac{2^{|\mathbf{k}|} \prod_{1 \leq i < j \leq r_2} (2\lambda - 1 - d(i+j-1))_{k_i+k_j}}{\prod_{1 \leq i < j \leq r_2+1} (2\lambda - 1 - d(i+j-2))_{k_i+k_j}} \frac{\prod_{i=1}^{r_2} (\lambda - \frac{1}{2} - \frac{d}{2}(2i-1))_{k_i}}{\prod_{i=1}^{r_2} (\lambda - d(i-1))_{k_i}} \Delta_{\mathbf{k}}^{\mathfrak{n}_2^+}(z_2),
\end{aligned}$$

where we have used the induction hypothesis and Proposition 3.1 at the 2nd equality, and (2.6) at the 3rd equality. Hence the theorem follows. \blacksquare

Proof of Theorem 6.2. Again it is enough to prove when both \mathfrak{p}^+ and \mathfrak{p}_2^+ are of tube type (i.e., $(\mathfrak{p}^+, \mathfrak{p}_2^+) = (\mathbb{C}^n, \mathbb{C}^{n''})$, $(\text{Sym}(2r_2, \mathbb{C}), \text{M}(r_2, \mathbb{C}))$, $(\text{Alt}(2r_2, \mathbb{C}), \text{M}(r_2, \mathbb{C}))$, $(\text{M}(r_2, \mathbb{C}), \text{Sym}(r_2, \mathbb{C}))$, $(\text{M}(2r_2, \mathbb{C}), \text{Alt}(2r_2, \mathbb{C}))$, $(\text{Herm}(3, \mathbb{O})^{\mathbb{C}}, \text{Alt}(6, \mathbb{C}))$), and may assume $f(x_2) = \Delta_{\mathbf{k}}^{\mathfrak{n}_2^+}(x_2)$. When $(\mathfrak{p}^+, \mathfrak{p}_2^+) = (\mathbb{C}^n, \mathbb{C}^{n''})$, the theorem follows from (6.7) and (6.8). When $(\mathfrak{p}^+, \mathfrak{p}_2^+) = (\text{Sym}(2r_2, \mathbb{C}), \text{M}(r_2, \mathbb{C}))$, $(\text{Alt}(2r_2, \mathbb{C}), \text{M}(r_2, \mathbb{C}))$, (6.6) follows from (3.9) and (3.10), and hence the theorem holds.

Next we consider the cases $(\mathfrak{p}^+, \mathfrak{p}_2^+) = (\text{M}(r_2, \mathbb{C}), \text{Sym}(r_2, \mathbb{C}))$, $d = 2$, $\varepsilon_2 = 1$, and $(\mathfrak{p}^+, \mathfrak{p}_2^+) = (\text{M}(2r_2, \mathbb{C}), \text{Alt}(2r_2, \mathbb{C}))$, $d = 2$, $\varepsilon_2 = 2$. We prove the theorem by induction on r_2 . When $r_2 = 1$, if $\varepsilon_2 = 1$, then since $\mathfrak{p}^+ = \mathfrak{p}_2^+$ holds, the theorem follows from Corollary 2.4. Similarly, if $\varepsilon_2 = 2$, then the theorem follows from (6.8). Next we assume the theorem for $r_2 - 1$, and prove it for r_2 . By (3.3), we have

$$\begin{aligned}
& \det_{\mathfrak{n}^+}(z)^{-\lambda + \frac{n}{r}} \det_{\mathfrak{n}_2^+} \left(\frac{1}{\varepsilon_2} \frac{\partial}{\partial z_2} \right)^{k_{r_2}} \det_{\mathfrak{n}^+}(z)^{\lambda + \frac{2k_{r_2}}{\varepsilon_2} - \frac{n}{r}} \\
& \times \left(\lambda + \frac{2k_{r_2}}{\varepsilon_2} \right)_{\phi_{\varepsilon_2}(\mathbf{k}-\underline{k}_{r_2, r_2}), d} \left\langle \Delta_{\mathbf{k}-\underline{k}_{r_2, r_2}}^{\mathfrak{n}_2^+}(x_2), e^{(x|\bar{z})_{\mathfrak{p}^+}} \right\rangle_{\lambda + \frac{2k_{r_2}}{\varepsilon_2}, x}
\end{aligned}$$

$$\begin{aligned}
&= (\lambda)_{\frac{2k_{r_2}}{\varepsilon_2} \varepsilon_2 r_2, d} \left(\lambda + \frac{2k_{r_2}}{\varepsilon_2} \right)_{\phi_{\varepsilon_2}(\mathbf{k} - \underline{k}_{r_2, r_2}), d} \langle \Delta_{\mathbf{k}}^{n_2^+}(x_2), e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\lambda, x} \\
&= (\lambda)_{\phi_{\varepsilon_2}(\mathbf{k} - \underline{k}_{r_2, r_2}) + \frac{2k_{r_2}}{\varepsilon_2} \varepsilon_2 r_2, d} \langle \Delta_{\mathbf{k}}^{n_2^+}(x_2), e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\lambda, x},
\end{aligned}$$

and by Proposition 3.1 and the induction hypothesis, this is holomorphic for all $\lambda \in \mathbb{C}$. Now we have

$$\begin{aligned}
\phi_1(\mathbf{k} - \underline{k}_{r_2, r_2})_a + 2k_{r_2} &= \min\{k_i + k_j \mid 1 \leq i < j \leq r_2, i + j = 2a + 1\}, \quad 1 \leq a \leq r_2, \\
\phi_2(\mathbf{k} - \underline{k}_{r_2, r_2})_a + k_{r_2} &= \min \left\{ \left\lfloor \frac{k_i + k_j}{2} \right\rfloor \mid 1 \leq i \leq j \leq r_2, i + j = a + 1 \right\}, \quad 1 \leq a \leq 2r_2,
\end{aligned}$$

and therefore by Corollary 2.4, we have

$$\begin{aligned}
\mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+) &\subset \bigoplus_{\substack{\mathbf{m} \in \mathbb{Z}_{++}^{r_2} \\ m_a \leq k_i + k_j, 1 \leq i < j \leq r_2, i + j = 2a + 1}} \mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+), \quad \varepsilon_2 = 1, \\
\mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+) &\subset \bigoplus_{\substack{\mathbf{m} \in \mathbb{Z}_{++}^{2r_2} \\ m_a \leq \left\lfloor \frac{k_i + k_j}{2} \right\rfloor, 1 \leq i \leq j \leq r_2, i + j = a + 1}} \mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+), \quad \varepsilon_2 = 2.
\end{aligned}$$

On the other hand, by (3.7) and (3.8),

$$\begin{aligned}
\text{Hom}_{\mathbb{U}(r_2)}(\mathcal{P}_{\mathbf{k}}(\text{Sym}(r_2, \mathbb{C})), \mathcal{P}_{\mathbf{m}}(\text{M}(r_2, \mathbb{C}))) \neq \{0\} &\quad \text{implies} \quad m_{r_2-i} \leq k_{r_2-2i}, \\
\text{Hom}_{\mathbb{U}(2r_2)}(\mathcal{P}_{\mathbf{k}}(\text{Alt}(2r_2, \mathbb{C})), \mathcal{P}_{\mathbf{m}}(\text{M}(2r_2, \mathbb{C}))) \neq \{0\} &\quad \text{implies} \quad m_{2r_2-i} \leq \left\lfloor \frac{k_{r_2-i}}{2} \right\rfloor,
\end{aligned}$$

and combining with the above formulas, we get

$$\begin{aligned}
\mathcal{P}_{\mathbf{k}}(\text{Sym}(r_2, \mathbb{C})) &\subset \bigoplus_{\substack{\mathbf{m} \in \mathbb{Z}_{++}^{r_2} \\ m_a \leq k_i + k_j, 1 \leq i < j \leq r_2 + 1, i + j = 2a + 1}} \mathcal{P}_{\mathbf{m}}(\text{M}(r_2, \mathbb{C})), \\
\mathcal{P}_{\mathbf{k}}(\text{Alt}(2r_2, \mathbb{C})) &\subset \bigoplus_{\substack{\mathbf{m} \in \mathbb{Z}_{++}^{2r_2} \\ m_a \leq \left\lfloor \frac{k_i + k_j}{2} \right\rfloor, 1 \leq i \leq j \leq r_2 + 1, i + j = a + 1}} \mathcal{P}_{\mathbf{m}}(\text{M}(2r_2, \mathbb{C})),
\end{aligned}$$

with $k_{r_2+1} = 0$. Hence by Corollary 2.4, $(\lambda)_{\phi_{\varepsilon_2}(\mathbf{k}), d} \langle \Delta_{\mathbf{k}}^{n_2^+}(x_2), e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\lambda, x}$ is holomorphic for all $\lambda \in \mathbb{C}$.

Now the case $(\mathfrak{p}^+, \mathfrak{p}_2^+) = (\text{Herm}(3, \mathbb{O})^{\mathbb{C}}, \text{Alt}(6, \mathbb{C}))$ is remaining, but we postpone the proof for this case until Section 7. \blacksquare

By Theorem 6.1 and Proposition 2.6(1), we get the following.

Corollary 6.4. *Let $\text{Re } \lambda > p - 1$, $\mathbf{k} \in \mathbb{Z}_{++}^{r_2}$, and let $f(x_2) \in \mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+)$. Then we have*

$$\|f(x_2)\|_{\lambda, x}^2 = C_{\varepsilon_2}^{d, d_2}(\lambda, \mathbf{k}) \|f(z_2)\|_{F, z}^2,$$

where $C_{\varepsilon_2}^{d, d_2}(\lambda, \mathbf{k})$ is as in (6.1) and (6.2).

Next we give a rough estimate of zeroes of $(\lambda)_{\phi_{\varepsilon_2}(\mathbf{k}), d} C_{\varepsilon_2}^{d, d_2}(\lambda, \mathbf{k})$.

Proposition 6.5. For $\mathbf{k} \in \mathbb{Z}_{++}^{r_2}$, let $C_{\varepsilon_2}^{d,d_2}(\lambda, \mathbf{k})$, $\phi_{\varepsilon_2}(\mathbf{k})$ be as in (6.1)–(6.4).

(1) Let $\varepsilon_2 = 1$, $r_2 = 2$. Then we have

$$\{\lambda \in \mathbb{C} \mid (\lambda)_{\phi_1(\mathbf{k}),d} C_1^{d,d_2}(\lambda, \mathbf{k}) = 0\} \subset \begin{cases} [\frac{d-d_2}{2} - (k_1 + k_2) + 1, \frac{d-d_2}{2} - k_1], & k_2 > 0, \\ \emptyset, & k_2 = 0. \end{cases}$$

(2) Let $\varepsilon_2 = 1$, $d_2 = d/2$. For $2 \leq m_1 \leq m_2 \leq r_2$, if $k_2 = \cdots = k_{m_1}$ and $k_{m_2+1} = 0$, then we have

$$\begin{aligned} & \{\lambda \in \mathbb{C} \mid (\lambda)_{\phi_1(\mathbf{k}),d} C_1^{d,d_2}(\lambda, \mathbf{k}) = 0\} \\ & \subset \begin{cases} [\frac{d}{4}(m_1 - 1) - (k_1 + k_2) + 1, \frac{d}{4}(2m_2 - 3) - k_{m_2-1}], & k_1 > k_2 > 0, \\ [\frac{d}{4}(2 \lceil \frac{m_1}{2} \rceil - 1) - (k_1 + k_2) + 1, \frac{d}{4}(2m_2 - 3) - k_{m_2-1}], & k_1 = k_2 > 0, \\ \emptyset, & k_2 = 0. \end{cases} \end{aligned}$$

(3) Let $\varepsilon_2 = 2$. For $1 \leq m_1, m_2, m_3 \leq r_2$ with $m_1, m_2 \leq m_3$, if $k_1 - k_{m_1} \leq 1$, $k_a = 1$ for $m_2 + 1 \leq a \leq m_3$ and $k_{m_3+1} = 0$, then we have

$$\begin{aligned} & \{\lambda \in \mathbb{C} \mid (\lambda)_{\phi_2(\mathbf{k}),d} C_2^{d,d_2}(\lambda, \mathbf{k}) = 0\} \\ & \subset \begin{cases} [\frac{d}{2}m_1 - k_1 + \frac{3}{2}, \frac{d}{2}(m_2 + m_3 - 1) - \frac{k_{m_2}}{2} + \frac{1}{2}], & k_1 \geq 2, m_2 < m_3, \\ [\frac{d}{2}m_1 - k_1 + \frac{3}{2}, \frac{d}{2}(2m_3 - 1) - \lceil \frac{k_{m_3}}{2} \rceil + \frac{1}{2}], & k_1 \geq 2, m_2 = m_3, \\ \emptyset, & k_1 \leq 1. \end{cases} \end{aligned}$$

Epecially, for $m = 1, 2, \dots, \varepsilon_2 r_2 - 1$, if $\phi_{\varepsilon_2}(\mathbf{k})_{m+1} = 0$ and $\phi_{\varepsilon_2}(\mathbf{k})_m \neq 0$, then $C_{\varepsilon_2}^{d,d_2}(\lambda, \mathbf{k})$ is non-zero holomorphic at $\lambda = \frac{d}{2}m$, and has a pole at $\lambda = \frac{d}{2}(m - 1)$.

Proof. (1) Clear from $(\lambda)_{\phi_1(\mathbf{k}),d} C_1^{d,d_2}(\lambda, \mathbf{k}) = (\lambda + k_1 - \frac{d-d_2}{2})_{k_2}$.

(2) By direct computation, we have

$$\phi_1(\mathbf{k})_a = \begin{cases} k_1 + k_2, & a = 1, \\ 2k_2, & 2 \leq a \leq \lfloor m_1/2 \rfloor, \\ \min\{k_1 + k_{m_1+1}, 2k_2\}, & a = \frac{m_1+1}{2}, \\ 0, & a \geq m_2 + 1, \end{cases}$$

and

$$\begin{aligned} C_1^{d,d/2}(\lambda, \mathbf{k}) &= C_1^{d,d/2}(\lambda, (k_1, \dots, k_{m_2})) \\ &= \frac{1}{\prod_{a=1}^{\lfloor m_1/2 \rfloor} (\lambda - \frac{d}{2}(a-1))_{\phi_1(\mathbf{k})_a}} \frac{\prod_{a=2}^{2m_2-2} \prod_{i=\max\{1, a+1-m_2\}}^{\lfloor a/2 \rfloor} (\lambda - \frac{d}{4}(a-1))_{k_i+k_{a+1-i}}}{\prod_{a=2}^{2m_2-1} \prod_{i=\max\{1, a+1-m_2\}}^{\lfloor a/2 \rfloor} (\lambda - \frac{d}{4}(a-1))_{k_i+k_{a+2-i}}} \\ & \quad \times \begin{cases} \frac{(\lambda - \frac{d}{4}(m_1-1))_{k_1+k_2}}{(\lambda - \frac{d}{4}(m_1-1))_{k_1+k_{m_1+1}} (\lambda - \frac{d}{4}(m_1-1))_{2k_2}}, & m_1: \text{ odd}, \\ 1, & m_1: \text{ even}. \end{cases} \end{aligned}$$

For $2 \leq a \leq 2m_2 - 2$, let

$$\begin{aligned} \phi'(\mathbf{k})_a &:= \begin{cases} \min\{k_i + k_j \mid 1 \leq i < j \leq m_2 + 1, i + j = a + 2\}, & a: \text{ even}, \\ \min_2\{k_i + k_j \mid 1 \leq i < j \leq m_2 + 1, i + j = a + 2\}, & a: \text{ odd}, \end{cases} \\ \psi'(\mathbf{k})_a &:= \max\{k_i + k_j \mid 1 \leq i < j \leq m_2, i + j = a + 1\}, \end{aligned}$$

where \min_2 denotes the second smallest element. Then we have

$$\begin{aligned}
& \{ \lambda \in \mathbb{C} \mid (\lambda)_{\phi_1(\mathbf{k}), d} C_1^{d, d/2}(\lambda, \mathbf{k}) = 0 \} \\
&= \bigcup_{a=2\lceil m_1/2 \rceil + 1, \text{odd}}^{2m_2-3} \left\{ \lambda \in \mathbb{C} \mid \frac{(\lambda - \frac{d}{4}(a-1))_{\phi_1(\mathbf{k})_{\lceil a/2 \rceil}}}{\prod_{i=\max\{1, a+1-r_2\}}^{\lceil a/2 \rceil} (\lambda - \frac{d}{4}(a-1))_{k_i+k_{a+1-i}}} = 0 \right\} \\
&\cup \bigcup_{a=2\lceil m_1/2 \rceil, \text{even}}^{2m_2-2} \left\{ \lambda \in \mathbb{C} \mid \frac{\prod_{i=\max\{1, a+1-r_2\}}^{\lceil a/2 \rceil} (\lambda - \frac{d}{4}(a-1))_{k_i+k_{a+1-i}}}{\prod_{i=\max\{1, a+1-r_2\}}^{\lceil a/2 \rceil} (\lambda - \frac{d}{4}(a-1))_{k_i+k_{a+2-i}}} = 0 \right\} \\
&\cup \left\{ \begin{array}{l} \left\{ \lambda \in \mathbb{C} \mid \frac{(\lambda - \frac{d}{4}(m_1-1))_{k_1+k_2}}{(\lambda - \frac{d}{4}(m_1-1))_{\max\{k_1+k_{m_1+1}, 2k_2\}}} = 0 \right\}, \quad m_1: \text{odd}, \\ \emptyset, \quad m_1: \text{even} \end{array} \right. \\
&\subset \bigcup_{a=2\lceil m_1/2 \rceil}^{2m_2-2} \left\{ \lambda \in \mathbb{C} \mid \frac{(\lambda - \frac{d}{4}(a-1))_{\psi'(\mathbf{k})_a}}{(\lambda - \frac{d}{4}(a-1))_{\phi'(\mathbf{k})_a}} = 0 \right\} \\
&\cup \left\{ \begin{array}{l} \left\{ \lambda \in \mathbb{C} \mid \frac{(\lambda - \frac{d}{4}(m_1-1))_{k_1+k_2}}{(\lambda - \frac{d}{4}(m_1-1))_{\max\{k_1+k_{m_1+1}, 2k_2\}}} = 0 \right\}, \quad \begin{pmatrix} m_1: \text{odd}, \\ k_1 > k_2 \end{pmatrix}, \\ \emptyset, \quad \text{otherwise} \end{array} \right. \\
&\subset \bigcup_{\substack{a=m_1 \ (k_1 > k_2) \\ a=2\lceil m_1/2 \rceil \ (k_1 = k_2)}}^{2m_2-2} \left\{ \frac{d}{4}(a-1) - j \mid j \in \mathbb{Z}, \phi'(\mathbf{k})_a \leq j \leq \psi'(\mathbf{k})_a - 1 \right\} \\
&\subset \begin{cases} [\frac{d}{4}(m_1-1) - (k_1+k_2) + 1, \frac{d}{4}(2m_2-3) - k_{m_2-1}], & k_1 > k_2, \\ [\frac{d}{4}(2\lceil \frac{m_1}{2} \rceil - 1) - (k_1+k_2) + 1, \frac{d}{4}(2m_2-3) - k_{m_2-1}], & k_1 = k_2. \end{cases}
\end{aligned}$$

When $k_2 = 0$, by direct computation we have

$$(\lambda)_{\phi_1(\mathbf{k}), d} C_1^{d, d/2}(\lambda, \mathbf{k}) = \frac{(\lambda)_{k_1}}{(\lambda)_{k_1}} = 1,$$

and this is non-zero everywhere.

If $\phi_1(\mathbf{k})_{m+1} = 0$ and $\phi_1(\mathbf{k})_m \neq 0$, then $k_{m+1} = 0$ and $k_m \neq 0$ hold, and by the above formula $(\lambda)_{\phi_1(\mathbf{k}), d} C_1^{d, d/2}(\lambda, \mathbf{k})$ is non-zero at $\lambda = \frac{d}{2}m$, $\frac{d}{2}(m-1)$. Hence we get the last claim.

(3) It is enough to prove for the following three cases:

- (i) $1 \leq m_1 \leq m_2 \leq m_3 \leq r_2$, $k_{m_2} \geq 2$.
- (ii) $1 \leq m_2 < m_1 = m_3 \leq r_2$, $k_{m_2} \geq 2$, i.e., $\mathbf{k} = (\underbrace{2, \dots, 2}_{m_2}, \underbrace{1, \dots, 1}_{m_3-m_2}, \underbrace{0, \dots, 0}_{r_2-m_3})$.
- (iii) $k_1 \leq 1$, i.e., $\mathbf{k} = (1, \dots, 1, 0, \dots, 0)$.

First we consider Case (i). We take $1 \leq m_0 \leq m_1$ such that $k_a = k_1$ for $1 \leq a \leq m_0$ and $k_a = k_1 - 1$ for $m_0 + 1 \leq a \leq m_1$. By direct computation we have

$$\phi_2(\mathbf{k})_a = \begin{cases} k_1, & 1 \leq a \leq m_0, \\ k_1 - 1, & m_0 + 1 \leq a \leq m_1, \\ \lfloor \frac{k_1 + k_{m_1+1}}{2} \rfloor, & a = m_1 + 1, \\ 1, & 2m_2 + 1 \leq a \leq m_2 + m_3, \\ 0, & m_2 + m_3 + 1 \leq a, \end{cases}$$

and

$$\begin{aligned}
C_2^{d,2d}(\lambda, \mathbf{k}) &= C_2^{d,2d}(\lambda, (k_1, \dots, k_{m_3})) \\
&= \frac{1}{\prod_{a=1}^{m_1+1} (\lambda - \frac{d}{2}(a-1))_{\phi_0(\mathbf{k})_a}} \frac{(\lambda - \frac{1}{2} - \frac{d}{2}m_1)_{k_1}}{(\lambda - \frac{1}{2} - \frac{d}{2}m_1)_{\lceil \frac{k_1+k_{m_1}+1}{2} \rceil}} \\
&\quad \times \prod_{a=m_1+2}^{\min\{2m_2, 2m_3-1\}} \left(\frac{\prod_{i=\max\{1, a-m_3\}}^{\lceil a/2 \rceil - 1} (\lambda - \frac{d}{2}(a-1))_{\lfloor \frac{k_i+k_{a-i}}{2} \rfloor}}{\prod_{i=\max\{1, a-m_3\}}^{\lceil a/2 \rceil} (\lambda - \frac{d}{2}(a-1))_{\lfloor \frac{k_i+k_{a+1-i}}{2} \rfloor}} \right. \\
&\quad \quad \quad \times \frac{\prod_{i=\max\{1, a-m_3\}}^{\lceil a/2 \rceil} (\lambda - \frac{1}{2} - \frac{d}{2}(a-1))_{\lfloor \frac{k_i+k_{a-i}}{2} \rfloor}}{\prod_{i=\max\{1, a-m_3\}}^{\lceil a/2 \rceil} (\lambda - \frac{1}{2} - \frac{d}{2}(a-1))_{\lfloor \frac{k_i+k_{a+1-i}}{2} \rfloor}} \left. \right) \\
&\quad \times \begin{cases} \frac{\prod_{a=\max\{2m_2, m_3\}+1}^{m_2+m_3} (\lambda - \frac{d}{2}(a-1) + \frac{k_{a-m_3}-1}{2})}{\prod_{a=2m_2+1}^{m_2+m_3} (\lambda - \frac{d}{2}(a-1))}, & m_2 < m_3, \\ \frac{1}{(\lambda - \frac{d}{2}(2m_3-1))_{\lfloor k_{m_3}/2 \rfloor}} \frac{(\lambda - \frac{1}{2} - \frac{d}{2}(2m_3-1))_{k_{m_3}}}{(\lambda - \frac{1}{2} - \frac{d}{2}(2m_3-1))_{\lfloor k_{m_3}/2 \rfloor}}, & m_2 = m_3. \end{cases}
\end{aligned}$$

Let

$$\begin{aligned}
\phi(\mathbf{k})_a &:= \min\{k_i + k_j \mid 1 \leq i \leq j \leq m_3 + 1, i + j = a + 1\}, & 1 \leq a \leq 2m_3, \\
\phi'(\mathbf{k})_a &:= \min_2\{k_i + k_j \mid 1 \leq i \leq j \leq m_3 + 1, i + j = a + 1\}, & 3 \leq a \leq 2m_3 - 1, \\
\phi''(\mathbf{k})_a &:= \min\{k_i + k_j \mid 1 \leq i < j \leq m_3 + 1, i + j = a + 1\}, & 2 \leq a \leq 2m_3, \\
\psi(\mathbf{k})_a &:= \max\{k_i + k_j \mid 1 \leq i \leq j \leq m_3, i + j = a\}, & 2 \leq a \leq 2m_3, \\
\psi''(\mathbf{k})_a &:= \max\{k_i + k_j \mid 1 \leq i < j \leq m_3, i + j = a\}, & 3 \leq a \leq 2m_3 - 1,
\end{aligned}$$

where \min_2 denotes the second smallest element, so that $\phi_2(\mathbf{k})_a = \lfloor \phi(\mathbf{k})_a/2 \rfloor$ holds for $1 \leq a \leq 2m_3$. Then since the zeroes of

$$\begin{aligned}
&\frac{\prod_{i=\max\{1, a-m_3\}}^{\lceil a/2 \rceil - 1} (\lambda - \frac{d}{2}(a-1))_{\lfloor \frac{k_i+k_{a-i}}{2} \rfloor}}{\prod_{i=\max\{1, a-m_3\}}^{\lceil a/2 \rceil} (\lambda - \frac{d}{2}(a-1))_{\lfloor \frac{k_i+k_{a+1-i}}{2} \rfloor}} \\
&\quad \times \frac{\prod_{i=\max\{1, a-m_3\}}^{\lceil a/2 \rceil} (\lambda - \frac{1}{2} - \frac{d}{2}(a-1))_{\lfloor \frac{k_i+k_{a-i}}{2} \rfloor}}{\prod_{i=\max\{1, a-m_3\}}^{\lceil a/2 \rceil} (\lambda - \frac{1}{2} - \frac{d}{2}(a-1))_{\lfloor \frac{k_i+k_{a+1-i}}{2} \rfloor}} \times \left(\lambda - \frac{d}{2}(a-1) \right)_{\phi_2(\mathbf{k})_a}
\end{aligned}$$

are contained in

$$\begin{aligned}
&\left\{ \lambda \in \mathbb{C} \mid \frac{(\lambda - \frac{d}{2}(a-1))_{\lfloor \psi''(\mathbf{k})_a/2 \rfloor}}{(\lambda - \frac{d}{2}(a-1))_{\lfloor \phi'(\mathbf{k})_a/2 \rfloor}} \frac{(\lambda - \frac{1}{2} - \frac{d}{2}(a-1))_{\lfloor \psi(\mathbf{k})_a/2 \rfloor}}{(\lambda - \frac{1}{2} - \frac{d}{2}(a-1))_{\lfloor \phi''(\mathbf{k})_a/2 \rfloor}} = 0 \right\} \\
&\quad \subset \left\{ \lambda \in \mathbb{C} \mid \frac{(2\lambda - 1 - d(a-1))_{\psi(\mathbf{k})_a}}{(2\lambda - 1 - d(a-1))_{\phi(\mathbf{k})_a}} = 0 \right\} \\
&\quad = \left\{ \frac{d}{2}(a-1) - \frac{j}{2} + \frac{1}{2} \mid j \in \mathbb{Z}, \phi(\mathbf{k})_a \leq j \leq \psi(\mathbf{k})_a - 1 \right\}
\end{aligned}$$

for $m_1 + 2 \leq a \leq \min\{2m_2, 2m_3 - 1\}$, we have

$$\left\{ \lambda \in \mathbb{C} \mid (\lambda)_{\phi_2(\mathbf{k}), d} C_2^{d,2d}(\lambda, \mathbf{k}) = 0 \right\}$$

$$\begin{aligned}
& \subset \left\{ \frac{d}{2}m_1 - j + \frac{1}{2} \mid j \in \mathbb{Z}, \left\lceil \frac{k_1 + k_{m_1+1}}{2} \right\rceil \leq j \leq k_1 - 1 \right\} \\
& \cup \bigcup_{a=m_1+2}^{\min\{2m_2, 2m_3-1\}} \left\{ \frac{d}{2}(a-1) - \frac{j}{2} + \frac{1}{2} \mid j \in \mathbb{Z}, \phi(\mathbf{k})_a \leq j \leq \psi(\mathbf{k})_a - 1 \right\} \\
& \cup \begin{cases} \left\{ \frac{d}{2}(a-1) - \frac{k_{a-m_3}}{2} + \frac{1}{2} \mid a \in \mathbb{Z}, \max\{2m_2, m_3\} + 1 \leq a \leq m_2 + m_3 \right\}, & m_2 < m_3, \\ \left\{ \frac{d}{2}(2m_3 - 1) - j + \frac{1}{2} \mid j \in \mathbb{Z}, \left\lceil \frac{k_{m_3}}{2} \right\rceil \leq j \leq k_{m_3} - 1 \right\}, & m_2 = m_3 \end{cases} \\
& \subset \begin{cases} \left[\frac{d}{2}m_1 - k_1 + \frac{3}{2}, \frac{d}{2}(m_2 + m_3 - 1) - \frac{k_{m_2}}{2} + \frac{1}{2} \right], & m_2 < m_3, \\ \left[\frac{d}{2}m_1 - k_1 + \frac{3}{2}, \frac{d}{2}(2m_3 - 1) - \left\lceil \frac{k_{m_3}}{2} \right\rceil + \frac{1}{2} \right], & m_2 = m_3. \end{cases}
\end{aligned}$$

Next we consider Case (ii). By direct computation we have

$$\phi_2(\mathbf{k}) = \left(\underbrace{2, \dots, 2}_{m_2}, \underbrace{1, \dots, 1}_{m_3}, \underbrace{0, \dots, 0}_{2r_2 - m_2 - m_3} \right), \quad C_1^{d, 2d}(\lambda, \mathbf{k}) = \frac{\prod_{a=m_3+1}^{m_2+m_3} \left(\lambda + \frac{1}{2} - \frac{d}{2}(a-1) \right)}{(\lambda)_{\phi_2(\mathbf{k}), d}},$$

and since $m_1 = m_3$, we get

$$\begin{aligned}
\{ \lambda \in \mathbb{C} \mid (\lambda)_{\phi_2(\mathbf{k}), d} C_2^{d, 2d}(\lambda, \mathbf{k}) = 0 \} &= \left\{ \frac{d}{2}(a-1) - \frac{1}{2} \mid a \in \mathbb{Z}, m_3 + 1 \leq a \leq m_2 + m_3 \right\} \\
&\subset \left[\frac{d}{2}m_1 - 2 + \frac{3}{2}, \frac{d}{2}(m_2 + m_3 - 1) - \frac{2}{2} + \frac{1}{2} \right].
\end{aligned}$$

Similarly, for Case (iii), when $k_1 = \dots = k_{m_3} = 1$ and $k_{m_3+1} = 0$, $0 \leq m_3 \leq r_2$, by direct computation we have

$$(\lambda)_{\phi_2(\mathbf{k}), d} C_1^{d, 2d}(\lambda, \mathbf{k}) = \frac{\prod_{a=1}^{m_3} \left(\lambda - \frac{d}{2}(a-1) \right)}{\prod_{a=1}^{m_3} \left(\lambda - \frac{d}{2}(a-1) \right)} = 1,$$

and this is non-zero everywhere.

If $\phi_2(\mathbf{k})_{m+1} = 0$ and $\phi_2(\mathbf{k})_m \neq 0$, then $k_{m_2} \geq 2$, $k_{m_2+1} = \dots = k_{m_3} = 1$ and $k_{m_3+1} = 0$ hold for some $m_2 \leq m_3$ with $m_2 + m_3 = m$, and by the above formula $(\lambda)_{\phi_2(\mathbf{k}), d} C_2^{d, 2d}(\lambda, \mathbf{k})$ is non-zero at $\lambda = \frac{d}{2}m$, $\frac{d}{2}(m-1)$. Hence we get the last claim. \blacksquare

Epecially, if $\phi_{\varepsilon_2}(\mathbf{k})_{a+1} = 0$ and $\phi_{\varepsilon_2}(\mathbf{k})_a \neq 0$, then for a non-zero polynomial $f(x_2) \in \mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+)$, $\langle f(x_2), e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\lambda, x}$ has a pole at $\lambda = \frac{d}{2}(a-1)$, and combining with Corollary 2.4 and (6.6), we have

$$\mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+) \subset \bigoplus_{\substack{\mathbf{m} \in \mathbb{Z}_{++}^{\varepsilon_2 r_2} \\ m_{a+1} = 0}} \mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+), \quad \mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+) \not\subset \bigoplus_{\substack{\mathbf{m} \in \mathbb{Z}_{++}^{\varepsilon_2 r_2} \\ m_a = 0}} \mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+),$$

that is, for $a = 0, 1, \dots, r-1$, we have

$$\begin{aligned}
\mathcal{P}(\mathfrak{p}_2^+) \cap \bigoplus_{\substack{\mathbf{m} \in \mathbb{Z}_{++}^r \\ m_{a+1} = 0}} \mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+) &= \bigoplus_{\substack{\mathbf{k} \in \mathbb{Z}_{++}^{r_2} \\ \phi_{\varepsilon_2}(\mathbf{k})_{a+1} = 0}} \mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+) \\
&= \begin{cases} \bigoplus_{\substack{\mathbf{k} \in \mathbb{Z}_{++}^{r_2} \\ k_{a+1} = 0}} \mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+), & \varepsilon_2 = 1, \\ \bigoplus_{\substack{0 \leq a_1 \leq a_2 \leq r_2 \\ a_1 + a_2 \leq a}} \bigoplus_{\substack{\mathbf{k} \in \mathbb{Z}_{++}^{a_1} \\ k_{a_1} \geq 2}} \mathcal{P}_{(\mathbf{k}, \mathbf{1}_{a_2 - a_1}, \mathbf{0}_{r_2 - a_2})}(\mathfrak{p}_2^+), & \varepsilon_2 = 2, \end{cases} \quad (6.9)
\end{aligned}$$

where we set $\phi_{\varepsilon_2}(\mathbf{k})_{\varepsilon_2 r_2 + 1} = \dots = \phi_{\varepsilon_2}(\mathbf{k})_r := 0$ and $k_0 := +\infty$.

Next we consider general $\mathbf{k} \in \mathbb{Z}_{++}^{r_2}$. Then again by the above proposition, $\frac{1}{C_{\varepsilon_2}^{d,d_2}(\lambda, \mathbf{k})} \langle f(x_2), e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\lambda, x}$ is holomorphic for $\operatorname{Re} \lambda \geq \frac{d}{2}(\varepsilon_2 r_2 - 1) + 1 - \lfloor \frac{k_{r_2}}{\varepsilon_2} \rfloor$. Then by Theorem 3.5, comparing the top terms, we get the following.

Theorem 6.6. *Suppose \mathfrak{p}^+ , \mathfrak{p}_2^+ are of tube type, and consider the meromorphic continuation of $\langle \cdot, \cdot \rangle_{\lambda}$. Then for $\mathbf{k} \in \mathbb{Z}_{++}^{r_2}$, $a = 1, 2, \dots, \lfloor k_{r_2}/\varepsilon_2 \rfloor$ and for $f(x_2) \in \mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+)$, we have*

$$\begin{aligned} \frac{1}{C_{\varepsilon_2}^{d,d_2}(\lambda, \mathbf{k})} \langle f(x_2), e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\lambda, x} \Big|_{\lambda = \frac{n}{r} - a} \\ = \frac{\det_{\mathfrak{n}^+}(z)^a}{C_{\varepsilon_2}^{d,d_2}(\frac{n}{r} + a, \mathbf{k} - \varepsilon_2 a_{r_2})} \langle \det_{\mathfrak{n}_2^+}(x_2)^{-\varepsilon_2 a} f(x_2), e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\frac{n}{r} + a, x}. \end{aligned}$$

6.2 Results on restriction of $\mathcal{H}_{\lambda}(D)$ to subgroups

Next we consider the decomposition of the holomorphic discrete series representation of scalar type $\mathcal{H}_{\lambda}(D)$ under the subgroup $\tilde{G}_1 \subset \tilde{G}$. By Theorem 2.5, we have

$$\mathcal{H}_{\lambda}(D)|_{\tilde{G}_1} \simeq \sum_{\mathbf{k} \in \mathbb{Z}_{++}^{r_2}}^{\oplus} \mathcal{H}_{\varepsilon_1 \lambda}(D_1, \mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+)),$$

where

$$\begin{aligned} & \mathcal{H}_{\varepsilon_1 \lambda}(D_1, \mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+)) \\ & \simeq \begin{cases} \mathcal{H}_{\lambda+|\mathbf{k}|}(D_{\mathrm{SO}_0(2, n')}) \boxtimes V_{(k_1-k_2, 0, \dots, 0)}^{[n']\vee} & \text{(Case 1),} \\ \mathcal{H}_{\lambda}(D_{\mathrm{Sp}(r', \mathbb{R})}, V_{\mathbf{k}}^{(r')\vee}) \hat{\boxtimes} \mathcal{H}_{\lambda}(D_{\mathrm{Sp}(r'', \mathbb{R})}, V_{\mathbf{k}}^{(r'')\vee}) & \text{(Case 2),} \\ \mathcal{H}_{\lambda}(D_{\mathrm{SO}^*(2s')}, V_{\mathbf{k}}^{(s')\vee}) \hat{\boxtimes} \mathcal{H}_{\lambda}(D_{\mathrm{SO}^*(2s'')}, V_{\mathbf{k}}^{(s'')\vee}) & \text{(Case 3),} \\ \mathcal{H}_{2\lambda}(D_{\mathrm{SO}^*(2r)}, V_{2\mathbf{k}}^{(r)\vee}) & \text{(Case 4),} \\ \mathcal{H}_{\lambda}(D_{\mathrm{Sp}(r, \mathbb{R})}, V_{\mathbf{k}^2}^{(r)\vee}) & \text{(Case 5),} \\ \mathcal{H}_{\lambda}(D_{\mathrm{SU}(2, 6)}, \mathbb{C} \boxtimes V_{(k_1+k_2, k_1+k_3, k_2+k_3)^2}^{(6)}) & \text{(Case 6),} \\ \mathcal{H}_{\lambda}(D_{\mathrm{SO}^*(12)}, V_{(\frac{k_1+k_2}{2}, \frac{k_1+k_2}{2}, \frac{k_1+k_2}{2}, \frac{k_1+k_2}{2}, \frac{k_1-k_2}{2}, \frac{-k_1+k_2}{2})}^{(6)\vee}) \boxtimes V_{(k_1, k_2)}^{(2)\vee} & \text{(Case 7),} \\ \mathcal{H}_{\lambda+|\mathbf{k}|}(D_{\mathrm{SL}(2, \mathbb{R})}) \hat{\boxtimes} \mathcal{H}_{\lambda+|\mathbf{k}|}(D_{\mathrm{Spin}_0(2, 10)}, V_{(\frac{k_1+k_2}{2}, \frac{k_1-k_2}{2}, \frac{k_1-k_2}{2}, \frac{k_1-k_2}{2}, \frac{k_1-k_2}{2})}^{[10]\vee}) & \text{(Case 8),} \\ \mathcal{H}_{\lambda}(D_{\mathrm{SU}(2, 4)}, \mathbb{C} \boxtimes V_{(k_1+k_2, k_1+k_2, k_1, k_2)}^{(4)}) \boxtimes V_{(k_1, k_2)}^{(2)\vee} & \text{(Case 9),} \\ \mathcal{H}_{\lambda+\frac{|\mathbf{k}|}{2}}(D_{\mathrm{SL}(2, \mathbb{R})}) \hat{\boxtimes} \mathcal{H}_{\lambda}(D_{\mathrm{SU}(1, 5)}, \mathbb{C} \boxtimes V_{(k_1+k_2, k_1, k_1, k_2, k_2)}^{(5)}) & \text{(Case 10).} \end{cases} \end{aligned}$$

For each $\mathbf{k} \in \mathbb{Z}_{++}^{r_2}$, let $V_{\mathbf{k}}$ be an abstract K_1 -module isomorphic to $\mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+)$, let $\|\cdot\|_{\varepsilon_1 \lambda, \mathbf{k}}$ be the \tilde{G}_1 -invariant norm on $\mathcal{H}_{\varepsilon_1 \lambda}(D_1, V_{\mathbf{k}})$ normalized such that $\|v\|_{\varepsilon_1 \lambda, \mathbf{k}} = \|v\|_{V_{\mathbf{k}}}$ holds for all constant functions $v \in V_{\mathbf{k}}$, and for $\lambda > p - 1$ let

$$\mathcal{F}_{\lambda, \mathbf{k}}^{\downarrow}: \mathcal{H}_{\lambda}(D)|_{\tilde{G}_1} \longrightarrow \mathcal{H}_{\varepsilon_1 \lambda}(D_1, V_{\mathbf{k}})$$

be the symmetry breaking operator given in (2.18) and (2.19), using a vector-valued polynomial $\mathbf{K}_{\mathbf{k}}(x_2) \in (\mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+) \otimes \overline{V_{\mathbf{k}}})^{K_1}$ satisfying

$$|\langle f(x_2), \mathbf{K}_{\mathbf{k}}(x_2) \rangle_{F, x}|_{V_{\mathbf{k}}} = \|f(z_2)\|_{F, z}, \quad f(x_2) \in \mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+),$$

so that

$$\|\mathcal{F}_{\lambda, \mathbf{k}}^{\downarrow}(f(x_2))\|_{\varepsilon_1 \lambda, \mathbf{k}} = \|f(z_2)\|_{F, z}, \quad f(x_2) \in \mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+)$$

holds. Also, when \mathfrak{p}_2^+ is of tube type, we fix $[K_1, K_1]$ -isomorphisms $V_{\mathbf{k}+a_{r_2}} \simeq V_{\mathbf{k}}$ for each $a \in \mathbb{Z}_{>0}$. Then by Proposition 2.6 (3) and Theorem 6.1, the following Parseval–Plancherel-type formula holds.

Corollary 6.7. *For $\lambda > p - 1$ and for $f \in \mathcal{H}_\lambda(D)$, we have*

$$\|f\|_\lambda^2 = \sum_{\mathbf{k} \in \mathbb{Z}_{++}^{r_2}} C_{\varepsilon_2}^{d, d_2}(\lambda, \mathbf{k}) \|\mathcal{F}_{\lambda, \mathbf{k}}^\downarrow f\|_{\varepsilon_1 \lambda, \mathbf{k}}^2,$$

where $C_{\varepsilon_2}^{d, d_2}(\lambda, \mathbf{k})$ is as in (6.1) and (6.2).

Next we consider the meromorphic continuation for smaller λ . Then by Propositions 2.7, 6.5, Theorems 6.2, 6.6 and the formula (6.9), we have the following.

Corollary 6.8. *For $\mathbf{k} \in \mathbb{Z}_{++}^{r_2}$, let $\phi_{\varepsilon_2}(\mathbf{k}) \in \mathbb{Z}_{++}^{\varepsilon_2 r_2}$ be as in (6.3) and (6.4), and set $\phi_{\varepsilon_2}(\mathbf{k})_{\varepsilon_2 r_2 + 1} = \cdots = \phi_{\varepsilon_2}(\mathbf{k})_r := 0$.*

(1) *For $a = 1, 2, \dots, r$,*

$$d\tau_\lambda(\mathcal{U}(\mathfrak{g}_1))\mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+) \subset M_a^{\mathfrak{g}}(\lambda)$$

holds if

$$\lambda \in \frac{d}{2}(a - 1) - \phi_{\varepsilon_2}(\mathbf{k})_a - \mathbb{Z}_{\geq 0},$$

where $M_a^{\mathfrak{g}}(\lambda) \subset \mathcal{O}_\lambda(D)_{\tilde{K}}$ is the $(\mathfrak{g}, \tilde{K})$ -submodule given in (2.12).

(2) *For $a = 0, 1, \dots, r - 1$, we have*

$$\begin{aligned} \mathcal{H}_{\frac{d}{2}a}(D)|_{\tilde{G}_1} &\simeq \sum_{\substack{\mathbf{k} \in \mathbb{Z}_{++}^{r_2} \\ \phi_{\varepsilon_2}(\mathbf{k})_{a+1} = 0}}^\oplus \mathcal{H}_{\varepsilon_1 \frac{d}{2}a}(D_1, \mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+)) \\ &= \begin{cases} \sum_{\substack{\mathbf{k} \in \mathbb{Z}_{++}^{r_2} \\ k_{a+1} = 0}}^\oplus \mathcal{H}_{\varepsilon_1 \frac{d}{2}a}(D_1, \mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+)), & \varepsilon_2 = 1, \\ \sum_{\substack{0 \leq a_1 \leq a_2 \leq r_2 \\ a_1 + a_2 \leq a}}^\oplus \sum_{\substack{\mathbf{k} \in \mathbb{Z}_{++}^{a_1} \\ k_{a_1} \geq 2}}^\oplus \mathcal{H}_{\varepsilon_1 \frac{d}{2}a}(D_1, \mathcal{P}_{(\mathbf{k}, \perp_{a_2 - a_1}, \mathfrak{p}_{r_2 - a_2})}(\mathfrak{p}_2^+)), & \varepsilon_2 = 2, \end{cases} \end{aligned}$$

where we set $k_0 := +\infty$.

(3) *For $a = 0, 1, \dots, r - 1$, if $\phi_{\varepsilon_2}(\mathbf{k})_{a+1} = 0$, then $\mathcal{F}_{\lambda, \mathbf{k}}^\downarrow$ is holomorphic at $\lambda = \frac{d}{2}a$, and its restriction gives the symmetry breaking operator*

$$\mathcal{F}_{\frac{d}{2}a, \mathbf{k}}^\downarrow : \mathcal{H}_{\frac{d}{2}a}(D)|_{\tilde{G}_1} \longrightarrow \mathcal{H}_{\varepsilon_1 \frac{d}{2}a}(D_1, V_{\mathbf{k}}).$$

(4) *Suppose \mathfrak{p}^+ , \mathfrak{p}_2^+ are of tube type. For $a = 1, 2, \dots, \lfloor k_{r_2}/\varepsilon_2 \rfloor$, if*

$$K_{\mathbf{k}}(x_2) = c \det_{\mathfrak{n}_2^+}(x_2)^{\varepsilon_2 a} K_{\mathbf{k} - \varepsilon_2 a_{r_2}}(x_2),$$

then we have

$$\begin{aligned} \mathcal{F}_{\frac{n}{r} - a, \mathbf{k}}^\downarrow &= c \mathcal{F}_{\frac{n}{r} + a, \mathbf{k} - \varepsilon_2 a_{r_2}}^\downarrow \circ \det_{\mathfrak{n}^-} \left(\frac{\partial}{\partial x} \right)^a : \\ \mathcal{O}_{\frac{n}{r} - a}(D)|_{\tilde{G}_1} &\longrightarrow \mathcal{O}_{\varepsilon_1(\frac{n}{r} - a)}(D_1, V_{\mathbf{k}}) \simeq \mathcal{O}_{\varepsilon_1(\frac{n}{r} + a)}(D_1, V_{\mathbf{k} - \varepsilon_2 a_{r_2}}). \end{aligned}$$

If \mathbf{k} satisfies the condition in Remark 6.3 (1), then “only if” in Corollary 6.8 (1) also holds. Also, Corollary 6.8 (2) for $\lambda = \frac{d}{2}$ ($a = 1$ case) is earlier given in [41], and that for $(G, G_1) = (\mathrm{SU}(r, r), \mathrm{SO}^*(2r))$ is earlier given in [55]. See also [30] for $(G, G_1) = (\mathrm{O}(p, q), \mathrm{O}(p, q') \times \mathrm{O}(q''))$ case. The parameter sets in Corollary 6.8 (2) also appear in Howe’s correspondence for the dual pairs $(\mathrm{SO}^*(2r_2), \mathrm{Sp}(a))$, $(\mathrm{Sp}(r_2, \mathbb{R}), \mathrm{O}(a))$ (see, e.g., [1, 24, 52]), and especially, we can prove (2) for Cases 2–5 by using the seesaw dual pair theory (see, e.g., [18, Section 3], [37]) as in [41].

7 Case \mathfrak{p}_2^+ is simple of rank 3

In the previous section, we skipped the proof of Theorem 6.2 for $(\mathfrak{p}^+, \mathfrak{p}_2^+) = (\mathrm{Herm}(3, \mathbb{O})^{\mathbb{C}}, \mathrm{Alt}(6, \mathbb{C}))$. In this section we assume that \mathfrak{p}_2^+ is simple, \mathfrak{p}^+ and \mathfrak{p}_2^+ are both of tube type, and $\mathrm{rank} \mathfrak{p}^+ = \mathrm{rank} \mathfrak{p}_2^+ = 3$, that is, we treat the cases

$$\begin{aligned} (\mathfrak{p}^+, \mathfrak{p}_1^+, \mathfrak{p}_2^+) &= (\mathfrak{p}^+, (\mathfrak{p}^+)^{\sigma}, (\mathfrak{p}^+)^{-\sigma}) \\ &= (\mathrm{Herm}(3, \mathbb{F})^{\mathbb{C}}, \mathrm{Alt}(3, \mathbb{F}')^{\mathbb{C}}, \mathrm{Herm}(3, \mathbb{F}')^{\mathbb{C}}), \quad (\mathbb{F}, \mathbb{F}') = (\mathbb{C}, \mathbb{R}), (\mathbb{H}, \mathbb{C}), (\mathbb{O}, \mathbb{H}) \\ &\simeq \begin{cases} (\mathrm{M}(3, \mathbb{C}), \mathrm{Alt}(3, \mathbb{C}), \mathrm{Sym}(3, \mathbb{C})) & (\text{Case 1, } (\mathbb{F}, \mathbb{F}') = (\mathbb{C}, \mathbb{R})), \\ (\mathrm{Alt}(6, \mathbb{C}), \mathrm{Alt}(3, \mathbb{C}) \oplus \mathrm{Alt}(3, \mathbb{C}), \mathrm{M}(3, \mathbb{C})) & (\text{Case 2, } (\mathbb{F}, \mathbb{F}') = (\mathbb{H}, \mathbb{C})), \\ (\mathrm{Herm}(3, \mathbb{O})^{\mathbb{C}}, \mathrm{M}(2, 6; \mathbb{C}), \mathrm{Alt}(6, \mathbb{C})) & (\text{Case 3, } (\mathbb{F}, \mathbb{F}') = (\mathbb{O}, \mathbb{H})). \end{cases} \end{aligned}$$

Then the corresponding symmetric pairs are

$$(G, G_1) = \begin{cases} (\mathrm{SU}(3, 3), \mathrm{SO}^*(6)) & (\text{Case 1}), \\ (\mathrm{SO}^*(12), \mathrm{SO}^*(6) \times \mathrm{SO}^*(6)) & (\text{Case 2}), \\ (E_{7(-25)}, \mathrm{SU}(2, 6)) & (\text{Case 3}), \end{cases}$$

and for these cases we have $d = \dim_{\mathbb{R}} \mathbb{F}$, $d_2 = \dim_{\mathbb{R}} \mathbb{F}' = \frac{d}{2}$, $\varepsilon_2 = 1$. The purpose of this section is to prove Theorem 6.2 for these cases, that is, for $\mathbf{k} \in \mathbb{Z}_{+++}^3$, $f(x_2) \in \mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+)$, we prove that

$$(\lambda)_{\phi_1(\mathbf{k}), d} \langle f(x_2), e^{(x|\bar{x})_{\mathfrak{p}^+}} \rangle_{\lambda, x}$$

is holomorphically continued for all $\lambda \in \mathbb{C}$, where

$$\phi_1(\mathbf{k}) := (k_1 + k_2, \min\{k_1, k_2 + k_3\}, k_3) \in \mathbb{Z}_{+++}^3.$$

7.1 Preliminaries

First we prepare some notations. We fix a maximal tripotent $e \in \mathfrak{p}_2^+ \subset \mathfrak{p}^+$, and regard \mathfrak{p}^+ , \mathfrak{p}_2^+ as Jordan algebras of rank 3. Let $\mathfrak{n}_2^+ \subset \mathfrak{n}^+$ be the Euclidean real forms, let $\Omega \subset \mathfrak{n}^+$ be the symmetric cone, and consider the \mathbb{C} -bilinear form $(\cdot|\cdot)_{\mathfrak{n}^+} := (\cdot|Q(\bar{e})\cdot)_{\mathfrak{p}^+} : \mathfrak{p}^+ \times \mathfrak{p}^+ \rightarrow \mathbb{C}$, and the determinant polynomials $\det_{\mathfrak{n}^+}(x)$, $\det_{\mathfrak{n}_2^+}(x_2)$ on \mathfrak{p}^+ , \mathfrak{p}_2^+ . Then $\det_{\mathfrak{n}_2^+}$ coincides with the restriction of $\det_{\mathfrak{n}^+}$ on \mathfrak{n}_2^+ . Let $K \subset G$, $K_1 \subset G_1$ be the maximal compact subgroups, with the complexification $K^{\mathbb{C}}$, $K_1^{\mathbb{C}}$, and let $L_2 \subset K_1^{\mathbb{C}}$ be the subgroup with the Lie algebra $\mathfrak{l}_2 = [\mathfrak{n}_2^+, \mathfrak{n}_2^-] \simeq \mathbb{R} \oplus \mathfrak{sl}(3, \mathbb{F}')$, defined as in Section 2.2.

Next we fix a Jordan frame $\{e_1, e_2, e_3\} \subset \mathfrak{p}_2^+ \subset \mathfrak{p}^+$ with $\sum_{j=1}^3 e_j = e$, set $e^k := \sum_{j=1}^k e_j$, $\check{e}^k := \sum_{j=4-k}^3 e_j$, $k = 1, 2, 3$, and let $\Delta_k(x) := \det_{\mathfrak{n}^+(e^k)}(x)$, $\check{\Delta}_k(x) := \det_{\mathfrak{n}^+(\check{e}^k)}(x)$ be the determinant polynomials on $\mathfrak{p}^+(e^k)_2$, $\mathfrak{p}^+(\check{e}^k)_2 \subset \mathfrak{p}^+$, regarded as polynomials on \mathfrak{p}^+ , as in Section 2.4. Especially we have $\Delta_3(x) = \check{\Delta}_3(x) = \det_{\mathfrak{n}^+}(x)$. Also, for $\mathbf{m} \in \mathbb{Z}_{++}^3$, $x_2 \in \mathfrak{p}_2^+$, as in (2.5) let

$$\begin{aligned} \Delta_{\mathbf{m}}^{\mathfrak{n}_2^+}(x_2) &= \Delta_1(x_2)^{m_1-m_2} \Delta_2(x_2)^{m_2-m_3} \Delta_3(x_2)^{m_3}, \\ \check{\Delta}_{\mathbf{m}}^{\mathfrak{n}_2^+}(x_2) &= \check{\Delta}_1(x_2)^{m_1-m_2} \check{\Delta}_2(x_2)^{m_2-m_3} \check{\Delta}_3(x_2)^{m_3}. \end{aligned}$$

Let $M_{L_2}A_{L_2}N_{L_2}^\top \subset L_2$ be the minimal parabolic subgroup defined as in Section 2.2, so that $\Delta_{\mathfrak{m}}^{\mathfrak{n}^+}(x_2)$ is relatively invariant under the action of $M_{L_2}A_{L_2}N_{L_2}^\top$.

Next, for $x \in \mathfrak{p}^+$, let $x^\sharp \in \mathfrak{p}^+$ be the adjugate element, which is characterized by

$$x^\sharp := \det_{\mathfrak{n}^+}(x)x^{-1}$$

for invertible x , so that

$$\Delta_2(x) = \check{\Delta}_1(x^\sharp), \quad \check{\Delta}_2(x) = \Delta_1(x^\sharp)$$

hold. Then the following holds.

Lemma 7.1. *Let $\mathfrak{p}^+ = \mathfrak{p}_1^+ \oplus \mathfrak{p}_2^+$ be as above.*

- (1) *For $x, y \in \mathfrak{p}^+$, we have $\det_{\mathfrak{n}^+}(x+y) = \det_{\mathfrak{n}^+}(x) + (x^\sharp|y)_{\mathfrak{n}^+} + (x|y^\sharp)_{\mathfrak{n}^+} + \det_{\mathfrak{n}^+}(y)$.*
- (2) *For $x_1 \in \mathfrak{p}_1^+$, $x_2 \in \mathfrak{p}_2^+$, we have $(x_j)^\sharp \in \mathfrak{p}_2^+$, and $(x_1)^\sharp$ is at most of rank 1.*
- (3) *For $x_1 \in \mathfrak{p}_1^+$, $x_2 \in \mathfrak{p}_2^+$, we have $\det_{\mathfrak{n}^+}(x_1+x_2) = \det_{\mathfrak{n}_2^+}(x_2) + (x_2|(x_1)^\sharp)_{\mathfrak{n}^+}$.*

Proof. (1) Since $\det_{\mathfrak{n}^+}$ is homogeneous of degree 3, $\det_{\mathfrak{n}^+}(sx+ty)$, $s, t \in \mathbb{C}$ is of the form

$$\det_{\mathfrak{n}^+}(sx+ty) = f_3(x, y)s^3 + f_2(x, y)s^2t + f_1(x, y)st^2 + f_0(x, y)t^3,$$

with $f_j(x, y) \in \mathcal{P}(\mathfrak{p}^+ \oplus \mathfrak{p}^+)$. Then clearly we have

$$\begin{aligned} f_3(x, y) &= \det_{\mathfrak{n}^+}(sx+ty)|_{s=1, t=0} = \det_{\mathfrak{n}^+}(x), \\ f_0(x, y) &= \det_{\mathfrak{n}^+}(sx+ty)|_{s=0, t=1} = \det_{\mathfrak{n}^+}(y), \end{aligned}$$

and by [13, Proposition III.4.2 (ii)], we have

$$\begin{aligned} f_2(x, y) &= \frac{\partial}{\partial t} \det_{\mathfrak{n}^+}(sx+ty) \Big|_{s=1, t=0} = \det_{\mathfrak{n}^+}(x)(x^{-1}|y)_{\mathfrak{n}^+} = (x^\sharp|y)_{\mathfrak{n}^+}, \\ f_1(x, y) &= \frac{\partial}{\partial s} \det_{\mathfrak{n}^+}(sx+ty) \Big|_{s=0, t=1} = \det_{\mathfrak{n}^+}(y)(x|y^{-1})_{\mathfrak{n}^+} = (x|y^\sharp)_{\mathfrak{n}^+}. \end{aligned}$$

Then by putting $s = t = 1$, we get the desired formula.

(2) Since $\sigma_- := -\sigma$ acts on \mathfrak{p}^+ as a Jordan algebra automorphism, for $x_j \in \mathfrak{p}_j^+$ we have

$$\sigma_-((x_j)^\sharp) = (\sigma_-(x_j))^\sharp = (\mp x_j)^\sharp = (\mp 1)^2(x_j)^\sharp = (x_j)^\sharp,$$

and hence $(x_j)^\sharp \in \mathfrak{p}_2^+$ holds. Also, for $x_1 \in \mathfrak{p}_1^+$ we have

$$\begin{aligned} \det_{\mathfrak{n}^+}(x_1) &= \frac{1}{2}(\det_{\mathfrak{n}^+}(x_1) + \det_{\mathfrak{n}^+}(\sigma_-(x_1))) = \frac{1}{2}(\det_{\mathfrak{n}^+}(x_1) + \det_{\mathfrak{n}^+}(-x_1)) = 0, \\ ((x_1)^\sharp)^\sharp &= \det_{\mathfrak{n}^+}(x_1)x_1 = 0, \end{aligned}$$

and hence for any $l \in K_1^{\mathbb{C}}$ we have

$$\Delta_2(l(x_1)^\sharp) = \check{\Delta}_1((l(x_1)^\sharp)^\sharp) = \check{\Delta}_1(l^\sharp((x_1)^\sharp)^\sharp) = 0,$$

where $l^\sharp := \chi(l)^2 l^{\top-1}$. Thus $(x_1)^\sharp$ is at most of rank 1.

(3) For $x = x_2 \in \mathfrak{p}_2^+$, $y = x_1 \in \mathfrak{p}_1^+$, we have $((x_2)^\sharp|x_1)_{\mathfrak{n}^+} = 0$ since $(x_2)^\sharp \in \mathfrak{p}_2^+$ is orthogonal to $x_1 \in \mathfrak{p}_1^+$, and we have $\det_{\mathfrak{n}^+}(x_1) = 0$ by the above argument. Hence the desired formula follows from (1). ■

7.2 Proof of theorem on poles

Now we give the proof of Theorem 6.2 when \mathfrak{p}^+ , \mathfrak{p}_2^+ are of tube type and of rank 3.

Proof of Theorem 6.2 for rank 3 cases. By the K_1 -equivariance, it is enough to prove the theorem when $f(x_2) = \Delta_{\mathbf{k}}^{\mathfrak{n}_2^+}(x_2) \in \mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+)$, $\mathbf{k} \in \mathbb{Z}_{++}^3$. By (3.3), for $z = z_1 + z_2 \in \Omega \subset \mathfrak{n}^+ \subset \mathfrak{p}^+$, we have

$$\begin{aligned}
& \langle \Delta_{\mathbf{k}}^{\mathfrak{n}_2^+}(x_2), e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\lambda, x} \\
&= \frac{\det_{\mathfrak{n}^+}(z)^{-\lambda+d+1}}{(\lambda)_{2k_3, d}} \det_{\mathfrak{n}_2^+} \left(\frac{\partial}{\partial z_2} \right)^{k_3} \det_{\mathfrak{n}^+}(z)^{\lambda+2k_3-d-1} \langle \Delta_{(k_1-k_3, k_2-k_3, 0)}^{\mathfrak{n}_2^+}(x_2), e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\lambda+2k_3, x} \\
&= \frac{1}{(\lambda)_{2k_3, d}} \det_{\mathfrak{n}^+}(z)^{-\lambda+d+1} \det_{\mathfrak{n}_2^+} \left(\frac{\partial}{\partial z_2} \right)^{k_3} (\det_{\mathfrak{n}_2^+}(z_2) + (z_2|z_1^{\sharp})_{\mathfrak{n}^+})^{\lambda+2k_3-d-1} \\
&\quad \times \frac{(\lambda + k_1 + k_3 - \frac{d}{4})_{k_2-k_3}}{(\lambda + 2k_3)_{(k_1+k_2-2k_3, k_2-k_3), d}} {}_2F_1 \left(\begin{matrix} -k_2 + k_3, -k_1 + k_3 - \frac{d}{4}, -\frac{\Delta_2(z_1)}{\Delta_2(z_2)} \\ -\lambda - k_1 - k_2 + \frac{d}{4} + 1 \end{matrix}, -\frac{\Delta_2(z_1)}{\Delta_2(z_2)} \right) \Delta_{(k_1-k_3, k_2-k_3, 0)}^{\mathfrak{n}_2^+}(z_2) \\
&= \frac{(-1)^{k_2-k_3} (-\lambda - k_1 - k_2 + \frac{d}{4} + 1)_{k_2-k_3}}{(\lambda)_{(k_1+k_2, k_2+k_3, 2k_3), d}} \det_{\mathfrak{n}^+}(z)^{-\lambda+d+1} \det_{\mathfrak{n}_2^+} \left(\frac{\partial}{\partial z_2} \right)^{k_3} \\
&\quad \times \det_{\mathfrak{n}_2^+}(z_2)^{\lambda+k_3-d-1} \sum_{m=0}^{\infty} \frac{(-\lambda - 2k_3 + d + 1)_m}{m!} \left(-\frac{(z_2|z_1^{\sharp})_{\mathfrak{n}^+}}{\det_{\mathfrak{n}_2^+}(z_2)} \right)^m \\
&\quad \times \sum_{n=0}^{k_2-k_3} \frac{(-k_2 + k_3)_n (-k_1 + k_3 - \frac{d}{4})_n}{(-\lambda - k_1 - k_2 + \frac{d}{4} + 1)_n n!} \left(-\frac{\Delta_2(z_1)}{\Delta_2(z_2)} \right)^n \Delta_{(k_1, k_2, k_3)}^{\mathfrak{n}_2^+}(z_2),
\end{aligned}$$

where we have used Proposition 3.1, (6.7) and Lemma 7.1 (3) at the 2nd equality, and used the binomial formula and the definition of ${}_2F_1$ at the 3rd equality. Now we have

$$\begin{aligned}
& \left(-\frac{(z_2|z_1^{\sharp})_{\mathfrak{n}^+}}{\det_{\mathfrak{n}_2^+}(z_2)} \right)^m \in \mathcal{P}_{2m}(\mathfrak{p}_1^+) \otimes \mathcal{P}_{(0, -m, -m)}(\mathfrak{p}_2^+), \\
& \left(-\frac{\Delta_2(z_1)}{\Delta_2(z_2)} \right)^n \Delta_{(k_1, k_2, k_3)}^{\mathfrak{n}_2^+}(z_2) \in \mathcal{P}_{2n}(\mathfrak{p}_1^+) \otimes \mathcal{P}_{(k_1-n, k_2-n, k_3)}(\mathfrak{p}_2^+),
\end{aligned}$$

where $\mathcal{P}_m(\mathfrak{p}_1^+)$ denotes the space of all homogeneous polynomials on \mathfrak{p}_1^+ of degree m , since z_1^{\sharp} is at most of rank 1. For $\mathbf{l} \in \mathbb{Z}^3$ with $l_1 \geq 0$, $0 \leq l_2 \leq k_1 - k_2$, $n \leq l_3 \leq k_2 - k_3$, $|\mathbf{l}| = m + n$, we define the polynomials

$$\begin{aligned}
F_{\mathbf{k}, \mathbf{l}}^n(z_1, z_2) &\in \mathcal{P}_{2(m+n)}(\mathfrak{p}_1^+) \otimes \mathcal{P}_{(k_1+l_1-m-n, k_2+l_2-m-n, k_3+l_3-m-n)}(\mathfrak{p}_2^+) \\
&= \mathcal{P}_{2|\mathbf{l}|}(\mathfrak{p}_1^+) \otimes \mathcal{P}_{(k_1-l_2-l_3, k_2-l_1-l_3, k_3-l_1-l_2)}(\mathfrak{p}_2^+)
\end{aligned}$$

such that

$$\sum_{\substack{\mathbf{l} \in \mathbb{Z}^3, |\mathbf{l}|=m+n \\ 0 \leq l_1 \\ 0 \leq l_2 \leq k_1-k_2 \\ n \leq l_3 \leq k_2-k_3}} F_{\mathbf{k}, \mathbf{l}}^n(z_1, z_2) = \frac{1}{m!n!} \left(-\frac{(z_2|z_1^{\sharp})_{\mathfrak{n}^+}}{\det_{\mathfrak{n}_2^+}(z_2)} \right)^m \left(-\frac{\Delta_2(z_1)}{\Delta_2(z_2)} \right)^n \Delta_{(k_1, k_2, k_3)}^{\mathfrak{n}_2^+}(z_2) \quad (7.1)$$

holds. Then we have

$$\langle \Delta_{\mathbf{k}}^{\mathfrak{n}_2^+}(x_2), e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\lambda, x}$$

$$\begin{aligned}
&= \frac{(-1)^{k_2-k_3}}{(\lambda)_{(k_1+k_2, k_2+k_3, 2k_3), d}} \det_{\mathfrak{n}^+}(z)^{-\lambda+d+1} \det_{\mathfrak{n}_2^+} \left(\frac{\partial}{\partial z_2} \right)^{k_3} \det_{\mathfrak{n}_2^+}(z_2)^{\lambda+k_3-d-1} \\
&\quad \times \sum_{\substack{\mathbf{l} \in \mathbb{Z}^3, 0 \leq l_1 \\ 0 \leq l_2 \leq k_1-k_2 \\ 0 \leq l_3 \leq k_2-k_3}} \sum_{n=0}^{l_3} (-\lambda - 2k_3 + d + 1)_{|\mathbf{l}|-n} \left(-\lambda - k_1 - k_2 + n + \frac{d}{4} + 1 \right)_{k_2-k_3-n} \\
&\quad \quad \quad \times (-k_2 + k_3)_n \left(-k_1 + k_3 - \frac{d}{4} \right)_n F_{\mathbf{k}, \mathbf{l}}^n(z_1, z_2) \\
&= \frac{(-1)^{k_2-k_3}}{(\lambda)_{(k_1+k_2, k_2+k_3, 2k_3), d}} \det_{\mathfrak{n}^+}(z)^{-\lambda+d+1} \det_{\mathfrak{n}_2^+} \left(\frac{\partial}{\partial z_2} \right)^{k_3} \det_{\mathfrak{n}_2^+}(z_2)^{\lambda+k_3-d-1} \\
&\quad \times \sum_{\substack{\mathbf{l} \in \mathbb{Z}^3, 0 \leq l_1 \\ 0 \leq l_2 \leq k_1-k_2 \\ 0 \leq l_3 \leq k_2-k_3}} (-\lambda - 2k_3 + d + 1)_{l_1+l_2} \left(-\lambda - k_1 - k_2 + l_3 + \frac{d}{4} + 1 \right)_{k_2-k_3-l_3} \\
&\quad \times \sum_{n=0}^{l_3} (-k_2 + k_3)_n \left(-k_1 + k_3 - \frac{d}{4} \right)_n \left(-\lambda - k_1 - k_2 + n + \frac{d}{4} + 1 \right)_{l_3-n} \\
&\quad \times (-\lambda - 2k_3 + l_1 + l_2 + d + 1)_{l_3-n} F_{\mathbf{k}, \mathbf{l}}^n(z_1, z_2).
\end{aligned}$$

Now we set

$$\begin{aligned}
\tilde{F}_{\mathbf{k}, \mathbf{l}}(\lambda; z_1, z_2) &:= \sum_{n=0}^{l_3} (-k_2 + k_3)_n \left(-k_1 + k_3 - \frac{d}{4} \right)_n \left(-\lambda - k_1 - k_2 + n + \frac{d}{4} + 1 \right)_{l_3-n} \\
&\quad \times (-\lambda - 2k_3 + l_1 + l_2 + d + 1)_{l_3-n} F_{\mathbf{k}, \mathbf{l}}^n(z_1, z_2) \\
&\in \mathbb{C}[\lambda]_{\leq 2l_3} \otimes \mathcal{P}_{2|\mathbf{l}|}(\mathfrak{p}_1^+) \otimes \mathcal{P}_{(k_1-l_2-l_3, k_2-l_1-l_3, k_3-l_1-l_2)}(\mathfrak{p}_2^+), \tag{7.2}
\end{aligned}$$

where $\mathbb{C}[\lambda]_{\leq l}$ denotes the space of polynomials in λ of degree at most l . Then since

$$\begin{aligned}
&\det_{\mathfrak{n}_2^+} \left(\frac{\partial}{\partial z_2} \right)^{k_3} \det_{\mathfrak{n}_2^+}(z_2)^{\lambda+k_3-d-1} \tilde{F}_{\mathbf{k}, \mathbf{l}}(\lambda; z_1, z_2) \\
&= (-1)^{k_3} \left(-\lambda - k_1 - k_3 + l_2 + l_3 + \frac{d}{2} + 1 \right)_{k_3} \left(-\lambda - k_2 - k_3 + l_1 + l_3 + \frac{3}{4}d + 1 \right)_{k_3} \\
&\quad \times (-\lambda - 2k_3 + l_1 + l_2 + d + 1)_{k_3} \det_{\mathfrak{n}_2^+}(z_2)^{\lambda-d-1} \tilde{F}_{\mathbf{k}, \mathbf{l}}(\lambda; z_1, z_2)
\end{aligned}$$

holds by (2.6), we have

$$\begin{aligned}
&\langle \Delta_{\mathbf{k}}^{\mathfrak{n}_2^+}(x_2), e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\lambda, x} \\
&= \frac{(-1)^{k_2}}{(\lambda)_{(k_1+k_2, k_2+k_3, 2k_3), d}} \det_{\mathfrak{n}^+}(z)^{-\lambda+d+1} \det_{\mathfrak{n}_2^+}(z_2)^{\lambda-d-1} \\
&\quad \times \sum_{\substack{\mathbf{l} \in \mathbb{Z}^3, 0 \leq l_1 \\ 0 \leq l_2 \leq k_1-k_2 \\ 0 \leq l_3 \leq k_2-k_3}} (-\lambda - 2k_3 + d + 1)_{l_1+l_2} \left(-\lambda - k_1 - k_2 + l_3 + \frac{d}{4} + 1 \right)_{k_2-k_3-l_3} \\
&\quad \times \left(-\lambda - k_1 - k_3 + l_2 + l_3 + \frac{d}{2} + 1 \right)_{k_3} \left(-\lambda - k_2 - k_3 + l_1 + l_3 + \frac{3}{4}d + 1 \right)_{k_3} \\
&\quad \times (-\lambda - 2k_3 + l_1 + l_2 + d + 1)_{k_3} \tilde{F}_{\mathbf{k}, \mathbf{l}}(\lambda; z_1, z_2) \\
&= \frac{(-1)^{k_2} (-\lambda - 2k_3 + d + 1)_{k_3}}{(\lambda)_{(k_1+k_2, k_2+k_3, 2k_3), d}} \left(1 + \frac{(z_2|z_1^{\sharp})_{\mathfrak{n}_2^+}}{\det_{\mathfrak{n}_2^+}(z_2)} \right)^{-\lambda+d+1}
\end{aligned}$$

$$\begin{aligned}
& \times \sum_{\substack{\mathbf{l} \in \mathbb{Z}^3, 0 \leq l_1 \\ 0 \leq l_2 \leq k_1 - k_2 \\ 0 \leq l_3 \leq k_2 - k_3}} (-\lambda - k_3 + d + 1)_{l_1 + l_2} \left(-\lambda - k_1 - k_2 + l_3 + \frac{d}{4} + 1 \right)_{k_2 - k_3 - l_3} \\
& \times \left(-\lambda - k_1 - k_3 + l_2 + l_3 + \frac{d}{2} + 1 \right)_{k_3} \left(-\lambda - k_2 - k_3 + l_1 + l_3 + \frac{3}{4}d + 1 \right)_{k_3} \tilde{F}_{\mathbf{k},1}(\lambda; z_1, z_2) \\
& = \frac{(-1)^{k_2 - k_3}}{(\lambda)_{(k_1 + k_2, k_2 + k_3, k_3), d}} \left(1 + \frac{(z_2 | z_1^\sharp)_{\mathbf{n}^+}}{\det_{\mathbf{n}_2^+}(z_2)} \right)^{-\lambda + d + 1} \\
& \times \sum_{\substack{\mathbf{l} \in \mathbb{Z}^3, 0 \leq l_1 \\ 0 \leq l_2 \leq k_1 - k_2 \\ 0 \leq l_3 \leq k_2 - k_3}} (-\lambda - k_3 + d + 1)_{l_1 + l_2} \left(-\lambda - k_1 - k_2 + l_3 + \frac{d}{4} + 1 \right)_{k_2 - k_3 - l_3} \\
& \times \left(-\lambda - k_1 - k_3 + l_2 + l_3 + \frac{d}{2} + 1 \right)_{k_3} \left(-\lambda - k_2 - k_3 + l_1 + l_3 + \frac{3}{4}d + 1 \right)_{k_3} \tilde{F}_{\mathbf{k},1}(\lambda; z_1, z_2), \tag{7.3}
\end{aligned}$$

where we have used Lemma 7.1 (3) at the 2nd equality. Therefore,

$$(\lambda)_{(k_1 + k_2, k_2 + k_3, k_3), d} \langle \Delta_{\mathbf{k}}^{\mathbf{n}_2^+}(x_2), e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\lambda, x}$$

is holomorphic for all $\lambda \in \mathbb{C}$, and hence by Corollary 2.4, we have

$$\Delta_{\mathbf{k}}^{\mathbf{n}_2^+}(x_2) \in \bigoplus_{\substack{\mathbf{m} \in \mathbb{Z}_{++}^3, m_1 \leq k_1 + k_2 \\ m_2 \leq k_2 + k_3, m_3 \leq k_3}} \mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+). \tag{7.4}$$

On the other hand, if $k_1 = k_2 = k_3$, since

$$\tilde{F}_{\underline{k}_3, (l_1, 0, 0)}(\lambda; z_1, z_2) = F_{\underline{k}_3, (l_1, 0, 0)}^0(z_1, z_2) = \frac{1}{l_1!} \left(-\frac{(z_2 | z_1^\sharp)_{\mathbf{n}^+}}{\det_{\mathbf{n}_2^+}(z_2)} \right)^{l_1} \det_{\mathbf{n}_2^+}(z_2)^{k_1}$$

holds, we have

$$\begin{aligned}
& \langle \det_{\mathbf{n}_2^+}(x_2)^{k_1}, e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\lambda, x} \\
& = \frac{(-\lambda - 2k_1 + \frac{d}{2} + 1)_{k_1}}{(\lambda)_{(2k_1, 2k_1, k_1), d}} \left(1 + \frac{(z_2 | z_1^\sharp)_{\mathbf{n}^+}}{\det_{\mathbf{n}_2^+}(z_2)} \right)^{-\lambda + d + 1} \\
& \quad \times \sum_{l_1=0}^{\infty} (-\lambda - k_1 + d + 1)_{l_1} \left(-\lambda - 2k_1 + l_1 + \frac{3}{4}d + 1 \right)_{k_1} \frac{1}{l_1!} \left(-\frac{(z_2 | z_1^\sharp)_{\mathbf{n}^+}}{\det_{\mathbf{n}_2^+}(z_2)} \right)^{l_1} \det_{\mathbf{n}_2^+}(z_2)^{k_1} \\
& = \frac{(-1)^{k_1} (-\lambda - 2k_1 + \frac{3}{4}d + 1)_{k_1}}{(\lambda)_{(2k_1, k_1, k_1), d}} \left(1 + \frac{(z_2 | z_1^\sharp)_{\mathbf{n}^+}}{\det_{\mathbf{n}_2^+}(z_2)} \right)^{-\lambda + d + 1} \\
& \quad \times {}_2F_1 \left(\begin{matrix} -\lambda - k_1 + d + 1, -\lambda - k_1 + \frac{3}{4}d + 1 \\ -\lambda - 2k_1 + \frac{3}{4}d + 1 \end{matrix}; -\frac{(z_2 | z_1^\sharp)_{\mathbf{n}^+}}{\det_{\mathbf{n}_2^+}(z_2)} \right) \det_{\mathbf{n}_2^+}(z_2)^{k_1} \\
& = \frac{(\lambda + k_1 - \frac{3}{4}d)_{k_1}}{(\lambda)_{(2k_1, k_1, k_1), d}} {}_2F_1 \left(\begin{matrix} -k_1, -k_1 - \frac{d}{4} \\ -\lambda - 2k_1 + \frac{3}{4}d + 1 \end{matrix}; -\frac{(z_2 | z_1^\sharp)_{\mathbf{n}^+}}{\det_{\mathbf{n}_2^+}(z_2)} \right) \det_{\mathbf{n}_2^+}(z_2)^{k_1}
\end{aligned}$$

as in [44, Corollary 6.5], and hence

$$(\lambda)_{(2k_1, k_1, k_1), d} \langle \det_{\mathbf{n}_2^+}(x_2)^{k_1}, e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\lambda, x}$$

is holomorphic for all $\lambda \in \mathbb{C}$, that is, we have

$$\det_{\mathfrak{n}_2^+}(x_2)^{k_1} \in \bigoplus_{\substack{\mathbf{m} \in \mathbb{Z}_{++}^3, m_1 \leq 2k_1 \\ m_2 \leq k_1, m_3 \leq k_1}} \mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+).$$

Then for general $\mathbf{k} \in \mathbb{Z}_{++}^3$, by Lemma 2.1, we also have

$$\Delta_{\mathbf{k}}^{\mathfrak{n}_2^+}(x_2) \in \mathbb{C} \check{\Delta}_{(k_1-k_3, k_1-k_2, 0)}^{\mathfrak{n}_2^+} \left(\frac{\partial}{\partial x_2} \right) \det_{\mathfrak{n}_2^+}(x_2)^{k_1} \subset \bigoplus_{\substack{\mathbf{m} \in \mathbb{Z}_{++}^3, m_1 \leq 2k_1 \\ m_2 \leq k_1, m_3 \leq k_1}} \mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+), \quad (7.5)$$

and combining (7.4) and (7.5), we get

$$\Delta_{\mathbf{k}}^{\mathfrak{n}_2^+}(x_2) \in \bigoplus_{\substack{\mathbf{m} \in \mathbb{Z}_{++}^3, m_1 \leq k_1+k_2 \\ m_2 \leq \min\{k_1, k_2+k_3\}, m_3 \leq k_3}} \mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+).$$

Therefore, by Corollary 2.4,

$$(\lambda)_{(k_1+k_2, \min\{k_1, k_2+k_3\}, k_3), d} \langle \Delta_{\mathbf{k}}^{\mathfrak{n}_2^+}(x_2), e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\lambda, x}$$

is holomorphic for all $\lambda \in \mathbb{C}$. ■

7.3 Conjecture on weighted Bergman inner products

In the previous subsection we roughly computed the inner product $\langle \Delta_{\mathbf{k}}^{\mathfrak{n}_2^+}(x_2), e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\lambda, x}$ for $\mathbf{k} \in \mathbb{Z}_{++}^3$. By (7.3), we have

$$\begin{aligned} & \langle \Delta_{\mathbf{k}}^{\mathfrak{n}_2^+}(x_2), e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\lambda, x} \\ &= \frac{(-\lambda - k_1 - k_2 + \frac{d}{4} + 1)_{k_2-k_3} (-\lambda - k_1 - k_3 + \frac{d}{2} + 1)_{k_3} (-\lambda - k_2 - k_3 + \frac{3}{4}d + 1)_{k_3}}{(\lambda)_{(k_1+k_2, k_2+k_3, k_3), d}} \\ & \quad \times (-1)^{k_2-k_3} \left(1 + \frac{(z_2|z_1^{\#})_{\mathfrak{n}^+}}{\det_{\mathfrak{n}_2^+}(z_2)} \right)^{-\lambda+d+1} \sum_{\substack{\mathbf{l} \in \mathbb{Z}^3, 0 \leq l_1 \\ 0 \leq l_2 \leq k_1-k_2 \\ 0 \leq l_3 \leq k_2-k_3}} \frac{(-\lambda - k_3 + d + 1)_{l_1+l_2}}{(-\lambda - k_1 - k_2 + \frac{d}{4} + 1)_{l_3}} \\ & \quad \times \frac{(-\lambda - k_1 + \frac{d}{2} + 1)_{l_2+l_3}}{(-\lambda - k_1 - k_3 + \frac{d}{2} + 1)_{l_2+l_3}} \frac{(-\lambda - k_2 + \frac{3}{4}d + 1)_{l_1+l_3}}{(-\lambda - k_2 - k_3 + \frac{3}{4}d + 1)_{l_1+l_3}} \tilde{F}_{\mathbf{k}, \mathbf{l}}(\lambda; z_1, z_2) \\ &= C_1^{d, d/2}(\lambda, \mathbf{k}) \left(1 + \frac{(z_2|z_1^{\#})_{\mathfrak{n}^+}}{\det_{\mathfrak{n}_2^+}(z_2)} \right)^{-\lambda+d+1} \sum_{\substack{\mathbf{l} \in \mathbb{Z}^3, 0 \leq l_1 \\ 0 \leq l_2 \leq k_1-k_2 \\ 0 \leq l_3 \leq k_2-k_3}} \frac{(-\lambda - k_3 + d + 1)_{l_1+l_2}}{(-\lambda - k_1 - k_2 + \frac{d}{4} + 1)_{l_3}} \\ & \quad \times \frac{(-\lambda - k_1 + \frac{d}{2} + 1)_{l_2+l_3}}{(-\lambda - k_1 - k_3 + \frac{d}{2} + 1)_{l_2+l_3}} \frac{(-\lambda - k_2 + \frac{3}{4}d + 1)_{l_1+l_3}}{(-\lambda - k_2 - k_3 + \frac{3}{4}d + 1)_{l_1+l_3}} \tilde{F}_{\mathbf{k}, \mathbf{l}}(\lambda; z_1, z_2), \end{aligned}$$

where

$$C_1^{d, d/2}(\lambda, \mathbf{k}) = \frac{\prod_{1 \leq i < j \leq 3} (\lambda - \frac{d}{4}(i+j-2))_{k_i+k_j}}{\prod_{1 \leq i < j \leq 4} (\lambda - \frac{d}{4}(i+j-3))_{k_i+k_j}}$$

$$\begin{aligned}
&= \frac{(\lambda + k_1 + k_3 - \frac{d}{4})_{k_2 - k_3} (\lambda + k_1 - \frac{d}{2})_{k_3} (\lambda + k_2 - \frac{3}{4}d)_{k_3}}{(\lambda)_{(k_1 + k_2, k_2 + k_3, k_3), d}} \\
&= \frac{(\lambda + k_1 + k_3 - \frac{d}{4})_{k_2 - k_3} (\lambda + \max\{k_1, k_2 + k_3\} - \frac{d}{2})_{\min\{k_3, k_1 - k_2\}} (\lambda + k_2 - \frac{3}{4}d)_{k_3}}{(\lambda)_{(k_1 + k_2, \min\{k_1, k_2 + k_3\}, k_3), d}}
\end{aligned} \tag{7.6}$$

with $k_4 := 0$. Now by the definition of $\tilde{F}_{\mathbf{k}, \mathbf{l}}(\lambda; z_1, z_2)$ (7.1) and (7.2), we can show that

$$\tilde{F}_{\mathbf{k}, \mathbf{l}}(\lambda; z_1, z_2) \in \mathbb{C}[\lambda]_{\leq 2l_3} \otimes (\mathcal{P}_{2|\mathbf{l}|}(\mathfrak{p}_1^+) \otimes \mathcal{P}_{(k_1 - l_2 - l_3, k_2 - l_1 - l_3, k_3 - l_1 - l_2)}(\mathfrak{p}_2^+))_{\mathbf{k}}^{M_{L_2} N_{L_2}^\top},$$

holds, where for $m \in \mathbb{Z}_{\geq 0}$, $\mathbf{k}, \mathbf{n} \in \mathbb{Z}_+^3$, $\mathcal{P}_m(\mathfrak{p}_1^+)$ denotes the space of all homogeneous polynomials on \mathfrak{p}_1^+ of degree m , and

$$\begin{aligned}
&(\mathcal{P}_m(\mathfrak{p}_1^+) \otimes \mathcal{P}_{\mathbf{n}}(\mathfrak{p}_2^+))_{\mathbf{k}}^{M_{L_2} N_{L_2}^\top} \\
&:= \left\{ f(x_1, x_2) \in \mathcal{P}_m(\mathfrak{p}_1^+) \otimes \mathcal{P}_{\mathbf{n}}(\mathfrak{p}_2^+) \mid \begin{array}{l} f(\text{man.}x_1, \text{man.}x_2) = e^{2(t_1 k_1 + t_2 k_2 + t_3 k_3)} f(x_1, x_2) \\ (m \in M_{L_2}, a = e^{t_1 h_1 + t_2 h_2 + t_3 h_3} \in A_{L_2}, n \in N_{L_2}^\top) \end{array} \right\}.
\end{aligned}$$

For Cases 1 and 2, $\mathcal{P}_m(\mathfrak{p}_1^+) \otimes \mathcal{P}_{\mathbf{n}}(\mathfrak{p}_2^+)$ are given by

$$\begin{aligned}
\mathcal{P}_m(\mathfrak{p}_1^+) \otimes \mathcal{P}_{\mathbf{n}}(\mathfrak{p}_2^+) &= \mathcal{P}_m(\text{Alt}(3, \mathbb{C})) \otimes \mathcal{P}_{\mathbf{n}}(\text{Sym}(3, \mathbb{C})) \\
&\simeq V_{(m, m, 0)}^{(3)\vee} \otimes V_{(2n_1, 2n_2, 2n_3)}^{(3)\vee} \tag{Case 1}, \\
\mathcal{P}_m(\mathfrak{p}_1^+) \otimes \mathcal{P}_{\mathbf{n}}(\mathfrak{p}_2^+) &= \bigoplus_{l=0}^m (\mathcal{P}_l(\text{Alt}(3, \mathbb{C})) \boxtimes \mathcal{P}_{m-l}(\text{Alt}(3, \mathbb{C}))) \otimes \mathcal{P}_{\mathbf{n}}(\text{M}(3, \mathbb{C})) \\
&\simeq \bigoplus_{l=0}^m \left(V_{(l, l, 0)}^{(3)\vee} \boxtimes V_{(m-l, m-l, 0)}^{(3)} \right) \otimes \left(V_{(n_1, n_2, n_3)}^{(3)\vee} \boxtimes V_{(n_1, n_2, n_3)}^{(3)} \right) \tag{Case 2},
\end{aligned}$$

and these decompose multiplicity-freely under $K_1^{\mathbb{C}}$ by the Pieri rule. Hence the space $(\mathcal{P}_m(\mathfrak{p}_1^+) \otimes \mathcal{P}_{\mathbf{n}}(\mathfrak{p}_2^+))_{\mathbf{k}}^{M_{L_2} N_{L_2}^\top}$ is at most 1-dimensional, and thus there exist polynomials

$$f_{\mathbf{k}, \mathbf{l}}(\lambda) \in \mathbb{C}[\lambda]_{\leq 2l_3}, \quad F_{\mathbf{k}, \mathbf{l}}(z_1, z_2) \in (\mathcal{P}_{2|\mathbf{l}|}(\mathfrak{p}_1^+) \otimes \mathcal{P}_{(k_1 - l_2 - l_3, k_2 - l_1 - l_3, k_3 - l_1 - l_2)}(\mathfrak{p}_2^+))_{\mathbf{k}}^{M_{L_2} N_{L_2}^\top}$$

such that

$$\tilde{F}_{\mathbf{k}, \mathbf{l}}(\lambda; z_1, z_2) = f_{\mathbf{k}, \mathbf{l}}(\lambda) F_{\mathbf{k}, \mathbf{l}}(z_1, z_2)$$

holds. We expect that this also holds for Case 3. In addition, we conjecture the following.

Conjecture 7.2. *For each $\mathbf{k} \in \mathbb{Z}_{++}^3$, $\mathbf{l} \in \mathbb{Z}^3$ with $l_1 \geq 0$, $0 \leq l_2 \leq k_1 - k_2$, $0 \leq l_3 \leq k_2 - k_3$, there exists a polynomial*

$$F_{\mathbf{k}, \mathbf{l}}(z_1, z_2) \in (\mathcal{P}_{2|\mathbf{l}|}(\mathfrak{p}_1^+) \otimes \mathcal{P}_{(k_1 - l_2 - l_3, k_2 - l_1 - l_3, k_3 - l_1 - l_2)}(\mathfrak{p}_2^+))_{\mathbf{k}}^{M_{L_2} N_{L_2}^\top}$$

such that for $\text{Re } \lambda > p - 1$ and for $z = z_1 + z_2 \in \Omega \subset \mathfrak{n}^+ \subset \mathfrak{p}^+$, we have

$$\begin{aligned}
&\langle \Delta_{\mathbf{k}}^{\mathfrak{n}_2^+}(x_2), e^{(x|\bar{z})_{\mathfrak{p}^+}} \rangle_{\lambda, x} \\
&= C_1^{d, d/2}(\lambda, \mathbf{k}) \left(1 + \frac{(z_2 | z_1^\sharp)_{\mathfrak{n}_2^+}}{\det_{\mathfrak{n}_2^+}(z_2)} \right)^{-\lambda + d + 1} \sum_{\substack{\mathbf{l} \in \mathbb{Z}^3, 0 \leq l_1 \\ 0 \leq l_2 \leq k_1 - k_2 \\ 0 \leq l_3 \leq k_2 - k_3}} \frac{(-\lambda - k_3 + d + 1)_{l_1 + l_2}}{(-\lambda - k_1 - k_2 + \frac{d}{4} + 1)_{l_3}}
\end{aligned}$$

$$\times \frac{(-\lambda - k_2 + \frac{3}{4}d + 1)_{l_1+l_3}}{(-\lambda - k_1 - k_3 + \frac{d}{2} + 1)_{l_2}} \frac{(-\lambda - k_1 + \frac{d}{2} + 1)_{l_2+l_3}}{(-\lambda - k_2 - k_3 + \frac{3}{4}d + 1)_{l_1}} F_{\mathbf{k},1}(z_1, z_2) \quad (7.7)$$

$$= C_1^{d,d/2}(\lambda, \mathbf{k}) \sum_{\substack{\mathbf{l} \in \mathbb{Z}^3, 0 \leq l_1 \\ 0 \leq l_2 \leq k_1 - k_2 \\ 0 \leq l_3 \leq k_2 - k_3}} \frac{(-k_3)_{l_1+l_2}}{(-\lambda - k_1 - k_2 + \frac{d}{4} + 1)_{l_3}} \\ \times \frac{(-k_2 - \frac{d}{4})_{l_1+l_3}}{(-\lambda - k_1 - k_3 + \frac{d}{2} + 1)_{l_2}} \frac{(-k_1 - \frac{d}{2})_{l_2+l_3}}{(-\lambda - k_2 - k_3 + \frac{3}{4}d + 1)_{l_1}} F_{\mathbf{k},1}(z_1, z_2), \quad (7.8)$$

where $C_1^{d,d/2}(\lambda, \mathbf{k})$ is as in (7.6).

By [44, Theorem 6.3 (1)], both (7.7) and (7.8) hold if $k_2 = k_3$. In general, (7.7) is equivalent to

$$f_{\mathbf{k},1}(\lambda) = \left(-\lambda - k_1 - k_3 + l_2 + \frac{d}{2} + 1\right)_{l_3} \left(-\lambda - k_2 - k_3 + l_1 + \frac{3}{4}d + 1\right)_{l_3}.$$

Also, if (7.8) is true, then the symmetry breaking operator

$$\mathcal{F}_{\lambda, \mathbf{k}}^\downarrow: \mathcal{H}_\lambda(D)|_{\tilde{G}_1} \longrightarrow \mathcal{H}_{\varepsilon_1 \lambda}(D_1, V_{\mathbf{k}})$$

with $V_{\mathbf{k}} := \mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+)$ given in (2.18) and (2.19), where

$$\mathcal{H}_{\varepsilon_1 \lambda}(D_1, V_{\mathbf{k}}) \simeq \begin{cases} \mathcal{H}_{2\lambda}(D_{\text{SO}^*(6)}, V_{2\mathbf{k}}^{(3)\vee}) & \text{(Case 1),} \\ \mathcal{H}_\lambda(D_{\text{SO}^*(6)}, V_{\mathbf{k}}^{(3)\vee}) \hat{\boxtimes} \mathcal{H}_\lambda(D_{\text{SO}^*(6)}, V_{\mathbf{k}}^{(3)\vee}) & \text{(Case 2),} \\ \mathcal{H}_\lambda(D_{\text{SU}(2,6)}, \mathbb{C} \boxtimes V_{(k_1+k_2, k_1+k_3, k_2+k_3)^2}^{(6)}) & \text{(Case 3),} \end{cases}$$

is given by the differential operator of the form

$$(\mathcal{F}_{\lambda, \mathbf{k}}^\downarrow f)(x_1) = \sum_{\substack{\mathbf{l} \in \mathbb{Z}^3, 0 \leq l_1 \\ 0 \leq l_2 \leq k_1 - k_2 \\ 0 \leq l_3 \leq k_2 - k_3}} \frac{(-k_3)_{l_1+l_2}}{(-\lambda - k_1 - k_2 + \frac{d}{4} + 1)_{l_3}} \frac{(-k_2 - \frac{d}{4})_{l_1+l_3}}{(-\lambda - k_1 - k_3 + \frac{d}{2} + 1)_{l_2}} \\ \times \frac{(-k_1 - \frac{d}{2})_{l_2+l_3}}{(-\lambda - k_2 - k_3 + \frac{3}{4}d + 1)_{l_1}} K_{\mathbf{k},1} \left(\frac{\partial}{\partial x} \right) f(x) \Big|_{x_2=0},$$

where $K_{\mathbf{k},1}(z) = K_{\mathbf{k},1}(z_1, z_2) \in \mathcal{P}(\mathfrak{p}^-, V_{\mathbf{k}})^{K_1}$ satisfies

$$(v, K_{\mathbf{k},1}(\overline{z_1}, \overline{z_2}))_{V_{\mathbf{k}}} = F_{\mathbf{k},1}(z_1, z_2)$$

for a suitable fixed lowest weight vector $v \in V_{\mathbf{k}}$. By the analytic continuation, this gives the symmetry breaking operator between the spaces of all holomorphic functions even for $\text{Re } \lambda \leq p-1$ except for the poles. Moreover, the equality (7.7) = (7.8) at $\lambda = \frac{n}{r} - a$, $a = 1, 2, \dots, k_3$ corresponds to Theorem 6.6 or Corollary 6.8 (4),

$$\mathcal{F}_{\frac{n}{r}-a, \mathbf{k}}^\downarrow = c \mathcal{F}_{\frac{n}{r}+a, \mathbf{k}-\mathbf{a}_3}^\downarrow \circ \det_{\mathfrak{n}^-} \left(\frac{\partial}{\partial x} \right)^a : \\ \mathcal{O}_{\frac{n}{r}-a}(D)|_{\tilde{G}_1} \longrightarrow \mathcal{O}_{\varepsilon_1(\frac{n}{r}-a)}(D_1, V_{\mathbf{k}}) \simeq \mathcal{O}_{\varepsilon_1(\frac{n}{r}+a)}(D_1, V_{\mathbf{k}-\mathbf{a}_3})$$

for some $c \in \mathbb{C}$.

8 Tensor product case

In this section we consider the direct sum $\mathfrak{p}^+ \oplus \mathfrak{p}^+$ with \mathfrak{p}^+ simple, and the involution $\sigma: (x, y) \mapsto (y, x)$, so that $\mathfrak{p}_1^+ = \{(x, x) \mid x \in \mathfrak{p}^+\}$, $\mathfrak{p}_2^+ = \{(x, -x) \mid x \in \mathfrak{p}^+\}$. Then the corresponding symmetric pair is of the form $(G \times G, \Delta(G))$. Let $\dim \mathfrak{p}^+ =: n$, $\text{rank } \mathfrak{p}^+ =: r$, and let d be the number defined in (2.4). This section contains some overlap with [43, Section 5.2]. See also [51].

8.1 Results on weighted Bergman inner products

We consider the outer tensor product $\mathcal{H}_\lambda(D) \hat{\boxtimes} \mathcal{H}_\mu(D)$ of holomorphic discrete series representations of scalar type, and let $\langle \cdot, \cdot \rangle_{\lambda \otimes \mu}$ denote its inner product. For $f \in \mathcal{P}_{\mathbf{k}}(\mathfrak{p}^+)$, we want to compute the inner product $\langle f(x - y), e^{(x|\bar{z})_{\mathfrak{p}^+} + (y|\bar{w})_{\mathfrak{p}^+}} \rangle_{\lambda \otimes \mu, (x, y)}$, with the variable of integration (x, y) . To do this, for $f \in \mathcal{P}(\mathfrak{p}^+)$, $\mathbf{m}, \mathbf{n} \in \mathbb{Z}_{++}^r$, define $\tilde{f}_{\mathbf{m}, \mathbf{n}}(x, y) \in \mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+) \otimes \mathcal{P}_{\mathbf{n}}(\mathfrak{p}^+)$ by

$$f(x + y) = \sum_{\mathbf{m} \in \mathbb{Z}_{++}^r} \sum_{\mathbf{n} \in \mathbb{Z}_{++}^r} \tilde{f}_{\mathbf{m}, \mathbf{n}}(x, y) \in \bigoplus_{\mathbf{m} \in \mathbb{Z}_{++}^r} \bigoplus_{\mathbf{n} \in \mathbb{Z}_{++}^r} \mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+) \otimes \mathcal{P}_{\mathbf{n}}(\mathfrak{p}^+).$$

If $f \in \mathcal{P}_{\mathbf{k}}(\mathfrak{p}^+)$, then $\tilde{f}_{\mathbf{m}, \mathbf{n}}(x, y) \neq 0$ holds only if $m_j \leq k_j$, $n_j \leq k_j$ hold for all $j = 1, \dots, r$, since the map $\mathcal{P}_{\mathbf{k}}(\mathfrak{p}^+) \rightarrow \mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+) \otimes \mathcal{P}_{\mathbf{n}}(\mathfrak{p}^+)$, $f \mapsto \tilde{f}_{\mathbf{m}, \mathbf{n}}$ is K -equivariant, and the restricted weights of $\mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+)$, $\mathcal{P}_{\mathbf{n}}(\mathfrak{p}^+)$ sit in $(\frac{1}{2}\mathbb{Z}_{\leq 0})^r$ under a suitable identification $(\mathfrak{a}_l)^\vee \simeq \mathbb{R}^r$, where $\mathfrak{a}_l \subset \mathfrak{k}^\mathbb{C}$ is as in Section 2.3. Moreover, we have $\tilde{f}_{\mathbf{k}, \mathbf{0}_r}(x, y) = f(x)$, $\tilde{f}_{\mathbf{0}_r, \mathbf{k}}(x, y) = f(y)$. Therefore, by Corollary 2.4, we easily get the following. Here, for $\lambda \in \mathbb{C}$, $\mathbf{m}, \mathbf{n} \in (\mathbb{Z}_{\geq 0})^r$, let $(\lambda + \mathbf{m})_{\mathbf{n}, d} := \prod_{j=1}^r (\lambda + m_j - \frac{d}{2}(j-1))_{n_j}$.

Theorem 8.1. *Let $\text{Re } \lambda, \text{Re } \mu > p - 1$, $\mathbf{k} \in \mathbb{Z}_{++}^r$, and let $f \in \mathcal{P}_{\mathbf{k}}(\mathfrak{p}^+)$. Then we have*

$$\langle f(x - y), e^{(x|\bar{z})_{\mathfrak{p}^+} + (y|\bar{w})_{\mathfrak{p}^+}} \rangle_{\lambda \otimes \mu, (x, y)} = \sum_{\substack{\mathbf{m} \in \mathbb{Z}_{++}^r \\ m_j \leq k_j}} \sum_{\substack{\mathbf{n} \in \mathbb{Z}_{++}^r \\ n_j \leq k_j}} \frac{1}{(\lambda)_{\mathbf{m}, d} (\mu)_{\mathbf{n}, d}} \tilde{f}_{\mathbf{m}, \mathbf{n}}(z, -w).$$

Epecially, as a function of (λ, μ) ,

$$\begin{aligned} & (\lambda)_{\mathbf{k}, d} (\mu)_{\mathbf{k}, d} \langle f(x - y), e^{(x|\bar{z})_{\mathfrak{p}^+} + (y|\bar{w})_{\mathfrak{p}^+}} \rangle_{\lambda \otimes \mu, (x, y)} \\ &= \sum_{\substack{\mathbf{m} \in \mathbb{Z}_{++}^r \\ m_j \leq k_j}} \sum_{\substack{\mathbf{n} \in \mathbb{Z}_{++}^r \\ n_j \leq k_j}} (\lambda + \mathbf{m})_{\mathbf{k} - \mathbf{m}, d} (\mu + \mathbf{n})_{\mathbf{k} - \mathbf{n}, d} \tilde{f}_{\mathbf{m}, \mathbf{n}}(z, -w) \end{aligned} \quad (8.1)$$

is holomorphically continued for all \mathbb{C}^2 .

If \mathfrak{p}^+ is of tube type, then for particular $\lambda, \mu \in \mathbb{C}$, the analytic continuation of (8.1) is factorized as follows.

Theorem 8.2. *Suppose \mathfrak{p}^+ is of tube type, and consider the meromorphic continuation of $\langle \cdot, \cdot \rangle_{\lambda \otimes \mu}$ for $\lambda, \mu \in \mathbb{C}$. Then for $\mathbf{k} \in \mathbb{Z}_{++}^r$, $a = 1, 2, \dots, k_r$ and for $f \in \mathcal{P}_{\mathbf{k}}(\mathfrak{p}^+)$, we have*

$$\begin{aligned} & (\lambda)_{\mathbf{k}, d} \langle f(x - y), e^{(x|\bar{z})_{\mathfrak{p}^+} + (y|\bar{w})_{\mathfrak{p}^+}} \rangle_{\lambda \otimes \mu, (x, y)} \Big|_{\lambda = \frac{n}{r} - a} \\ &= \left(\frac{n}{r} + a \right)_{\mathbf{k} - \mathbf{a}_r, d} \det_{\mathfrak{n}^+}(z)^a \langle \det_{\mathfrak{n}^+}(x - y)^{-a} f(x - y), e^{(x|\bar{z})_{\mathfrak{p}^+} + (y|\bar{w})_{\mathfrak{p}^+}} \rangle_{(\frac{n}{r} + a) \otimes \mu, (x, y)}, \\ & (\mu)_{\mathbf{k}, d} \langle f(x - y), e^{(x|\bar{z})_{\mathfrak{p}^+} + (y|\bar{w})_{\mathfrak{p}^+}} \rangle_{\lambda \otimes \mu, (x, y)} \Big|_{\mu = \frac{n}{r} - a} \\ &= \left(\frac{n}{r} + a \right)_{\mathbf{k} - \mathbf{a}_r, d} \det_{\mathfrak{n}^+}(-w)^a \langle \det_{\mathfrak{n}^+}(x - y)^{-a} f(x - y), e^{(x|\bar{z})_{\mathfrak{p}^+} + (y|\bar{w})_{\mathfrak{p}^+}} \rangle_{\lambda \otimes (\frac{n}{r} + a), (x, y)}. \end{aligned}$$

Proof. Put $g(x) := \det_{\mathbf{n}^+}(x)^{-a} f(x) \in \mathcal{P}_{\mathbf{k}-\underline{a}_r}(\mathfrak{p}^+) = \mathcal{P}_{\mathbf{k}-\overbrace{(a, \dots, a)}^r}(\mathfrak{p}^+)$. Then by [44, Proposition 5.5], we have

$$\begin{aligned} \left(\frac{n}{r} - a + \mathbf{k}\right)_{\underline{a}_r, d} \tilde{g}_{\mathbf{m}, \mathbf{n}}(x, y) &= \left(\frac{n}{r} + \mathbf{m}\right)_{\underline{a}_r, d} \det_{\mathbf{n}^+}(x)^{-a} \tilde{f}_{\mathbf{m}+\underline{a}_r, \mathbf{n}}(x, y) \\ &= \left(\frac{n}{r} + \mathbf{n}\right)_{\underline{a}_r, d} \det_{\mathbf{n}^+}(y)^{-a} \tilde{f}_{\mathbf{m}, \mathbf{n}+\underline{a}_r}(x, y). \end{aligned}$$

Using this, we get

$$\begin{aligned} &(\lambda)_{\mathbf{k}, d} \langle f(x-y), e^{(x|\bar{z})_{\mathfrak{p}^+} + (y|\bar{w})_{\mathfrak{p}^+}} \rangle_{\lambda \otimes \mu, (x, y)} \Big|_{\lambda = \frac{n}{r} - a} \\ &= \sum_{\substack{\mathbf{m} \in \mathbb{Z}_{++}^r \\ m_j \leq k_j}} \sum_{\substack{\mathbf{n} \in \mathbb{Z}_{++}^r \\ n_j \leq k_j}} \frac{(\lambda + \mathbf{m})_{\mathbf{k}-\mathbf{m}, d}}{(\mu)_{\mathbf{n}, d}} \tilde{f}_{\mathbf{m}, \mathbf{n}}(z, -w) \Big|_{\lambda = \frac{n}{r} - a} \\ &= \sum_{\substack{\mathbf{m} \in \mathbb{Z}_{++}^r \\ a \leq m_j \leq k_j}} \sum_{\substack{\mathbf{n} \in \mathbb{Z}_{++}^r \\ n_j \leq k_j}} \frac{\left(\frac{n}{r} - a + \mathbf{m}\right)_{\mathbf{k}-\mathbf{m}, d}}{(\mu)_{\mathbf{n}, d}} \tilde{f}_{\mathbf{m}, \mathbf{n}}(z, -w) \\ &= \sum_{\substack{\mathbf{m} \in \mathbb{Z}_{++}^r \\ m_j \leq k_j - a}} \sum_{\substack{\mathbf{n} \in \mathbb{Z}_{++}^r \\ n_j \leq k_j}} \frac{\left(\frac{n}{r} + \mathbf{m}\right)_{\mathbf{k}-\underline{a}_r - \mathbf{m}, d}}{(\mu)_{\mathbf{n}, d}} \tilde{f}_{\mathbf{m}+\underline{a}_r, \mathbf{n}}(z, -w) \\ &= \sum_{\substack{\mathbf{m} \in \mathbb{Z}_{++}^r \\ m_j \leq k_j - a}} \sum_{\substack{\mathbf{n} \in \mathbb{Z}_{++}^r \\ n_j \leq k_j - a}} \frac{\left(\frac{n}{r} + \mathbf{m}\right)_{\mathbf{k}-\underline{a}_r - \mathbf{m}, d}}{(\mu)_{\mathbf{n}, d}} \frac{\left(\frac{n}{r} - a + \mathbf{k}\right)_{\underline{a}_r, d}}{\left(\frac{n}{r} + \mathbf{m}\right)_{\underline{a}_r, d}} \det_{\mathbf{n}^+}(z)^a \tilde{g}_{\mathbf{m}, \mathbf{n}}(z, -w) \\ &= \sum_{\substack{\mathbf{m} \in \mathbb{Z}_{++}^r \\ m_j \leq k_j - a}} \sum_{\substack{\mathbf{n} \in \mathbb{Z}_{++}^r \\ n_j \leq k_j - a}} \frac{\left(\frac{n}{r} + a + \mathbf{m}\right)_{\mathbf{k}-\underline{a}_r - \mathbf{m}, d}}{(\mu)_{\mathbf{n}, d}} \det_{\mathbf{n}^+}(z)^a \tilde{g}_{\mathbf{m}, \mathbf{n}}(z, -w) \\ &= \left(\frac{n}{r} + a\right)_{\mathbf{k}-\underline{a}_r, d} \det_{\mathbf{n}^+}(z)^a \langle \det_{\mathbf{n}^+}(x-y)^{-a} f(x-y), e^{(x|\bar{z})_{\mathfrak{p}^+} + (y|\bar{w})_{\mathfrak{p}^+}} \rangle_{\left(\frac{n}{r} + a\right) \otimes \mu, (x, y)}. \end{aligned}$$

The 2nd formula is also proved similarly. We can also prove this by an argument similar to Proposition 2.8 (2) and Theorem 3.5 by using Theorem 3.4. \blacksquare

We note that (8.1) may vanish for some $(\lambda, \mu) \in \mathbb{C}^2$ in general. For such $(\lambda, \mu) \in \mathbb{C}^2$ the following holds. For rank 1 case see also [33, Sections 8, 9].

Proposition 8.3. *Suppose $f \in \mathcal{P}_{\mathbf{k}}(\mathfrak{p}^+)$ is non-zero, and we consider the analytic continuation of (8.1). If (8.1) vanishes at $(\lambda, \mu) = (\lambda_0, \mu_0) \in \mathbb{C}^2$, then there exist $l_1, l_2 \in \mathbb{Z}_{>0}$ such that*

$$\begin{aligned} &\lim_{\lambda \rightarrow \lambda_0} \frac{1}{(\lambda - \lambda_0)^{l_1}} \left((\lambda)_{\mathbf{k}, d} (\mu)_{\mathbf{k}, d} \langle f(x-y), e^{(x|\bar{z})_{\mathfrak{p}^+} + (y|\bar{w})_{\mathfrak{p}^+}} \rangle_{\lambda \otimes \mu, (x, y)} \Big|_{\mu = \mu_0} \right), \\ &\lim_{\mu \rightarrow \mu_0} \frac{1}{(\mu - \mu_0)^{l_2}} \left((\lambda)_{\mathbf{k}, d} (\mu)_{\mathbf{k}, d} \langle f(x-y), e^{(x|\bar{z})_{\mathfrak{p}^+} + (y|\bar{w})_{\mathfrak{p}^+}} \rangle_{\lambda \otimes \mu, (x, y)} \Big|_{\lambda = \lambda_0} \right) \end{aligned}$$

converge to non-zero polynomials. These are linearly independent in $\mathcal{P}(\mathfrak{p}^+ \oplus \mathfrak{p}^+)$.

Proof. Let $\Lambda(\mathbf{k}) := \{(\mathbf{m}, \mathbf{n}) \in \mathbb{Z}_{++}^r \times \mathbb{Z}_{++}^r \mid \tilde{f}_{\mathbf{m}, \mathbf{n}}(x, y) \neq 0\}$. Then since

$$\begin{aligned} &(\lambda)_{\mathbf{k}, d} (\mu)_{\mathbf{k}, d} \langle f(x-y), e^{(x|\bar{z})_{\mathfrak{p}^+} + (y|\bar{w})_{\mathfrak{p}^+}} \rangle_{\lambda \otimes \mu, (x, y)} \Big|_{\mu = \mu_0} \\ &= \sum_{\substack{(\mathbf{m}, \mathbf{n}) \in \Lambda(\mathbf{k}) \\ (\mu_0 + \mathbf{n})_{\mathbf{k}-\mathbf{n}, d} \neq 0}} (\lambda + \mathbf{m})_{\mathbf{k}-\mathbf{m}, d} (\mu_0 + \mathbf{n})_{\mathbf{k}-\mathbf{n}, d} \tilde{f}_{\mathbf{m}, \mathbf{n}}(x, -y) \end{aligned}$$

vanishes at $\lambda = \lambda_0$, this is divisible by $(\lambda - \lambda_0)^{l_1}$ for some $l_1 \in \mathbb{Z}_{>0}$. Let l_1 be the maximum integer satisfying this. Then

$$\begin{aligned} & \lim_{\lambda \rightarrow \lambda_0} \frac{1}{(\lambda - \lambda_0)^{l_1}} \left((\lambda)_{\mathbf{k},d} (\mu)_{\mathbf{k},d} \langle f(x-y), e^{(x|\bar{z})_{\mathfrak{p}^+} + (y|\bar{w})_{\mathfrak{p}^+}} \rangle_{\lambda \otimes \mu, (x,y)} \Big|_{\mu=\mu_0} \right) \\ &= \lim_{\lambda \rightarrow \lambda_0} \sum_{\substack{(\mathbf{m}, \mathbf{n}) \in \Lambda(\mathbf{k}) \\ (\mu_0 + \mathbf{n})_{\mathbf{k}-\mathbf{n},d} \neq 0}} \frac{(\lambda + \mathbf{m})_{\mathbf{k}-\mathbf{m},d}}{(\lambda - \lambda_0)^{l_1}} (\mu_0 + \mathbf{n})_{\mathbf{k}-\mathbf{n},d} \tilde{f}_{\mathbf{m},\mathbf{n}}(x, -y) \end{aligned}$$

is non-zero. Similarly,

$$\begin{aligned} & \lim_{\mu \rightarrow \mu_0} \frac{1}{(\mu - \mu_0)^{l_2}} \left((\lambda)_{\mathbf{k},d} (\mu)_{\mathbf{k},d} \langle f(x-y), e^{(x|\bar{z})_{\mathfrak{p}^+} + (y|\bar{w})_{\mathfrak{p}^+}} \rangle_{\lambda \otimes \mu, (x,y)} \Big|_{\lambda=\lambda_0} \right) \\ &= \lim_{\mu \rightarrow \mu_0} \sum_{\substack{(\mathbf{m}, \mathbf{n}) \in \Lambda(\mathbf{k}) \\ (\lambda_0 + \mathbf{m})_{\mathbf{k}-\mathbf{m},d} \neq 0}} (\lambda_0 + \mathbf{m})_{\mathbf{k}-\mathbf{m},d} \frac{(\mu + \mathbf{n})_{\mathbf{k}-\mathbf{n},d}}{(\mu - \mu_0)^{l_2}} \tilde{f}_{\mathbf{m},\mathbf{n}}(x, -y) \end{aligned}$$

is non-zero for some $l_2 \in \mathbb{Z}_{>0}$. Now since $\tilde{f}_{\mathbf{m},\mathbf{n}}(x, -y)$ are linearly independent for all $(\mathbf{m}, \mathbf{n}) \in \Lambda(\mathbf{k})$, and since

$$\begin{aligned} & \{(\mathbf{m}, \mathbf{n}) \in \Lambda(\mathbf{k}) \mid (\mu_0 + \mathbf{n})_{\mathbf{k}-\mathbf{n},d} \neq 0\} \cap \{(\mathbf{m}, \mathbf{n}) \in \Lambda(\mathbf{k}) \mid (\lambda_0 + \mathbf{m})_{\mathbf{k}-\mathbf{m},d} \neq 0\} \\ &= \{(\mathbf{m}, \mathbf{n}) \in \Lambda(\mathbf{k}) \mid (\lambda_0 + \mathbf{m})_{\mathbf{k}-\mathbf{m},d} (\mu_0 + \mathbf{n})_{\mathbf{k}-\mathbf{n},d} \neq 0\} = \emptyset \end{aligned}$$

holds, the above two limits are linearly independent. ■

Next we compute the value of the inner product at $w = -z$.

Theorem 8.4. *Let $\operatorname{Re} \lambda, \operatorname{Re} \mu > p - 1$, $\mathbf{k} \in \mathbb{Z}_{++}^r$, and let $f \in \mathcal{P}_{\mathbf{k}}(\mathfrak{p}^+)$. Then we have*

$$\langle f(x-y), e^{(x-y|\bar{z})_{\mathfrak{p}^+}} \rangle_{\lambda \otimes \mu, (x,y)} = \frac{\tilde{C}^d(\lambda, \mu, \mathbf{k})}{(\lambda)_{\mathbf{k},d} (\mu)_{\mathbf{k},d}} f(z),$$

where

$$\begin{aligned} \tilde{C}^d(\lambda, \mu, \mathbf{k}) &= \frac{\prod_{1 \leq i \leq j \leq r} (\lambda + \mu - 1 - \frac{d}{2}(i+j-2))_{k_i+k_j}}{\prod_{1 \leq i < j \leq r+1} (\lambda + \mu - 1 - \frac{d}{2}(i+j-3))_{k_i+k_j}} \\ &= \frac{\prod_{a=1}^{2r-1} \prod_{i=\max\{1, a+1-r\}}^{\lceil a/2 \rceil} (\lambda + \mu - 1 - \frac{d}{2}(a-1))_{k_i+k_{a+1-i}}}{\prod_{a=1}^{2r-1} \prod_{i=\max\{1, a+1-r\}}^{\lceil a/2 \rceil} (\lambda + \mu - 1 - \frac{d}{2}(a-1))_{k_i+k_{a+2-i}}}. \end{aligned} \tag{8.2}$$

Here we put $k_{r+1} := 0$.

Proof. The 2nd equality of (8.2) is easy. For the 1st equality, by the last paragraph of Section 3.1, we may assume \mathfrak{p}^+ is of tube type, and by the K -equivariance, may assume $f(x) = \Delta_{\mathbf{k}}^{\mathfrak{n}^+}(x)$. We prove the theorem by induction on r . When $\mathbf{k} = (0, \dots, 0)$ (“ $r = 0$ case”), this is clear. Next we assume the theorem for $r - 1$, and prove for r . Since we have defined $k_{r+1} = 0$, by (3.6), we have

$$\begin{aligned} & \langle \Delta_{\mathbf{k}}^{\mathfrak{n}^+}(x-y), e^{(x-y|\bar{z})_{\mathfrak{p}^+}} \rangle_{\lambda \otimes \mu, (x,y)} \\ &= \frac{\det_{\mathfrak{n}^+}(z)^{-\lambda-\mu+\frac{2n}{r}}}{(\lambda)_{\underline{k}_r, d} (\mu)_{\underline{k}_r, d}} \det_{\mathfrak{n}^+} \left(\frac{\partial}{\partial z} \right)^{k_r} \det_{\mathfrak{n}^+}(z)^{\lambda+\mu+2k_r-\frac{2n}{r}} \left\langle \Delta_{\mathbf{k}-\underline{k}_r}^{\mathfrak{n}^+}(x-y), e^{(x-y|\bar{z})_{\mathfrak{p}^+}} \right\rangle_{\substack{(\lambda+k_r) \otimes \\ (\mu+k_r), (x,y)}} \end{aligned}$$

$$\begin{aligned}
&= \frac{\det_{\mathfrak{n}^+}(z)^{-\lambda-\mu+d(r-1)+2}}{(\lambda)_{\mathbf{k}_r,d}(\mu)_{\mathbf{k}_r,d}(\lambda+k_r)_{\mathbf{k}-\mathbf{k}_r,d}(\mu+k_r)_{\mathbf{k}-\mathbf{k}_r,d}} \det_{\mathfrak{n}^+} \left(\frac{\partial}{\partial z} \right)^{k_r} \det_{\mathfrak{n}^+}(z)^{\lambda+\mu+2k_r-d(r-1)-2} \\
&\quad \times \frac{\prod_{1 \leq i \leq j \leq r-1} ((\lambda+k_r) + (\mu+k_r) - 1 - \frac{d}{2}(i+j-2))_{(k_i-k_r)+(k_j-k_r)}}{\prod_{1 \leq i < j \leq r} ((\lambda+k_r) + (\mu+k_r) - 1 - \frac{d}{2}(i+j-3))_{(k_i-k_r)+(k_j-k_r)}} \Delta_{\mathbf{k}-\mathbf{k}_r}^{\mathfrak{n}^+}(z) \\
&= \frac{1}{(\lambda)_{\mathbf{k},d}(\mu)_{1,d}} \frac{\prod_{1 \leq i \leq j \leq r-1} (\lambda+\mu+2k_r-1-\frac{d}{2}(i+j-2))_{k_i+k_j-2k_r}}{\prod_{1 \leq i < j \leq r} (\lambda+\mu+2k_r-1-\frac{d}{2}(i+j-3))_{k_i+k_j-2k_r}} \\
&\quad \times \prod_{i=1}^r \left(\lambda+\mu+k_i-1-d(r-1)+\frac{d}{2}(r-i) \right)_{k_r} \det_{\mathfrak{n}^+}(z)^{k_r} \Delta_{\mathbf{k}-\mathbf{k}_r}^{\mathfrak{n}^+}(z) \\
&= \frac{1}{(\lambda)_{\mathbf{k},d}(\mu)_{1,d}} \frac{\prod_{1 \leq i \leq j \leq r-1} (\lambda+\mu-1-\frac{d}{2}(i+j-2))_{k_i+k_j}}{\prod_{1 \leq i < j \leq r} (\lambda+\mu-1-\frac{d}{2}(i+j-3))_{k_i+k_j}} \Delta_{\mathbf{k}}^{\mathfrak{n}^+}(z) \\
&\quad \times \frac{\prod_{1 \leq i < j \leq r} (\lambda+\mu-1-\frac{d}{2}(i+j-3))_{2k_r}}{\prod_{1 \leq i \leq j \leq r-1} (\lambda+\mu-1-\frac{d}{2}(i+j-2))_{2k_r}} \frac{\prod_{i=1}^r (\lambda+\mu-1-\frac{d}{2}(i+r-2))_{k_i+k_r}}{\prod_{i=1}^r (\lambda+\mu-1-\frac{d}{2}(i+(r+1)-3))_{k_i}} \\
&= \frac{1}{(\lambda)_{\mathbf{k},d}(\mu)_{1,d}} \frac{\prod_{1 \leq i \leq j \leq r} (\lambda+\mu-1-\frac{d}{2}(i+j-2))_{k_i+k_j}}{\prod_{1 \leq i < j \leq r+1} (\lambda+\mu-1-\frac{d}{2}(i+j-3))_{k_i+k_j}} \Delta_{\mathbf{k}}^{\mathfrak{n}^+}(z),
\end{aligned}$$

where we have used the induction hypothesis and Proposition 3.1 at the 2nd equality, and (2.6) at the 3rd equality. Therefore, the theorem holds for all r . \blacksquare

By Theorem 8.4, we get the following.

Corollary 8.5. *Let $\operatorname{Re} \lambda, \operatorname{Re} \mu > p-1$, $\mathbf{k} \in \mathbb{Z}_{++}^r$, and let $f \in \mathcal{P}_{\mathbf{k}}(\mathfrak{p}^+)$. Then we have*

$$\|f(x-y)\|_{\lambda \otimes \mu, (x,y)}^2 = \frac{\tilde{C}^d(\lambda, \mu, \mathbf{k})}{(\lambda)_{\mathbf{k},d}(\mu)_{\mathbf{k},d}} \|f\|_{F, \mathfrak{p}^+}^2,$$

where $\tilde{C}^d(\lambda, \mu, \mathbf{k})$ is as in (8.2).

Proof. As in the proof of Proposition 2.6 (1), we have

$$\begin{aligned}
\|f(x-y)\|_{\lambda \otimes \mu, (x,y)}^2 &= \left\langle \left\langle f(x-y), e^{\frac{1}{2}(x+y|\overline{z+w})} \right\rangle_{\lambda \otimes \mu, (x,y)}, \left\langle f(z-w), e^{\frac{1}{2}(x-y|\overline{z-w})} \right\rangle_{F, (z,w)} \right\rangle_{F, (z,w)} \\
&= \left\langle \left\langle f(x-y), e^{\frac{1}{2}(x-y|\overline{z-w})} \right\rangle_{\lambda \otimes \mu, (x,y)}, \left\langle f(z-w), e^{\frac{1}{2}(x-y|\overline{z-w})} \right\rangle_{F, (z,w)} \right\rangle_{F, (z,w)} \\
&= \frac{\tilde{C}^d(\lambda, \mu, \mathbf{k})}{(\lambda)_{\mathbf{k},d}(\mu)_{\mathbf{k},d}} \left\langle f \left(\frac{z-w}{2} \right), f(z-w) \right\rangle_{F, (z,w)} \\
&= \frac{\tilde{C}^d(\lambda, \mu, \mathbf{k})}{(\lambda)_{\mathbf{k},d}(\mu)_{\mathbf{k},d}} \|f\|_{F, \mathfrak{p}^+}^2. \quad \blacksquare
\end{aligned}$$

Next we give a rough estimate of zeroes of $\tilde{C}^d(\lambda, \mu, \mathbf{k})$.

Proposition 8.6. *For $\mathbf{k} \in \mathbb{Z}_{++}^r$, let $\tilde{C}^d(\lambda, \mu, \mathbf{k})$ be as in (8.2). For $1 \leq m_1 \leq m_2 \leq r$, if $k_1 = \dots = k_{m_1}$ and $k_{m_2+1} = 0$, then*

$$\begin{aligned}
&\{(\lambda, \mu) \in \mathbb{C}^2 \mid \tilde{C}^d(\lambda, \mu, \mathbf{k}) = 0\} \\
&\subset \begin{cases} \{(\lambda, \mu) \in \mathbb{C}^2 \mid \frac{d}{2}(m_1-1) - 2k_1 + 2 \leq \lambda + \mu \leq d(m_2-1) - k_{m_2} + 1\}, & k_1 \geq 1, \\ \emptyset, & k_1 = 0. \end{cases}
\end{aligned}$$

Epecially, for $m = 0, 1, \dots, r-1$, if $k_{m+1} = 0$, then $\tilde{C}^d(\lambda, \mu, \mathbf{k}) \neq 0$ for $\lambda, \mu \geq \frac{d}{2}m$.

Proof. By direct computation, we have

$$\begin{aligned}\tilde{C}^d(\lambda, \mu, \mathbf{k}) &= \tilde{C}^d(\lambda, \mu, (k_1, \dots, k_{m_2})) \\ &= \prod_{a=m_1}^{2m_2-1} \prod_{i=\max\{1, a+1-m_2\}}^{\lceil a/2 \rceil} \frac{(\lambda + \mu - 1 - \frac{d}{2}(a-1))_{k_i+k_{a+1-i}}}{(\lambda + \mu - 1 - \frac{d}{2}(a-1))_{k_i+k_{a+2-i}}}.\end{aligned}$$

For $1 \leq a \leq 2m_2 - 1$, let

$$\begin{aligned}\phi(\mathbf{k})_a &:= \min\{k_i + k_j \mid 1 \leq i < j \leq m_2 + 1, i + j = a + 2\}, \\ \psi(\mathbf{k})_a &:= \max\{k_i + k_j \mid 1 \leq i \leq j \leq m_2, i + j = a + 1\}.\end{aligned}$$

Then we have

$$\begin{aligned}\{(\lambda, \mu) \in \mathbb{C}^2 \mid \tilde{C}^d(\lambda, \mu, \mathbf{k}) = 0\} &= \bigcup_{a=m_1}^{2m_2-1} \left\{ (\lambda, \mu) \in \mathbb{C}^2 \mid \prod_{i=\max\{1, a+1-m_2\}}^{\lceil a/2 \rceil} \frac{(\lambda + \mu - 1 - \frac{d}{2}(a-1))_{k_i+k_{a+1-i}}}{(\lambda + \mu - 1 - \frac{d}{2}(a-1))_{k_i+k_{a+2-i}}} = 0 \right\} \\ &\subset \bigcup_{a=m_1}^{2m_2-1} \left\{ (\lambda, \mu) \in \mathbb{C}^2 \mid \frac{(\lambda + \mu - 1 - \frac{d}{2}(a-1))_{\psi(\mathbf{k})_a}}{(\lambda + \mu - 1 - \frac{d}{2}(a-1))_{\phi(\mathbf{k})_a}} = 0 \right\} \\ &= \bigcup_{a=m_1}^{2m_2-1} \left\{ (\lambda, \mu) \in \mathbb{C}^2 \mid \lambda + \mu = \frac{d}{2}(a-1) - j + 1, j \in \mathbb{Z}, \phi(\mathbf{k})_a \leq j \leq \psi(\mathbf{k})_a - 1 \right\} \\ &\subset \left\{ (\lambda, \mu) \in \mathbb{C}^2 \mid \frac{d}{2}(m_1 - 1) - 2k_1 + 2 \leq \lambda + \mu \leq \frac{d}{2}(2m_2 - 2) - k_{m_2} + 1 \right\}.\end{aligned}$$

When $k_1 = 0$, we have $\tilde{C}^d(\lambda, \mu, \mathbf{k}) = 1$, and this is non-zero everywhere. The last statement is clear. \blacksquare

8.2 Results on restriction of $\mathcal{H}_\lambda(D) \hat{\boxtimes} \mathcal{H}_\mu(D)$ to subgroups

Next we consider the decomposition of the tensor product representation $\mathcal{H}_\lambda(D) \hat{\boxtimes} \mathcal{H}_\mu(D)|_{\Delta(\tilde{G})} = \mathcal{H}_\lambda(D) \hat{\otimes} \mathcal{H}_\mu(D)$ under the diagonal subgroup $\Delta(\tilde{G}) \subset \tilde{G} \times \tilde{G}$. By Theorem 2.5, we have

$$\mathcal{H}_\lambda(D) \hat{\otimes} \mathcal{H}_\mu(D) \simeq \sum_{\mathbf{k} \in \mathbb{Z}_{++}^r}^{\oplus} \mathcal{H}_{\lambda+\mu}(D, \mathcal{P}_{\mathbf{k}}(\mathfrak{p}^+)).$$

For each $\mathbf{k} \in \mathbb{Z}_{++}^r$, let $V_{\mathbf{k}}$ be the abstract K -module isomorphic to $\mathcal{P}_{\mathbf{k}}(\mathfrak{p}^+)$, let $\|\cdot\|_{\nu, \mathbf{k}}$ be the \tilde{G} -invariant norm on $\mathcal{H}_\nu(D, V_{\mathbf{k}})$ normalized such that $\|v\|_{\nu, \mathbf{k}} = |v|_{V_{\mathbf{k}}}$ holds for all constant functions $v \in V_{\mathbf{k}}$, and for $\lambda, \mu > p - 1$, we consider the symmetry breaking operator

$$\begin{aligned}\hat{\mathcal{F}}_{\lambda, \mu, \mathbf{k}}^\downarrow &: \mathcal{H}_\lambda(D) \hat{\otimes} \mathcal{H}_\mu(D) \longrightarrow \mathcal{H}_{\lambda+\mu}(D, V_{\mathbf{k}}), \\ (\hat{\mathcal{F}}_{\lambda, \mu, \mathbf{k}}^\downarrow f)(x) &:= \hat{F}_{\lambda, \mu, \mathbf{k}} \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) f(x, y) \Big|_{y=x}, \\ \hat{F}_{\lambda, \mu, \mathbf{k}}(z, w) &:= (\lambda)_{\mathbf{k}, d} (\mu)_{\mathbf{k}, d} \langle e^{(x|z)_{\mathfrak{p}^+} + (y|w)_{\mathfrak{p}^+}}, \mathbf{K}_{\mathbf{k}}(x - y) \rangle_{\lambda \otimes \mu, (x, y)} \in \mathcal{P}(\mathfrak{p}^- \oplus \mathfrak{p}^-, V_{\mathbf{k}}),\end{aligned}$$

where $\mathbf{K}_{\mathbf{k}}(x) \in \mathcal{P}(\mathfrak{p}^+, \overline{V_{\mathbf{k}}})^K$ is normalized such that

$$|\langle f(x), \mathbf{K}_{\mathbf{k}}(x) \rangle_{F, \mathfrak{p}^+}|_{V_{\mathbf{k}}}^2 = \|f\|_{F, \mathfrak{p}^+}^2, \quad f(x) \in \mathcal{P}_{\mathbf{k}}(\mathfrak{p}^+).$$

This operator coincides with the one given in [51] up to constant multiple. In our normalization, $\hat{\mathcal{F}}_{\lambda,\mu,\mathbf{k}}^\downarrow: \mathcal{O}_\lambda(D) \hat{\otimes} \mathcal{O}_\mu(D) \rightarrow \mathcal{O}_{\lambda+\mu}(D, V_{\mathbf{k}})$ is holomorphically continued for all $(\lambda, \mu) \in \mathbb{C}^2$. We note that the normalization of $\hat{\mathcal{F}}_{\lambda,\mu,\mathbf{k}}^\downarrow$ is different from (2.18) and (2.19), that is, we have

$$\hat{\mathcal{F}}_{\lambda,\mu,\mathbf{k}}^\downarrow(f(x-y)) = \tilde{C}^d(\lambda, \mu, \mathbf{k}) \langle f(x), K_{\mathbf{k}}(x) \rangle_{F, \mathfrak{p}^+} \in V_{\mathbf{k}}, \quad f(x) \in \mathcal{P}_{\mathbf{k}}(\mathfrak{p}^+).$$

Also, when \mathfrak{p}^+ is of tube type, we fix $[K, K]$ -isomorphisms $V_{\mathbf{k}+a_r} \simeq V_{\mathbf{k}}$ for each $a \in \mathbb{Z}_{>0}$. Then as in Proposition 2.6, by Theorems 8.1, 8.2, Propositions 8.3, 8.6 and Corollary 8.5, the following hold.

Corollary 8.7. *For $\lambda, \mu > p - 1$ and for $f \in \mathcal{H}_\lambda(D) \hat{\otimes} \mathcal{H}_\mu(D)$, we have*

$$\|f\|_{\lambda \otimes \mu}^2 = \sum_{\mathbf{k} \in \mathbb{Z}_{++}^r} \frac{1}{(\lambda)_{\mathbf{k},d} (\mu)_{\mathbf{k},d} \tilde{C}^d(\lambda, \mu, \mathbf{k})} \|\hat{\mathcal{F}}_{\lambda,\mu,\mathbf{k}}^\downarrow f\|_{\lambda+\mu,\mathbf{k}}^2,$$

where $\tilde{C}^d(\lambda, \mu, \mathbf{k})$ is as in (8.2).

Corollary 8.8. *Let $\mathbf{k} \in \mathbb{Z}_{++}^r$.*

(1) *For $a_1, a_2 = 1, 2, \dots, r$,*

$$d(\tau_\lambda \hat{\otimes} \tau_\mu)(\mathcal{U}(\mathfrak{g}))\{f(x-y) \mid f \in \mathcal{P}_{\mathbf{k}}(\mathfrak{p}^+)\} \subset M_{a_1}^{\mathfrak{g}}(\lambda) \otimes M_{a_2}^{\mathfrak{g}}(\mu)$$

holds if and only if

$$(\lambda, \mu) \in \left(\frac{d}{2}(a_1 - 1) - k_{a_1} - \mathbb{Z}_{\geq 0} \right) \times \left(\frac{d}{2}(a_2 - 1) - k_{a_2} - \mathbb{Z}_{\geq 0} \right),$$

where $M_a^{\mathfrak{g}}(\lambda) \subset \mathcal{O}_\lambda(D)_{\tilde{K}}$ is the $(\mathfrak{g}, \tilde{K})$ -submodule given in (2.12).

(2) *Let $\lambda, \mu \in \{0, \frac{d}{2}, \dots, \frac{d}{2}(r-1)\} \cup (\frac{d}{2}(r-1), \infty)$. If $\min\{\lambda, \mu\} = \frac{d}{2}a$, $a \in \{0, 1, \dots, r-1\}$, then we have*

$$\mathcal{H}_\lambda(D) \hat{\otimes} \mathcal{H}_\mu(D) \simeq \sum_{\substack{\mathbf{k} \in \mathbb{Z}_{++}^r \\ k_{a+1}=0}}^{\oplus} \mathcal{H}_{\lambda+\mu}(D, \mathcal{P}_{\mathbf{k}}(\mathfrak{p}^+)).$$

(3) *For $a = 0, 1, \dots, r-1$, if $k_{a+1} = 0$, then for $\lambda, \mu \in \{\frac{d}{2}a, \frac{d}{2}(a+1), \dots, \frac{d}{2}(r-1)\} \cup (\frac{d}{2}(r-1), \infty)$, the restriction of $\hat{\mathcal{F}}_{\lambda,\mu,\mathbf{k}}^\downarrow$ gives the symmetry breaking operator*

$$\hat{\mathcal{F}}_{\lambda,\mu,\mathbf{k}}^\downarrow: \mathcal{H}_\lambda(D) \hat{\otimes} \mathcal{H}_\mu(D) \longrightarrow \mathcal{H}_{\lambda+\mu}(D, V_{\mathbf{k}}).$$

(4) *Suppose \mathfrak{p}^+ is of tube type. For $a = 1, 2, \dots, k_r$, if*

$$K_{\mathbf{k}}(x) = c \det_{\mathfrak{n}^+}(x)^a K_{\mathbf{k}-a_r}(x),$$

then we have

$$\begin{aligned} \hat{\mathcal{F}}_{\frac{n}{r}-a,\mu,\mathbf{k}}^\downarrow &= c \frac{(\mu)_{\mathbf{k},d}}{(\mu)_{\mathbf{k}-a_r,d}} \hat{\mathcal{F}}_{\frac{n}{r}+a,\mu,\mathbf{k}-a_r}^\downarrow \circ \left(\det_{\mathfrak{n}^-} \left(\frac{\partial}{\partial x} \right)^a \otimes 1 \right): \\ \mathcal{O}_{\frac{n}{r}-a}(D) \hat{\otimes} \mathcal{O}_\mu(D) &\longrightarrow \mathcal{O}_{\frac{n}{r}-a+\mu}(D, V_{\mathbf{k}}) \simeq \mathcal{O}_{\frac{n}{r}+a+\mu}(D, V_{\mathbf{k}-a_r}), \\ \hat{\mathcal{F}}_{\lambda,\frac{n}{r}-a,\mathbf{k}}^\downarrow &= c \frac{(\lambda)_{\mathbf{k},d}}{(\lambda)_{\mathbf{k}-a_r,d}} \hat{\mathcal{F}}_{\lambda,\frac{n}{r}+a,\mathbf{k}-a_r}^\downarrow \circ \left(1 \otimes \det_{\mathfrak{n}^-} \left(-\frac{\partial}{\partial y} \right)^a \right): \\ \mathcal{O}_\lambda(D) \hat{\otimes} \mathcal{O}_{\frac{n}{r}-a}(D) &\longrightarrow \mathcal{O}_{\lambda+\frac{n}{r}-a}(D, V_{\mathbf{k}}) \simeq \mathcal{O}_{\lambda+\frac{n}{r}+a}(D, V_{\mathbf{k}-a_r}). \end{aligned}$$

(5) Suppose $\hat{\mathcal{F}}_{\lambda_0, \mu_0, \mathbf{k}}^\downarrow = 0$. Then there exist $l_1, l_2 \in \mathbb{Z}_{>0}$ such that

$$\lim_{\lambda \rightarrow \lambda_0} \frac{\hat{\mathcal{F}}_{\lambda, \mu_0, \mathbf{k}}^\downarrow}{(\lambda - \lambda_0)^{l_1}}, \lim_{\mu \rightarrow \mu_0} \frac{\hat{\mathcal{F}}_{\lambda_0, \mu, \mathbf{k}}^\downarrow}{(\mu - \mu_0)^{l_2}} : \mathcal{O}_{\lambda_0}(D) \hat{\otimes} \mathcal{O}_{\mu_0}(D) \longrightarrow \mathcal{O}_{\lambda_0 + \mu_0}(D, V_{\mathbf{k}})$$

converge to non-zero operators. These are linearly independent in $\text{Hom}_{\tilde{G}}(\mathcal{O}_{\lambda_0}(D) \hat{\otimes} \mathcal{O}_{\mu_0}(D), \mathcal{O}_{\lambda_0 + \mu_0}(D, V_{\mathbf{k}}))$, and especially, this space is at least 2-dimensional.

Corollary 8.8 (2) and (3) are earlier given in [51, Theorems 3.3 and 4.4]. As an example of (5), by (4) we easily get the following.

Corollary 8.9. *Suppose \mathfrak{p}^+ is of tube type, and let $\mathbf{k} \in \mathbb{Z}_{++}^r$. For $a_1, a_2 = 1, 2, \dots, k_r$ with $a_1 + a_2 \geq k_r + 1$, we have*

$$\begin{aligned} \lim_{\mu \rightarrow \frac{n}{r} - a_2} \frac{\hat{\mathcal{F}}_{\frac{n}{r} - a_1, \mu, \mathbf{k}}^\downarrow}{(\mu - a_1 + \mathbf{k})_{\underline{a}_{1,r}, d}} &= c \hat{\mathcal{F}}_{\frac{n}{r} + a_1, \frac{n}{r} - a_2, \mathbf{k} - \underline{a}_{1,r}}^\downarrow \circ \left(\det_{\mathfrak{n}^-} \left(\frac{\partial}{\partial x} \right)^{a_1} \otimes 1 \right), \\ \lim_{\lambda \rightarrow \frac{n}{r} - a_1} \frac{\hat{\mathcal{F}}_{\lambda, \frac{n}{r} - a_2, \mathbf{k}}^\downarrow}{(\lambda - a_2 + \mathbf{k})_{\underline{a}_{2,r}, d}} &= c \hat{\mathcal{F}}_{\frac{n}{r} - a_1, \frac{n}{r} + a_2, \mathbf{k} - \underline{a}_{2,r}}^\downarrow \circ \left(1 \otimes \det_{\mathfrak{n}^-} \left(-\frac{\partial}{\partial y} \right)^{a_2} \right); \\ \mathcal{O}_{\frac{n}{r} - a_1}(D) \hat{\otimes} \mathcal{O}_{\frac{n}{r} - a_2}(D) &\longrightarrow \mathcal{O}_{2\frac{n}{r} - a_1 - a_2}(D, V_{\mathbf{k}}), \end{aligned}$$

and these are linearly independent.

8.3 Example: $\mathbf{k} = (k, \dots, k)$ case

In this subsection we assume $\mathfrak{p}^+ = \mathfrak{n}^{\mathbb{C}}$ is of tube type, and consider the case $\mathbf{k} = \underline{k}_r = (k, \dots, k)$, so that $\mathcal{P}_{\underline{k}_r}(\mathfrak{p}^+) = \mathbb{C} \det_{\mathfrak{n}^+}(x)^k$ holds. For $\lambda, \mu \in \mathbb{C}$, $k \in \mathbb{Z}_{\geq 0}$, we define the polynomial

$RC_{\lambda, \mu, k}^{\mathfrak{n}^+}(z, w) \in \mathcal{P}(\mathfrak{p}^+ \oplus \mathfrak{p}^+)$ by

$$RC_{\lambda, \mu, k}^{\mathfrak{n}^+}(z, w) := (\lambda)_{\underline{k}_r, d} (\mu)_{\underline{k}_r, d} \langle \det_{\mathfrak{n}^+}(x - y)^k, e^{(x|\bar{z})_{\mathfrak{p}^+} + (y|\bar{w})_{\mathfrak{p}^+}} \rangle_{\lambda \otimes \mu, (x, y)}.$$

This is originally defined for $\text{Re } \lambda, \text{Re } \mu > p - 1 = \frac{2n}{r} - 1$, and holomorphically continued for all $\lambda, \mu \in \mathbb{C}$. To describe this explicitly, for $\mathbf{m} \in \mathbb{Z}_{++}^r$ let

$$\Phi_{\mathbf{m}}^{\mathfrak{n}^+}(x) := \int_{K_L} \Delta_{\mathbf{m}}^{\mathfrak{n}^+}(kx) dk \in \mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+)^{K_L}, \quad d_{\mathbf{m}}^{\mathfrak{p}^+} := \dim \mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+),$$

where $K_L \subset K \subset G$ is the subgroup given in Section 2.2, which acts on \mathfrak{n}^+ as Jordan algebra automorphisms. Also, for $\alpha, \beta, \gamma \in \mathbb{C}$ let

$${}_2F_1^{\mathfrak{n}^+} \left(\begin{matrix} \alpha, \beta \\ \gamma \end{matrix}; z \right) := \sum_{\mathbf{m} \in \mathbb{Z}_{++}^r} \frac{(\alpha)_{\mathbf{m}, d} (\beta)_{\mathbf{m}, d}}{(\gamma)_{\mathbf{m}, d}} \frac{d_{\mathbf{m}}^{\mathfrak{p}^+}}{\binom{n}{r}_{\mathbf{m}, d}} \Phi_{\mathbf{m}}^{\mathfrak{n}^+}(z).$$

Then the following holds.

Theorem 8.10. *For $\lambda, \mu \in \mathbb{C}$, $k \in \mathbb{Z}_{\geq 0}$, $z, w \in \mathfrak{p}^+$, we have*

$$\begin{aligned} RC_{\lambda, \mu, k}^{\mathfrak{n}^+}(z, w) &= (\mu)_{\underline{k}_r, d} \det_{\mathfrak{n}^+}(z)^k {}_2F_1^{\mathfrak{n}^+} \left(\begin{matrix} -k, -\lambda - k + \frac{n}{r} \\ \mu \end{matrix}; -P(z^{-1/2})w \right) \\ &= (\lambda)_{\underline{k}_r, d} \det_{\mathfrak{n}^+}(-w)^k {}_2F_1^{\mathfrak{n}^+} \left(\begin{matrix} -k, -\mu - k + \frac{n}{r} \\ \lambda \end{matrix}; -P(w^{-1/2})z \right). \end{aligned}$$

Proof. By [13, Proposition XII.1.3 (ii)], we have

$$\det_{\mathfrak{n}^+}(x-y)^k = \det_{\mathfrak{n}^+}(x)^k \sum_{\substack{\mathbf{m} \in \mathbb{Z}_{++}^r \\ m_1 \leq k}} \frac{(-k)_{\mathbf{m},d} d_{\mathbf{m}}^{\mathfrak{p}^+}}{\binom{n}{r}_{\mathbf{m},d}} \Phi_{\mathbf{m}}^{\mathfrak{n}^+}(P(x^{-1/2})y) \quad (8.3)$$

$$= \det_{\mathfrak{n}^+}(-y)^k \sum_{\substack{\mathbf{m} \in \mathbb{Z}_{++}^r \\ m_1 \leq k}} \frac{(-k)_{\mathbf{m},d} d_{\mathbf{m}}^{\mathfrak{p}^+}}{\binom{n}{r}_{\mathbf{m},d}} \Phi_{\mathbf{m}}^{\mathfrak{n}^+}(P(y^{-1/2})x), \quad (8.4)$$

and by [13, Lemma XIV.1.2], we have

$$\det_{\mathfrak{n}^+}(x)^k \Phi_{\mathbf{m}}^{\mathfrak{n}^+}(P(x^{-1/2})y) = \det_{\mathfrak{n}^+}(y)^k \Phi_{\underline{k}_r - \mathbf{m}^\vee}^{\mathfrak{n}^+}(P(y^{-1/2})x) \in \mathcal{P}_{\underline{k}_r - \mathbf{m}^\vee}(\mathfrak{p}^+) \otimes \mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+),$$

where $\mathbf{m}^\vee := (m_r, \dots, m_1)$. Then by Corollary 2.4 and (8.3), we get

$$\begin{aligned} & RC_{\lambda, \mu, k}^{\mathfrak{n}^+}(z, w) \\ &= (\lambda)_{\underline{k}_r, d} (\mu)_{\underline{k}_r, d} \sum_{\substack{\mathbf{m} \in \mathbb{Z}_{++}^r \\ m_1 \leq k}} \frac{(-k)_{\mathbf{m},d} d_{\mathbf{m}}^{\mathfrak{p}^+}}{\binom{n}{r}_{\mathbf{m},d}} \langle \det_{\mathfrak{n}^+}(x)^k \Phi_{\mathbf{m}}^{\mathfrak{n}^+}(P(x^{-1/2})y), e^{(x|\bar{z})_{\mathfrak{p}^+} + (y|\bar{w})_{\mathfrak{p}^+}} \rangle_{\lambda \otimes \mu, (x, y)} \\ &= (\lambda)_{\underline{k}_r, d} (\mu)_{\underline{k}_r, d} \sum_{\substack{\mathbf{m} \in \mathbb{Z}_{++}^r \\ m_1 \leq k}} \frac{(-k)_{\mathbf{m},d}}{(\lambda)_{\underline{k}_r - \mathbf{m}^\vee, d} (\mu)_{\mathbf{m},d}} \frac{d_{\mathbf{m}}^{\mathfrak{p}^+}}{\binom{n}{r}_{\mathbf{m},d}} \det_{\mathfrak{n}^+}(z)^k \Phi_{\mathbf{m}}^{\mathfrak{n}^+}(P(z^{-1/2})w) \\ &= (\mu)_{\underline{k}_r, d} \det_{\mathfrak{n}^+}(z)^k \sum_{\substack{\mathbf{m} \in \mathbb{Z}_{++}^r \\ m_1 \leq k}} \frac{(-k)_{\mathbf{m},d} (-\lambda - k + \frac{n}{r})_{\mathbf{m},d}}{(\mu)_{\mathbf{m},d}} \frac{d_{\mathbf{m}}^{\mathfrak{p}^+}}{\binom{n}{r}_{\mathbf{m},d}} (-1)^{|\mathbf{m}|} \Phi_{\mathbf{m}}^{\mathfrak{n}^+}(P(z^{-1/2})w) \\ &= (\mu)_{\underline{k}_r, d} \det_{\mathfrak{n}^+}(z)^k {}_2F_1^{\mathfrak{n}^+} \left(\begin{matrix} -k, -\lambda - k + \frac{n}{r} \\ \mu \end{matrix}; -P(z^{-1/2})w \right). \end{aligned}$$

The 2nd equality is also proved similarly by using (8.4). ■

For special $\lambda, \mu \in \mathbb{C}$, $RC_{\lambda, \mu, k}^{\mathfrak{n}^+}$ is factorized as follows.

Theorem 8.11. *Let $a = 1, 2, \dots, k$.*

- (1) $RC_{\frac{n}{r}-a, \mu, k}^{\mathfrak{n}^+}(z, w) = (\mu + k - a)_{\underline{a}, d} \det_{\mathfrak{n}^+}(z)^a RC_{\frac{n}{r}+a, \mu, k-a}^{\mathfrak{n}^+}(z, w)$.
- (2) $RC_{\lambda, \frac{n}{r}-a, k}^{\mathfrak{n}^+}(z, w) = (\lambda + k - a)_{\underline{a}, d} \det_{\mathfrak{n}^+}(-w)^a RC_{\lambda, \frac{n}{r}+a, k-a}^{\mathfrak{n}^+}(z, w)$.
- (3) *Suppose $\lambda + \mu = \frac{n}{r} - 2k + a$. Then*

$$\begin{aligned} RC_{\lambda, \mu, k}^{\mathfrak{n}^+}(z, w) &= (\mu + k - a)_{\underline{a}, d} \det_{\mathfrak{n}^+}(z+w)^a RC_{\lambda, \mu, k-a}^{\mathfrak{n}^+}(z, w) \\ &= (-1)^{ar} (\lambda + k - a)_{\underline{a}, d} \det_{\mathfrak{n}^+}(z+w)^a RC_{\lambda, \mu, k-a}^{\mathfrak{n}^+}(z, w). \end{aligned}$$

Proof. (1) and (2) follow from Theorem 8.2.

For (3), by [13, Proposition XV.3.4 (ii)], we have

$$\begin{aligned} RC_{\lambda, \mu, k}^{\mathfrak{n}^+}(z, w) &= (\mu)_{\underline{k}_r, d} \det_{\mathfrak{n}^+}(z)^k {}_2F_1^{\mathfrak{n}^+} \left(\begin{matrix} -k, -\lambda - k + \frac{n}{r} \\ \mu \end{matrix}; -P(z^{-1/2})w \right) \\ &= (\mu)_{\underline{k}_r, d} \det_{\mathfrak{n}^+}(z)^k \det_{\mathfrak{n}^+}(e + P(z^{-1/2})w)^{\lambda + \mu + 2k - \frac{n}{r}} \end{aligned}$$

$$\begin{aligned}
 & \times {}_2F_1^{\mathfrak{n}^+} \left(\lambda + \mu + k - \frac{n}{r}, \mu + k; -P(z^{-1/2})w \right) \\
 & = (\mu)_{\underline{k}_r, d} \det_{\mathfrak{n}^+}(z)^k \det_{\mathfrak{n}^+}(e + P(z^{-1/2})w)^a {}_2F_1^{\mathfrak{n}^+} \left(-k + a, \mu + k; -P(z^{-1/2})w \right) \\
 & = (\mu)_{\underline{k}-a_r, d} (\mu + k - a)_{\underline{a}_r, d} \det_{\mathfrak{n}^+}(z)^{k-a} \det_{\mathfrak{n}^+}(z+w)^a \\
 & \quad \times {}_2F_1^{\mathfrak{n}^+} \left(-k + a, -\lambda - k + a + \frac{n}{r}; -P(z^{-1/2})w \right) \\
 & = (\mu + k - a)_{\underline{a}_r, d} \det_{\mathfrak{n}^+}(z+w)^a RC_{\lambda, \mu, k-a}^{\mathfrak{n}^+}(z, w),
 \end{aligned}$$

and we have $(\mu + k - a)_{\underline{a}_r, d} = (-\lambda - k + \frac{n}{r})_{\underline{a}_r, d} = (-1)^{ar} (\lambda + k - a)_{\underline{a}_r, d}$. \blacksquare

The polynomial $RC_{\lambda, \mu, k}^{\mathfrak{n}^+}(z, w)$ gives the symmetry breaking operator

$$\begin{aligned}
 \mathcal{RC}_{\lambda, \mu, k}^\downarrow &: \mathcal{O}_\lambda(D) \hat{\otimes} \mathcal{O}_\mu(D) \longrightarrow \mathcal{O}_{\lambda+\mu+2k}(D), \\
 (\mathcal{RC}_{\lambda, \mu, k}^\downarrow f)(x) &:= RC_{\lambda, \mu, k}^{\mathfrak{n}^+} \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) f(x, y) \Big|_{y=x},
 \end{aligned}$$

where we normalize $\frac{\partial}{\partial x}$ with respect to the bilinear form $(\cdot|\cdot)_{\mathfrak{n}^+} = (\cdot|Q(\bar{e})\cdot)_{\mathfrak{p}^+}$ on $\mathfrak{p}^+ = \mathfrak{n}^{+\mathbb{C}}$. We note that $\mathcal{O}_{\lambda+\mu}(D, \mathcal{P}_{\underline{k}_r}(\mathfrak{p}^+)) \simeq \mathcal{O}_{\lambda+\mu+2k}(D)$ holds if \mathfrak{p}^+ is of tube type. When $\mathfrak{p}^+ = \mathbb{C}$, $G = \mathrm{SL}(2, \mathbb{R})$, this is proportional to the Rankin–Cohen bidifferential operator (see, e.g., [8, 53] and [33, Sections 8 and 9]). For general \mathfrak{p}^+ see also, e.g., [6, 7, 49, 50]. In our normalization we have the following.

Proposition 8.12. $\mathcal{RC}_{\lambda, \mu, k}^\downarrow(\det_{\mathfrak{n}^+}(x-y)^k) = \left(\lambda + \mu + k - \frac{n}{r}\right)_{\underline{k}_r, d} \binom{n}{r}_{\underline{k}_r, d}$.

Proof. As in Proposition 2.6 (2), by Corollary 8.5 we have

$$\begin{aligned}
 \mathcal{RC}_{\lambda, \mu, k}^\downarrow(\det_{\mathfrak{n}^+}(x-y)^k) &= (\lambda)_{\underline{k}_r, d} (\mu)_{\underline{k}_r, d} \|\det_{\mathfrak{n}^+}(x-y)^k\|_{\lambda \otimes \mu, (x, y)}^2 \\
 &= \tilde{C}^d(\lambda, \mu, \underline{k}_r) \|\det_{\mathfrak{n}^+}(x)^k\|_{F, \mathfrak{p}^+}^2,
 \end{aligned}$$

and we have

$$\tilde{C}^d(\lambda, \mu, \underline{k}_r) = \left(\lambda + \mu + k - \frac{n}{r}\right)_{\underline{k}_r, d}, \quad \|\det_{\mathfrak{n}^+}(x)^k\|_{F, \mathfrak{p}^+}^2 = \binom{n}{r}_{\underline{k}_r, d}$$

(for the 2nd equality see [13, Proposition XI.4.1 (ii)]). Hence we get the proposition. \blacksquare

Next we consider the zeroes of $RC_{\lambda, \mu, k}^{\mathfrak{n}^+}$. If this vanishes at (λ, μ) , then by Corollary 8.8 (4), $\mathrm{Hom}_{\tilde{G}}(\mathcal{O}_\lambda(D) \hat{\otimes} \mathcal{O}_\mu(D), \mathcal{O}_{\lambda+\mu+2k}(D))$ is at least 2-dimensional. More precisely, the following holds.

Theorem 8.13. For $k \in \mathbb{Z}_{\geq 1}$, $1 \leq i, j \leq r$, let

$$Z_{i, j}^{k, d} := \left\{ \left(\frac{d}{2}(i-1) + 1 - a_1, \frac{d}{2}(j-1) + 1 - a_2 \right) \mid \begin{array}{l} a_1, a_2 \in \mathbb{Z}, 1 \leq a_1, a_2 \leq k, \\ a_1 + a_2 \geq k + 1 \end{array} \right\}.$$

(1) $RC_{\lambda, \mu, k}^{\mathfrak{n}^+} = 0$ holds if

$$(\lambda, \mu) \in \bigcup_{\substack{1 \leq i, j \leq r \\ i+j \geq r+1}} Z_{i, j}^{k, d}.$$

(2) For (λ, μ) in the above set, let

$$l := \min \{i + j \mid 1 \leq i, j \leq r, i + j \geq r + 1, (\lambda, \mu) \in Z_{i,j}^{k,d}\},$$

$$\alpha := \#\{(i, j) \in \{1, \dots, r\}^2 \mid i + j = l, (\lambda, \mu) \in Z_{i,j}^{k,d}\}.$$

Then we have

$$\dim \operatorname{Hom}_{\widehat{G}}(\mathcal{O}_\lambda(D) \widehat{\otimes} \mathcal{O}_\mu(D), \mathcal{O}_{\lambda+\mu+2k}(D)) \geq \alpha + 1.$$

Proof. For $1 \leq i, j \leq r$, let $Z'_{i,j} := Z_{r-i+1,j}^{k,d}$, so that we have

$$\bigcup_{\substack{1 \leq i, j \leq r \\ i+j \geq r+1}} Z_{i,j}^{k,d} = \bigcup_{1 \leq i \leq j \leq r} Z'_{i,j}.$$

(1) For $\mathbf{m} \in \mathbb{Z}_{++}^r$, $m_1 \leq k$, the coefficient of $\det_{n^+}(z)^k \Phi_{\mathbf{m}}^{n^+}(P(z^{-1/2})w)$ in $RC_{\lambda, \mu, k}^{n^+}(z, w)$ is given by

$$F_{\mathbf{m}}(\lambda, \mu) := (-1)^{kr} (-k)_{\mathbf{m},d} \left(-\lambda - k + \frac{n}{r}\right)_{\mathbf{m},d} \left(-\mu - k + \frac{n}{r}\right)_{\underline{k}, -\mathbf{m}^\vee, d} \frac{d_{\mathbf{m}}^{p^+}}{\left(\frac{n}{r}\right)_{\mathbf{m},d}}$$

$$= \frac{(-k)_{\mathbf{m},d} d_{\mathbf{m}}^{p^+}}{(-1)^{kr} \left(\frac{n}{r}\right)_{\mathbf{m},d}} \prod_{i=1}^r \left(-\lambda - k + 1 + \frac{d}{2}(r-i)\right)_{m_i} \prod_{j=1}^r \left(-\mu - k + 1 + \frac{d}{2}(j-1)\right)_{k-m_j}.$$

We fix $(\lambda_0, \mu_0) \in \mathbb{C}^2$. For $1 \leq i, j \leq r$, let $\frac{d}{2}(r-i) + 1 - \lambda_0 =: a_i^{(1)}$, $\frac{d}{2}(j-1) + 1 - \mu_0 =: a_j^{(2)}$. If $a_i^{(1)} \in \{1, \dots, k\}$, then

$$\left(-\lambda_0 - k + 1 + \frac{d}{2}(r-i)\right)_{m_i} = (-k + a_i^{(1)})_{m_i} = 0 \quad \text{holds if} \quad k - a_i^{(1)} + 1 \leq m_i \leq k.$$

Similarly, if $a_j^{(2)} \in \{1, \dots, k\}$, then

$$\left(-\mu_0 - k + 1 + \frac{d}{2}(j-1)\right)_{k-m_j} = (-k + a_j^{(2)})_{k-m_j} = 0 \quad \text{holds if} \quad 0 \leq m_j < a_j^{(2)}.$$

If $(\lambda_0, \mu_0) \in Z'_{i,j}$ holds for some $1 \leq i \leq j \leq r$, then we have $k - a_i^{(1)} + 1 \leq a_j^{(2)}$, and at least one of the two above formulas vanishes for $k \geq m_i \geq m_j \geq 0$. Hence $F_{\mathbf{m}}(\lambda_0, \mu_0)$ vanishes for all \mathbf{m} , and $RC_{\lambda, \mu, k}^{n^+} = 0$ holds.

(2) Again we fix (λ_0, μ_0) , define $a_i^{(1)}, a_j^{(2)}$ as above, and let

$$\{i \in \{1, \dots, r\} \mid a_i^{(1)} \in \{1, \dots, k\}\} =: \{i(1), \dots, i(r')\}, \quad i(1) < \dots < i(r'),$$

$$\{j \in \{1, \dots, r\} \mid a_j^{(2)} \in \{1, \dots, k\}\} =: \{j(1), \dots, j(r'')\}, \quad j(1) < \dots < j(r'').$$

Then for $0 \leq u \leq r'$, $0 \leq v \leq r''$, $F_{\mathbf{m}}(\lambda, \mu)$ has a zero of order u at $\lambda = \lambda_0$ and of order $r'' - v$ at $\mu = \mu_0$ if

$$\mathbf{m} \in W(u, v) := \left\{ \mathbf{m} \in \mathbb{Z}_{++}^r \mid \begin{array}{l} 0 \leq m_j \leq k, \quad j = 1, \dots, r, \\ m_{i(u)} \geq k - a_{i(u)}^{(1)} + 1, \quad m_{i(u+1)} \leq k - a_{i(u+1)}^{(1)}, \\ m_{j(v)} \geq a_{j(v)}^{(2)}, \quad m_{j(v+1)} \leq a_{j(v+1)}^{(2)} - 1 \end{array} \right\},$$

where we ignore the conditions for $m_{i(0)}$, $m_{i(r'+1)}$, $m_{j(0)}$, $m_{j(r''+1)}$. For $\mathbf{m} \in W(u, v)$, let

$$\hat{F}_{\mathbf{m}}(\lambda, \mu) := (\lambda - \lambda_0)^{-u} (\mu - \mu_0)^{-r''+v} F_{\mathbf{m}}(\lambda, \mu),$$

so that $\hat{F}_{\mathbf{m}}(\lambda_0, \mu_0) \neq 0$ holds. Then we have

$$\begin{aligned} RC_{\lambda, \mu, k}^{\mathbf{n}^+}(z, w) &= \sum_{\substack{\mathbf{m} \in \mathbb{Z}_{++}^r \\ m_1 \leq k}} F_{\mathbf{m}}(\lambda, \mu) \det_{\mathbf{n}^+}(z)^k \Phi_{\mathbf{m}}^{\mathbf{n}^+}(P(z^{-1/2})w) \\ &= \sum_{\substack{0 \leq u \leq r' \\ 0 \leq v \leq r''}} \sum_{\mathbf{m} \in W(u, v)} (\lambda - \lambda_0)^u (\mu - \mu_0)^{r''-v} \hat{F}_{\mathbf{m}}(\lambda, \mu) \det_{\mathbf{n}^+}(z)^k \Phi_{\mathbf{m}}^{\mathbf{n}^+}(P(z^{-1/2})w). \end{aligned}$$

Now let

$$l' := \max \left\{ u - v \mid \begin{array}{l} 1 \leq u \leq r', 1 \leq v \leq r'', \\ i(u) \leq j(v), a_{i(u)}^{(1)} + a_{j(v)}^{(2)} \geq k + 1 \end{array} \right\}.$$

If $u - v \leq l' - 1$, then since $i(u + 1) \leq j(v)$ and $k - a_{i(u+1)}^{(1)} + 1 \leq a_{j(v)}^{(2)}$ we have $W(u, v) = \emptyset$. Then for $(s, t) \in \mathbb{C}^2$, we can take the limit

$$\begin{aligned} &\lim_{\nu \rightarrow 0} \frac{1}{\nu^{l'+r''}} RC_{\lambda_0 + \nu s, \mu_0 + \nu t, k}^{\mathbf{n}^+}(z, w) \\ &= \lim_{\nu \rightarrow 0} \sum_{u-v \geq l'} \sum_{\mathbf{m} \in W(u, v)} \frac{(\nu s)^u (\nu t)^{r''-v}}{\nu^{l'+r''}} \hat{F}_{\mathbf{m}}(\lambda_0 + \nu s, \mu_0 + \nu t) \det_{\mathbf{n}^+}(z)^k \Phi_{\mathbf{m}}^{\mathbf{n}^+}(P(z^{-1/2})w) \\ &= \sum_{u-v=l'} s^u t^{r''-v} \sum_{\mathbf{m} \in W(u, v)} \hat{F}_{\mathbf{m}}(\lambda_0, \mu_0) \det_{\mathbf{n}^+}(z)^k \Phi_{\mathbf{m}}^{\mathbf{n}^+}(P(z^{-1/2})w). \end{aligned}$$

By varying (s, t) , we get $\alpha + 1$ linearly independent polynomials. By substituting $(z, w) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right)$ and restricting to $y = x$, these polynomials give symmetry breaking operators in $\text{Hom}_{\tilde{G}}(\mathcal{O}_{\lambda}(D) \hat{\otimes} \mathcal{O}_{\mu}(D), \mathcal{O}_{\lambda+\mu+2k}(D))$, and hence this space is at least $(\alpha + 1)$ -dimensional. ■

Example 8.14.

(1) Suppose $d \in 2\mathbb{Z}$, $k \geq \frac{n}{r} = \frac{d}{2}(r-1) + 1$. If

$$(\lambda, \mu) \in \bigcap_{j=1}^r Z_{r-j+1, j}^{k, d} = \left\{ (\lambda, \mu) \in \mathbb{Z}^2 \mid \begin{array}{l} \frac{n}{r} - k \leq \lambda, \mu \leq 0, \\ \lambda + \mu \leq \frac{n}{r} - k \end{array} \right\},$$

then we have

$$\dim \text{Hom}_{\tilde{G}}(\mathcal{O}_{\lambda}(D) \hat{\otimes} \mathcal{O}_{\mu}(D), \mathcal{O}_{\lambda+\mu+2k}(D)) \geq r + 1.$$

(2) Suppose $d = 1$, i.e., $\mathfrak{p}^+ = \text{Sym}(r, \mathbb{C})$, and let $\delta \in \{0, 1\}$. For $k \geq \lfloor \frac{r+\delta}{2} \rfloor$, if

$$\begin{aligned} (\lambda, \mu) &\in \bigcap_{\substack{1 \leq j \leq r \\ j \in 2\mathbb{Z} + \delta}} Z_{r-j+1, j}^{k, 1} \\ &= \left\{ (\lambda, \mu) \in \left(\mathbb{Z} + \frac{r+\delta}{2} \right) \times \left(\mathbb{Z} + \frac{1-\delta}{2} \right) \mid \begin{array}{l} \frac{1}{2}r - k \leq \lambda, \mu \leq \frac{1}{2}, \\ \lambda + \mu \leq \frac{1}{2}(r+1) - k \end{array} \right\}, \end{aligned}$$

then we have

$$\dim \operatorname{Hom}_{\tilde{G}}(\mathcal{O}_\lambda(D) \hat{\otimes} \mathcal{O}_\mu(D), \mathcal{O}_{\lambda+\mu+2k}(D)) \geq \left\lfloor \frac{r+\delta}{2} \right\rfloor + 1.$$

Similarly, for $k \geq \lfloor \frac{r-\delta}{2} \rfloor$, if

$$\begin{aligned} (\lambda, \mu) &\in \bigcap_{\substack{2 \leq j \leq r \\ j \in 2\mathbb{Z} + \delta}} Z_{r-j+2, j}^{k, 1} \\ &= \left\{ (\lambda, \mu) \in \left(\mathbb{Z} + \frac{r+1+\delta}{2} \right) \times \left(\mathbb{Z} + \frac{1-\delta}{2} \right) \mid \begin{array}{l} \frac{1}{2}r - k \leq \lambda, \mu \leq 1, \\ \lambda + \mu \leq \frac{1}{2}r + 1 - k \end{array} \right\}, \end{aligned}$$

then we have

$$\dim \operatorname{Hom}_{\tilde{G}}(\mathcal{O}_\lambda(D) \hat{\otimes} \mathcal{O}_\mu(D), \mathcal{O}_{\lambda+\mu+2k}(D)) \geq \left\lfloor \frac{r-\delta}{2} \right\rfloor + 1.$$

(3) Suppose $d = 2$, i.e., $\mathfrak{p}^+ = M(r, \mathbb{C})$. For $\alpha = 1, 2, \dots, \min\{k, r\}$, if

$$(\lambda, \mu) \in \left\{ (\lambda, \mu) \in \mathbb{Z}^2 \mid \begin{array}{l} \alpha - k \leq \lambda, \mu \leq r - \alpha, \\ r + \alpha - 2k \leq \lambda + \mu \leq 2r - \alpha - k \end{array} \right\},$$

then we have

$$\dim \operatorname{Hom}_{\tilde{G}}(\mathcal{O}_\lambda(D) \hat{\otimes} \mathcal{O}_\mu(D), \mathcal{O}_{\lambda+\mu+2k}(D)) \geq \alpha + 1.$$

When $\mathfrak{p}^+ = \mathbb{C}$, it is proved in [33, Theorem 9.1] by the F-method that $\operatorname{Hom}_{\tilde{G}}(\mathcal{O}_\lambda(D) \hat{\otimes} \mathcal{O}_\mu(D), \mathcal{O}_{\lambda+\mu+2k}(D))$ is precisely 2-dimensional if $(\lambda, \mu) \in Z_{1,1}^{k,d}$, and precisely 1-dimensional otherwise. For general \mathfrak{p}^+ , we need further study to determine the precise dimension of this space. Also, for $(\lambda, \mu) \in Z_{r,r}^{k,d} \cup Z_{r,1}^{k,d} \cup Z_{1,r}^{k,d}$, we can consider an analogue of [33, Theorem 9.2] as follows. We note that the result on linear independence for $\frac{n}{r} \in \mathbb{Z}_{\geq 2}$ case differs from $\mathfrak{p}^+ = \mathbb{C}$ case.

Theorem 8.15.

(1) If $\frac{n}{r} - \lambda_0 =: a_1 \in \{1, 2, \dots, k\}$, then we have

$$\lim_{\mu \rightarrow \mu_0} \frac{\mathcal{RC}_{\frac{n}{r}-a_1, \mu, k}^\downarrow}{(\mu + k - a_1)_{\underline{a_1}, d}} = \mathcal{RC}_{\frac{n}{r}+a_1, \mu_0, k-a_1}^\downarrow \circ \left(\det_{\mathfrak{n}^+} \left(\frac{\partial}{\partial x} \right)^{a_1} \otimes 1 \right). \quad (8.5)$$

(2) If $\frac{n}{r} - \mu_0 =: a_2 \in \{1, 2, \dots, k\}$, then we have

$$\lim_{\lambda \rightarrow \lambda_0} \frac{\mathcal{RC}_{\lambda, \frac{n}{r}-a_2, k}^\downarrow}{(\lambda + k - a_2)_{\underline{a_2}, d}} = \mathcal{RC}_{\lambda_0, \frac{n}{r}+a_2, k-a_2}^\downarrow \circ \left(1 \otimes \det_{\mathfrak{n}^+} \left(-\frac{\partial}{\partial y} \right)^{a_2} \right). \quad (8.6)$$

(3) If $\lambda_0 + \mu_0 - \frac{n}{r} + 2k =: a_3 \in \{1, 2, \dots, k\}$, then we have

$$\lim_{\nu \rightarrow 0} \frac{\mathcal{RC}_{\lambda_0-\nu, \mu_0+\nu, k}^\downarrow}{(\mu_0 + \nu + k - a_3)_{\underline{a_3}, d}} = \det_{\mathfrak{n}^+} \left(\frac{\partial}{\partial x} \right)^{a_3} \circ \mathcal{RC}_{\lambda_0, \mu_0, k-a_3}^\downarrow. \quad (8.7)$$

(4) If $(\lambda_0, \mu_0) \in Z_{r,r}^{k,d}$, then (8.5) and (8.6) are linearly independent.

(5) If $(\lambda_0, \mu_0) \in Z_{r,1}^{k,d}$, then (8.5) and (8.7) are linearly independent.

- (6) If $(\lambda_0, \mu_0) \in Z_{1,r}^{k,d}$, then (8.6) and (8.7) are linearly independent.
- (7) Suppose $\frac{n}{r} \in \mathbb{Z}_{\geq 2}$. If $(\lambda_0, \mu_0) \in Z_{r,r}^{k,d} \cap (Z_{r,1}^{k,d} \cup Z_{1,r}^{k,d})$, then (8.5), (8.6) and (8.7) are linearly independent.

Proof. (1), (2), (3) follow from Theorem 8.11. (4) follows from Corollary 8.9.

For (5), we have

$$\begin{aligned} & RC_{\lambda,\mu,k}^{\mathbf{n}^+}(z, w) \\ &= \det_{\mathbf{n}^+}(z)^k \sum_{\substack{\mathbf{m} \in \mathbb{Z}_{++}^r \\ m_1 \leq k}} (-k)_{\mathbf{m},d} \binom{-\lambda - k + \frac{n}{r}}{\mathbf{m},d} (\mu + \mathbf{m})_{\underline{k}_r - \mathbf{m},d} \frac{d_{\mathbf{m}}^{\mathbf{p}^+}}{\binom{\frac{n}{r}}{\mathbf{m},d}} \Phi_{\mathbf{m}}^{\mathbf{n}^+}(-P(z^{-1/2})w). \end{aligned}$$

Put $\frac{n}{r} - \lambda_0 =: a_1 \in \{1, \dots, k\}$, $1 - \mu_0 =: a'_2 \in \{1, \dots, k\}$, so that $a_3 = 2k - a_1 - a'_2 + 1 \in \{1, \dots, k\}$. Then we have

$$\begin{aligned} & \det_{\mathbf{n}^+}(z)^{-k} \lim_{\mu \rightarrow 1 - a'_2} \frac{RC_{\frac{n}{r} - a_1, \mu, k}^{\mathbf{n}^+}(z, w)}{(\mu + k - a_1)_{\underline{a}_1, d}} \\ &= \lim_{\mu \rightarrow 1 - a'_2} \sum_{\substack{\mathbf{m} \in \mathbb{Z}_{++}^r \\ m_1 \leq k - a_1}} (-k)_{\mathbf{m},d} (a_1 - k)_{\mathbf{m},d} (\mu + \mathbf{m})_{\underline{k} - a_1 - \mathbf{m},d} \frac{d_{\mathbf{m}}^{\mathbf{p}^+}}{\binom{\frac{n}{r}}{\mathbf{m},d}} \Phi_{\mathbf{m}}^{\mathbf{n}^+}(-P(z^{-1/2})w) \\ &= \sum_{\substack{\mathbf{m} \in \mathbb{Z}_{++}^r \\ m_1 \leq k - a_1}} (-k)_{\mathbf{m},d} (a_1 - k)_{\mathbf{m},d} (1 - a'_2 + \mathbf{m})_{\underline{k} - a_1 - \mathbf{m},d} \frac{d_{\mathbf{m}}^{\mathbf{p}^+}}{\binom{\frac{n}{r}}{\mathbf{m},d}} \Phi_{\mathbf{m}}^{\mathbf{n}^+}(-P(z^{-1/2})w), \\ & \det_{\mathbf{n}^+}(z)^{-k} \lim_{\nu \rightarrow 0} \frac{RC_{\frac{n}{r} - a_1 - \nu, 1 - a'_2 + \nu, k}^{\mathbf{n}^+}(z, w)}{(\nu - k + a_1)_{2k - a_1 - a'_2 + 1, d}} \\ &= \lim_{\nu \rightarrow 0} \sum_{\substack{\mathbf{m} \in \mathbb{Z}_{++}^r \\ m_1 \leq k}} (-k)_{\mathbf{m},d} \frac{(\nu + a_1 - k)_{\mathbf{m},d} (\nu + 1 - a'_2 + \mathbf{m})_{\underline{k}_r - \mathbf{m},d}}{(\nu - k + a_1)_{2k - a_1 - a'_2 + 1, d}} \frac{d_{\mathbf{m}}^{\mathbf{p}^+}}{\binom{\frac{n}{r}}{\mathbf{m},d}} \Phi_{\mathbf{m}}^{\mathbf{n}^+}(-P(z^{-1/2})w) \\ &= \lim_{\nu \rightarrow 0} \sum_{\substack{\mathbf{m} \in \mathbb{Z}_{++}^r \\ m_1 \leq k}} (-k)_{\mathbf{m},d} (\nu + 1 - a'_2 + \mathbf{m})_{\underline{a_1 + a'_2 - k - 1}_r, d} \frac{d_{\mathbf{m}}^{\mathbf{p}^+}}{\binom{\frac{n}{r}}{\mathbf{m},d}} \Phi_{\mathbf{m}}^{\mathbf{n}^+}(-P(z^{-1/2})w) \\ &= \sum_{\substack{\mathbf{m} \in \mathbb{Z}_{++}^r \\ m_1 \leq k}} (-k)_{\mathbf{m},d} (1 - a'_2 + \mathbf{m})_{\underline{a_1 + a'_2 - k - 1}_r, d} \frac{d_{\mathbf{m}}^{\mathbf{p}^+}}{\binom{\frac{n}{r}}{\mathbf{m},d}} \Phi_{\mathbf{m}}^{\mathbf{n}^+}(-P(z^{-1/2})w). \end{aligned}$$

Then the coefficient for $\mathbf{m} = (k, \underline{0}_{r-1})$ is zero for the former formula, and is non-zero for the latter one, and hence these are linearly independent.

(6) Proved similarly by comparing the coefficients for $\mathbf{m} = (\underline{k}_{r-1}, 0)$.

(7) By the assumption $\frac{n}{r} \geq 2$, we have $r \geq 2$, and thus \underline{k}_r , $(k, \underline{0}_{r-1})$, $\underline{0}_r$ (for $(\lambda_0, \mu_0) \in Z_{r,r}^{k,d} \cap Z_{r,1}^{k,d}$ case) or \underline{k}_r , $(\underline{k}_{r-1}, 0)$, $\underline{0}_r$ (for $(\lambda_0, \mu_0) \in Z_{r,r}^{k,d} \cap Z_{1,r}^{k,d}$ case) are distinct. Hence, comparing the coefficients for these \mathbf{m} , we get the linear independence of three formulas. \blacksquare

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